Representation Theory (Fall 2004)
Lecture 14

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Induced Representations

Recall from the previous lecture that if G is a group, $H \leq G$, and V a representation of G where there exists some $W \subseteq V$ an H-stable subset of V, we can write

$$V \cong \bigoplus_{\sigma \in G/H} \sigma W$$

where $\sigma W = g_\sigma W$, where g_σ is any representative of the coset σ. We then call V the *induced representation*, and write $V = \mathrm{Ind}_H^G(W)$.

An Example: $G = S_3$, $H = \langle (123) \rangle$

Let W be the one dimensional representation of H given by

$$(123) \mapsto \zeta_3 = e^{2\pi i/3}.$$

We want to find the induced representation of W in S_3, $\mathrm{Ind}_H^G(W)$. To do this, we must determine how G acts on our representation

$$V := \bigoplus_{\sigma \in G/H} W^\sigma.$$

Think of $W^\sigma = g_\sigma W$, where $g(g_\sigma w) = (g g_\sigma) w = g_\tau (hw)$, $h \in H$, where g_τ is the new coset representative of H and hw represents the action of $h \in H$ on $w \in W$. In our example, let

$$g_\sigma \in \{1, (12)\}.$$

We can then compute g_τ and h for each g, g_σ to obtain the following chart of $g_\tau \cdot h$:

$g_\sigma \rightarrow$ $g \downarrow$	1	(12)
1	$1 \cdot 1$	$(12) \cdot 1$
(12)	$(12) \cdot 1$	$1 \cdot 1$
(13)	$(12) \cdot (132)$	$1 \cdot (123)$
(23)	$(12) \cdot (123)$	$1 \cdot (132)$
(123)	$1 \cdot (123)$	$(12) \cdot (132)$
(132)	$1 \cdot (132)$	$(12) \cdot (123)$

Notes by Pippa Charters; edited by Bill Kalahurka

Recall that G acts on the cosets of G/H by $gg_\sigma H = g_\tau H$. This action would simply be a permutation representation on G if H acts trivially. However, since in this case H does not act trivially, we'll compute the explicit induced representation. For each element $g \in G$, first note whether or not it flips the cosets. This will tell us whether our matrix is diagonal or flipped. Then note the element of H which appears. This will tell us what the actual entries are for our matrix. Thus we end up with

g		Matrix	Trace	g		Matrix	Trace
1	$\mapsto$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	2	(23)	$\mapsto$	$\begin{pmatrix} 0 & \zeta_3^{-1} \\ \zeta_3 & 0 \end{pmatrix}$	0
(12)	$\mapsto$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	0	(123)	$\mapsto$	$\begin{pmatrix} \zeta_3 & 0 \\ 0 & \zeta_3^{-1} \end{pmatrix}$	-1
(13)	$\mapsto$	$\begin{pmatrix} 0 & \zeta_3 \\ \zeta_3^{-1} & 0 \end{pmatrix}$	0	(132)	$\mapsto$	$\begin{pmatrix} \zeta_3^{-1} & 0 \\ 0 & \zeta_3 \end{pmatrix}$	-1

Thus the induced representation V is the standard representation of S_3.

Induced Representation vs. Permutation Representation

We can write the above action on V as a large matrix where each entry is zero except for blocks representing the action of G on each of the W^σ. In other words, it is a matrix for the permutation representation of G on the cosets of H together with the action of each block on the individual W^σ. This looks a lot like the tensor product.

In other words, let μ be the permutation representation of G on G/H. Then

$$\mu \otimes \rho w$$

has the same dimension as the induced representation, but it is not in general the same. The matrix does have the same shape - i.e. all of the blocks are in the same place - but in the tensor product each of the blocks is the same as all the others, while in the induced representation, as seen explicitly in our S_3 example, the contents of each block depends on the element $h \in H$ gotten by $gg_\sigma = g_\tau h$.

