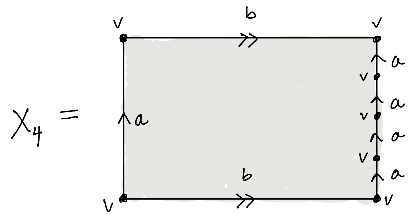
Preliminary Examination: Algebraic topology. August 20, 2024

**Instructions:** Answer all three questions. Question 1 is worth 4 points. Questions 2 is worth 2 points. Question 3 is worth 2 points.

**Time limit:** 2 hours.

**1.** For each $d \in \mathbb{Z}$, consider the map $f_d : S^1 \to S^1$ that wraps the circle around itself $d$ times: $f_d(e^{i\theta}) = e^{di\theta}$. Let $X_d = S^1 \times I/\sim$ be the mapping torus of $f_d$, defined by the relation $(x, 0) \sim (f_d(x), 1)$. The figure suggests a cell structure on $X_d$.

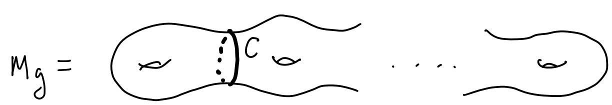



**(a)**: For each $d \neq 1$, compute the homology groups $H_*(X_d)$.

**(b)**: For which values of $d$ does $X_d$ retract onto the circle $S^1 \times \{1/2\}$?

**(c)**: For which values of $d$ is $X_d$ homotopy equivalent to a surface?

**(d):** Now let the space $Y = Y_{d,g}$ be obtained from $X_d$ and a closed orientable surface $M_g$ of genus $g \geq 2$ by identifying $S^1 \times \{1/2\} \subset X_d$ with the curve $C \subset M_g$ shown in the figure. Use a Mayer-Vietoris sequence to compute the homology groups $H_*(Y)$.




**2.** Let $X = S^1 \vee S^1$ be the wedge of two circles, and let $x \in X$ be the point where the two circles meet (the "wedge" point). Then $\pi_1(X, x) \cong F_2 = \langle a, b \rangle$ is the free group on two generators.

**(a)**: Describe all path-connected covering spaces of $X$ of degree two. [*Note:* it is acceptable to describe a covering space by drawing a well-labeled picture.]

**(b)**: Using your answer to (a), list all subgroups of index two in $F_2$ and give a geometric proof that every such subgroup is normal. [*Note:* It is a general fact that any index two subgroup in an arbitrary group $G$ is normal, but do not use this general fact. The point is to prove this fact using covering space theory in the case $G = F_2$.]

**3.** Let $G$ be a group.

**(a)**: Briefly explain the general construction of a basepointed 2-dimensional cell complex $(X_G, x_0)$, so that $\pi_1(X, x_0) = G$. Demonstrate the construction in the case $G = \langle a, b, c : a^2 b, c^3 \rangle$.

**(b)**: Let $(Y, y_0)$ be a path connected topological space with basepoint. Let $h : G \to \pi_1(Y, y_0)$ be a group homomorphism. Briefly describe a general construction of a continuous map $f : (X_G, x_0) \to (Y, y_0)$ such that $f_* = h$, where here $f_*$ denotes the induced map on fundamental group.