Schrödinger operators with decaying potentials: some counterexamples

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March 29, 1999

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1991 AMS Subject Classification: primary 34L40, 81Q10; secondary 34L20, 81Q20

Abstract

Consider the one-dimensional Schrödinger equation $-y'' + Vy = Ey$ with potential $|V(x)| \leq C(1+x)^{-\alpha}$. We construct examples which show that some recently obtained results on the embedded singular spectrum of the corresponding operators $H = -d^2/dx^2 + V(x)$ are optimal.

1 Introduction

In this paper, I'm interested in one-dimensional Schrödinger equations,

$$-y''(x) + V(x)y(x) = Ey(x), \tag{1}$$

with potentials V bounded by a decaying power:

$$|V(x)| \leq \frac{C}{(1+x)^{\alpha}} \qquad (\alpha > 0). \tag{2}$$

More specifically, we'll study the spectral properties of the associated self-adjoint operators $H_\beta = -\frac{d^2}{dx^2} + V(x)$, acting on the Hilbert space $L_2(0,\infty)$. The index $\beta \in [0,\pi)$ refers to the boundary condition $y(0)\cos\beta + y'(0)\sin\beta = 0$. For the general theory, see, for example, [29]. These questions are of relevance in quantum mechanics; for more background information on this topic refer to, e.g., [18].

Since V tends to zero as $x \to \infty$, it follows that the essential spectrum satisfies $\sigma_{ess} = [0,\infty)$ (this is a classical result going back to Weyl). However, this doesn't say much about the physics of the corresponding system since it doesn't give information on the type of the spectrum on $(0,\infty)$ (absolutely continuous, singular continuous or point spectrum). There has been some progress on this question recently, and several new positive results have been obtained. In this paper, we launch the counterattack: By constructing suitable examples, we show that these results are in fact optimal.

We're interested in situations where $\sigma_{ac} = [0,\infty)$ with possibly also some embedded singular spectrum on $(0,\infty)$. The corresponding range of exponents in (2) is $1/2 < \alpha \leq 1$. If $\alpha > 1$ or if only $V(x) = o(x^{-1})$, then the spectrum is purely absolutely continuous on $(0,\infty)$. See [20] for the proof of this under the weaker assumption $V(x) = o(x^{-1})$; if $\alpha > 1$ is assumed, the result is classical and easy to prove. On the other hand, there are examples of (random) potentials $V(x) = O(x^{-1/2})$ with purely singular spectrum [6, 24], so there need not be any absolutely continuous spectrum if $\alpha \leq 1/2$. That $\alpha > 1/2$ does imply presence of absolutely continuous spectrum was first shown in [1, 3, 20] (see also [10, 11] for earlier work in this direction). Deift and Killip then proved the most satisfactory result that it already suffices to assume $V \in L_2$ [4]. Actually, in case V satisfies (2), one can say still more on the structure of the spectrum. To formulate this precisely, we first require a definition. The solutions of (1) are said to be of WKB form if there is a solution $y(x,E)$ satisfying the asymptotic formula

$$\begin{pmatrix} y(x,E) \\ y'(x,E) \end{pmatrix} = \begin{pmatrix} 1 \\ i\sqrt{E} \end{pmatrix} \exp\left(i \int_0^x \sqrt{E - V(t)}\, dt \right) + o(1) \qquad (x \to \infty). \tag{3}$$

Note that in this case, we have control over *all* solutions of (1), because $\overline{y}$ is a linearly independent second solution. The so-called WKB methods give solutions satisfying (3), provided V tends to zero and doesn't oscillate too much (see, for instance, [8]). Of course, this latter assumption need not be satisfied if we only suppose (2), but it turns out that we still have WKB asymptotics for "most" energies E. Namely, let S denote the exceptional set, i.e.

$$S = \{E > 0 : \text{No solution of (1) satisfies (3).}\}. \tag{4}$$

Then we have the following bound on the Hausdorff dimension of S.

Theorem 1.1 ([22]) *Suppose* (2) *holds. Then* $\dim S \leq 2(1-\alpha)$.

This strengthens the result from [1, 20] that S is of Lebesgue measure zero if $\alpha > 1/2$. Note that $\sigma_{ac} = [0,\infty)$, in turn, follows from $|S| = 0$ by the general results of [25, 27]. In fact, we get the stronger statement that $(0,\infty)$ is an essential support of the absolutely continuous part of the spectral measure.

It is not, in general, possible to exclude embedded singular spectrum (cf. [16, 26, 28]), but the bound on $\dim S$ leads to the following restrictions.

Theorem 1.2 ([22]) *Suppose* (2) *holds. Then: a) The singular continuous part of the spectral measure is supported by a set of dimension* $\leq 2(1-\alpha)$.

b) There is an exceptional set $B \subset [0, \pi)$ of boundary conditions β with $\dim B \leq 2(1-\alpha)$*, so that the spectrum of H_β is* purely *absolutely continuous on* $(0, \infty)$ *if* $\beta \in [0, \pi) \setminus B$.

Part a) is an immediate corollary to Theorem 1.1, because S itself always supports the singular part of the spectral measure on $(0, \infty)$. For the proof of part b) (which is based on methods developed in [5]), see again [22].

In all these statements, the bound $2(1-\alpha)$ gives the correct values in the borderline cases $\alpha = 1/2$ and $\alpha = 1$, and it looks natural. So the obvious guess is that it's optimal. Our first construction addresses this issue; it may be summarized as follows.

Theorem 1.3 *Suppose $\alpha > 2/3$. Then there is a potential V so that* (2) *holds and* $\dim S = 2(1-\alpha)$.

In other words, Theorem 1.1 is sharp, and, in fact, the bound $2(1-\alpha)$ is attained. A variation on the construction behind Theorem 1.3 will give an analogous statement (= Theorem 7.1) which shows that Theorem 1.2b) is optimal, too (again, if $\alpha \geq 2/3$). On the other hand, no such claim can be made as to part a) of Theorem 1.2. The situation is even worse: It's an open question if there are potentials satisfying (2) with $\alpha > 1/2$ and non-empty singular continuous spectrum. In fact, examples with singular continuous spectrum embedded in the absolutely continuous spectrum were constructed only very recently in [15, 21] (based in part on ideas from [12]), using so-called sparse potentials. In these examples, the essential support of the absolutely continuous part of the spectral measure does *not* have full measure in σ_{ac}, in contrast to the situation under consideration here. That makes it possible to deduce existence of singular continuous spectrum by indirect arguments which are not available here.

The assumption $\alpha > 2/3$ in Theorem 1.3 is of course of a technical character and thus somewhat annoying. On the other hand, I'll try to argue below that at least the method used here cannot work for $1/2 < \alpha \leq 2/3$.

The borderline case $\alpha = 1$ is particularly subtle. We're now interested in the set

$$P = \{E > 0 : (1) \text{ has an } L_2 \text{ solution.}\}.$$

This is the set of positive energies which are eigenvalues for some boundary condition β. Clearly, $P \subset S$, so $\dim P = 0$ by Theorem 1.1. Kiselev, Last, and Simon showed that much more is true:

Theorem 1.4 ([12]) *Suppose* (2) *holds with $\alpha = 1$. Then P is countable and* $\sum_{E \in P} E < \infty$.

We will prove that Theorem 1.4 is sharp in the sense that there are no further asymptotic conditions on the energies from the set P:

Theorem 1.5 *For any sequence $e_n > 0$ with $\sum e_n < \infty$, there are energies $E_n \geq e_n$ and a potential $V(x) = O((1+x)^{-1})$ so that $P \supset \{E_n\}$.*

This answers a question asked in [12]. Earlier, Simon has constructed potentials with the property that the point spectrum contains an arbitrary prescribed sequence $E_n > 0$ with $\sum E_n^{1/2} < \infty$ [26]. One can use a slight variant of this construction and combine it with an additional argument to get examples with the following properties:

Corollary 1.6 *There are potentials $V(x) = O((1+x)^{-1})$ with $\sum_{E\in P} E^p = \infty$ for every $p < 1$.*

Of course, this is weaker than Theorem 1.5. We have stated it separately, because the proof of the Corollary based on [26] is elegant and different from the construction we will use to prove Theorem 1.5.

The potentials we construct here will be obtained by pasting together carefully chosen periodic pieces. Everything relies on two nice features of periodic potentials: First of all, they are good at localizing particles even at high energies, and second, there are efficient tools for an accurate analysis in this high energy regime. In fact, I used similar ideas already in [19] in a somewhat different context. It will also be important that certain "inverse" problems can be solved. Namely, it is possible to prescribe lower bounds on the gap lengths and on the Lyapunov exponent. We will attack these problems by reducing them to a similar problem on the Fourier coefficients of bounded functions, and the solution to this problem can be found in the literature (see Theorem 5.1 below).

Here's a more detailed overview on what is done in this paper: The proof of Theorem 1.3 is given in Sections 2–6. In Section 2, we collect some simple preliminary observations. Then we interrupt the formal proof; in Section 3, we give a heuristic discussion, in order to explain and motivate what happens in the following sections. We also try to show here that the assumption $\alpha > 2/3$ is inevitable with the method used. Section 3 doesn't contain rigorous arguments and may thus be skipped by the confident reader. In Section 4, we prove asymptotic estimates on the trace of the transfer matrix (also known as Lyapunov function) for periodic potentials. Although this subject is classical, there are several new features in the analysis we give: We deal with different potentials simultaneously, and we use recent results from [1] on multilinear operators to study the remainder terms for potentials that are not smooth more carefully than usual. What we state in Theorem 4.2 is what is needed later, but perhaps the ideas used in the proof are also of some independent interest.

The main part of the proof of Theorem 1.3 is then presented in Sections 5 and 6. We construct the exceptional set S and discuss its properties in Section 5. Finally, the analysis of the solutions of (1) for $E \in S$ given in Section 6 concludes the proof of Theorem 1.3. Section 7 contains the modification of this construction relevant to Theorem 1.2b). Theorem 1.5 can now also be proved with the ideas developed in the preceding sections. This is done in Section 8. In Section 9, we present the independent proof of Corollary 1.6. Finally, in the Appendix, we sketch the proof of the result of [1] that we use in Section 4. We do this partly because this result is rather recent and partly because the treatment of [1] simplifies considerably in the special case needed here.

Acknowledgment: I'd like to thank Sasha Kiselev for most useful notes on the proof of Theorem 4.1. I'd like to thank the Heisenberg program of the Deutsche Forschungsgemeinschaft for financial support.

2 Preliminaries

We will work with wavenumbers $k = \sqrt{E}$ instead of energies E. Of course, the transformed set $\{k > 0 : k^2 \in S\}$ has the same Hausdorff dimension as S. Slightly abusing notation, we will denote this set also by S. Let $y(x,k)$ be the solution of (1) with $E = k^2$ with the initial values $y(0,k) = 1, y'(0,k) = 0$ (say). It will be convenient to use modified Prüfer variables $R(x,k) > 0, \varphi(x,k)$ defined by the relations $y = R\sin\varphi, y' = kR\cos\varphi$ and by demanding that φ be continuous in x. The transfer matrix $T(x,t;k)$ is the 2×2-matrix that takes solution vectors $\left(\begin{smallmatrix} y(s,k) \\ y'(s,k)/k \end{smallmatrix}\right)$ from $s = t$ to $s = x$. In particular,

$$R(x,k) = R(t,k)\|T(x,t;k)e_{\varphi(t,k)}\|, \quad e_\varphi \equiv \begin{pmatrix} \sin\varphi \\ \cos\varphi \end{pmatrix}.$$

Also, if u, v solve (1) and $u(t,k) = v'(t,k) = 1, u'(t,k) = v(t,k) = 0$, then

$$T(x,t;k) = \begin{pmatrix} u(x,k) & kv(x,k) \\ u'(x,k)/k & v'(x,k) \end{pmatrix}. \tag{5}$$

If $V = 0$ on an interval (a,b), then the evolution of R, φ is very simple: $R(x,k) \equiv R(a,k)$ and $\varphi(x,k) = \varphi(a,k) + k(x-a)$ for all $x \in (a,b)$.

