

Representation Theory (Fall 2004)

Lecture 2

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Representations of Abelian Groups

Let G be abelian. How many 1-dimensional representations does it have? We are looking for all homomorphisms $\rho : G \rightarrow \mathbb{C}^\times$. By the Fundamental Theorem of Abelian Groups, G decomposes as direct sum of cyclic groups

$$G = \mathbb{Z}/m_1\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/m_N\mathbb{Z}.$$

Denote the generator of $\mathbb{Z}/m_i\mathbb{Z}$ by x_i . This yields a representation

$$\rho = \rho_1 \oplus \dots \oplus \rho_N$$

where each $\rho_i : \mathbb{Z}/m_i\mathbb{Z} \rightarrow \mathbb{C}^\times$ is defined by sending the generator x_i to some m_i -th root of unity ζ_i (so $\zeta_i^{m_i} = 1$). Observe that there are m_i choices for each ρ_i , and so there are $\prod_{i=1}^N m_i = |G|$ one-dimensional representations of G .

More on Irreducible Representations of S_3

In the previous lecture we found three irreducible representations of S_3 :

		dimension
U	Trivial	1
U'	Sign	1
W	Standard	2

We will now prove that these are the only irreducible representations of S_3 .

Theorem. *The only irreducible representations of S_3 are the trivial, sign, and standard representations.*

Proof. Recall that S_3 is generated by the two elements $\sigma = (1, 2)$ and $\tau = (1, 2, 3)$, which satisfy the relation

$$\sigma\tau^{-1} = \tau\sigma.$$

Let $H = \langle \tau \rangle$ be the cyclic group of order 3 generated by τ , and let V be an arbitrary representation of S_3 . Then, V is certainly a representation of H (i.e., the representation of H over V is the restriction of the representation $\rho : S_3 \rightarrow GL(V)$ to H).

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Let V_λ be the subspace of V corresponding to the eigenvalue λ of τ , i.e.,

$$V_\lambda = \{v \in V : \tau v = \lambda v\}.$$

Since H is cyclic of order 3, as discussed above, the eigenvalues of τ are the third roots of unity, namely, 1, ζ , and ζ^2 (where $\zeta = \zeta_3$ is a primitive third root of unity). So we may write

$$V = V_1 \oplus V_\zeta \oplus V_{\zeta^2}.$$

How does σ act on each of the pieces of V ? Let $v \in V_\lambda$ with $\lambda = 1, \zeta$, or ζ^2 . Then, $\tau v = \lambda v$, and

$$\tau(\sigma v) = \sigma(\tau^{-1}v) = \sigma(\lambda^{-1}v) = \lambda^{-1}(\sigma v),$$

that is, $\sigma v \in V_{\lambda^{-1}}$. So, σ maps V_1 onto V_1 , V_ζ onto V_{ζ^2} (since $\zeta^{-1} = \zeta^2$) and V_{ζ^2} onto V_ζ . In particular, if $\{v_1, \dots, v_m\}$ be a basis for V_ζ , then $\{\sigma v_1, \dots, \sigma v_m\}$ is a basis for V_{ζ^2} . We further have:

$$\tau v_i = \zeta v_i \quad \text{and} \quad \tau(\sigma v_i) = \zeta^2(\sigma v_i).$$

Thus, for each i , the representation generated by v_i and σv_i is isomorphic to the standard representation W :

$$\langle v_i, \sigma v_i \rangle \simeq W,$$

and so

$$V_\zeta \oplus V_{\zeta^2} \simeq W^m,$$

for some integer m .

Now, consider again V_1 ; it is fixed by both σ and τ . The element σ is of order 2, so V_1 decomposes as

$$V_1 = V^+ \oplus V^-,$$

with

$$V^+ = \{v \in V : \sigma v = v\} \quad \text{and} \quad V^- = \{v \in V : \sigma v = -v\},$$

and τ still acting trivially on both pieces (i.e., for all $v \in V, \tau v = v$). But this implies that V^+ is just some number, say k , copies of the trivial representation U , and V^- is some number, say l , copies of the sign representation U' .

Evidently, we have just shown that

$$V \simeq V_1 \oplus V_\zeta \oplus V_{\zeta^2} \simeq U^k \oplus (U')^l \oplus W^m.$$

Since U, U' , and W are irreducible, there can be no further decomposition, thus showing that the only irreducible representations of S_3 are the trivial one U , the sign one U' , and the standard one W . □

Characters

Definition. Let $\rho : G \rightarrow GL(V)$ be a representation. The *character* associated to ρ (or V) is map $\chi_\rho : G \rightarrow \mathbb{C}$ defined by

$$\chi_\rho(g) = \chi_V(g) = \text{Tr}(g)$$

where $\text{Tr}(g)$ is the trace of $\rho(g)$ as an element of $GL(V)$.

