# Kinetical theory beyond conventional approximations and 1/f-noise

Yu.E.Kuzovlev
Donetsk Physics and Technology Institute of NASU
340114, Donetsk, Ukraine
e-mails: kuzovlev@kinetic.ac.donetsk.ua
medvedev@host.dipt.donetsk.ua

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**Abstract**

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# 1 INTRODUCTION

The low-frequency 1/f-noise (flicker noise) was discovered by Johnson in 1925 in electronic lamps. At present 1/f-noise is known in giant variety of systems and processes in physics, chemistry, biology, medicine, cosmic and Earth sciences and in human activity. Perhaply, thousands of experiments were performed during many years in order to comprehend this apparently mysterious phenomenon. Howevere, a conventional theory of 1/f-noise is still lacked.

It seems that the situation testifies the absence of true ideas. May be, that is why there are no new substancial reviews on 1/f-noise like ones [1,2] published in 1981. As before, usually 1/f-noise is believed be produced by some slow processes with a broad distribution of long relaxation times (life-times), and 1/f spectrum is thought as superposition of Lorentzians. But in most particular cases the hypothetical slow processes are not identified.

The popular idea about long life-times is evoked by common opinion that any long-range statistical correlations associated with 1/f spectrum reflect some real long-living causal connections between the events which took place in the past and which could occur in the future. But as long ago as in 1950 N.S.Krylov [3] argued the necessity to make difference between statistical correlations and factual causality.

In view of this distinction, the non-trivial explanation of 1/f-noise is possible [4-11] attributing its origin just to absence of long-living causality. More concretely, to processes which currently forget their own history and thus could not keep an eye on its characteristics. The flow of incident events produced by such a process have no certain "number of events per unit time", and just this circumstance implies 1/f-noise. Indeed, if a process remembers nothing about old events and thus about previous number of events per unit time, how could it know about the future one? If the past is of no importance, all the more the future is too.

From this viewpoint, the long-range 1/f correlations are only the manifestation of a freedom of the random process. Correspondingly, the shorter is life-time of real causal memory the larger is magnitude of 1/f fluctuations [4-8], i.e. just the inverse rule takes place, and only if the number of events per unit time is somehow anticipated by underlying dynamics one gets a process without 1/f-noise.

For a first look, this picture is in contradiction with traditional hopes that absence of long-living memory should result in white noise, Marcovian behaviour and Poissonian flows of events. But in fact the latter is only very special variant of pure randomness. One has not to break foundations, merely the central limit theorem is not valid, and so another chapters of the probability theory [12] and theory of randomness in dynamical systems [13] should be involved.

The main purpose of this maniscript is to review our own works on this subject because their English translations are hardly available. We do not try to review either the infinity of experimental data or recently suggested models of 1/f-noise (although some models may be in fact more close to our ideology than to conventional tendency to long life-times). Instead, we would like to show that if one avoids the limitations and approximations standardly attracted when constructing models of noise then both the phenomenology [4-8,10] and approaches based on statistical mechanics [9,11] naturally predict 1/f contribution to whole noise without long life-times and slow processes.

In particular, the correct procedure of derivation of gas kinetics from exact BBGKY equations demonstrates [9] that 1/f-noise exists even in gas in the Boltzmann-Grad limit, that is in the system where nothing like slow processes can exist.

Unfortunately, up to now those who deal with rigorous statistical mechanics payed almost no attention to 1/f-noise and likely had no suspicions that its nature could be as fundamental as that of relaxation and irreversibility in Hamiltonian systems. In our opinion, 1/f-noise is typical companion of relaxation and dissipation phenomena in time-reversible dynamics, and its sense is the absence of absolutely certain (well time-averagable) quantitative rates of relaxation and irreversibility.

# 2 THE PROBLEM OF 1/F-NOISE

## 2.1 FORMAL DEFINITION Of 1/F-NOISE

Let $X(t)$, $\langle X\rangle$ and $\delta X(t) \equiv (X(t) - \langle X\rangle)/\langle X\rangle$ be a random process, its mean value and its relative variation, respectively. The process will be called 1/f-noise if the power spectrum $S_{\delta X}(f)$ of $\delta X(t)$ looks approximately as $1/f$ at low frequencies and has no tendency to saturation at $f \rightarrow 0$ . At high frequences, in practice $S_{\delta X}(f) \rightarrow S_\infty \neq 0$ (the white noise level). The brackets $\langle\rangle$ denote the ensemble averaging. Sometimes spectra like $S_{\delta X}(f) \propto f^{-\gamma}$ are observed with $\gamma$ noticably different from unit, but we shall not discuss this unprincipal difference, except the meaning of formal non-stationarity at $\gamma > 1$.

The distinctive temporal property of 1/f-noise is the impossibility of precise measuring its mean value $\langle X \rangle$ with the help of time smoothing procedure. Indeed, the relative deviation

$$\delta x(T) \equiv [\frac{1}{T}\int_0^T X(t)dt - \langle X\rangle]/\langle X\rangle$$

of the time-averaged value from $\langle X \rangle$ has the variance

$$\left\langle \delta x^2(T) \right\rangle \approx \int_{f_0}^{\infty} [\sin(\pi f T)/\pi f T]^2 S_{\delta X}(f) df$$

with $f_0$ being the inverse duration of observations. When $T$ increases, first this integral decreases as $\langle \delta x^2(T)\rangle \approx S_\infty/T$ , in accordance with the large number law, but later turns into nearly constant.

Formally, $\delta x(T)$ could either decrease in an extremely slow logarithmic way or even slowly grow if $\gamma > 1$ . But practically any case results in existence of nonzero limit of $\sqrt{\langle \delta x^2(T)\rangle}$ at large intervals $T$ . Therefore, some characteristical uncertainty of the mean value of $X(t)$ does exist, the so-called 1/f-floor or flicker floor.

## 2.2 EXAMPLES OF THE FLICKER FLOOR

a) Let $X(t)$ be the number of alpha-particle decays per unit time in a piece of radioactive matter (for instance, in $Am$ with life-time of order of 1000 years). The experimentally discovered 1/f fluctuations of the rate of decays give the evidence that the decay probability possesses relative uncertainty $\sim 10^{-4}$ .

b) Let $X(t)$ be the electron drift velocity in a current-carrying slightly doped semiconductor. In accordance with the empirical Hooge's formula, $S_{\delta X}(f) \approx \frac{\alpha}{f}$ , where $\alpha \approx 2 \cdot 10^{-3}$ , at frequences lower than tens kHz [1,2]. The corresponding relative uncertainty of drift velocity and mobility is of order of 0.1 , i.e. surprisingly large.

c) The giant relative uncertainty, $\sim 1$ or even larger, characterizes 1/f-fluctuations of the inner friction and dissipation rate in excited quartz crystals [14,15]. In this case $X(t)$ can denote the energy dissipated per unit time.