As already mentioned in the introduction, the potential we're after is built out of periodic pieces. The nth piece has period g_n^{-1} and size $\sim c_n g_n^2$, where $g_n > 0$ is small. The number of periods is equal to $L_n \in \mathbb{N}$. We also need intervals of length Δ_n with zero potential between the blocks in order to randomize the phases φ.

So, V will be of the following form: Put $a_1 = 0$ and $b_n = a_n + L_n g_n^{-1}, a_{n+1} = b_n + \Delta_n$, where $L_n \in \mathbb{N}$, $\Delta_n \in [0, \pi/2]$, and define

$$V(x) = \begin{cases} c_n W_{g_n}(x - a_n) & x \in (a_n, b_n) \\ 0 & x \in (b_n, a_{n+1}) \end{cases}.$$

Here, W_g denotes the rescaled function $W_g(x) = g^2 W(gx)$, and the basic potential W (as well as the other parameters) will be chosen later. The following elementary observation is crucial to our construction. Namely, the rescaling $W \to W_g$ has the same effect as replacing k by k/g. More precisely, write T_W for the transfer matrix associated with $-y'' + Wy = k^2 y$. Then we have

Lemma 2.1 $T_{W_g}(1/g, 0; k) = T_W(1, 0; k/g)$.

Sketch of the proof. Introduce the new variable $t = gx$ in the Schrödinger equation and use (5). (See also [19, Lemma 3.1].) □

Of course, we need a few simple facts about periodic potentials; a more comprehensive treatment can be found in [7]. In the sequel, $T(k) = T_W(1,0;k)$ is short-hand for the transfer matrix over one period of some (unspecified, but fixed) potential W with period 1. It suffices to study this object since $T_W(N,0;k) = T(k)^N$ by periodicity. Then, since $\det T(k) = 1$, the eigenvalues of $T(k)$ can be computed from its trace

$$D(k) \equiv \mathrm{tr}\; T(k) = u(1,k) + v'(1,k)$$

(u, v solve $-y'' + Wy = k^2 y$, and $u(0) = v'(0) = 1, u'(0) = v(0) = 0$). Namely, the eigenvalues are λ, λ^{-1}, where

$$\lambda(k) = \frac{D(k)}{2} \pm \sqrt{\frac{D(k)^2}{4} - 1}. \tag{6}$$

Here, we choose the sign so that $|\lambda| \geq 1$. If $|D| > 2$, then λ is real and $|\lambda| > 1$; if $|D| < 2$, then $|\lambda| = 1, \lambda \notin \mathbb{R}$. The set $\{E \in \mathbb{R} : |D(E)| \leq 2\}$ (this notation is a little sloppy since we've switched from k to E, but self-explanatory) is a (possibly infinite) union of closed intervals. These intervals are called bands. The intervals where $|D| > 2$ are called gaps. The spectrum of the whole-line operator with potential W is purely absolutely continuous and consists precisely of the bands. If k is in a gap, the Lyapunov exponent is positive. So solutions that are not very close to the decaying solution are exponentially increasing. Here's the precise statement we'll use:

Lemma 2.2 *If $|D(k)| > 2$, there is an angle $\theta(k) \in [0, \pi)$ so that for all $n \in \mathbb{N}$,*

$$\|T(k)^n e_\varphi\| \geq |\lambda(k)|^n \sin|\theta(k) - \varphi|. \tag{7}$$

Here, we use the ℓ_2 norm on $\mathbb{C}^2$, and, as above, $e_\varphi = (\sin\varphi, \cos\varphi)^t$.

Proof. Fix k, and pick $\psi, \theta \in [0, \pi)$ so that e_ψ, e_θ are eigenvectors of T corresponding to the (real and distinct!) eigenvalues λ and λ^{-1}, respectively. Actually, we may assume $\psi = 0$ here without loss of generality. Then $\theta \neq 0$, and a calculation shows that

$$\begin{aligned}\|T^n e_\varphi\|^2 &= \sin^{-2}\theta\left[\lambda^{2n}\sin^2(\theta-\varphi) + \lambda^{-2n}\sin^2\varphi + 2\cos\theta\sin(\theta-\varphi)\sin\varphi\right]\\ &= \sin^{-2}\theta\left[\lambda^{-n}\sin\varphi + \lambda^n\sin(\theta-\varphi)\cos\theta\right]^2 + \lambda^{2n}\sin^2(\theta-\varphi)\\ &\geq \lambda^{2n}\sin^2(\theta-\varphi),\end{aligned}$$

as claimed. □

We conclude this section with an asymptotic estimate of the usual type (compare, e.g., [7, 14, 17]). This result will be used in Section 8, and, incidentally, it also suffices to prove Theorem 1.3 for $\alpha > 4/5$. However, to go further, we'll need the more accurate analysis of Section 4. Therefore, we will omit the proof of the following Proposition. We use the notations $\widehat{W}_n = \int_0^1 W(x)e^{2\pi i n x}\,dx$ and $\|W\|_1 = \int_0^1 |W(x)|\,dx$.

Proposition 2.3 *There is a constant $C_0 > 0$ such that the following holds. If $W \in L_2(0,1)$, $\int_0^1 W = 0$, $j \geq C_0\|W\|_1$, $j|\widehat{W}_j|^2 \geq C_0\|W\|_1^3$, and*

$$|k - j\pi| \leq \frac{|\widehat{W}_j|}{10j},$$

then

$$|\lambda(k)| \geq 1 + \frac{|\widehat{W}_j|}{10j}.$$

3 A guide to the proof

The set S from (4) will be constructed as a Cantor type set. More precisely, at step n, we'll use some of the gaps of the potential $c_n W_{g_n}$, and we let then T be the intersection over all n of the sets thus obtained. There is definite hope that S, as defined in (4), can indeed contain such a set T, because by Lemma 2.2, we expect that the solutions of (1) grow if E is in a gap of $c_n W_{g_n}$ for all n.

Now, to analyze this situation quantitatively, we will have to apply Proposition 2.3 to potentials of the form $c_n g_n^2 W(g_n x)$; by Lemma 2.1 we may as well apply the Proposition to $c_n W(x)$ if we also replace k by k/g_n. In this way, we see that the gaps are located at the points $k_j = j\pi g_n$; their size is of the order $l_n \approx c_n g_n^2 |\widehat{W}_j|$. We proceed inductively and we use only those gaps which are contained in a gap that was used in the preceding step. Let N_n denote the number of new gaps contained in some fixed gap constructed in the $(n-1)$st step, and write $P_n \equiv N_1 \cdots N_n$ for the total number of gaps used in step n. Because of the condition $\sum_j |\widehat{W}_j|^2 < \infty$, there are restrictions on the size of $|\widehat{W}_j|$. We will pick a W that satisfies $|\widehat{W}_j| \approx P_n^{-1/2}$ for the indices j that are used in the nth step. Then $l_n \approx c_n g_n^2 P_n^{-1/2}$. Since adjacent gaps are separated by a distance of order $\approx g_n$, we obtain the relation $l_{n-1} \approx N_n g_n = P_n P_{n-1}^{-1} g_n$. Finally, in order to get a set of dimension D, we should have a scaling of the type $P_n l_n^D \approx 1$. Solving these latter three relations for l_n, g_n, c_n, we obtain

$$l_n \approx P_n^{-1/D},$$

$$g_n \approx P_n^{-1} P_{n-1}^{1-1/D}, \tag{8}$$

$$c_n \approx P_n^{5/2-1/D} P_{n-1}^{2/D-2}. \tag{9}$$

By Proposition 2.3 again, the large eigenvalue λ_n of the transfer matrix is of the order $|\lambda_n| \approx 1 + c_n|\widehat{W}_j|/j \approx 1 + c_n g_n P_n^{-1/2}$. By Lemma 2.2, we expect that the solution grows by a factor $\approx |\lambda_n|^{L_n}$ across the nth piece. Thus, to get substantial increase, we must take

$$L_n \approx P_n^{1/2}(c_n g_n)^{-1} \approx (P_n/P_{n-1})^{1/D-1}.$$

It remains to estimate the rate of decay of V we get with these parameters: In the nth piece, we have that $x \approx \sum_{m=1}^n L_m/g_m$. If L_m/g_m increases rapidly, this is $\approx L_n/g_n \approx P_n^{1/D}$. On the other hand, $|V(x)| \leq Cc_n g_n^2 \approx P_n^{1/2-1/D}$ for these x, so $V(x) = O(x^{-\alpha})$ with $\alpha = 1 - D/2$ or $D = 2(1-\alpha)$, as desired.

We made heavy use of asymptotic expansions in these heuristic considerations, so we will certainly need that $k/g_n \gg \|c_n W\|_1$ (the basic assumption in these expansions). That is, we need that $c_n g_n \ll 1$, and since $c_n g_n \approx P_n^{3/2-1/D} P_{n-1}^{1/D-1}$, our method can, unfortunately, only work if $D < 2/3$ (or, equivalently, $\alpha > 2/3$).

4 Asymptotic expansions

We need asymptotic formulae more precise than Proposition 2.3. We begin by introducing two maximal functions which will be used to control the remainders. For $f, g \in L_1(0,1)$, put

$$M_f(k) = \max_{0\leq b\leq 1} \left| \int_0^b dx\, f(x) e^{2\pi i k x} \right|,$$

$$M_{f,g}(k) = \max_{0\leq b\leq 1} \left| \int_0^b dx\, f(x) e^{2\pi i k x} \int_0^x dt\, g(t) e^{-2\pi i k t} \right|.$$

Then the results of Christ and Kiselev [1] on multilinear operators, specialized to the case at hand, give the following norm bounds.

Theorem 4.1 ([1]) *If $p \in (1,2)$ and $1/p + 1/q = 1$, then*

$$\left(\sum_{n=-\infty}^{\infty} (M_f(n))^q \right)^{1/q} \leq C_p \|f\|_p, \quad \left(\sum_{n=-\infty}^{\infty} (M_{f,g}(n))^{q/2} \right)^{2/q} \leq C_p \|f\|_p \|g\|_p.$$

The *proof* will be sketched in the Appendix.

Actually, we will only need the weak type bounds that follow from these estimates. These bounds will let us prove

Theorem 4.2 *There is a constant $C_0 > 0$ such that the following holds. If $W \in L_2(0,1), \int_0^1 W = 0$, $j \geq C_0 \|W\|_1$, and*

$$|k - j\pi| \leq \frac{|\widehat{W}_j|}{10j},$$

then

$$|D(k)| \geq 2 + \frac{|\widehat{W}_j|^2}{70j^2} - R(j),$$

where, for every $q > 2$, $R(j)$ satisfies the weak type estimate

$$\#\{j \in \mathbb{N} : j \geq C_0 \|W\|_1 \text{ and } R(j) > \lambda \|W\|_1 j^{-3}\} \leq C_q \lambda^{-q/2} \|W\|_2^q.$$

Remark. Note that the constants C_0, C_q are independent of W. This will be important in the application of Theorem 4.2.