It is easy to check that χ_ρ is well-defined and depends only on the isomorphism class of ρ . Notice in particular that if ρ is one-dimensional, then $\chi_\rho(g) = \rho(g)$.

Further, the character is constant on conjugacy classes of G , for

$$\chi_\rho(g_1 g g_1^{-1}) = \text{Tr}(g_1 g g_1^{-1}) = \text{Tr}(g).$$

Definition. A function that is constant on conjugacy classes of G is called a *class function*.

Definition. A *character table* for a group G is the list of (values of) the characters of all the irreducible representations of G .

We can easily compute the character table for S_3 . S_3 has three conjugacy classes: one corresponding to the identity, one corresponding to the two-cycles, and one corresponding to the three-cycles. For the 1-dimensional representations, the characters are just the values of the representation. For the standard representation, we recall that it maps $\sigma = (1, 2)$ and $\tau = (1, 2, 3)$ to the matrices

$$\begin{bmatrix} 0 & \zeta \\ \zeta^2 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^2 \end{bmatrix},$$

respectively. Thus, the character table is

	1	(1,2)	(1,2,3)
U	1	1	1
U'	1	-1	1
W	2	0	-1

Character Table of S_3

First, notice that the value of the character at the identity is always the degree of the representation, or, equivalently, the dimension of the vector space. Further, observe that the table is apparently square, implying that the number of irreducible representations is equal to the number of conjugacy classes. Moreover, the "matrix" obtained from the table has non-zero determinant, implying that the rows are linearly independent. This means that *all* class functions are linear combinations of the characters, and that every representation is completely determined by its characters. All the necessary information is contained in the character table.

Suppose G acts on some set X , and ρ is a permutation representation. Then $\text{Tr}(\rho(g))$ is the number of points of X fixed by g . To see this, notice that 1 appears on the diagonal of the matrix corresponding to g if and only if g fixes that coordinate, i.e., 1 appears in position k if and only if $g e_k = e_{gk} = e_k$. Thus, in a permutation representation, the character entries are all nonnegative integers.

For example, let S_3 act on $\{1, 2, 3\}$. Then, the permutation representation has character χ :

$$\begin{array}{c|c|c|c} & 1 & (1,2) & (1,2,3) \\ \hline \chi & 3 & 1 & 0 \end{array}$$

Using the table for irreducible representation of S_3 above, it's easy to check that $\chi = \chi_U + \chi_W$ and so $V = U \oplus W$.

Tensor Products

Two representations V and W of a group G define a representation $V \otimes W$ defined by

$$g(v \otimes w) = g(v) \otimes g(w).$$

Using the properties of the tensor product,

$$\chi_{V \otimes W} = \chi_V \chi_W.$$

To see this, if $\rho_V(g) = [a_{ij}]$ is the matrix for g under V and $\rho_W(g) = [b_{ij}]$ is the matrix for g under W , then the matrix for g in $V \otimes W$ is

$$\left[\begin{array}{c|c|c|c} a_{11}[b_{ij}] & a_{12}[b_{ij}] & \dots & a_{1n}[b_{ij}] \\ \hline a_{21}[b_{ij}] & a_{22}[b_{ij}] & \dots & a_{2n}[b_{ij}] \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline a_{n1}[b_{ij}] & a_{n2}[b_{ij}] & \dots & a_{nn}[b_{ij}] \end{array} \right],$$

and its trace is precisely

$$a_{11}\text{Tr}([b_{ij}]) + \dots + a_{nn}\text{Tr}([b_{ij}]) = \text{Tr}([a_{ij}])\text{Tr}([b_{ij}]) = \text{Tr}(\rho_V(g))\text{Tr}(\rho_W(g)).$$

We can work out $W \otimes W$ for S_3 . Since W is 2-dimensional, $W \otimes W$ is 4-dimensional. Using the character table for S_3 and the fact that $\chi_{W \otimes W} = \chi_W^2$, we find

$$\begin{array}{c|c|c|c} & 1 & (1,2) & (1,2,3) \\ \hline \chi & 4 & 0 & 1 \end{array}$$

To illustrate how the tensor product works, we can construct the representation $W \otimes W$ explicitly. Recall that σ and τ correspond to

$$\begin{bmatrix} 0 & \zeta \\ \zeta^2 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^2 \end{bmatrix},$$

respectively, in $GL(W)$. Consequently, in $GL(W \otimes W)$, σ and τ correspond to

$$\left[\begin{array}{c} 0 \begin{bmatrix} 0 & \zeta \\ \zeta^2 & 0 \end{bmatrix} \\ \zeta^2 \begin{bmatrix} 0 & \zeta \\ \zeta^2 & 0 \end{bmatrix} \end{array} \quad \zeta \begin{bmatrix} 0 & \zeta \\ \zeta^2 & 0 \end{bmatrix} \quad 0 \begin{bmatrix} 0 & \zeta \\ \zeta^2 & 0 \end{bmatrix} \right] = \begin{bmatrix} 0 & 0 & 0 & \zeta^2 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \zeta & 0 & 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} \zeta \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^2 \end{bmatrix} & 0 \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^2 \end{bmatrix} \\ 0 \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^2 \end{bmatrix} & \zeta^2 \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \zeta^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \zeta \end{bmatrix},$$

respectively. Taking the traces of these matrices we obtain the character above.