## 2.3 IT IS EVERYWHERE! MORE EXAMPLES

i) Different solid, liquid and gaseous electric conductors [1,2]. $X(t)$ is the charge transported per unit time, or resistance. Many of experiments are well described by $S_{\delta X}(f) \approx \alpha / nf$ , where $\alpha = 10^{-7} - 10^{1}$ for various media and $n$ is the number of charge carriers in a sample under observation.

ii) Highway traffic. $X(t)$ is the number of cars passing per unit time.

iii) Laser light Raman scattering [16]. $X(t)$ is the number of photons scattered per unit time.

iv) Chemical reactions in small volumes. $X(t)$ is the number of reaction events per unit time.

v) Definite type of diabetics. $X(t)$ is the consumption of insuline per unit time.

vi) Cosmic rays. $X(t)$ is the current intensity of cosmic radiation.

vii) Music. $X(t)$ is the volume or dominating tone or other property of a sound track.

viii) Earth's rotation. $X(t)$ is the angular velocity of rotation.

ix) DNA. $X(t)$ is the number of definite complementary pairs per unit length of DNA sequence. In this case the distance along the sequence serves instead of time (for review and some original ideas see [17,18]).

x) Thermal equilibrium voltage noise of a currentless disconnected conductor. $X(t)$ is the instantaneous power spectrum $S$ of the noise (that is the spectrum currently obtained by filtering, squaring and smoothing over a finite time interval). Voss and Clarke [19] supposed that the resistance 1/f-noise in nonequlibrium current-carrying conductors (the example i) ) means the existence of similar 1/f-fluctuations of the power spectrum $S$ of equilibrium white noise, in accordance with Nyquist formula if extended to slowly varying resistivity, and experimentally confirmed their hypothesis.

## 2.4 1/F-NOISE OF EQUILIBRIUM WHITE NOISE AND HIGHER-ORDER CUMULANTS

The example x) is of special importance for us, because it demonstrates that 1/f-noise in disturbed thermodynamical systems can be theoretically investigated with the help of four-point equilibrium correlators. Indeed, the spectrum $S$ of equilibrium noise $u(t)$ is measured as a quadratic functional of $u(t)$ , so the correlation function and spectrum of fluctuations of $S$ can be expressed with four-point correlators of $u(t)$ [20].

To be more precise, the information about 1/f-noise is hidden in those specific part of four-point correlators which can not be reduced to lower order correlations, i.e. in the four-point cumulants (semi-invariants) $\langle u_1, u_2, u_3, u_4 \rangle$ defined by

$$\langle u_1 u_2 u_3 u_4 \rangle = \langle u_1 u_2 \rangle \langle u_3 u_4 \rangle + \langle u_1 u_3 \rangle \langle u_2 u_4 \rangle + \langle u_1 u_4 \rangle \langle u_2 u_3 \rangle + \langle u_1, u_2, u_3, u_4 \rangle$$

where $u_i \equiv u(t_i)$ and the equaliy $\langle u(t) \rangle = 0$ is supposed. In most practically interesting cases the equilibrium noise $u(t)$ looks as white noise with a single characteristic relaxation time $\tau_c$ which determines its average spectrum, $\langle S \rangle = \int \langle u(\tau) u(0) \rangle \, d\tau \approx 2 \langle u^2 \rangle \tau_c$ . The first three terms on the right-hand side reflect only this short-range relaxation and say nothing about low-frequency fluctuations of $S$ described by the last cumulant term.

If the equilibrium noise $u(t)$ was purely Gaussian, this term would turn to zero. So the existence of 1/f-noise of white noise inevitably means that $u(t)$ possesses non-Gaussian statistics.

## 2.5 WHAT IS THE QUESTION?

Though being everywhere the 1/f-noise phenomenon still stays without a commonly accepted explanation. Therefore the natural question arises: could it be explained in some general manner or not?

At present, the most popular approaches to this issue are based on the idea that 1/f-noise is caused by some "slow processes" having large life-times [1,2]. For instance, in case of electric 1/f-noise in solids the slow processes could be associated with scattering of electrons by metastable movable structural defects or with their trapping by long-living structural inhomogenities. With no doubts, some part of low-frequency noise in solids is produced by such mechanisms and reflects real slow processes. It is interesting that formal summation of Lorentzians corresponding to a distribution of activation energies can reproduce factually observed connections between frequency and temperature dependencies of 1/f-noise in metals [1].

But, at the same time, there are no doubts that this aproach is unsufficient for complete explanation of 1/f-noise even in solids, not to mention the electric 1/f-noise in liquid metals and other liquids, all the more, various 1/f-noises as in the above examples. May me, Tandon and Bilger [21] were right that the nice idea about summation of Lorentzians perhaply is a disaster for building a true 1/f-noise theory.

# 3 PHILOSOPHY AND PHENOMENOLOGY OF 1/F-NOISE

## 3.1 1/F WITHOUT SLOWNESS

The non-standard idea we are carrying from 1982 [4-11] is that 1/f-noise in general has no relation to actual (physical, chemical, etc.) slow processes, long-living memory and causal correlations, but, in opposite, manifests the absence of thats.

To feel the idea let us consider the example v) . Assume that after taking a doze of insuline a diabetics completely restores the basic normal state of his organism and takes next doze only under the necessity, not under a timetable. By definition, the basic state does not depend on how frequently he taked the drag during last day, last week, etc., all the more during last year. The past is constantly forgotten! If it is so, there are neither reasons nor ways to establish some definite "doze per unit time". And one has no grounds to hope that the time avering will bring him some predictably certain result, just because of absence of long-living memory! The events are accumulated following the principle "let it be what occurs to be". The future is as unknown as the past.

But such a free behaviour, without certain limit of the time smoothed doze, means the existence of 1/f-fluctuations of "doze per unit time". Why 1/f , not other? If the smoothed doze had some definite limit but dependent on the occasion, the low-frequency spectrum of the doze would acquire the delta-function $\delta(f)$ . In absence of the limit the spectrum should have the same dimensionality on the frequency scale, that is be nothing but $1/f$ [4].

By its nature, the 1/f-noise produced in such a way has no saturation at zero frequency, and one can not saturate it by a noisy perturbation.

## 3.2 STATISTICAL LONG CORRELATIONS WITHOUT LONG-LIVING CAUSALITY

The existence of the 1/f floor means that in a random flow of events the probable random deviation of the number of events from its ensemble-averaged value grows nearly as the average, i.e. is approximately proportional to time. The loss of information about old events gives the natural explanation of this property : merely the system can not distinguish what part of happened events could be qualified as a proper part and what as a deviation from it. Any present result equally may serve as the starting point for further accumulation of events.

The remarkable paradox is that this memoryless behavior can not be statisticaslly described somehow except the language of long-living correlations!

All the events which belong to one and the same time piece seem mutually correlated, because all thats equally participate in the result. But there are no actual long-living causality beyond the corresponding statistical correlations [4-10].

Thus slow processes and slow statistical correlations are not one and the same thing. This important circumsance was underlined by N.Krylov [3] in 1950, in the frame of his critical revision of statistical thermodynamics. But he pointed out no examples. We supposed in 1982 that 1/f-noise is just the case [4].