Proof. The following expansion holds:

$$D(k) = 2\cos k + \sum_{n=2}^{\infty} \frac{D_n(k)}{k^n},$$

where

$$D_n(k) = \int_0^1 dt_1\, W(t_1) \int_0^{t_1} dt_2\, W(t_2) \dots \int_0^{t_{n-1}} dt_n\, W(t_n) \times \\ \sin k(1 + t_n - t_1) \sin k(t_1 - t_2) \cdots \sin k(t_{n-1} - t_n). \quad (10)$$

This follows from the corresponding expansions of the solutions u, v obtained from iterating the variation of constants formula (see, e.g., [17, Chapter 1]). The details of these routine computations can be left to the reader. Note also that the series for D converges absolutely and uniformly in $k \geq 0$ (say).

We want to treat $\sum_{n\geq 3} k^{-n} D_n$ as a remainder. Of course, there's the obvious bound $(\|W\|_1/k)^3 e^{\|W\|_1/k}$, and, indeed, this estimate lets one prove Proposition 2.3. We give a more careful analysis in this proof.

It's useful to rewrite the D_n's using the complex representation of the sine. We get

$$D_n(k) = (2i)^{-n} \sum_{\sigma_1, \dots, \sigma_n = \pm 1} \sigma_1 \cdots \sigma_n e^{i\sigma_1 k} S_n(\sigma) \quad (11)$$

with

$$S_n(\sigma) \equiv \int_0^1 dt_1\, W(t_1) \int_0^{t_1} dt_2\, W(t_2) \dots \int_0^{t_{n-1}} dt_n\, W(t_n) \times \\ e^{ik(\sigma_2 - \sigma_1)t_1} e^{ik(\sigma_3 - \sigma_2)t_2} \dots e^{ik(\sigma_1 - \sigma_n)t_n}.$$

If $\sigma_1 = \ldots = \sigma_n$, then $S_n(\sigma) = \left(\int_0^1 W\right)^n / n! = 0$. On the other hand, if not all the σ's are equal, we can find $m, l \in \{1, \dots, n\}$ with $\sigma_{m+1} - \sigma_m = 2$ and $\sigma_{l+1} - \sigma_l = -2$ (here, we have put $\sigma_{n+1} \equiv \sigma_1$). We have

$$|S_n(\sigma)| \leq \int dt_1\, |W(t_1)| \dots \int dt_{m-1}\, |W(t_{m-1})| \int dt_{m+1}\, |W(t_{m+1})| \dots \\ \int dt_{l-1}\, |W(t_{l-1})| \int dt_{l+1}\, |W(t_{l+1})| \dots \int dt_n\, |W(t_n)| \times \\ \left| \int dt_m\, W(t_m) e^{2ikt_m} \int dt_l\, W(t_l) e^{-2ikt_l} \right|.$$

The integration is over the region $\{(t_1, \dots, t_n) : 1 \geq t_1 \geq \dots \geq t_n \geq 0\}$. In particular, in the last integral t_l runs over $[t_{l+1}, t_{l-1}]$ (using the obvious convention $t_0 = 1, t_{n+1} = 0$). The structure of the double integral with respect to (t_m, t_l) depends on the distance between the indices m and l. We have to distinguish two cases: If $m - l$ (evaluated modulo n) is not equal to ± 1, then the region of integration for the t_m-integral is $[t_{m+1}, t_{m-1}]$, and thus the double integral is equal to the product of the two single integrals. Since clearly

$$\max_{0 \le s,t \le 1} \left| \int_s^t dx\, W(x) e^{2ikx} \right| \le 2 M_W(k/\pi),$$

we see that in the case under consideration

$$|S_n(\sigma)| \le 4 \frac{\|W\|_1^{n-2}}{(n-2)!} M_W^2(k/\pi).$$

On the other hand, if $l = m+1$, say, then the limits of the double integral are (in self-explanatory notation)

$$\int_{t_{m+2}}^{t_{m-1}} dt_m \int_{t_{m+2}}^{t_m} dt_l.$$

Since

$$\int_a^b dt \int_a^t ds = \int_0^b \int_0^t - \int_0^a \int_0^t + \int_0^a \int_0^a - \int_0^b \int_0^a ,$$

we get this time that

$$|S_n(\sigma)| \le 2 \frac{\|W\|_1^{n-2}}{(n-2)!} \left(M_W^2(k/\pi) + M_{W,W}(k/\pi) \right). \tag{12}$$

In either case, it is true that $S_n(\sigma)$ can be estimated by the right-hand side of (12) with 2 replaced by 4. By (11), $|D_n|$ satisfies the same inequality.

Since we want to expand $D_n(k)$ around $k = j\pi$, we also need similar estimates on dD_n/dk. We have that

$$\begin{aligned} \frac{d}{dk} S_n(\sigma) = i \int_0^1 dt_1\, W(t_1) \int_0^{t_1} dt_2\, W(t_2) \ldots \int_0^{t_{n-1}} dt_n\, W(t_n) \times \\ e^{ik(\sigma_2 - \sigma_1)t_1} e^{ik(\sigma_3 - \sigma_2)t_2} \cdots e^{ik(\sigma_1 - \sigma_n)t_n} \left((\sigma_2 - \sigma_1)t_1 + \ldots + (\sigma_1 - \sigma_n)t_n \right). \end{aligned}$$

We pick m, l as above, and we treat separately the contributions coming from $2it_m$, $-2it_l$, and those coming from $\sum_{r \ne m,l} (\sigma_{r+1} - \sigma_r) t_r$. For these latter terms, it suffices to note that $\left| \sum_{r \ne m,l} (\sigma_{r+1} - \sigma_r) t_r \right| \le 2(n-2)$ and then apply the above estimates again. Dealing with the other two contributions, we also proceed as above, but now new maximal functions enter; for instance, in the case when $l = m+1$, we use that

$$\begin{aligned} \left| \int_{t_{m+2}}^{t_{m-1}} dt_m\, 2it_m W(t_m) e^{2ikt_m} \int_{t_{m+2}}^{t_m} dt_l\, W(t_l) e^{-2ikt_l} \right| \le \\ 4 \left(M_{tW}(k/\pi) M_W(k/\pi) + M_{tW,W}(k/\pi) \right). \end{aligned}$$

We omit the details; the final result is an inequality of the form

$$\left|\frac{dS_n(\sigma)}{dk}\right| \leq 8\frac{\|W\|_1^{n-2}}{(n-3)!}\left(M_W^2 + M_{tW}M_W + M_{W,W} + M_{tW,W} + M_{W,tW}\right)$$

(where all maximal functions are evaluated at k/π). Since the additional term coming from differentiating $e^{i\sigma_1 k}$ is easily controlled, it now follows from (11) that also

$$|D_n'(k)| \leq 12\frac{\|W\|_1^{n-2}}{(n-3)!}\left(M_W^2 + M_{tW}M_W + M_{W,W} + M_{tW,W} + M_{W,tW}\right).$$

For the second derivative, a crude estimate will be sufficient. It follows directly from (10) that

$$\left|\frac{d^2D_n(k)}{dk^2}\right| \leq 4\frac{\|W\|_1^n}{(n-2)!}.$$

We now write $k = j\pi \pm \delta$ with $0 \leq \delta \leq \pi/2$ (and, typically, j large). Then our results on D_n from above imply that

$$\begin{aligned}|D_n(k)| \leq &|D_n(j\pi)| + |D_n'(j\pi)|\,\delta + \max|D_n''(k)|\cdot\delta^2/2\\ \leq &C\frac{\|W\|_1^{n-2}}{(n-2)!}\left[M_W^2(j) + M_{W,W}(j)\right] + C\frac{\|W\|_1^{n-2}}{(n-3)!}\left[M_W^2(j) +\right.\\ &\left. M_{tW}(j)M_W(j) + M_{W,W}(j) + M_{tW,W}(j) + M_{W,tW}(j)\right]\delta +\\ &C\frac{\|W\|_1^n}{(n-2)!}\,\delta^2.\end{aligned}$$

Now if $\|W\|_1/j \leq 1$, say, then this inequality leads to

$$\left|\sum_{n=3}^{\infty}\frac{D_n(k)}{k^n}\right| \leq CM(j)\frac{\|W\|_1}{j^3} + C\frac{\|W\|_1^3}{j^3}\delta^2,$$

where $M \equiv M_W^2 + M_{tW}M_W + M_{W,W} + M_{tW,W} + M_{W,tW}$. Theorem 4.1 shows that for $q > 2, 1/p + 1/q = 1$,

$$\begin{aligned}\|M\|_{\ell_{q/2}} &\leq \|M_W\|_q^2 + \|M_{tW}\|_q\|M_W\|_q + \|M_{W,W}\|_{q/2} + \|M_{tW,W}\|_{q/2} + \|M_{W,tW}\|_{q/2}\\ &\leq C_q\left(\|W\|_p^2 + \|tW\|_p^2\right) \leq 2C_q\|W\|_p^2 \leq 2C_q\|W\|_2^2.\end{aligned}$$

This implies the weak type bound

$$\#\{j \in \mathbb{N} : M(j) > \lambda\} \leq C_q\lambda^{-q/2}\|W\|_2^q.$$

Finally, we need to take a closer look at $D_2(k)$. A calculation gives

$$D_2(k) = \frac{|\widehat{W}(k/\pi)|^2}{4}\cos k + \frac{1}{2}\,\mathrm{Im}\int_0^1 dt_1\,W(t_1)e^{2ikt_1}\int_0^{t_1} dt_2\,W(t_2)e^{-2ikt_2}\sin k,$$

where $\widehat{W}(s) = \int_0^1 W(x)e^{2\pi i s x}\, dx$. We again use Taylor expansions about $k = j\pi$ to control the second term on the right-hand side; also using the fact that $|\sin k| \le \delta$, we get the bound $CM(j)\delta + C\|W\|_1^2\delta^2$. Therefore, putting everything together, we obtain

$$D(k) = 2\cos k \left(1 + \frac{|\widehat{W}(k/\pi)|^2}{8k^2} \right) + O\left(M(j)(\|W\|_1/j^3 + \delta/j^2) + (\|W\|_1\delta/j)^2 \right).$$

The constant implicit in the error term $O(\dots)$ does not depend on anything. To conclude the proof, we also expand the leading term:

$$\begin{aligned} \cos k &= (-1)^j \left(1 - \delta^2/2 + O(\delta^4) \right), \\ \widehat{W}(k/\pi) &= \widehat{W}_j + O(\|W\|_1\delta). \end{aligned}$$

Using this and noting that, by hypothesis, $\delta \le |\widehat{W}_j|/(10j)$, it is now routine, if somewhat tedious, to check that the assertion holds if $\|W\|_1/j$ is small enough, that is, if we take C_0 sufficiently large. The role of $R(j)$ is taken by a remainder of the order $O(M(j)\|W\|_1/j^3)$. □

5 Construction of the exceptional set

We now make the program outlined in Section 3 rigorous. We start by constructing a Cantor type set (which will basically be the set S from (4)) and, simultaneously, the basic periodic potential W.

So, fix $\alpha \in (2/3, 1)$, and let $D = 2(1-\alpha)$. We want to construct a potential V of the order $V(x) = O(x^{-\alpha})$ so that $\dim S = D$. It will be more convenient to work with still another parameter a related to D by

$$a = \frac{4(1-D)}{2-3D}; \tag{13}$$

notice that $a > 2$. Pick $b \in (1, a)$, and put

$$g_n = \exp\left(-a^{n+1} \frac{2a-3}{(2a-4)(a-1)} \right), \tag{14}$$

$$c_n = 10 \exp\left(\frac{a^{n+1}}{a-1} - b^{n+1} \right). \tag{15}$$

The motivation for these choices comes from the arguments of Section 3: We make the ansatz $N_n = \exp(a^n)$, so $P_n \approx \exp(a^{n+1}/(a-1))$, then use (8), (9) and finally pick a so that $c_n g_n \ll 1$. The use of the rapidly increasing sequence $\exp(a^n)$ is not a whim but is necessary to be able to satisfy this latter condition. Finally, the additional parameter b is introduced for technical reasons; it allows us to slightly improve the decay rate of the V to be constructed. In fact, Theorem 1.3 can be proved with $b = 0$, but in the proof of Theorem 7.1 below, it will be essential that we squeeze out a little more than what Theorem 1.3 states.