More on Induced Representations

Suppose that $V = \bigoplus_i W_i$ such that G permutes the W_i transitively.
Claim: $V = \mathrm{Ind}_{H_i}^G W_i$ where $H_i := \mathrm{Stab}_G(W_i)$. Then W_i is an H_i-representation.

The proof of this fact is omitted.

Note: If V is irreducible then transitivity is automatic, since the orbit of any subspace of V is a G-invariant subspace. Hence

$$\bigoplus_{W_j \in \text{ orbit of } W_i} W_j$$

is a G-stable nonzero subspace of V, so if V is irreducible it must be the whole space. Hence G acts transitively on an irreducible representation.

GAP and the Rubic's Cube as an Induced Representation

GAP is a software package [available on our computers here, in fact] that allows you to type in generators for a group and will then tell you how large the group is, as well as some other properties.

For example, consider the $2 \times 2 \times 2$ Rubic's Cube. By putting into GAP the permutations described by rotating the various faces, we find that the group represented by the various legitimate rotations of this cube has size $|G| = 88179840$.

GAP can also tell us which subsets of the 24 total cubes making up the 6 faces of our cubes are permuted as sets. It turns out to be exactly the eight corners.

Hence our representation on 24 blocks - a 24-dimensional representation - is actually an induced representation of the subspaces generated by the 8 corners, each of which is a 3-dimensional representation.

Another Way to Look at Induced Representations

Recall that formally, we can think of an induced representation V as

$$V = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W.$$

Alternatively, let

$$V = \{f : G \to W \mid f(hx) = hf(x) \text{ for all } x \in G\}.$$

We want G to act on V; how can we define the action? Let's try

$$(gf)(x) = f(g^{-1}x).$$

Then

$$g_1(g_2(f(x)) = g_2(f(g_1^{-1}x)) = f(g_2^{-1}g_1^{-1}x) = f((g_1g_2)^{-1}x) = (g_1g_2)f(x)$$

so this is indeed a left action. But there is one problem - this action is not closed. That is, $f \in V$ does not necessarily imply that $gf \in V$, as

$$gf(hx) = f(g^{-1}hx), \text{ while}$$

$$h(gf)(x) = (gf)(h^{-1}x) \neq (gf)(hx),$$

so gf is not H-linear. So how can we get an H-linear action which also preserves V? It turns out that defining the action of $g \in G$ to be

$$g(f(x)) = f(xg)$$

is a left action which preserves V as desired.

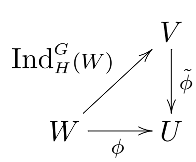
Claim: $V = \text{Ind}_H^G W$. That is, G acts on the space of H-invariant functions.

A Universal Property

Let U be a G-representation, W an H-representation and $\phi : W \to U$ an H-linear map. Then there exists a unique G-linear extension $\tilde{\phi}$,

$$\tilde{\phi} : \text{Ind}_H^G W (= V) \to U.$$

We can picture this as:



More formally,

$$\begin{array}{rcl} \mathrm{Hom}_H(W, \mathrm{Res}_H^G(U)) & = & \mathrm{Hom}_G(\mathrm{Ind}_H^G(W), U) \\ \phi & \leftrightarrow & \tilde{\phi} \end{array}$$

Proof. Let $V = \bigoplus_{\sigma \in G/H} \sigma W$. Then

$$\sigma W \xrightarrow{g_\sigma^{-1}} W \xrightarrow{\phi} U \xrightarrow{g_\sigma} U$$

defines $\tilde{\phi}\,|_{\sigma W}$. Thus

$$\tilde{\phi} := \bigoplus_{\sigma \in G/H} \tilde{\phi}\,|_{\sigma W}$$

where $g_1 := 1 \in G$. □

The Frobenious Reciprocity Law

Letting U and W be as above,

$$\left(\chi_{\mathrm{Ind}_H^G(W)}, \chi_U\right)_G = \left(\chi_W, \chi_{\mathrm{Res}_H^G(U)}\right)_H$$

Proof. [Aside: $\mathrm{Ind}_H^G(\bigoplus_i W_i) = \bigoplus_i \mathrm{Ind}_H^G(W_i)$.]

