

Representation Theory (Fall 2004) Lecture 1

Fernando Rodriguez-Villegas

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References

The classical references are:

1. Fulton and Harris, *Representation Theory: A First Course*, Chapters 1 and 2
2. Serre, *Linear Representations of Finite Groups*
3. Curtis and Reiner, *Representation Theory of Finite Groups and Associative Algebras*

Definitions

Let G be a group, and let X be a set. We are interested in studying the actions of G on X , i.e., homomorphisms $G \rightarrow \text{Aut}(X)$. We begin with some examples.

1. Let X be any set. Then, $\text{Aut}(X) = S(X)$ is the set of permutations of elements of X , and G acts by permuting the elements of X . If $|X| = n$, then $S(X) \simeq S_n(X)$, the symmetric group on n letters.

Note: Isomorphism is not the same as equality; $\simeq \neq =$. For example, the isomorphism of $S(X)$ and $S_n(X)$ depends on the labeling of the elements of X ; the isomorphism is not canonical.

2. Let $X = V$ be a vector space over a field K . Then, $\text{Aut}(V) = GL(V)$ is the group of K -linear, invertible transformations on V , and G acts by invertible, linear transformations of V .

If the dimension of V over K is n , then $GL(V) \simeq GL_n(K)$, the group of invertible $n \times n$ matrices with entries in K . This can be seen by picking a basis for V ; again, this isomorphism is not canonical and depends on the choice of basis for V .

Convention. Unless otherwise noted, from now on, the field K is the field of complex numbers, \mathbb{C} ; the group G is finite; and the vector space V is finite-dimensional (over \mathbb{C}).

Definition. The map $\rho : G \rightarrow GL(V)$ is a *representation* of G over V if it is a group homomorphism, i.e., if for all $g_1, g_2 \in G$ and all $v \in V$, $\rho(g_2 g_1)(v) = \rho(g_2)(\rho(g_1)(v))$. To ease notation, for a fixed representation ρ , we will write $g \cdot v$ or even gv for $\rho(g)(v)$.

The set

$$\mathbb{C}[G] = \left\{ \sum_{g \in G} a_g g : a_g \in \mathbb{C} \right\},$$

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endowed with the natural operations of addition and multiplication,

$$\sum_{g \in G} a_g g + \sum_{g \in G} b_g g = \sum_{g \in G} (a_g + b_g) g$$

and

$$\left(\sum_{g \in G} a_g g \right) \times \left(\sum_{g \in G} b_g g \right) = \sum_{g \in G} \left(\sum_{g_1 g_2 = g} (a_{g_1} b_{g_2}) g \right),$$

is a ring. It is called the *group algebra*. Clearly $\mathbb{C}[G]$ is commutative if and only if G is commutative. An element of $\mathbb{C}[G]$ acts on V by

$$\left(\sum_{g \in G} a_g g \right) (v) = \sum_{g \in G} a_g g v,$$

thus showing that V is a left $\mathbb{C}[G]$ -module where the action of G is specified by ρ .

On the other hand, if some vector space V is a left $\mathbb{C}[G]$ -module, then by definition of module, the action of G on V corresponds to some representation ρ . Thus, we will often speak of V as a representation.

Definition. The *degree* of a representation $\rho : G \rightarrow GL(V)$ is the dimension of V over K .

Definition. Let ρ_1 and ρ_2 be two representation of a group G with associated vector spaces V_1 and V_2 . A map $\varphi : V_1 \rightarrow V_2$ is a *map of representations* if (1) it is K -linear and (2) it makes the following diagram commute:

$$\begin{array}{ccc} V_1 & \xrightarrow{\varphi} & V_2 \\ \rho_1(g) \downarrow & & \downarrow \rho_2(g) \\ V_1 & \xrightarrow{\varphi} & V_2 \end{array} .$$

That is, φ is a K -linear map that satisfies

$$\varphi(\rho_1(g)(v)) = (\rho_2(g))(\varphi(v))$$

for all $g \in G$ and all $v \in V$. An *isomorphism of representations* is a map of representations that is invertible.

Examples of Representations

Degree 1 Representations

Let V be of dimension 1 over \mathbb{C} ; so $GL(V) = \mathbb{C}^\times$. A representation ρ is then a homomorphism $G \rightarrow \mathbb{C}^\times$. First observe that since ρ is a homomorphism, denoting the identity of G by 1,

$$\rho(1) = \rho(1^2) = \rho(1)^2,$$

so $\rho(1) = 1$. Further, because G is finite, for each $g \in G$, $g^n = 1$ for some n and so

$$1 = \rho(1) = \rho(g^n) = \rho(g)^n,$$

implying that $\rho(g)$ is a root of unity for every g .

Definition. The *trivial representation* is the representation $\rho : G \rightarrow \mathbb{C}^\times$ defined by $\rho(g) = 1$ for all g , i.e., for all $g \in G$ and all $v \in V$, $gv = v$.

Representation Over a Finite Set

Let X be a finite set; $|X| = n$. Then, a representation in this case is a map $G \rightarrow S(X)$. Now define an n -dimensional vector space spanned by the vectors e_x , one for each element of X :

$$V = \left\{ \sum_{x \in X} a_x e_x : a_x \in \mathbb{C} \right\} \simeq \mathbb{C}^n.$$

To get a representation $G \rightarrow GL(V)$, let G act on V by

$$g \left(\sum_{x \in X} a_x e_x \right) = \sum_{x \in X} a_x e_{gx}.$$

It is easy to check that this is indeed an action:

$$\begin{aligned} g_2 \left(g_1 \left(\sum_{x \in X} a_x e_x \right) \right) &= g_2 \left(\sum_{x \in X} a_x e_{g_1 x} \right) \\ &= \sum_{x \in X} a_x e_{g_2(g_1 x)} \\ &= \sum_{x \in X} a_x e_{(g_2 g_1)x} = (g_2 g_1) \left(\sum_{x \in X} a_x e_x \right). \end{aligned}$$

Definition. This representation is called a *permutation representation*.