In order to get gaps at the right places (by an application of Theorem 4.2), we need to make sure that certain Fourier coefficients of W are large enough. More precisely, we construct a Cantor type set $\bigcap F_n$, and we simultaneously try to find a function W satisfying the following conditions:

1. Put $F_0 = [1,2]$.

2. The set F_1 will be a union of intervals of the form
$$J_j^{(1)} = [j\pi g_1 - l_1/2, j\pi g_1 + l_1/2],$$
where
$$l_1 = \frac{\pi}{10} c_1 g_1^2 \exp\left(-\frac{a^{1+1}}{2(a-1)}\right)$$
(the motivation for this choice will become clear shortly – see Lemma 5.2). In fact, we use only those $J_j^{(1)}$ which are contained in F_0. For the corresponding indices j, we require that
$$|\widehat{W}_j| \geq \exp\left(-\frac{a^{1+1}}{2(a-1)}\right).$$
Let F_1 be the union of the $J_j^{(1)}$'s with $J_j^{(1)} \subset F_0$.

3. Similarly, F_2 is a union of intervals
$$J_j^{(2)} = [j\pi g_2 - l_2/2, j\pi g_2 + l_2/2]$$
with
$$l_2 = \frac{\pi}{10} c_2 g_2^2 \exp\left(-\frac{a^{2+1}}{2(a-1)}\right).$$
This time, we use only those j for which $J_j^{(2)} \subset F_1$, and we require that then
$$|\widehat{W}_j| \geq \exp\left(-\frac{a^{2+1}}{2(a-1)}\right).$$

Continuing in this way, we obtain a sequence of sets $F_n \subset F_{n-1}$. Every F_n is a union of intervals $J_j^{(n)}$ of the form
$$\begin{aligned} J_j^{(n)} &= [j\pi g_n - l_n/2, j\pi g_n + l_n/2], \\ l_n &= \frac{\pi}{10} c_n g_n^2 \exp\left(-\frac{a^{n+1}}{2(a-1)}\right). \end{aligned} \tag{16}$$

Actually, we can't be sure at this point that $F_n \neq \emptyset$ for all n. However, it's easy to repair this: We start out with a collection of $J_j^{(n_0)} \subset [1,2]$ instead of F_0, where we take $n_0 \in \mathbb{N}$ large enough, and then proceed as described above. We will see shortly that then $F_n \neq \emptyset$ for all $n \geq n_0$.

In any event, we're also trying to find a function W with the property that

$$|\widehat{W}_j| \geq \exp\left(-\frac{a^{n+1}}{2(a-1)}\right) \tag{17}$$

whenever j is such that $J_j^{(n)} \subset F_n$. The existence of such a W will follow from a result of de Leeuw, Kahane, and Katznelson:

Theorem 5.1 *Suppose $w_j \geq 0$, $\sum w_j^2 < \infty$. Then there exists a bounded function $f : [0,1] \to \mathbb{R}$ with $|\widehat{f}_j| \geq w_j$ for all j. Moreover, it is possible to choose f so that $\|f\|_\infty \leq C\left(\sum w_j^2\right)^{1/2}$.*

See [9, Section 5.9] for the *proof.* The bound on $\|f\|_\infty$ is also derived in the proof given there, although it is then not mentioned in the theorem of that section. Incidentally, we'll need this bound only in Section 8.

Before applying Theorem 5.1, we can simplify things a little by passing to subsets of the F_n's which are, in a sense, more symmetric. To this end, we first observe that every fixed $J_j^{(n-1)} \subset F_{n-1}$ contains at least

$$\frac{l_{n-1}}{\pi g_n} - 2 = \frac{\pi}{10} c_{n-1} g_{n-1}^2 \exp\left(-\frac{a^n}{2(a-1)}\right) \frac{1}{\pi g_n} - 2$$

intervals $J_k^{(n)}$. A computation using (14), (15) shows that this number is equal to $\exp(a^n - b^n) - 2$. In particular, for large n, it is certainly larger than $(1/2)\exp(a^n - b^n)$, say. We fix once and for all integers

$$N_n \in [\frac{1}{3}\exp(a^n - b^n), \frac{1}{2}\exp(a^n - b^n)] \cap \mathbb{N} \quad (n \geq n_0 + 1). \tag{18}$$

By the above remarks, we can now (inductively) pass to subsets $G_n \subset F_n$ ($n \geq n_0$) as follows. Put $G_{n_0} = F_{n_0}$; then G_{n_0} is the union of certain intervals $J_j^{(n_0)}$. Generally, if G_{n-1} is such a union, pick precisely N_n subintervals $J_k^{(n)} \subset J_j^{(n-1)}$ for every interval $J_j^{(n-1)} \subset G_{n-1}$, and let G_n be the union of these $J_k^{(n)}$'s.

Now, if we denote the right-hand side of (17) by w_n and if N_{n_0} is the number of intervals contained in G_{n_0}, then

$$\begin{aligned}
\sum_{n=n_0}^{\infty} \sum_{\{j : J_j^{(n)} \subset G_n\}} w_n^2 &= \sum_{n=n_0}^{\infty} N_{n_0} N_{n_0+1} \cdots N_n w_n^2 \\
&\leq C \sum_{n=n_0}^{\infty} 2^{-n} \exp\left(\frac{a^{n+1}}{a-1} - \frac{b^{n+1}}{b-1}\right) w_n^2 \\
&= C \sum_{n=n_0}^{\infty} 2^{-n} \exp\left(-\frac{b^{n+1}}{b-1}\right) < \infty.
\end{aligned}$$

So Theorem 5.1 applies: There is a bounded function W so that for every $n \geq n_0$, condition (17) holds for all $j \in \mathbb{N}$ with $J_j^{(n)} \subset G_n$. Since we want to apply Theorem 4.2 to (multiples of) W later on, we replace this $W(x)$ by $W(x) - \int_0^1 W$ to make sure that $\int_0^1 W = 0$. Of course, this doesn't change $\widehat{W}_j$ for $j \neq 0$.

Let $T = \bigcap_{n \geq n_0} G_n$. In the sequel, we assume that $n_0 = 0$ and $G_0 = F_0 = [1,2]$. This is done in order to simplify the notation; the following arguments are of course also valid in the general case. We summarize the properties of T, and we also add an important additional observation:

Lemma 5.2 *G_n is a (disjoint) union of precisely $N_1 N_2 \cdots N_n$ intervals $J_j^{(n)}$ of the form*

$$J_j^{(n)} = [j\pi g_n - l_n/2, j\pi g_n + l_n/2] .$$

Every $J_j^{(n)} \subset G_n$ contains exactly N_{n+1} intervals $J_i^{(n+1)} \subset G_{n+1}$. If $J_j^{(n)} \subset G_n$, then (17) *holds. Moreover, if $k \in J_j^{(n)} \subset G_n$, then*

$$\left| \frac{k}{g_n} - j\pi \right| \leq \frac{c_n |\widehat{W}_j|}{10 j}.$$

Proof. It remains to check the last statement. So suppose $k \in J_j^{(n)} \subset G_n$. Then, by (16) and (17),

$$\left| \frac{k}{g_n} - j\pi \right| \leq \frac{l_n}{2g_n} = \frac{\pi}{20} c_n g_n \exp\left(-\frac{a^{n+1}}{2(a-1)} \right) \leq \frac{\pi}{20} c_n g_n |\widehat{W}_j|.$$

Since $j\pi g_n \in G_n \subset F_0 = [1,2]$, we have $g_n \leq 2/(\pi j)$, proving the assertion. □

There's a natural Borel (probability) measure μ supported by T. Namely, we define μ by the following two properties: $\mu(\mathbb{R} \setminus T) = 0$, and for every n, μ puts equal weight on the subintervals of G_n, that is

$$\mu(J_j^{(n)}) = \frac{1}{N_1 N_2 \cdots N_n}$$

for every $J_j^{(n)} \subset G_n$. It's easy to see that μ is indeed well defined by these requirements.

We can now complete our analysis of T by showing that T itself and certain subsets are not too small. More precisely, we have:

Lemma 5.3 *If $\mu(M) > 0$, then* $\dim M \geq D$.

Proof. We will prove the following: For all $\gamma < D$, we have that

$$\lim_{\delta \to 0+} \sup_{|I| \leq \delta} \frac{\mu(I)}{|I|^\gamma} = 0.$$

Then, by general results on Hausdorff measures [23, Section 3.4, Theorem 67], it will follow that $\mu(M) = 0$ for all sets M with $\dim M < D$, and this is exactly what we have to prove.

So fix $\gamma < D$, let $\delta > 0$, and let I be an interval with $|I| \leq \delta$. Define $n \in \mathbb{N}$ by $l_n < |I| \leq l_{n-1}$. First, we consider the case when $l_n < |I| \leq \pi g_n$. Then I intersects at most two of the intervals $J_j^{(n)}$ that build up G_n, so

$$\frac{\mu(I)}{|I|^\gamma} \leq \frac{2}{N_1 N_2 \cdots N_n} |I|^{-\gamma} \leq \frac{2 l_n^{D-\gamma}}{N_1 N_2 \cdots N_n l_n^D}.$$

Next, if $\pi g_n < |I| \leq l_{n-1}$, then I intersects at most $|I|/(\pi g_n) + 1 \leq 2|I|/(\pi g_n)$ of the intervals $J_j^{(n)}$. Thus

$$\frac{\mu(I)}{|I|^\gamma} \leq \frac{2}{\pi g_n N_1 N_2 \cdots N_n} |I|^{1-\gamma} \leq \frac{2 l_{n-1}}{\pi N_n g_n} \frac{l_{n-1}^{D-\gamma}}{N_1 N_2 \cdots N_{n-1} l_{n-1}^D}.$$

By (18), we have that

$$N_1 \cdots N_n \geq C 3^{-n} \exp\left(\frac{a^{n+1}}{a-1} - \frac{b^{n+1}}{b-1} \right),$$

and a computation using (13), (14), (15), and (16) shows that

$$l_n = \pi \exp\left(-\frac{a^{n+1}}{D(a-1)} - b^{n+1} \right).$$

Therefore, $N_1 \cdots N_n l_n^D \geq C_1 \exp(-C_2 b^n)$. Similarly, one sees that $l_{n-1}/(N_n g_n) \leq C_1 \exp(C_2 b^n)$. So, since $D - \gamma > 0$ by assumption, we get, in either case, an inequality of the form

$$\frac{\mu(I)}{|I|^\gamma} \leq C_1 \exp\left(C_2 b^n - \epsilon a^n \right)$$

with $\epsilon = \epsilon(D, \gamma) > 0$. Our claim follows, because n goes to infinity as δ tends to zero. $\square$

6 Asymptotics of the solutions

We know from Section 3 that we can expect things to work out provided nothing special happens, but there are two kinds of bad luck we have to be prepared against: First of all, the Prüfer angle φ could be close to the exceptional phase θ from Lemma 2.2, and second, the error term $R(j)$ from Theorem 4.2 could be large.

The first problem is overcome by randomizing the Prüfer angles with the help of the parameters Δ_n. More precisely, we show

Lemma 6.1 *Given measurable functions $\theta_n(k)$, it is possible to choose $\Delta_n \in [0, \pi/2]$ in such a way that there is a subset $T_1 \subset T$ of measure $\mu(T_1) \geq 1/2$ with the following property: If $k \in T_1$, then*

$$|\varphi(a_n, k) - \theta_n(k)| \geq \frac{\pi}{8} \tag{19}$$

for infinitely many n.