## 3.3 SELF-FORGETTING FLOWS OF EVENTS AND CAUCHY STATISTICS

Let us consider a stationary random process of accumulation of some quantity $Q(t)$ , with instantaneous speed $u(t)$ and $\Delta Q(t) \equiv Q(t) - Q(0) = \int_0^t u(t')dt'$ being the increment during time interval $(0,t)$ , $t > 0$ . For instance, $\Delta Q$ may be an amount of drag consumpted by diabetics. In other example $\Delta Q$ is spatial displacement of Brownian particle subjected to a constant force $x$ [6]. In this case we take in mind so large $t$ that the drift component of displacement much exceeds its diffusional component, that is $U_0 t >> \sqrt{2D_0 t}$ (here $U_0$ , $D_0$ and $\mu = D_0/T$ are the average drift velocity, diffusivity and mobility, defined by the relation $U_0 \equiv \langle u(t)\rangle = \mu x$, with $u(t)$ and $T$ being velocity and temperature, respectively). One more example is the charge transport in electric junction, if applied voltage much exceeds $T/e$ and so almost all electron jumps are in one and the same direction. Then the number of transported electrons $\Delta Q(t)$ also can be treated as a random flow of events (random walk) with definitely (let positively) directed steps, $\Delta Q(t) \geq 0$ .

Let $\tau_0$ be the maximum of time scales characterizing causal correlations of $u(t)$ , i.e. the life-time of dependence of the process on its history. We want to analyse the asymptotical statistics of $\Delta Q(t)$ and of the time-averaged speed (rate) of the walk $U \equiv \Delta Q/t$ (doze per unit time, drift velocity, current, etc.) at $t >> \tau_0$ , when all the real memory effects should disappear [6,8].

Consider the amounts accumulated during two successive time intervals each longer than $\tau_0$ , $\Delta Q(t) \equiv Q(t) - Q(0)$ and $\Delta Q'(t) \equiv Q(2t) - Q(t)$ , the summary amount $\Delta Q(2t) = \Delta Q(t) + \Delta Q'(t)$ and the corresponding time-smoothed rates $U(t) \equiv \Delta Q(t)/t$ , $U'(t) \equiv \Delta Q'(t)/t$ and $U(2t) \equiv \Delta Q(2t)/2t = [U(t) + U'(t)]/2$ .

Because of $t >> \tau_0$ we believe that $U$ and $U'$ are independent random values. From the other hand, at $t >> \tau_0$ the result should be insensitive to $t$ , because the system forgets when the measurement started and how large is $t$ as compared with $\tau_0$ . Consequently , $U(2t)$ should coinside with $U(t)$ in the statistical sense [6], i.e. $2U(2t) = U(t) + U'(t)$ is equivalent to $2U(t) = U(t) + U(t)$ . We come to conclusion that the sum $U + U'$ of two independent random values behaves as the sum $U + U$ of two completely dependent values! In other words, $\Delta Q(2t)$ does not statistically differ from $2\Delta Q(t)$ .

That is nothing but the distinctive feature of Cauchy statistics. It is easy to verify this in case of a random value $z$ subjected to symmetrical Cauchy distribution whose probability density is $W(z, w) = \frac{w}{\pi}(w^2 + z^2)^{-1}$ , with $w$ being the width of distribution. Indeed, the convolution of $W(z, w)$ with itself equals to $W(z, 2w)$ which is nothing but the probability density of the doubled variable $2z$ .

The same is true also with respect to maximally asymmetrical Cauchy distribution [12] whose characteristic function (CF), $\int \exp(ikz)W(z, w)dz$ , looks as $\exp[-ikw \ln(-ikcw)]$ , where the width parameter $w$ determines the magnitude of random value, $z \propto w$ , and $c \sim 1$ .

Obviously, if $z$ is identified with $\Delta Q$ then the width $w$ must be proportional to time, $w \propto t$ . Hence, the statistical equivalence of $U(2t)$ and $U(t)$ implies both the probabilistic and temporal properties of $\Delta Q(t)$ .

We see that Cauchy statistics makes no difference between dependent and independent contributions to a random walk. Therefore, it is naturally suitable statistics to describe correlations what have no actual causality beyond them. Its characteristical temporal property is that the magnitude of accumulated fluctuations is proportional to time, $\Delta Q(t) \propto t$ (that is the central limit theorem fails).

## 3.4 CAUCHY STATISTICS AND LONG CORRELATIONS

Now we come to important point. The formal defect of the ideal Cauchy statistics is the infinity of statistical moments. Indeed, the identity of $U+U'$ and $U+U$ in the sense of moments is possible only if the variance of $U$ is infinite. This means that probability density has a long power-law tail.

In practice one hopes that random rates are somehow bounded and have finite cumulants and moments. From the other hand, even if a statistical ensemble has infinite moments, this infinity never could be observed, at least while finite time intervals. Any moment practically determined with the help of time averaging always is finite with unit probability.

But no theory can do without ensemble language. Therefore, we need in an ensemble which would be close to the Cauchy ensemble but possess finite moments and be applicable to real time-limited observations.

Clearly, such an ensemble will allow for as large $\Delta Q(t)$ deviations from its most probable behaviour as long is the observation time, in accordance with the temporal scaling, $\Delta Q(t) \propto t$ , of the ideal Cauchy random walk. Hence, this ensemble inevitably includes long-living statistical correlations although thats express nothing mote than the perfect randomness of the process.

## 3.5 SELF-FORGETTING AND SCALE INVARIANCE

The universal features of such statistics can be derived from following requirements [6,8].

a). The ensemble average value of the accumulated amount is strictly proportional to time, $\langle \Delta Q(t) \rangle = U_0 t$ , that is the mean value of time-averaged rate does not depend on $t$ , $\langle U(t) \rangle = U_0$ .

b). In the limit $t/\tau_0 \to \infty$ , the quantities $\Delta Q$ and $U$ are are constituted by many causally independent incidents, so the probability distribution of $\Delta Q$ should acquire an infinitely divisible form. In accordance with the general Levy-Khinchin representation of infinitely divisible distributions (for one-directed asymmetrical walks with $\Delta Q \geq 0$ ) [12], the CF of $\Delta Q$ can be written as

$$\langle \exp\{ik\Delta Q(t)\} \rangle = \exp\{tU_0 \int_0^\infty (e^{ikq} - 1) G(q,t) \frac{dq}{q}\}$$

The essence of this formula is that function $G$ is non-negative, $G(q,t) \geq 0$

. Besides, because of condition a) , the equality $\int_0^\infty G(q,t)dq = 1$ should be satisfied.

c). Due to $t >> \tau_0$ , $\Delta Q(t)$ is believed to asymptotically behave as a random walk with independent increments. This means that logarithm of CF is approximatelly proportional to time, $\ln \langle \exp\{ik\Delta Q\}\rangle \propto t$ , i.e. $G(q,t)$ becomes almost time independent at large $t$ .

d). $\Delta Q(t)$ should asymptotically behave in a scale-invariant way, because of indifference to the short-time scale $\tau_0$ and absence of larger characteristic scales. The concrete form of scale invariance is prompted by the mean law $\langle \Delta Q\rangle \propto t$ and looks as

$$\Delta Q(\lambda t) \sim \lambda \Delta Q(t)$$

where symbol $\sim$ denotes the approximate statistical equivalence. Thus though being random $\Delta Q(t)$ grows nearly proportionally to time.