WLOG, assume that U and W are irreducible [which we can do by the above aside]. Then

$$\dim(\underbrace{\mathrm{Hom}_H(W, \mathrm{Res}_H^G(U))}_{\substack{\text{multiplicity of } W \text{ in } \mathrm{Res}_H^G(U) \\ \text{since } W \text{ irreducible}}}) = \dim(\underbrace{\mathrm{Hom}_G(\mathrm{Ind}_H^G(W), U)}_{\substack{\text{multiplicity of } U \text{ in } \mathrm{Ind}_H^G(W) \\ \text{since } U \text{ irreducible}}})$$

□

In our example, recall that we had $H = \langle(123)\rangle$, $G = S_3$, W was irreducible, and U was the standard representation. Then

$$\left(\chi_{\mathrm{Ind}(W)}, \chi_U\right) = \left(\chi_W, \chi_{\mathrm{Res}(U)}\right) = 1$$

hence we can conclude that U appears once in $\mathrm{Ind}_H^G(W)$. So

$$\mathrm{Res}_H^G(U) = W \oplus \overline{W}$$

Exercise: Show that $\mathrm{Ind}_H^G((\text{Regular rep.})_H) = (\text{Regular rep.})_G$. In fact, we actually have

$$\mathrm{Ind}_H^G(W) = \mathrm{Ind}_K^G(\mathrm{Ind}_H^K(W)), \quad H \le K \le G.$$

The Heisenberg Group

Let

$$H = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{F}_p \right\}$$

Denote an element of this group as (a, b, c). Then from regular matrix multiplication, we get the relationship

$$(a, b, c)(x, y, z) = (a + x, b + y, ay + c + z).$$

We can then compute the commutator of any two elements of H, namely

$$[(a, b, c), (x, y, z)] = (0, 0, ay - bx).$$

Thus conclude that the center of H is

$$Z(H) = \{(0, 0, c) \mid c \in \mathbb{F}_p\}.$$

Moding H out by its center then gives us an Abelian group, namely $\mathbb{F}_p \oplus \mathbb{F}_p$. This gives us the short exact sequence:

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{F}_p & \to & H & \to & \mathbb{F}_p \oplus \mathbb{F}_p & \to & 0 \\ & & c & \mapsto & (0,0,c) & & & & \\ & & & & (a,b,c) & \mapsto & (a,b) & & \end{array}$$

Thus H is a *central extension* of $\mathbb{F}_p \oplus \mathbb{F}_p$ by $\mathbb{F}_p$.

Representations of H

It is possible to describe all representations of H for an arbitrary p. First note that

$$[(a, b, c), (x, y, z)] = \{(0, 0, ay - bx)\} = Z(H).$$

Hence

$$\{\text{characters of } \mathbb{F}_p \oplus \mathbb{F}_p\} \rightsquigarrow \{1 - \text{dimensional representations of } H\}$$

Thus H has p^2 1-dimensional representations. In order to determine the other representations of H, it is first necessary to find all of the conjugacy classes. Certainly each of the elements $(0, 0, c) \in Z(H)$, $c \in \mathbb{F}_p$ has its own conjugacy class. Further,

$$(a, b, c)(x, y, z)(a, b, c)^{-1} = (0, 0, ay - bx)(x, y, z) = (x, y, z + ay - bx).$$

Thus we have conjugacy classes

$$\{(x, y, c) \mid c \in \mathbb{F}_p, (x, y) \neq (0, 0)\}.$$

Thus we have $p + (p^2 - 1) = p^2 + (p - 1)$ conjugacy classes, and hence there are a total of $p^2 + (p - 1)$ irreducible representations. We already know that there are p^2 1-dimensional ones, hence we are missing $p - 1$ additional irreducible representations.