Remarks. 1. In this Lemma and its proof, all angles are evaluated modulo π.
2. As is probably already clear from the remarks preceding the Lemma, θ_n will eventually be the exceptional phase defined by Lemma 2.2. However, the following proof clearly also works for arbitrary (measurable) functions θ_n.
3. Since $\varphi(a_n, k)$ depends on the values of $V(x)$ for $x \leq a_n$, it's clear that we should first fix c_n, g_n, L_n, W, and only then can Lemma 6.1 be used to finally pick the Δ_n's. (On top of that, the exceptional angle θ_n from Lemma 2.2 of course depends on c_n, g_n, W.) Having said that, we won't find it necessary to follow this correct order also in the representation of our arguments, because that would involve making rather unmotivated choices.

Proof. We begin by showing that the Δ_n's can be (inductively) chosen so that

$$\mu\left(\{k : |\varphi(a_n, k) - \theta_n(k)| < \pi/8\}\right) \leq 1/2 \tag{20}$$

for every n. Since $V = 0$ on (b_{n-1}, a_n), we have $\varphi(a_n, k) = \varphi(b_{n-1}, k) + k\Delta_{n-1}$. So, if we denote the set defined in (20) by $M(\Delta_{n-1})$, then

$$\begin{aligned}
&\int_0^{\pi/2} \mu(M(\Delta_{n-1}))\, d\Delta_{n-1} \\
&\qquad = \int d\mu(k) \int_0^{\pi/2} d\Delta_{n-1}\, \chi_{(-\pi/8, \pi/8)}(\varphi(b_{n-1}, k) - \theta_n(k) + k\Delta_{n-1}) \\
&\qquad \leq \int d\mu(k)\, \frac{\pi}{4k} \leq \frac{\pi}{4}.
\end{aligned}$$

Hence (20) must indeed hold for some $\Delta_{n-1} \in [0, \pi/2]$.

Now (20) implies the statement of the Lemma by an elementary probabilistic argument. Namely, let T' be the set of k's for which (19) holds only for finitely many n. Also, for $k \in T'$, let $N(k)$ be the smallest integer with the property that (19) fails for all $n \geq N(k)$. Then the sets $T'_n \equiv \{k \in T' : N(k) \leq n\}$ increase to T', thus $\mu(T') = \lim \mu(T'_n)$. But (20) in particular implies that $\mu(T'_n) \leq 1/2$ for every n, so the proof is complete. $\square$

Next, we show that the error term $R(j)$ from Theorem 4.2 can cause problems only on a set of μ measure zero. To formulate this precisely, we need some notation: Of course, eventually, we want to get information on the trace of the transfer matrix associated with the potential $c_n W_{g_n}$. By Lemma 2.1, we can instead apply Theorem 4.2 to $c_n W$ if we also replace k by k/g_n. We thus denote the trace and the largest eigenvalue (in absolute value) of the transfer matrix associated with $c_n W$ by D_n and λ_n, respectively. Our goal is to establish the following estimate for $k \in T$:

$$|\lambda_n(k/g_n)| \geq 1 + \frac{1}{6} c_n g_n \exp\left(-\frac{a^{n+1}}{2(a-1)}\right). \tag{21}$$

Indeed, we have

Lemma 6.2 *Let $T_0 = \{k \in T : (21)$ fails for infinitely many $n \in \mathbb{N}\}$. Then $\mu(T_0) = 0$.*

Proof. In view of (6), inequality (21) is implied by

$$|D_n(k/g_n)| \geq 2 + \left(\frac{c_n g_n}{6}\right)^2 \exp\left(-\frac{a^{n+1}}{a-1}\right), \tag{22}$$

so it suffices to show that for μ-almost all k, this latter condition holds for all but finitely many n. Now if $k \in T$, then, for every n, there's a unique $j = j(n,k)$ so that $k \in J_j^{(n)}$. Moreover, $1 \leq j\pi g_n \leq 2$; in particular, $j \geq 1/(\pi g_n)$, and this is bigger than $C_0 c_n \|W\|_1$ for n large enough, because $c_n g_n \to 0$. Using this and invoking Lemma 5.2, we see that for $n \geq n_0$, Theorem 4.2 applies to k/g_n and $c_n W$. Here, n_0 is independent of $k \in T$. Thus

$$\begin{aligned} |D_n(k/g_n)| &\geq 2 + \frac{c_n^2 |\widehat{W}_j|^2}{70 j^2} - R_n(j) \\ &\geq 2 + \frac{\pi^2}{280} c_n^2 g_n^2 \exp\left(-\frac{a^{n+1}}{a-1}\right) - R_n(j). \end{aligned} \tag{23}$$

To pass to the second line, we used (17) and the fact that $j \leq 2/(\pi g_n)$. The remainder R_n satisfies, for every $q > 2$, the following weak type estimate. Here, we absorb some irrelevant constants by C_q, so C_q below is of course not the same as the one from Theorem 4.2. In particular, it now depends on W (but not on n!).

$$\#\{j \in \mathbb{N} : 1 \leq j\pi g_n \leq 2 \text{ and } R_n(j) > \lambda c_n g_n^3\} \leq C_q \lambda^{-q/2} c_n^q \tag{24}$$

We can now again reformulate the claim. Namely, let

$$S_n = \{j \in \mathbb{N} : 1 \leq j\pi g_n \leq 2 \text{ and } R_n(j) > \delta c_n^2 g_n^2 \exp(-a^{n+1}/(a-1))\}$$

with $\delta = \pi^2/280 - 1/36$ (this is positive); then if $k \in J_j^{(n)}$ with $j \notin S_n$, then (23) implies (22). In other words, it suffices to prove that for μ-almost all k, there are only finitely many n so that

$$k \in \bigcup_{j \in S_n} J_j^{(n)}.$$

But by definition of μ,

$$\mu\left(\bigcup_{j \in S_n} J_j^{(n)}\right) \leq \frac{|S_n|}{N_1 \cdots N_n},$$

and (24) with

$$\lambda = \delta \frac{c_n}{g_n} \exp\left(-\frac{a^{n+1}}{a-1}\right)$$

says that

$$|S_n| \leq C_q \delta^{-q/2} (c_n g_n)^{q/2} \exp\left(\frac{q}{2} \frac{a^{n+1}}{a-1}\right).$$

Using this and inserting the definitons (14), (15), and (18), we finally arrive at

$$\mu\left(\bigcup_{j\in S_n} J_j^{(n)}\right) \leq C'_q \exp\left[\frac{a^{n+1}}{a-1}\left(\frac{q}{2}\frac{2a-5}{2a-4} - 1\right) + Cb^n\right]. \tag{25}$$

We can now take $q = 4a/(2a-1)$, say, (obviously, this is bigger than 2, as required). Then the right hand side of (25) is summable, and hence an application of the Borel-Cantelli Lemma completes the proof. □

We are now in a position to show that the potential we've constructed (we still have to choose the L_n's, though) has the properties stated in Theorem 1.3. We begin by proving that the set S defined in (4) contains $T_1 \setminus T_0$. By Lemmas 5.3, 6.1, and 6.2, it must then indeed be true that $\dim S \geq D$.

So fix $k \in T_1 \setminus T_0$, and let n be such that (19) and (21) hold. There are infinitely many such n's. We want to show that the solution of (1) grows on the interval (a_n, b_n) (which corresponds to the nth step of the construction; please consult Section 2 again for this and other definitions). The Prüfer radius R satisfies

$$R(b_n,k) = R(a_n,k)\ \left\| \left(T_{c_n W_{g_n}}(1/g_n, 0; k)\right)^{L_n} e_{\varphi(a_n,k)} \right\|,$$

so Lemmas 2.1 and 2.2 together with (19) and (21) now show that

$$R(b_n,k) \geq R(a_n,k) \sin(\pi/8) \left(1 + \frac{1}{6} c_n g_n \exp\left(-\frac{a^{n+1}}{2(a-1)}\right)\right)^{L_n},$$

or, taking logarithms,

$$\ln(R(b_n,k)/R(a_n,k)) \geq C L_n c_n g_n \exp\left(-\frac{a^{n+1}}{2(a-1)}\right) + \ln(\sin(\pi/8)).$$

This suggests that we take

$$L_n = \left[A \frac{1}{c_n g_n} \exp\left(\frac{a^{n+1}}{2(a-1)}\right)\right], \tag{26}$$

where $[x]$ denotes the largest integer $\leq x$. By taking the constant A in this definition large enough, we can achieve that $\ln(R(b_n)/R(a_n)) \geq \ln 2$, say. Hence

$$\limsup_{n\to\infty} \frac{R(b_n,k)}{R(a_n,k)} \geq 2 \tag{27}$$

for all $k \in T_1 \setminus T_0$. On the other hand, it's easy to see that WKB asymptotics would imply that $R(x,k)$ tends to a positive limit as $x \to \infty$, and this is incompatible with (27). Thus $S \supset T_1 \setminus T_0$, as claimed.

It remains to check that the V constructed above has the right rate of decay, but this is easy: Clearly, if $a_n < x < a_{n+1}$, then $|V(x)| \le C c_n g_n^2$. On the other hand,

$$a_{n+1} = \sum_{r=1}^{n} \left(\Delta_r + \frac{L_r}{g_r} \right) \le \frac{n\pi}{2} + \sum_{r=1}^{n} \frac{L_r}{g_r} \le C \frac{L_n}{g_n}.$$

Here, the last estimate follows from the rapid increase of $L_r g_r^{-1}$. As usual, we plug in the definitions (14), (15), and (26) to obtain

$$\begin{aligned} |V(x)| &\le C_1 \exp\left(-\frac{a^{n+1}}{a-2} - b^{n+1} \right), \\ x &\le C_2 \exp\left(a^{n+1} \frac{3a-4}{(a-1)(2a-4)} + b^{n+1} \right). \end{aligned} \tag{28}$$

A calculation using (13) shows that $(2a-2)/(3a-4) = 1 - D/2 = \alpha$. So (28) implies that $|V(x)| \le C x^{-\alpha} \exp((\alpha - 1) b^{n+1})$; in particular, $V(x) = O(x^{-\alpha})$, as required. The proof of Theorem 1.3 is complete. □

Using (28) again to estimate n, we see that we've in fact proved that

$$|V(x)| \le C x^{-\alpha} \exp\left(-\delta (\ln x)^{1-\epsilon} \right).$$

Here, $\delta, \epsilon > 0$; moreover, we can get arbitrarily small ϵ by taking b sufficiently close to a.

I don't think this improvement of $V(x) = O(x^{-\alpha})$ is particularly interesting. We use Hausdorff dimensions to measure the size of the set S, so it's not natural to use a scale finer than the power scale to classify the potentials. The real point is the following: As we show in the next section, we can sacrifice a bit of the decay rate of V to get more detailed information on the solutions. Then, of course, it will be most useful to start out with a bound better than $O(x^{-\alpha})$.

7 Embedded eigenvalues

In this section, we show that it's also possible to get $\dim P = 2(1-\alpha)$. Here, P is the set of positive energies (or wavenumbers, depending on the context) with L_2 solutions, as defined in the Introduction. Recall also that $P \subset S$, so the following result is, in a sense, a strengthening of Theorem 1.3.

Theorem 7.1 *Suppose $\alpha > 2/3$. Then there is a potential so that* (2) *holds and* $\dim P = 2(1-\alpha)$.

