ALGEBRA PRELIMINARY EXAM: PART II

Problem 1

Let p be a prime, $\mathbb{F}_p$ be the field with p elements, and F a finite field of characteristic p.

a) Prove that F has p^n elements for some positive integer n.
b) Prove that $\prod_{\alpha \in F^\times} \alpha = -1$ where $F^\times = F \setminus \{0\}$.
c) For any $a \in \mathbb{F}_p$ we consider $f_a(x) := x^p - x + a \in \mathbb{F}_p[x]$. Prove that f_a is separable and irreducible over $\mathbb{F}_p$ for every $a \in \mathbb{F}_p \setminus 0$.

Problem 2

Consider the polynomial $f(x) = x^7 - 3 \in \mathbb{Q}[x]$. Let $E/\mathbb{Q}$ denote the splitting field of $f(x)$.

a) Determine the Galois group of $E/\mathbb{Q}$ as an abstract group and by describing its elements in terms of their action on the generators of E.
b) Determine a primitive generator for $E/\mathbb{Q}$.
c) Identify all subfields $E/\mathbb{Q}$ which are Galois over $\mathbb{Q}$.

Problem 3

Consider polynomial $f(x) = x^4 - 7x + 7 \in \mathbb{Q}[x]$.

a) Determine the Galois group of $f(x)$ over $\mathbb{Q}$.
b) Does there exist $n \in \mathbb{N}$ such that $f(x)$ has a root in $\mathbb{Q}(\zeta_n)$?

You may use without proof the fact that

$$\mathrm{Disc}(f) = (-1)^{(4\cdot 3/2)} 4^4 7^4 + (-1)^{(3\cdot 2/2)} 3^3 (-7)^4 = 7^3(4^4 - 3^3 \cdot 7) = 7^3 \cdot 67.$$

Date: August 21, 2024.