The physical meaning of this assertion was already discussed: a process what is currently forgetting its previous accumulations (number of events, Brownian displacement, etc.) is unable to make difference between its mean part and deviation part. Any result of previous behavior is equally acceptable, regardless of its difference from ensemble expectations! Therefore it is more correct to say that just the average $\langle \Delta Q(t)\rangle \propto t$ reflects the scaling $\Delta Q(t) \propto t$ of random increments. That is why we must not especialy carry out $ik\langle \Delta Q\rangle$ from $\ln \langle \exp\{ik\Delta Q\}\rangle$ .

e). In the limit $t/\tau_0 \to \infty$ the time-averaged rate tends to $U_0$ . Thus we even want the process be ergodic. We will see that the ergodicity automatically follows from a)-d) , but it is satisfied only in a logarithmically weak sense, as well as c) and d) do.

## 3.6 CHARACTERISTIC FUNCTION OF PURELY RANDOM FLOW OF EVENTS

If being formally punctual the scale-invariance condition d) would mean that $\lambda G(q,\lambda t) = G(q/\lambda, t)$ , that is $G(q,t) = q^{-1}\Phi(q/Vt)$ . Here $\Phi$ is formally arbitrary function and $V$ is some characteristical rate introduced to make the argument $q/Vt$ dimensionless. But this purely invariant form is incompatible with conditions b) and c). Indeed, if $\Phi(0) \neq 0$ then the integration diverges at $q \to 0$ , while if $\Phi(0) = 0$ then the condition c) will not be satisfied.

Hence we should allow for a slight violation of the scale invariance. We can expect that necessary violation will be really weak because the divergency is logarithmically weak.

The correction must cut the divergency at small values of $q$ , $q < Q_0$ , with $Q_0$ being some characteristical "microscopic" scale for measuring $\Delta Q$ . Such formal correction reflects a really inevitable violation of the invariance at small increments $< Q_0$ as well as at small time intervals. For instance, one could simply replace the integration from zero to infinity by integration starting at $q = Q_0$ (there are many other possibilities but results.will be practically the same). Then condition a) yields

$$1 = \int_{Q_0}^{\infty} \frac{1}{q} \Phi(\frac{q}{Vt}) dq \approx \Phi(Q_0/Vt) \ln(\frac{Vt}{Q_0}) \approx \Phi(0) \ln(\frac{t}{\tau_0})$$

where $\tau_0 = Q_0/V$ , and the inequality $t >> \tau_0$ is supposed.

From here it follows that the correction is forced to include additional logarithmical dependence of $\Phi(0)$ and so of $G(q,t)$ on time. This dependence can be separated as the multiplier

$$A(t) \approx [\ln \frac{t}{\tau_0}]^{-1}$$

if make the change $\Phi(\frac{q}{Vt}) \Rightarrow A(t)\Phi(\frac{q}{Vt})$ accompanied by the requirement $\Phi(0) = 1$ .

If all the cumulants and statistical moments of $\Delta Q$ are finite then $\Phi(x)$ should decrease from unit to zero in a sufficiently fast way. Details of this function are of no principal importance. For example, one can choose $\Phi(x) = \exp(-x)$ . Finally, the simple estimates of the integral give

$$\langle \exp\{ik\Delta Q\}\rangle \approx \exp\{-ikU_0 t A(t) \ln(\frac{\tau_0}{t} - ikQ_0)\} =$$

$$= \exp\{ikU_0 t - ikU_0 t A(t) \ln(1 - ikVt)\}$$

under the conditions $t >> e\tau_0$ and $|k|\, Q_0 << 1$ , with the latter corresponding to $\Delta Q >> Q_0$ .

Evidently, this is the CF of probability distribution which is as much close to ideal Cauchy distribution as the limitations of statistical moments do allow. There is also slightly other way to avoid the divergency [8], which results in the CF $\langle \exp\{ik\Delta Q\}\rangle \approx \exp\{ikU_0 t[1 - ikVt]^{-A(t)}\}$ . The corresponding probability distribution resembles Levy-Khinchin stable distributions with Levy's alpha-parameter close to 1, $\alpha_L = 1 - A(t)$ , and practically reduces to the above Cauchy type distribution if $A(t) << 1$ . One more possible modification was obtained in [10].

## 3.7 LONG-RANGE STATISTICS OF PURELY RANDOM FLOW OF EVENTS

The time scale $\tau_0$ should be treated just as life-time of the memory of random walk: at $t >> \tau_0$ the transition to almost scale invariant behaviour takes place. At once, the factor $\frac{\tau_0}{t}$ in logarithm serves as the cut parameter. If we neglected it the CF would transform into

$$\langle \exp\{ik\Delta Q\}\rangle \approx \exp\{-ikU_0 tA(t)\ln(-ikQ_0)\}$$

what corresponds to pure Cauchy statistics with infinite moments.

The probabilistic properties of $\Delta Q$ significantly depend on the parameter $\kappa(t) \equiv V/[U_0A(t)] = Q_0/[\tau_0 U_0 A(t)]$ . If $\kappa << 1$ then the distribution is almost Gaussian. In opposite case, $\kappa >> 1$ , it approaches to ideal asymmetric Cauchy distribution and its density has the long tail $\approx U_0 tA(t)\Delta Q^{-2}$ although cut at far $\Delta Q$ values [6,8]. In this case all the probabilistic properties become almost insensitive to how far the long tail is cut. For example, the estimates take place $P\{U > U_0\} \approx \frac{1}{\pi}\arctan(\pi/\ln\kappa)$ , and $P\{U > 2U_0\} \approx A(t)\exp(-U_0\tau_0/Q_0)$ where $P\{\}$ denotes the probability.

The essential feature of the statistics is a difference between the mean and most probable values of $\Delta Q$ as well of the rate $U = \Delta Q/t$ . In particular, at $\kappa > 1$ this difference is decsribed by $U_0 - U_{mp} \approx U_0 A(t)\ln\kappa(t)$ , with $U_{mp}$ being the most probable rate.

## 3.8 CAUCHY STATISTICS AND 1/F-NOISE

In contrary to probabilities, the second and higher statistical moments are exteremely sensitive to the long tail being determined mainly by bad untypical occurences. From the obtained CF one gets

$$\left\langle \Delta Q^2\right\rangle - \langle\Delta Q\rangle^2 \approx 2U_0\frac{Q_0}{\tau_0}t^2A(t)\ ,\ \left\langle U^2\right\rangle - \langle U\rangle^2 \approx 2U_0\frac{Q_0}{\tau_0}A(t)\quad (\,t >> \tau_0)$$

Therefore, the variance of time-averaged rate $U(t)$ tends to zero but only in logarithmically slow way. This means that the rate undergoes low-frequency 1/f fluctuations. It is easy to show that their relative spectrum is $S_{\delta U}(f) \approx (Q_0/U_0\tau_0)f^{-1}(\ln 2\pi f\tau_0)^{-2}$ , at $f\tau_0 << 1$ .

The only free dimensionless parameter of the statistics is $Q_0/U_0\tau_0$ . It determines both the 1/f-noise level and the degree of non-Gaussianity of the statistics. Therefore, the close relations between probabilistic properties and spectral properties of memoryless random flow of events do exist, and 1/f-noise is as strong as non-Gaussian. Both the 1/f spectrum and bad statistics mutually come from the absence of long-living memory. Their quantitative connection is the manifestation of highly irregular fractal structure of realizations of $\Delta Q(t)$ [8] similar to that of Levy flights with $\alpha_L \approx 1$ .