To explain the significance of this variation on Theorem 1.3, we combine it with a result of del Rio, Jitomirskaya, Last, and Simon:

Theorem 7.2 **(= [5, Theorem 5.1])** *Let* $B = \{\beta \in [0,\pi) : \sigma_{pp}(H_\beta) \cap (0,\infty) \neq \emptyset\}$. *Then* $\dim B = \dim P$.

In other words, Theorem 7.1 now shows that Theorem 1.2b) is optimal (for $\alpha \geq 2/3$).

Proof of Theorem 7.1. We use the results and notations of the preceding sections. The basic idea of the modification is as follows. We will try to get rapidly increasing solutions on the set T, and then prove that there are also decaying solutions which will eventually turn out to be square integrable.

To this end, we modify the construction at two places. The L_n's will be taken somewhat larger, and the Δ_n's will be picked such that eventually the Prüfer angle is *never* very close to the phase θ corresponding to the decaying direction. We discuss this latter point first. For technical reasons, it will be necessary to work with two linearly independent solutions y_1, y_2 in this proof. We can take, let's say, $y_1(0) = y_2'(0) = 1, y_1'(0) = y_2(0) = 0$. Denote the corresponding Prüfer angles by φ_1 and φ_2, respectively. The following statement will replace Lemma 6.1.

Lemma 7.3 *Given measurable functions* $\theta_n(k)$*, it is possible to choose* $\Delta_n \in [0,\pi/2]$ *such that the following holds on a set* $T_1 \subset T$ *of measure* $\mu(T_1) = 1$*. If* $k \in T_1$*, then there is an* $n_0 = n_0(k)$ *so that*

$$|\varphi_i(a_n,k) - \theta_n(k)| \geq 1/n^2$$

for all $n \geq n_0$ *and for* $i = 1,2$.

Proof. Let

$$M_n^{(i)} = \{k \in T : |\varphi_i(a_n,k) - \theta_n(k)| < 1/n^2\},$$

$M_n = M_n^{(1)} \cup M_n^{(2)}$, and argue as in the first part of the proof of Lemma 6.1; one sees that the Δ_n's can inductively be picked so that $\mu(M_n) \leq 8/(\pi n^2)$. This is summable, so the claim follows by the Borel-Cantelli Lemma. □

To go from increasing solutions to decaying solutions, we will need the following tool.

Proposition 7.4 **(= [13, Theorem 8.1])** *Let* $B_n \in \mathbb{R}^{2\times 2}$ *with* $\det B_n = 1$*, and put* $T_n = B_n B_{n-1} \cdots B_1$*. Suppose that*

$$\sum_{n=1}^{\infty} \frac{\|B_{n+1}\|^2}{\|T_n\|^2} < \infty.$$

Then there is a $u \in \mathbb{R}^2, \|u\| = 1$ *so that*

$$\|T_n u\|^2 \leq 8 \left[\|T_n\|^{-2} + \|T_n\|^2 \left(\sum_{m=n}^{\infty} \frac{\|B_{m+1}\|^2}{\|T_m\|^2} \right)^2 \right]. \tag{29}$$

Moreover, if $v \in \mathbb{R}^2$ *is linearly independent of* u*, then*

$$\inf_{n\in\mathbb{N}} \frac{\|T_n v\|}{\|T_n\|} > 0.$$

See [13] for the *proof.* The "moreover" part is not explicitly stated in [13], but is established in the proof given there.

In this section, it will be necessary to analyze the behavior of the solutions at the points $x_{n,j} \equiv a_n + j/g_n$. Lemma 7.3, however, gives control on the phases φ only at the points $x_{n,0}$. We will therefore need the following extension of Lemma 2.2. (We drop k in the notation.)

Lemma 7.5 *In the situation of Lemma 2.2, we also have that*

$$\|T^{m+n}e_\varphi\| \geq \|T^m e_\varphi\| \cdot |\lambda|^n \sin|\theta - \varphi|$$

for all $m, n \in \mathbb{N}_0$.

Clearly, in the special case $m = 0$, we recover Lemma 2.2.

Proof. We use the notation from the proof of Lemma 2.2. Of course, we can again assume that $\psi = 0$. In fact, we can further reduce the general case to the following situation: $\lambda > 1$, $0 < \theta \leq \pi/2$, and $0 < \varphi < \pi$ (the case $\varphi = 0$ is of course trivial).

We define $\omega \in (0, 2\pi)$ by $T^m e_\varphi = \|T^m e_\varphi\| e_\omega$. We must then show that $\|T^n e_\omega\| \geq \lambda^n \sin|\theta - \varphi|$. Intuitively, this holds because the position of ω is not worse than that of the original phase φ. More precisely, we have the following: If $0 < \varphi < \theta$, then $0 < \omega < \varphi$, and if $\theta < \varphi < \pi$, then $\varphi < \omega < \pi$. In other words, $T^m e_\varphi$ approaches the subspace corresponding to the large eigenvalue λ. Since this is geometrically obvious, we don't give the formal verification (which is carried out using the explicit form of T).

Now if $0 < \omega \leq \pi/2 + \theta$, then $\sin|\theta - \omega| \geq \sin|\theta - \varphi|$ by the discussion above, so applying Lemma 2.2 with ω taking the role of φ gives the assertion in this case. But if $\pi/2 + \theta < \omega < \pi$, things are even better, because then $\|T^n e_\omega\| \geq \lambda^n$. To prove this, it suffices to consider $n = 1$ because by the above remarks again, under repeated application of T, the phase will remain in the interval $(\pi/2 + \theta, \pi)$. By direct computation as in the proof of Lemma 2.2, we have that

$$\|Te_\omega\|^2 = \frac{\lambda^2}{\sin^2\theta} f(\lambda^{-2}), \tag{30}$$

where

$$f(x) \equiv x^2 \sin^2\omega + \sin^2(\omega - \theta) - 2x\cos\theta \sin\omega \sin(\omega - \theta).$$

The parabola f has its minimum at

$$x_0 = \frac{\cos\theta \sin(\omega - \theta)}{\sin\omega}.$$

Since $0 < \theta < \pi/2$ and $\pi/2 + \theta < \omega < \pi$, we see that

$$x_0 = \cos^2\theta + \sin\theta\cos\theta|\cot\omega| \geq \cos^2\theta + \sin\theta\cos\theta|\cot(\pi/2 + \theta)| = 1.$$

By assumption $\lambda^{-2} \leq 1$, so we get from (30)

$$\|Te_\omega\|^2 \geq \frac{\lambda^2}{\sin^2\theta} f(1) = \lambda^2,$$

as claimed. □

With these preparations out of the way, we can now begin the more careful analysis of the solutions, as outlined above. To this end, fix $k \in T_1 \setminus T_0$, where T_0, T_1 are the sets defined in Lemmas 6.2 and 7.3, respectively. (Although we're about to change Δ_n, L_n, we may still apply Lemma 6.2, because the set T_0 defined there is independent of these parameters.) By these Lemmas together with Lemma 5.3, $\dim T_1 \setminus T_0 \geq D$.

Recall that $x_{n,j} = a_n + j/g_n$, and abbreviate the corresponding transfer matrix by $T_{n,j} \equiv T(x_{n,j}, 0; k)$. Our first aim is to show that Proposition 7.4 applies to this situation. More precisely, we claim that for appropriate L_n's (see (32) below),

$$\sum_{n=1}^{\infty} \sum_{j=0}^{L_n} \frac{\|T(x_{n,j+1}, x_{n,j}; k)\|^2}{\|T_{n,j}\|^2} < \infty$$

(here, we set $x_{n,L_n+1} = x_{n+1,0}$): First of all, we observe that the sequence of integrals

$$\int_{x_{n,j}}^{x_{n,j+1}} |V(x)|\, dx = c_n g_n \|W\|_1$$

is bounded. Thus also $\|T(x_{n,j+1}, x_{n,j}; k)\| \leq C$ by a Gronwall estimate. Moreover, writing $R_{n,j} = R(x_{n,j}, k)$ for the Prüfer radius (corresponding to either y_1 or y_2) at $x_{n,j}$, we clearly have that $\|T_{n,j}\| \geq R_{n,j}$, so it suffices to prove that $\sum_{n,j} R_{n,j}^{-2} < \infty$. Now for n big enough and $l \geq j$, we can proceed as in the preceding section to estimate $R_{n,l}/R_{n,j}$. Namely, we have that $R_{n,l} = R_{n,0}\|T^l e_\varphi\|$, where $T \equiv T_{c_n W}(k/g_n)$ and $\varphi \equiv \varphi(a_n, k)$. Of course, the same relation holds if l is replaced by j. Thus $R_{n,l}/R_{n,j} = \|T^{l-j+j} e_\varphi\|/\|T^j e_\varphi\|$, and we can now invoke Lemma 7.5 together with Lemma 7.3 and inequality (21). We get

$$\ln(R_{n,l}/R_{n,j}) \geq C(l-j)c_n g_n \exp\left(-\frac{a^{n+1}}{2(a-1)}\right) - 2\ln n \qquad (C > 0). \tag{31}$$

The last term comes from the bound $1/n^2$ on $\sin|\theta - \varphi|$. In this section, we take

$$L_n = \left[A^n \frac{1}{c_n g_n} \exp\left(\frac{a^{n+1}}{2(a-1)}\right)\right], \tag{32}$$

where A is a constant with $A > a$. Then (31), when specialized to $j = 0, l = L_n$, says that

$$\ln(R_{n+1,0}/R_{n,0}) \geq CA^n. \tag{33}$$

Using (31), (33), it's now easy to obtain bounds on R^{-2} which show that indeed $\sum_{n,j} R_{n,j}^{-2} < \infty$. We omit the details, because a similar analysis will be carried out below.

We now want to show that the solution y obtained from the vector u from Proposition 7.4 as $(y(x), y'(x)/k)^t = T(x, 0; k)u$ is square integrable. So we must analyze the right-hand side of (29). To begin with, we notice that by the last statement of Proposition 7.4, we must have an estimate of the type $R_{n,j} \geq C\|T_{n,j}\|$ with $C > 0$ for at least one of the two solutions y_1, y_2 (recall that R involved a choice between y_1, y_2, even though this has not been made explicit in the notation). Fix such a solution once and for all. Since the reverse inequality $R \leq \|T\|$ is trivially true, we may replace $\|T\|$ by R whenever it occurs on the right-hand side of (29). Finally, we can of course again estimate $\|B_{m+1}\|$ by a constant. Thus, we can work with

$$\|T_{n,j}u\|^2 \leq CR_{n,j}^{-2} \left[1 + \left(\sum_{x_{m,l} \geq x_{n,j}} \frac{R_{n,j}^2}{R_{m,l}^2} \right)^2 \right].$$

We break this up according as $m = n$, $m = n+1$, or $m \geq n+2$:

$$\|T_{n,j}u\|^2 \leq CR_{n,j}^{-2} \left[1 + \left(\sum_{l=j}^{L_n} \frac{R_{n,j}^2}{R_{n,l}^2} + \sum_{l=0}^{L_{n+1}} \frac{R_{n,j}^2}{R_{n+1,l}^2} + \sum_{m=n+2}^{\infty} \sum_{l=0}^{L_m} \frac{R_{n,j}^2}{R_{m,l}^2} \right)^2 \right],$$

and we call these three sums Σ_1, Σ_2, and Σ_3, respectively. As for $\Sigma_{1,2}$, we use the crude estimates $\Sigma_1 \leq Cn^4 L_n$, $\Sigma_2 \leq Cn^4 L_{n+1}$ which follow at once from (31) and also (33) in the case of Σ_2. To analyze Σ_3, we write it in the form

$$\Sigma_3 = \sum_{m=n+2}^{\infty} \sum_{l=0}^{L_m} \frac{R_{n,j}^2}{R_{n+1,0}^2} \frac{R_{m,0}^2}{R_{m,l}^2} \frac{R_{n+1,0}^2}{R_{m,0}^2}.$$

Noting that $R_{n+1,0} = R_{n,L_n}$, we can again use (31) to bound the first two ratios by Cn^4 and Cm^4, respectively. The last ratio is controlled by iterating (33). This gives a bound of the type $\exp(-CA^m)$. Hence

$$\Sigma_3 \leq C_1 n^4 \sum_{m=n+2}^{\infty} m^4 L_m \exp(-C_2 A^m).$$

Collecting the bounds just proved and using a similar estimate on $R_{n,j}^{-2}$, we get

$$\|T_{n,j}u\|^2 \leq C_1 n^4 \exp(-C_2 A^n) \left(n^4 L_{n+1}^2 + n^4 \sum_{m=n+2}^{\infty} m^4 L_m \exp(-C_2 A^m) \right)^2,$$

with positive constants C_1, C_2. In view of the definitions (14), (15), and (32), it's now clear that the small factors $\exp(-C_2 A^n)$, $\exp(-C_2 A^m)$ dominate everything else. Straightforward manipulations (details are left to the reader) finally give an estimate of the form $\|T_{n,j}u\|^2 \leq C \exp(-\delta A^n)$, with $\delta > 0$.