We can decompose $W \otimes W$ into irreducible representations by noticing that $\chi_{W \otimes W} = \chi_U + \chi_{U'} + \chi_W$, and so $W \otimes W = U \oplus U' \oplus W$.

Dual Representation

Given a representation ρ of a group G over V we want to define a representation on the dual space V^* . (The dual space V^* is the space of all linear maps $V \rightarrow \mathbb{C}$.) Define the pairing

$$\langle , \rangle : V^* \times V \rightarrow \mathbb{C}$$

by $\langle v^*, v \rangle = v^*(v)$. It is clear that this is a bilinear map. We would like to make it G -stable, i.e., one that satisfies

$$\langle \rho^*(g)v^*, \rho(g)v \rangle = \langle v^*, v \rangle$$

for all $g \in G$, $v \in V$, and $v^* \in V^*$. This defines the action of G on V^* uniquely, as follows.

For any map $\phi : V \rightarrow V$, there exists the transpose map ${}^t\phi : V^* \rightarrow V^*$ (with ${}^t({}^t\phi) = \phi$). Then, the condition above implies that

$$\langle v^*, {}^t(\rho^*(g))\rho(g)v \rangle = \langle v^*, v \rangle$$

for all $v^* \in V^*$. But this means that ${}^t(\rho^*(g))\rho(g)$ is the identity map on V . So we can define

$$\rho^*(g) = ({}^t(\rho(g)))^{-1} = {}^t(\rho(g^{-1})).$$

To verify that the stability condition holds, notice that

$$\langle \rho^*(g)v^*, \rho(g)v \rangle = \langle {}^t(\rho(g^{-1}))v^*, \rho(g)v \rangle = \langle \rho(g^{-1})\rho(g)v \rangle = \langle v^*, v \rangle.$$

Definition. The representation $\rho^*(g)$ defined above is the *dual representation* for ρ .

We can easily find the character of ρ^* . First, notice that since $\text{Tr}(\rho)(g)$ is the sum of the eigenvalues λ_i of $\rho(g)$,

$$\chi_\rho(g) = \sum_i \lambda_i.$$

Now, the eigenvalues of $\rho^*(g) = {}^t(\rho(g^{-1}))$ are $\lambda_i^{-1} = \bar{\lambda}_i$, since λ_i are roots of unity. Therefore,

$$\chi_{\rho^*}(g) = \sum_i \bar{\lambda}_i = \overline{\chi_\rho(g)}.$$

For S_3 all the characters are real, so every representation is isomorphic to its dual.

Complete Reducibility

Now we would like to prove the following theorem that we used in finding the standard representation of S_3 :

Theorem. *Let V be a representation of a group G . If $W \subset V$ is a subrepresentation, then there exists a subrepresentation $W' \subset V$ such that $V = W \oplus W'$.*

Proof. Suppose that we have a G -stable inner product H , i.e., an inner product H satisfying

$$H(gv_1, gv_2) = H(v_1, v_2)$$

for all $g \in G$ and $v_1, v_2 \in V$. Then, put

$$W' = \{w' \in V : H(w', w) = 0 \text{ for all } w \in W\}.$$

Since H is G -stable and $W' \cap W = \emptyset$, we have W' also G -stable, and thus a subrepresentation of V , and $V = W \oplus W'$, verifying the claim in the theorem.

It remains to find the inner product H . Let $\{v_1, \dots, v_n\}$ be a basis for V . Let H_0 be the Hermitian inner product defined by

$$H_0\left(\sum_i a_i v_i, \sum_i b_i v_i\right) = \sum_i \bar{a}_i b_i.$$

Next, put

$$H(v, w) = \frac{1}{|G|} \sum_{g \in G} H_0(gv, gw).$$

Since H_0 is Hermitian, so is H . It is clear from the definition of H that it is also G -stable. □

Note: An alternative way of proving this theorem would be to find a G -stable projection $\pi : V \rightarrow W$. Then, $\ker \pi$ would be a complement of W and a subrepresentation. In general, the idea of averaging over the group to obtain a G -stable object will be very useful to us.