## 3.9 ABSOLUTE TIME SCALES AND SCALELESS FLUCTUATIONS

In general, any random walk subjected to a stable statistics [12] possesses self-similar fractal structure. But, unfortunately, a pure mathematical Levy flight has no intrinsic time scale at all, and so its rate looks as a white noise. Besides, at alpha-parameter $\alpha_L \leq 1$ even its mean value turns into infinity. This is too idealized model to apply it to real processes which always have some absolute lower time scale. We showed that in the case $\alpha_L = 1 - 0$ the existence of such a scale automatically leads to appearance of logarithmically time-dependent factors like $A(t)$ responsible for long-living scaleless statistical correlations. One comes to the idealized statistics only in the limit $\tau_0 \rightarrow 0$ , $U_0 \propto 1/A(t) \rightarrow \infty$ .

## 3.10 LOW-FREQUENCY NOISE AS DETERMINED BY SHORT-TIME SCALES

Formally $Q_0/U_0\tau_0$ is arbitrary parameter, but in concrete applications it is possible to suggest some natural estimates for it. For example, let $\Delta Q(t)$ be formed by well distnguishable identical discrete events (dozes, electron jumps, etc.). In this situation it is natural to choose $Q_0 = 1$ , and $U_0 \approx 1/\tau_c$ , with $\tau_c$ being the mean time separation of events, that is the correlation time of the Poissonian short-range statistics of $u(t)$ . Then $S_{\delta U}(f) \approx (\tau_c/\tau_0) f^{-1} (\ln 2\pi f \tau_0)^{-2}$ , and the relative 1/f-noise level is determined by ratio of mean distance between events and the memory duration, i.e. by the amount, $\sim \tau_0/\tau_c$ , of recent events which could be kept in memory.

The corresponding mean square relative uncertainty of the rate if measured during $t >> \tau_0$ is $\langle \delta U^2 \rangle \sim (\tau_c/\tau_0)(\ln t/\tau_0)^{-1}$ . Hence, the rate can not be predicted with much better presicion ( $<< \sqrt{\tau_c/\tau_0}$ ) than what is available for the system itself and is achievable by its self-observation during its memory life-time.

The particular case $\tau_0 \approx \tau_c$ corresponds to minimally short memory, with low-frequency noise $S_{\delta U}(f) \approx \frac{\alpha}{f}$ , where $\alpha = (\ln 2\pi f\tau_0)^{-2}$ , $\alpha \sim 0.02$ at $f\tau_0 \sim 10^{-4}$ and $\alpha \sim 0.002$ at $f\tau_0 \sim 10^{-9} \div 10^{-13}$ (what is typical for electric noises). It can be shown that at $\alpha > 0.0001$ the probability density of $U$ has noticable long tail.

The constructed model is able to give satisfactory quantitative estimates [6-8] of both the spectrum of 1/f-noise and its statistics which can be experimentally investigated [22-24] in small carbon resistors, thin metal whiskers and some other systems (in particular, a shape of experimental probabilistic histograms and the difference between average and most probable values can be correctly described).

## 3.11 1/F-NOISE IN DNA

Now we are able to hypothetically interprete also the example ix) [17,18]. The 1/f-noise in a distribution of any given sort of bases over DNA sequence could mean merely that only more or less localized pieces of DNA serve for encoding an information, while the amounts of bases accumulated by long pieces and total number of given bases in DNA are of no importance.

If this is true, we concern only statistical long correlations which themselves do not have any specific significance. But, in accordance with above phenomenology, the magnitude of 1/f-noise could contain an information about typical length of actually self-connected pieces.

In any case, the difference between statistical and actual (causal, structural) correlations should be taken into account. The paper [18] is one a very few works we know where this difference is realized.

## 3.12 COIN TOSSING AND LASER BEAM

Of course, the statistics under consideration is only the principal possibilty, not a necessity. Everybody knows about random flows of events which keep no place for 1/f-noise. For instance, the mathematical coin tossing first investigated by J.Bernoulli in his "Ars Conjectandi" in 1713.

However, what one could say about flipping a physical coin, not a mathematical one? Nothing, if do not assume that the events are "statisticaly independent" and the probabilities of heads are one half, thus coming back to the mathematical model. May be, the probabilities differ from one half, but this is of no importance if they are thought as purely certain. The essence is that one introduces "the probabilities per unit time" (i.e. per unit tossing) which eventually coinside with limits of averaging over many tossings.

Therefore, to say that events are statistically independent is the same as to assume that these limits exist (i.e. certain probabilities of heads exist) and besides are achieved in sufficiently fast way (in any case, not in logarithmical one). But if not say, then one returns to physical experiment. May be, in this case the flicker floor is really negligible. But the opposite reality does not imply that there are some slow processes, it means rather that the ideal certainty of the probabilities materialized in ideal constancy of the coin takes no place.

Hence, the only factual Bernoulli's achievement was the law what describes the approach to the limit, provided it does exist, not the proof of its existence.

In fact, the ideal constancy of coin serves as infinitely long-living causal correlation what just ensures the absence of 1/f-noise in the model. The Poissonian statistics of photons in an ideal laser beam is the more modern example of Bernoullian flow of events. The absence of 1/f-noise in photon flow in ideal beam also is due to hidden infinitely long causal correlation, namely, to what all photons belong to one and the same quantum state. Similarly, an electric supercurrent has no 1/f fluctuations, in contrary to normal current in superconductors.

## 3.13 QUANTUM DECAY

From the viewpoint of conventional statistical interpretation of quantum mechanics, the random flow of decays mentioned in the example a) is very similar to coin tossing. One needs in definite probability of decay per unit time and, besides, in statistical independency of decays in order to predict the Poissonian statistics of decays. The 1/f fluctuations (observed at frequencies below $10^{-3} \div 10^{-4}\,Hz$ ) are in principal contradiction to this theoretical scheme. So we can conclude that the hypothesis about the statistical independency is wrong. In other words, the decay probability has no absolutely definite value. The observed 1/f-noise is quite equivalent to 1/f fluctuations of probability of decay.

In fact, one lacks the statistical independency in order to connect the decay probability calculated in quantum mechanics and imaginary ensemble of identical experiments. But if we wanted to prove the statistical independency of two events we would need in the more complicated assumption that any two pairs of events are statistically independent, and so on up to infinity. Therefore the statistical independency can not be grounded in a logical way. This circumstance is noted in the book [25].

It is the open question whether an uncompleteness of quantum mechanics or primitivity of decay models is responsible for the discrepancy. Another problem was pointed out by Prigogine [26]. It is well known that time dependence of the decay probability includes non-exponential (power-law) tail. It relates to times much larger than the life-time and so it itself is practically unobservable. Nevertheless, it is inconsistent with the principle that quantum decay does not remember the past: if one knows that a nucleus has not yet decayed then any expectations for future should not depend on how long it was being safe. As a consequnce, the probability of no decay should be purely exponential, $P(t) = \exp(-\gamma t)$ .