It now follows easily that the solution y defined as the first component of $T(x)u$ is in L_2. Namely, since $\int_{x_{n,j}}^{x_{n,j+1}} |V|$ is bounded, a Gronwall type estimate lets us extend the inequality $\|T(x)u\|^2 \leq C \exp(-\delta A^n)$ to all of $x_{n,j} \leq x \leq x_{n,j+1}$ (the constant C here might be larger than the original one). Thus

$$\int_0^\infty |y(x)|^2 \, dx \leq C \sum_{n=1}^\infty \left(\Delta_n + L_n g_n^{-1}\right) \exp(-\delta A^n),$$

and this is finite since $\Delta_n + L_n g_n^{-1} \leq \exp(Ca^n)$.

Finally, we have to estimate V. We proceed as in the last part of Section 6. Instead of (28) we obtain this time that if $a_n < x < a_{n+1}$, then

$$|V(x)| \leq C_1 \exp\left(-\frac{a^{n+1}}{a-2} - b^{n+1}\right),$$
$$x \leq C_2 A^n \exp\left(a^{n+1} \frac{3a-4}{(a-1)(2a-4)} + b^{n+1}\right).$$

Now the precaution of introducing b at the beginning of the construction finally pays off: In spite of the additional large factor A^n, we still have $V(x) = O(x^{-\alpha})$ (actually, again a slightly better estimate holds), so the proof of Theorem 7.1 is complete. □

8 Proof of Theorem 1.5

We will again work with wavenumbers k instead of energies E. We use only k's of a rather special form because we want them to lie in the gaps of small periodic potentials, and by Proposition 2.3 (or Theorem 4.2), there's nothing we can do about the location of these gaps. Also, for technical reasons (see Lemma 8.1 below), we mustn't have too many rational relations among the k's (rationally independent k's would be best for this purpose). The choice below is a good compromise between these partly contradictory requirements.

So suppose we're given k_i's of the form $k_i = \pi j(i) 2^{-m(i)}$ with $j(i), m(i) \in \mathbb{N}$, $j(i)$ odd and with $\sum k_i^2 < \infty$. We further assume that $m(i+1) \geq m(i) + 2$ for all $i \in \mathbb{N}$. We will then construct a potential $V(x) = O(x^{-1})$ so that $P \supset \{k_i^2\}$. Clearly, this proves Theorem 1.5; in fact, it also shows that we can make $\sum(E_n - e_n)$ arbitrarily small.

The basic strategy of the proof is as follows: We place gaps at the k_i's and then use the arguments of the preceding section to deduce existence of L_2 solutions at these energies. At each step, we work with only finitely many k_i's (let's say with N_n of them), and we let this number N_n tend to infinity.

Again, several modifications of the original construction are necessary. First of all, we have to use different periodic potentials, depending on the interval

(a_n, b_n). So, here V has the form

$$V(x) = \begin{cases} W_{g_n}^{(n)}(x - a_n) & x \in (a_n, b_n) \\ 0 & x \in (b_n, a_{n+1}) \end{cases}.$$

Of course, the parameter c_n is now superfluous. Then, we must use g_n's which are only exponentially decreasing. We put $g_n = 2^{-n}$. Next, according to what has been said above, we pick functions $W^{(n)} \in L_\infty(0,1)$ which satisfy for all $i \in \mathbb{N}$ with $m(i) \leq n$ the inequality

$$\left| \widehat{W}_j^{(n)} \right| \geq k_i.$$

Here, $j = j(i,n)$ is defined as the unique integer with $\pi j 2^{-n} = k_i$. If $n < m(i)$, there is no condition. We further require that $\int_0^1 W^{(n)} = 0$ and $\sup_n \|W^{(n)}\|_\infty < \infty$. The existence of such functions $W^{(n)}$ follows from Theorem 5.1.

We need still another version of Lemma 6.1 in this section:

Lemma 8.1 *Given angles $\theta_1, \ldots, \theta_N, \varphi_1^{(s)}, \ldots, \varphi_N^{(s)}$ $(s = 1,2)$, there is $\Delta \in [0, 2^{m(N)}]$ so that*

$$\left| \varphi_i^{(s)} + k_i \Delta - \theta_i \right| \geq \frac{\pi}{8} \tag{34}$$

for $i = 1, \ldots, N$, $s = 1,2$.

In the formulation, we already anticipated the fact that in the application of Lemma 8.1 below, θ_i will be the exceptional angle from Lemma 2.2 and $\varphi_i^{(s)} = \varphi_s(b_n, k_i)$ will be the Prüfer angles corresponding to two different solutions. Also, as usual in such situations, all angles are evaluated modulo π.

The *proof* is by induction on N. The case $N = 1$ is trivial. For general N, let $F_N(\Delta) = (k_1\Delta, \ldots, k_N\Delta)$, viewed as a function with values in the torus $\mathbb{T}^N = [0,\pi)^N$. F_N is periodic with period $p_N = 2^{m(N)}/J_N$, where J_N is the greatest common factor of $j(1), \ldots, j(N)$. In particular, we see that $p_N \geq 4p_{N-1}$.

By induction hypothesis, there is a Δ_0 so that (34) holds for $\Delta = \Delta_0$ and $i \leq N - 1$. We may of course assume that $\Delta_0 \in [0, p_{N-1})$. We now look at $F_N(\Delta)$ at the points $\Delta_r = \Delta_0 + r p_{N-1}$, $r = 0, 1, \ldots, p_N/p_{N-1} - 1$. Since $F_N(\Delta_r) = (F_{N-1}(\Delta_0), k_N\Delta_r)$ by periodicity of F_{N-1}, it suffices to consider the last component of F_N. Now the $k_N\Delta_r$, $r = 0, 1, \ldots, p_N/p_{N-1} - 1$ are all distinct for otherwise the period of F_N would be smaller than p_N. It's easy to see that these values are in fact equally spaced when ordered appropriately, and there are at least four of them. So (34) with $i = N$ and $\Delta = \Delta_r$ must hold for some choice of r (in fact, there are at least two r's that will do the job). The estimate on Δ is of course immediate from $p_N \leq 2^{m(N)}$. $\square$

We now pick the numbers N_n which indicate how many k_i's are taken into account at step n of the construction. We fix once and for all $N_n \in \mathbb{N}_0$ with the following properties: $N_n \to \infty$, and $m(N_n) \leq n$ for all $n \in \mathbb{N}$.

We are now ready to show that (1) with $E = k_i^2$ indeed has an L_2 solution. Since the argument is reasonably close to the discussion of the preceding section, it will suffice to provide a sketch. The right choice of the parameters L_n will turn out to be $L_n = (15/\pi) \cdot 2^n$ ($15/\pi$ could be replaced by other sufficiently large constants). Now use Lemma 8.1 to find $\Delta_{n-1} \in [0, 2^{m(N_n)}]$ so that

$$|\varphi_s(a_n, k_i) - \theta_n(k_i)| \geq \frac{\pi}{8} \tag{35}$$

for $s = 1, 2$, $i = 1, \ldots, N_n$. As already explained above, θ denotes the exceptional angle defined via Lemma 2.2. Furthermore, we're again considering two different solutions of (1) here. We've indicated this by using the additional index s.

Fix i, and let n be large enough so that $i \leq N_n$. Then also $m(i) \leq n$; so, if $j = j(i, n)$ is (as above) defined by $j\pi = k_i/g_n$, then $|\widehat{W}_j^{(n)}| \geq k_i$ by construction of $W^{(n)}$. Thus Proposition 2.3 tells us that the eigenvalue λ_n associated with $W^{(n)}$ satisfies

$$|\lambda_n(k_i/g_n)| \geq 1 + \frac{k_i}{10j} = 1 + \frac{\pi}{10} g_n.$$

Actually, it may be necessary here to restrict n to even larger values, to make sure the hypotheses of Proposition 2.3 are satisfied.

With this and (35) as new input, we can now again run the machine based on Proposition 7.4 and Lemma 7.5. We omit the details, but we will give some intermediate results for better orientation. The notation is from Section 7. The analogs of (31), (33) are

$$\begin{aligned} \ln(R_{n,l}/R_{n,j}) &\geq (l-j)\frac{\pi}{5} g_n + \ln(\sin(\pi/8)), \\ \ln(R_{n+1,0}/R_{n,0}) &\geq 2. \end{aligned}$$

(Of course, the constants are anything but optimal.) Using this, one first shows that Proposition 7.4 applies. Then, the analysis of (29) shows that there is a decaying solution y of (1) with

$$|y(x)|^2 \leq C e^{-4n} 2^{2n}$$

for $x \in (a_n, a_{n+1})$. It follows that

$$\begin{aligned} \int_0^\infty |y(x)|^2 \, dx &\leq C \sum_n e^{-4n} 2^{2n} \left(\Delta_n + L_n/g_n\right) \\ &\leq C \sum_n e^{-4n} 2^{2n} \left(2^{n+1} + (15/\pi) 2^{2n}\right) < \infty, \end{aligned}$$

as desired. The estimate on Δ_n uses the fact that $m(N_n) \leq n$.

To conclude the proof, we note that for $x \in (a_n, a_{n+1})$, we have that $|V(x)| \leq C_1 g_n^2 = C_1 2^{-2n}$. On the other hand,

$$x \leq \sum_{r=1}^{n} (\Delta_r + L_r/g_r) \leq C_2 2^{2n}$$

for these x, thus $V(x) = O((1+x)^{-1})$. $\square$

9 Second proof of Corollary 1.6

In [26], Simon uses von Neumann-Wigner potentials (see [28])

$$W(x) = C\frac{\sin(2kx + \theta)}{1 + x}$$

in an original way to construct a V with prescribed point spectrum and "almost" $V(x) = O((1 + x)^{-1})$. (A different approach to this problem had earlier been presented in [16]; [26] also removes a technical condition from [16].)

The gist of Simon's work is contained in the following:

Theorem 9.1 ([26]) *Given $k_m^{(n)} > 0$ $(n \in \mathbb{N}, m = 1, \dots, N_n)$, put*

$$f_n(x;\theta) = \sum_{m=1}^{N_n} k_m^{(n)} \sin\left(2k_m^{(n)}x + \theta_m^{(n)}\right).$$

Then there are cut-offs $x_n \geq 0, x_n \to \infty$ so that for arbitrary $\theta_m^{(n)} \in [0, 2\pi)$, the Schrödinger equation (1) with potential

$$V(x) = \frac{4}{1 + x} \sum_{n=1}^{\infty} f_n(x;\theta)\chi_{(x_n,\infty)}(x)$$

has an L_2 solution for all $E \in \left\{\left(k_m^{(n)}\right)^2\right\}$.