Definition. Let $X = G$; then G acts on itself by left multiplication. The resulting permutation representation is called the *regular representation*. In this case the vector space V is the group algebra $\mathbb{C}[G]$.

Other Representations

The kernel and image of a representation map φ are also representations of G since the kernel and image are both left $\mathbb{C}[G]$ -modules. We will later see that a tensor product and the direct sum of representations are also representations.

Recall that a left $\mathbb{C}[G]$ -module is a representation of G .

Definition. A *subrepresentation* W of a representation V is a $\mathbb{C}[G]$ -submodule of V , i.e., for all $w \in W$ and $g \in G$, $gw \in W$.

Definition. A representation is *irreducible* if it contains no proper subrepresentations.

Irreducible Representations of S_3

S_3 is generated by the two elements $\sigma = (1, 2)$ and $\tau = (1, 2, 3)$; they satisfy the relation $\tau\sigma = \sigma\tau^{-1}$. We immediately obtain two degree 1 representations, the trivial representation and the sign representation, where for any element $\alpha \in S^3$,

$$\text{sgn}(\alpha) = \begin{cases} 1 & \alpha \text{ is even} \\ -1 & \alpha \text{ is odd} \end{cases}.$$

Since the trivial and sign representations are both degree 1, they're irreducible. We will denote the trivial representation by U and the sign representation by U' .

Since S_3 acts on the set $X = \{1, 2, 3\}$, we also have the permutation representation, which is a degree 3 representation. Let $\{e_1, e_2, e_3\}$ be the standard basis for \mathbb{C}^3 . Observe that

$$\tau(a_1e_1 + a_2e_2 + a_3e_3) = a_1e_2 + a_2e_3 + a_3e_1,$$

or equivalently

$$\tau(a_1, a_2, a_3) = (a_3, a_1, a_2).$$

That is, τ moves the first coordinate into the second coordinate, the second coordinate into the third coordinate, and the third coordinate into the first coordinate. This is often a point of confusion. Similarly, we find that

$$\sigma(a_1, a_2, a_3) = (a_2, a_1, a_3).$$

So in matrix form,

$$\tau = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \sigma = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Matrices like the ones above for σ and τ , containing exactly 1 one in each row and column and zeros otherwise, are called *permutation matrices*.

We claim that this representation is not irreducible. Indeed, let $W' \subset \mathbb{C}^3$ be the one-dimensional vector space spanned by $(1, 1, 1)$. Then W' is a submodule of \mathbb{C}^3 , i.e., for all $g \in S^3$ and $w \in W'$, $gw = w$:

$$\tau(1, 1, 1) = \sigma(1, 1, 1) = (1, 1, 1).$$

Indeed, W' is the trivial representation!

In the next lecture we will show that the complement of a subrepresentation is also a subrepresentation. So we need to find $W \subset \mathbb{C}^3$ such that $\mathbb{C}^3 = W \oplus W'$ as a representation. Let

$$W = \{(a_1, a_2, a_3) : a_1 + a_2 + a_3 = 0\}.$$

This is clearly stable under G (i.e., for all $g \in G$ and $w \in W$, $gw \in W$), and $W \cap W' = \{0\}$, so $\mathbb{C}^3 = W' \oplus W$. The representation W is called the *standard representation* of S_3 .

Claim. W is irreducible.

Proof. Let $w_1, w_2 \in W$ be the vectors

$$w_1 = (1, \zeta^2, \zeta) \quad \text{and} \quad w_2 = (1, \zeta, \zeta^2),$$

so that $W = W_1 \oplus W_2$ where W_1 is spanned by w_1 and W_2 is spanned by w_2 . Consider the action of τ on each of the two pieces:

$$\tau w_1 = \tau(1, \zeta^2, \zeta) = (\zeta, 1, \zeta^2) = \zeta(1, \zeta^2, \zeta) = \zeta w_1.$$

and

$$\tau w_2 = \tau(1, \zeta, \zeta^2) = (\zeta^2, 1, \zeta) = \zeta^2(1, \zeta, \zeta^2) = \zeta^2 w_2.$$

Similarly

$$\sigma w_1 = \zeta^2 w_2 \quad \text{and} \quad \sigma w_2 = \zeta w_1.$$

These calculations give the matrix representations for σ and τ for the basis $\{w_1, w_2\}$ as

$$\sigma = \begin{bmatrix} 0 & \zeta \\ \zeta^2 & 0 \end{bmatrix} \quad \text{and} \quad \tau = \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^2 \end{bmatrix}.$$

Thus, τ fixes W_1 and fixes W_2 while σ permutes W_1 and W_2 . This means that there is no proper subspace V of W that is fixed by *both* σ and τ . Since S_3 is generated by σ and τ , this means that there is no proper subspace of W that is S_3 -stable, and so there are no subrepresentations of W . \square

In the next lecture we will prove that U , U' , and W are the only irreducible representations of S_3 . We summarize our results below.

		dimension
U	Trivial	1
U'	Sign	1
W	Standard	2

Irreducible representations of S_3