In our opinion, it may be possible to avoid at once both the contradictions in the unified way if suppose (or better deduce from quantum mechanics) that the probability of decay per unit time, $\gamma$ , is not purely certain, and that its uncertainty is just as large as necessary in order to disguise the nonexponential tail. In other words, we should make difference between the most probable behaviour of experimentally registered flow of decays and its ensemble-averaged behaviour, as in general in memoryless flows of events. Then the quantitative measure of this difference is nothing but that of 1/f-noise.

## 3.14 LOSS OF MEMORY AND EQUILIBRIUM THERMAL 1/F-NOISE

The equilibrium voltage noise in a disconnected tunnel junction is constituted by random jumps of electrons from left to right-hand side of the junction and in opposite direction. Let $N_+(t)$ and $N_-(t)$ be the numbers of these jumps during some time interval $t$ . If $N_+ > N_-$ then the electric voltage arises between sides, $u = e(N_+ - N_-)/C$ , with $C$ being their electric capacity, which makes one sort of jumps more probable than another. Due to this thermodynamical opposite reaction, after a characteristic relaxation time of order of $\tau_c = RC$ , with $R$ being the junction resistance, the system returns to a vicinity of the chargeless state. Therefore the difference $N_+ - N_-$ always is kept near its average zero value, and the time-smoothed current $e(N_+ - N_-)/t$

turns to zero at $t >> \tau_c$ .

But let it happened that the sum $N(t) \equiv N_+ + N_- \approx 2N_+$ essentially deviates from its average value, while the difference remains close to zero. Such a fluctuation causes neither a charging nor any other thermodynamical consequences. All stay as before! So we can expect that later this fluctuation will not be "compensated and damped" by some oppositely aimed process. Therefore, in reality the time-smoothed value $N(t)/t$ has no reasons to tend to its "theoretical" ensemble averaged value, moreover, it have no explicitly predictable limit at all [10].

Now notice that both types of jumps give an equal positive contribution to the noise intensity. The latter is determined by the summary number of jumps per unit time, $S = e^2 N/t = e^2(N_+ + N_-)/t$ , with $S$ being the instantaneous (time-averaged) power spectrum. Hence, we come to white noise but whose intensity undergoes low-frequency 1/f-fluctuations, i.e. to flicker "noise of noise".

## 3.15 1/F-NOISE AS UNCERTAINTY OF KINETICAL VALUES

It is important to underline that in all the above examples the values subjected to 1/f-fluctuations could be qualified as kinetical values. This is principal peculiarity of 1/f-noise without slow processes. In contrary to dynamical and thermodynamical variables, kinetical values characterize the statistics of random transitions between various states of the system, not statistics of the states themselves.

In general, good ergodicity with respect to visiting the states does not imply the same with respect to transitions. Even in the simplest case, when a model distinquishes only two states and each of them occupies well certain portion of time, the rate of random jumps between the states may be uncertain.

None kinetical value could be referred to a definite time moment. The matter is that any kinetical value takes a shape only during a finite time interval longer than memory life-times. This circumstance means that kinetical values have no their own characteristic time scales. As a consequence, their fluctuations naturally acquire a scale invariance with a scaleless power-law low-frequency spectra.

### 3.16 1/F-NOISE IN NON-EQUILIBRIUM STATE

In thermodynamical systems every sort of kinetical events (random transitions from one state to another) is accompanied by time-reversed events, and a freedom of random behavior is restricted by definite ratio of amounts of opposite events (by detailed balance, in equilibrium case). This restriction arises from reversibility of underlying dynamics, always are under thermodynamical control and can be taken into account with the help of fluctuation-dissipation relations. The example is the ratio $N_-/N_+ \simeq 1$ in equilibrium electric junction.

But fluctuations in the summary rate of direct and reversed kinetical events does not destroy their balance and so does not cause complete counteraction.

If an external voltage $U$ is applied to the junction, then undumped fluctuations in the total rate of jumps result in not only noise of noise, but also strictly in the voltage and current. Now the relation $N_- \simeq N_+ \exp(-eU/T)$ is thermodynamically kept, instead of $N_- \simeq N_+$ , and the current, $e(N_+ - N_-)/t = e\tanh(eU/2T)N/t = \tanh(eU/2T)S/e \propto S$ , begins to reflect uncertainty of the total number of jumps per unit time. In this nonequilibrium state the voltage spectrum acquires 1/f contribution in addition to the white background, and 1/f-noise can be viewed directly in the two-point correlators as effective resistance fluctuations.

# 4 BROWNIAN MOTION REVISED

### 4.1 QUAZI-GAUSSIAN RANDOM WALKS

Let $u(t)$ be equilibrium noise short-correlated in the sense of two-point correlators. Then the integral $Q(t) = \int_0^t u(t)dt$ is symmetrical random walk like Brownian motion. Its characteristic fractal properties are determined by the scaling $Q(t) \propto \sqrt{t}$ , at least if $t$ sufficiently exceeds $\tau_0$, with $\tau_0$ being maximal correlation time of $u(t)$. As it was shown in [4,5], if only one excludes very formal case of ideally Gaussian $u(t)$ , then this scaling results in 1/f-noise of instantaneous power spectrum, $S(t)$ , of the noise $u(t)$ and in definite non-Gaussian long-range statistics of $Q(t)$ (termed quazi-Gaussian) which resembles the statistics of Levy flights with alpha-paremeter slightly less than 2, $\alpha_L = 2 - 0$ . One of many examples is the equilibrium voltage noise in disconnected electric junction. Another example is the equilibrium

Brownian motion, with $u(t)$, $Q(t)$ and $D(t) \equiv S(t)/2$ being the velocity, displacement of a Brownian particle and its current diffusivity, respectively.

Principally, the statistics of $Q(t)$, like that of the above analysed one-directed random walk, is derivable from rather trivial requirements to be satisfied asymptotically, at $t >> \tau_0$.

a) The standard diffusion law takes place, $\langle Q^2(t)\rangle = 2D_0 t$ .

b) $Q(t)$ behaves as infinitely divisible random value.

c) It approximately behaves as random walk with independent increments.

d) Due to forgetting the past the system does not distinguish what is a proper rate of the walk and what is deviation from it. Any currently happened location of the walk is equally permissible as starting point for further adventures! Therefore, the scaling $Q(t) \propto \sqrt{t}$ is peculiar for the statistics as a whole (not only for the mean square displacement). This can be expressed in the form of approximate statistical identity

$$Q(\lambda t) \sim \sqrt{\lambda} Q(t)$$

These requirements imply nearly stable probability distribution whose characteristic function (CF) looks as [5,7,8]

$$\langle \exp\{ikQ\}\rangle \approx \exp\{D_0 k^2 t A(t) \ln(\frac{\tau_0}{t} + k^2 q_0^2)\} =$$

$$= \exp\{-D_0 k^2 t + D_0 k^2 t A(t) \ln(1 + D' k^2 t)\}$$

Here $A(t) \approx 1/\ln(t/\tau_0)$ , $t >> \tau_0$ , $k^2 q_0^2 << 1$ , $D' = q_0^2/\tau_0$ , and the scales $q_0$ and $\tau_0$ serve as lower spatial and temporal bounds of the scale invariance of the random walk.