We refer to [26] for the *proof*, but a few comments may be helpful. First of all, note that the sum defining $V(x)$ is finite for each fixed x, so convergence is not an issue. Then, in [26], the k's are not grouped together as in the definition of the functions f_n above, but that doesn't make a real difference in the proof. The basic idea is that V_N defined as the finite sum $4/(1 + x)\sum_{n=1}^{N} \cdots$ would generate L_2 solutions at the k's occuring in the sum by the standard theory of von Neumann-Wigner potentials [8]. Then, with x_{N+1} chosen sufficiently large, one shows that things change only little at the energies considered so far when passing from V_N to V_{N+1}.

In [26], Simon uses the θ's to control the initial phases of the L_2 solutions. Here, we trade this additional information for bounds on $|V|$, thus obtaining an example that proves Corollary 1.6. We specialize to the situation where

$$k_m^{(n)} = \epsilon_n\left(1 + \frac{m}{N_n}\right), \qquad m = 1, \dots, N_n,$$

with $\epsilon_n > 0, N_n \in \mathbb{N}$. One could of course treat other examples as well, but I don't see how to obtain Theorem 1.5 in full generality with the method of this section. The crucial property of the choice above is that f_n becomes a periodic function with not too large period. To see that something of this sort is necessary to get reasonable bounds on f_n, consider the extreme case when the $k_m^{(n)}$ $(m = 1, \dots, N_n)$ are rationally independent: Then $\|f_n\|_\infty = \sum_m k_m^{(n)}$, independently of the θ's, and this is far too big for our purposes.

We will use the fact that random trigonometric polynomials are, up to logarithmic (in the degree) corrections, bounded pointwise by their L_2 norm. In the case at hand, this result takes the following form:

Proposition 9.2 *There exist* $\theta_m^{(n)} \in [0, 2\pi)$ *so that*

$$\|f_n(\cdot;\theta)\|_\infty \leq 4\left[\epsilon_n^2 N_n \ln(32\pi N_n^2)\right]^{1/2}.$$

Although this is merely an adaptation of classical methods to the situation under consideration here, we will give the full proof below. However, let us first show how the Proposition can be used to prove Corollary 1.6. To this end, we let $\epsilon_n = 2^{-n^2-n/2}, N_n = 2^{2n^2}$. Then

$$\sum_{m=1}^{N_n} \left(k_m^{(n)}\right)^{2-1/n} \geq \epsilon_n^{2-1/n} N_n = 2^{1/2},$$

so $E_{m,n} \equiv \left(k_m^{(n)}\right)^2$ is not in ℓ_p for any $p < 1$. On the other hand, with θ's picked according to Proposition 9.2, we have that

$$\|f_n\|_\infty \leq Cn2^{-n/2}$$

is summable, thus the V from Theorem 9.1 satisfies $V(x) = O((1+x)^{-1})$, as required.

Proof of Proposition 9.2. We basically follow the development of [9, Chapter 6]. Of course, n is fixed, so we can drop this index in the notation. We view the θ_m $(m = 1, \dots, N)$ as independent, identically distributed random variables with uniform distribution, and we show that the assertion in fact holds with large probability. The function f is periodic in x with period $\pi N/\epsilon$, so

$$M(\theta) \equiv \sup_x |f(x;\theta)| = \max_{0 \leq x \leq \pi N/\epsilon} |f(x;\theta)|.$$

Pick x_0 so that $|f(x_0)| = M$. Notice that

$$k_m = \frac{2\epsilon}{\pi N} \int_0^{\pi N/\epsilon} f(x) \sin(2k_m x + \theta_m)\, dx \leq 2M,$$

hence

$$|f'(x)| = 2\left|\sum_{m=1}^{N} k_m^2 \cos(2k_m x + \theta_m)\right| \leq 8\epsilon NM.$$

It follows that $|f(x)| \geq M - 8\epsilon NM|x - x_0|$. In particular, there's an interval $I = I(\theta) \subset [0, \pi N/\epsilon]$ of length

$$|I(\theta)| \geq \min\left\{\frac{1}{8\epsilon N}, \frac{\pi N}{\epsilon}\right\} = \frac{1}{8\epsilon N}$$

so that $|f(x)| \geq M/2$ for all $x \in I$.

We can now estimate the quantity $E(e^{\lambda M/2})$, where $E(\cdots)$ denotes expectation with respect to the probability measure introduced above, and $\lambda > 0$ will be chosen later. Namely, also using independence and the computation

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} e^{a\sin(\alpha+\theta)}\,d\theta &= \sum_{n=0}^{\infty}\frac{a^n}{n!}\,\frac{1}{2\pi}\int_0^{2\pi}\sin^n\theta\,d\theta\\ &= \sum_{n=0}^{\infty}\frac{a^{2n}}{(2n)!}4^{-n}\binom{2n}{n} \leq e^{a^2/4},\end{aligned}$$

we get

$$\begin{aligned}\frac{1}{8\epsilon N}E\left(e^{\lambda M/2}\right) \leq E\left(|I|e^{\lambda M/2}\right) &\leq E\left(\int_{I(\theta)}\left(e^{\lambda f(x;\theta)}+e^{-\lambda f(x;\theta)}\right)dx\right)\\ &\leq \int_0^{\pi N/\epsilon} E\left(e^{\lambda f(x)}+e^{-\lambda f(x)}\right)dx \leq \frac{2\pi N}{\epsilon}\,e^{\lambda^2\epsilon^2 N}.\end{aligned}$$

We can write this in the form

$$E\left(\exp\left(\frac{\lambda}{2}\left[M - 2\lambda\epsilon^2 N - \frac{2}{\lambda}\ln(32\pi N^2)\right]\right)\right) \leq \frac{1}{2}$$

and deduce that with probability at least $1/2$,

$$M \leq 2\lambda\epsilon^2 N + \frac{2}{\lambda}\ln(32\pi N^2).$$

With $\lambda = \epsilon^{-1}N^{-1/2}(\ln(32\pi N^2))^{1/2}$, we obtain the claim. $\square$

A Norm bounds

In this appendix, we sketch the proof of Theorem 4.1. We follow [1] and private notes of Kiselev. First of all, the bound on M_f follows from the fact that the Fourier transform itself is bounded as a map from $L_p(0,1)$ to ℓ_q if $1 \leq p \leq 2$ and $1/p + 1/q = 1$. Now such norm bounds automatically carry over to the corresponding maximal functions, provided that $p < q$, i.e. $p < 2$. See [1, Theorem 3.1] and also [2] for this remarkable general fact.

The proof of the bound on $M_{f,g}$ is more involved. We need the following result, which is essentially a special case of [1, Theorem 4.1].

Theorem A.1 *Let $K_i(n,\cdot) \in L_\infty(0,1)$ for every $n \in \mathbb{Z}$, and suppose that the corresponding integral operators*

$$\begin{aligned}T_i :& L_p(0,1) \to \ell_q \qquad (1 \leq p < 2, q \geq 2)\\ (T_i f)(n) &= \int_0^1 K_i(n,x)f(x)\,dx\end{aligned}$$

are norm bounded for $i = 1, 2$. Let

$$T(f, g)(n) = \int_0^1 dx\, K_1(n, x) f(x) \int_0^x dt\, K_2(n, t) g(t).$$

Then

$$\|T(f, g)\|_{q/2} \leq C \|f\|_p \|g\|_p$$

for all $f, g \in L_p$. Here, we may take a constant C that only depends on p and the norms of T_1, T_2, but not on the particular form of the kernels K_i.

Before proving Theorem A.1, let us first show how it is used to complete the proof of Theorem 4.1. Clearly, the operator T_1 with kernel $K_1(n, x) = \chi_{(0,b_n)}(x) e^{2\pi i n x}$ is dominated by the corresponding maximal function for arbitrary choice of the sequence $b_n \in [0, 1]$. We know already that this maximal function is norm bounded as a map from L_p to ℓ_q if $p \in [1, 2)$ and $1/p + 1/q = 1$, so $\|T_1\| \leq C$, independently of (b_n). Thus Theorem A.1 with this K_1 and $K_2(n, x) = e^{-2\pi i n x}$ yields

$$\left\| \int_0^{b_n} dx\, f(x) e^{2\pi i n x} \int_0^x dt\, g(t) e^{-2\pi i n t} \right\|_{q/2} \leq C \|f\|_p \|g\|_p.$$

Here, (b_n) is arbitrary and C is independent of (b_n), so $M_{f,g}$ satisfies the same inequality, as claimed.

Proof of Theorem A.1. We need a decomposition of the characteristic function of the triangle $0 \leq t \leq x \leq 1$ into an infinite sum of characteristic functions of rectangles. This decomposition is adapted to the functions f, g. Let $F = (|f|^p + |g|^p)^{1/p}$. Obviously, $F \in L_p$, and by bilinearity, we can normalize so that $\|f\|_p = \|g\|_p = 2^{-p}$, thus $\|F\|_p = 1$. For every $m \in \mathbb{N}$, decompose $[0, 1]$ into 2^m subintervals $I_i^{(m)}$ so that $\|F\chi_{I_i^{(m)}}\|_p^p = 2^{-m}$ for all $i \in \{1, \ldots, 2^m\}$. More specifically, we do this inductively, and at step m, we construct the intervals $I_i^{(m)}$ by subdividing every $I_j^{(m-1)}$ ($j = 1, \ldots, 2^{m-1}$) into two new intervals. Furthermore, we label in the natural way, that is, $I_{j+1}^{(n)}$ is the right neighbor of $I_j^{(n)}$. Then we have that for almost all $(x, t) \in [0, 1]^2$,

$$\chi_{(0,\infty)}(x - t) = \sum_{m=1}^{\infty} \sum_{i=1}^{2^{m-1}} \chi_{I_{2i-1}^{(m)}}(t) \chi_{I_{2i}^{(m)}}(x).$$

This is most easily seen from a sketch; for the formal proof, see [1].

Using this decomposition to evaluate $T(f,g)$, we get

$$
\begin{aligned}
\|T(f,g)\|_{q/2} &= \left\| \sum_{m=1}^{\infty} \sum_{i=1}^{2^{m-1}} T_1\left(f\chi_{I_{2i}^{(m)}}\right) T_2\left(g\chi_{I_{2i-1}^{(m)}}\right) \right\|_{q/2} \\
&\leq \sum_{m=1}^{\infty} \sum_{i=1}^{2^{m-1}} \left\| T_1\left(f\chi_{I_{2i}^{(m)}}\right)\right\|_q \left\| T_2\left(g\chi_{I_{2i-1}^{(m)}}\right)\right\|_q \\
&\leq \|T_1\|\,\|T_2\| \sum_{m=1}^{\infty} \sum_{i=1}^{2^{m-1}} \left\| f\chi_{I_{2i}^{(m)}}\right\|_p \left\| g\chi_{I_{2i-1}^{(m)}}\right\|_p \\
&\leq \|T_1\|\,\|T_2\| \sum_{m=1}^{\infty} \sum_{i=1}^{2^{m-1}} \left\| F\chi_{I_{2i}^{(m)}}\right\|_p \left\| F\chi_{I_{2i-1}^{(m)}}\right\|_p \\
&= \|T_1\|\,\|T_2\| \sum_{m=1}^{\infty} 2^{-2m/p} 2^{m-1} = \left(2^{2/p} - 2\right)^{-1} \|T_1\|\,\|T_2\|,
\end{aligned}
$$

as required. □

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