Under the formal limit $\tau_0 \rightarrow 0$ , $D_0 \propto 1/A(t) \rightarrow \infty$ , this CF turns into CF of pure quazi-Gaussian distribution, $\exp\{-Dtk^2 \ln(1/k^2 q_0^2)\}$ , with infinite second and higher even-order moments and without characteristic time at all. This random walk constructed in [4,5] is absent in standard classification of Levy flights [12]. Independently on us it was found in the mathematical work [27]. But many of applications need in random walk models which possess finite statistical moments and non-zero absolute lower time scale with consequently deformed scale invariance, as in the model under consideration.

## 4.2 FOUR-POINT CUMULANTS AND DIFFUSIVITY FLUCTUATIONS

The main information about 1/f-noise is contained in unusually behaving 4-order cumulant of the displacement what follows from the above CF,

$$\left\langle Q^{(4)}(t)\right\rangle \equiv \langle Q(t),Q(t),Q(t),Q(t)\rangle \propto t^2 A(t) \approx t^2[\ln(t/\tau_0)]^{-1}$$

In general, the standard variance $\langle Q^2(t)\rangle = 2D_0 t$ can be accompanied by non-standard 4-point cumulant like $\left\langle Q^{(4)}(t)\right\rangle \propto t^{1+\gamma}$ with $\gamma > 0$. At any $\gamma > 0$ such the asymptotics is formally equivalent to low-frequency fluctuations of instantaneous power spectrum (diffusivity) of $u(t)$ . Indeed, by the formal definition of cumulants,

$$\langle \exp[ikQ(t)]\rangle = \exp[-k^2\left\langle Q^2(t)\right\rangle/2 + k^4\left\langle Q^{(4)}(t)\right\rangle/24 - ...]$$

From the other hand, one can phenomenologically treat $u(t)$ as a shot noise what would be Gaussian if its power spectrum was constant, $S(t) = 2D_0$ , but becomes non-Gaussian because of its relatively slow random modulations, $S(t) = 2D(t)$. Correspondingly, the CF can be represented by double averaging, first over fast Gaussian noise under fixed realization $D(t)$ and then over random $D(t)$ :

$$\langle \exp[ikQ(t)]\rangle \approx \left\langle\left\langle \exp[-k^2\int_0^t D(t')dt']\right\rangle\right\rangle =$$

$$= \exp[-k^2 D_0 t + k^4\int_0^t\int_0^t K_D(t_1-t_2)dt_1 dt_2/2 - ...]$$

Here the doubled brackets denote the second stage of averaging, $D_0 \equiv \langle D(t)\rangle$ , the inequality $t >> \tau_c$ is implied, and $K_D(\tau)$ is the effective correlation function of fluctuating diffusivity $D(t)$ ,

$$K_D(t) = \frac{1}{24}\frac{d^2}{dt^2}\left\langle Q^{(4)}(t)\right\rangle = \frac{1}{2}\int_0^t\int_0^t \langle u(\tau),u(\tau'),u(\tau''),u(0)\rangle\, d\tau' d\tau''$$

The unusual 4-point cumulant means slow asymptotics $K_D(\tau) \propto \tau^{\gamma-1}$ and thus $1/f^\gamma$ low-frequency spectrum of diffusivity fluctuations. The case $\gamma > 1$ formally corresponds to non-integrable spectrum like that of nonstationary random process, but this is only imaginary non-stationarity because $u(t)$ is purely stationary random process (all its cumulants possesses time translation invariance).

## 4.3 MICROSCOPIC TIME SCALES AND MAGNITUDE OF 1/F-NOISE

In case of quazi-Gaussian random walk $\gamma \to 1$ , $K_D(\tau) \propto A(\tau)$ , and the low-frequency part of spectrum of relative diffusivity fluctuations is $S_{\delta D}(f) \approx \alpha(f)/f$ , where $\alpha(f) \equiv (q_0^2/D_0\tau_0)[\ln 2\pi f\tau_0]^{-2}$ .

Let us estimate this noise taking in mind literally Brownian motion of a microscopic particle. The mean diffusivity can be expressed as $D_0 \approx q_c^2/\tau_c$ with $q_c$ and $\tau_c$ being its mean free path and free flight time, respectively. Clearly, $\tau_c$ coinsides with correlation time of the velocity $u(t)$. It is quite reasonable to suppose that spatial scale invariance takes beginning just from one free path, that is $q_0 \approx q_c$ . Then magnitude of 1/f-noise is determined by the ratio $\tau_c/\tau_0$.

Again we see that 1/f-noise is as strong as long is the memory duration as compared characteristic time separation of elementary kinetical events (interactions, collisions, etc.). The smaller is a number of previous kinetical events remembered and taken under control by a mechanism what produces white noise the greater is the accompanying 1/f-noise. In fact, one can be convinced of validity of this statement [7] with respect to different particular cases of Brownian motion of charge carriers in semiconductors and metals, to charge transport in normal junctions and superconducting Josephson junctions, and to Brownian particle in liquids (in the latter cases the difference between $q_0$ and $q_c$ can be significant too).

The special case, $\tau_0 \approx \tau_c$ , when scale invariance starts immediately after one free flight time, corresponds to natural basic level of diffusivity noise. This is the case for electrons in a slightly doped semiconductor, due to inelasticity of collisions. Taking into account that $\tau_c \sim 10^{-12}\,s$ , at $f \sim 1\,Hz$ (as well as at frequencies a few orders larger or smaller) one finds $S_{\delta D}(f) \approx 0.0015/f$ . This estimate [4,5] is in good agreement with empirical Hooge constant $\alpha \approx 0.002$ [1,2].

## 4.4 PROBABILITY DISTRIBUTION OF BROWNIAN MOTION

In contrary to 4-th and higher cumulants, the probability density $W(Q,t)$ of quazi-Gaussian random walk always has only small quantitative differences from that of ideal Brownian motion, $W_G(Q,t) = (4\pi Dt)^{-1/2}\exp(-Q^2/4Dt)$. If $D' > D_0$ then $W(Q,t)$ has noticable power-law tails whose extent approximately equals to $\sqrt{2D't}$ but magnitude does not depend on $D'/D_0$ , i.e. on the 1/f-noise level. In the limit $D'/D_0 \rightarrow \infty$, for the region $Q^2 >> \langle Q^2 \rangle$, one can get $W(Q,t) \approx W_G(Q,t) + 2D_0 t A(t)/\left|Q\right|^3$ . From here the probability $P$ of that $Q^2$ is three times larger than $\langle Q^2 \rangle = 2D_0 t$ can be estimated as $P < P_G + [3\ln(t/\tau_0)]^{-1}$ , where $P_G$ means similar probability for ideal Brownian walk, $P_G \approx 0.1$ . Therefore, at $t/\tau_0 > 10^3$ the difference between $P$ and $P_G$ never could exceed 0.05. Analogouosly, one can find that at $t/\tau_0 > 10^3$ always $P\{|Q| > 4\sqrt{2D_0 t}\} \approx [16\ln t/\tau_0]^{-1} < 0.01$ .

### 4.5 STATISTICAL SENSE OF LONG-LIVING CORRELATIONS OF KINETICAL VALUES

Hence, the probability of noticably far deviations from standard behaviour does not exceed one percent even in the worst case (infinite 1/f-noise). Nevertheless, just such rather unprobable occasions determine the asymptotics of $K_D(t)$ and the level of diffusivity 1/f-noise.

Evidenly, the long tail $K_D(t) \propto A(t)$ reflects only the statistical weight of untypical realizations of the random walk, not some actual correlations inside realizations. More explicitly, any concrete realization looks as sometimes typical and sometimes rare, and because of its fractal structure every variant of its behaviour during longer time interval is constituted by both typical and rare contributions from shorter subintervals.

The same can be said also about fluctuations of other kinetical values. From the viewpoint of underlying dynamics, their correlation functions are roughly phenomenological characteristics. However, one can present rigorous definitions for these correlators in terms of four-point (or higher order) cumulants (or related ensemble averages) of purely dynamical variables.

## 5 DYNAMICS AND 1/F-NOISE

### 5.1 SOME QUESTIONS

Of course, we are highly interested in 1/f-noise derived from Hamiltonian dynamics. Unfortunately, none of realistic Hamiltonian models allows for explicite calculations of 4-point cumulants even for Hibbsian equilibrium ensemble. Moreover, this problem never was under careful consideration. Instead, everybody wanted to somehow reduce the statistics to two-point correlators thus diminishing specific contribution of 4-point cumulants. Various means were attracted to this aim, either ansatzes like molecular chaos, random phases, thermodynamical limit, continuous spectra of energy levels, diagonal singularity and others, or exclusion of some interactions what do not influence low-order correlations under interest.

For example, the solvable dynamical model of tunnel junction, so-called tunnel Hamiltonian approximation, can reproduce shot noise but nothing like 1/f-noise which could be expected in view of above phenomenological reasonings. Probably, the matter is that this model takes into account neither discreteness of energy levels nor Coulombian interaction of electrons.

Some of well investigated low-dimensional nonlinear mappings have the intermittency regimes accompanied by 1/f-noise, but this noise is connected with long periods of regular motion, that is with long memory and slow processes (for example, see [28]). Therefore it is not surprising that this 1/f-noise becomes destroyed (saturated at zero frequency) under influence of external white noise. So it is not the 1/f-noise we are interested in whose low-frequency behaviour should not be supressed by noisy perturbation.

At present the axiomatic theory of the so-called Anosov systems is developed [29,30] based on works by Sinay on scattering billiards and ideas by Anosov. These systems are characterized by exponential instability and divergency of phase trajectories and their mixing in phase space, possibly in company with time reversibility and phase volume conservation (what is most interesting for us). In such systems 1/f-noise could be connected with random temporal non-uniformity of the rate of exponential instability. As far as we know, this hypothetical possibility was not under investigation. In contrary, the attention was concentrated on the systems and phase space divisions which produce Bernoullian flows of events.

However, as it is was shown by Ornstein [13], in principle generalized random flows of events (K-flows) do exist which can not be recoded into Bernoullian flows. One could search for 1/f-noise in concrete realizations of such generalized flows, but this subject is still apart of achievements of the theory.

## 5.2 WHY KINETICS LOST 1/F-NOISE ?

Because none realistic dynamics can be exactly solved, physicists resort to kinetical models of relaxation, noise and irreversible phenomena. Everybody knows how this is carried out. First one chooses a reasonable coarsened characterization of the system, secondly, analyses what variants of its evolution may occur during some short time step $\Delta t$ if start from an uncompletely described state, then evaluates probabilities of the variants and finally keeps these probabilities as governing parameters for a long-time evolution during many steps.

This theoretical scheme was expressively criticized by M.Kac [31]. Clearly, its main defect is its last ansatz, because it means that probability of any chain of transitions between states is equalized to product of probabilities of marginal transitions, i.e. that successive transitions are considered as statistically independent events. One could say that such a model divides the real trajectories to short pieces and randomly permutes these pieces at each time step $\Delta t$. As a consequence, the model can correctly describe short correlations and relaxation, but in general is unable to correctly reproduce a long time behavior of trajectories.

Usually one sees nothing criminal in this simplification. But we insist that long-living statistical dependencies and non-Bernoullian behaviour may mean merely that trajectories take liberties just because of no really long-living memory, then the statistical independency looks as too rough approximation. Eventually, this approximation results in the loss of 1/f-noise.

We see that modern theory is not kind with respect to 1/f-noise. Nevertheless, we have pointed out two Hamiltonian systems, both being basic models of statistical mechanics, which produce 1/f-fluctuations of kinetical values, namely, a slightly non-ideal gas [9] and slightly unharmonic phononic system [11]. In our opinion, every many-particle Hamiltonian model which really bears a relaxation inevitably bears also 1/f fluctuations of the rate of relaxation.

## 5.3 EQUILIBRIUM $1/F^2$-NOISE IN THE KAC'S RING MODEL

Many years ago M.Kac [31] suggested the model which represents a good toy analogy of the problems always accompanying derivation of relaxation, noise and kinetics from dynamics. The system is the ring chain of $n >> 1$ boxes each containing the only black or white ball. At every moment of discrete time $t = .. -1, 0, 1, 2...$the balls displace to left along the ring. The boxes are characterized by time-independent parameters $\sigma_j = \pm 1$ . If $\sigma_j = -1$ then $j$-th box inverts the colour of any ball what leaves it. The dynamical variables

$\xi_j(t) = \pm 1$ are defined so that $\xi_j(t) = 1$ if $j$-th box contains a white ball at time moment $t$ and $\xi_j(t) = -1$ if it contains a black ball. The equations of motion are easily soluble: $\xi_j(t+1) = \sigma_{j-1}\xi_{j-1}(t)$ (the number $j = -1$ is equivalent to $j = n-1$). The equilibrium statistical ensemble is defined by conditions that $\xi_j$ are statistically independent if considered at one and the same time moment and $\xi_j = \pm 1$ with equal probabilities $1/2$ , and that constant parameters $\sigma_j$ equals to 1 or $-1$ with probabilities $p, 1-p$ also being independent one on another. Evidently, the dynamics is time-reversible and the ensemble is time-invariant. We confine ourselves by the simplest case $p = 1/2$ .

The quantity under macroscopic observation is the mixed colour $u(t) \equiv [\sum_j \xi_j(t)]/\sqrt{n}$ . It is easy to verify that $\langle u(t)\rangle = 0$ and that from the point of view of second-order correlators $u(t)$ represents a white noise: $\langle u(t_1)u(t_2)\rangle = \delta_{t_1 t_2}$ , with Kronecker symbol on the right-hand side.

Kac demonstrated [31] that this model allows for wide analogies with serious statistical mechanics and for illustrations of approximate methods usually attracted when building kinetics. But Kac did not note that $u(t)$ is very far from white noise in the sense of higher-order statistics.

Let us consider the discrete random walk $Q(t) = \sum_{k=t_0+1}^{t_0+t} u(k)$ at $1 \leq t < n$. Obviously, $\langle Q^2(t)\rangle = 2D_0 t$, with diffusivity $D_0 = 1/2$. If this process was asymptotically gaussian and subjected to the central limit theorem at $t >> 1$, the four-order cumulant of $Q(t)$ would grow also as a linear function of time, $\left\langle Q^{(4)}(t)\right\rangle \propto t$. For instance, if $u(t)$ was purely white noise, one would get $\left\langle Q^{(4)}(t)\right\rangle = -2t/n$ . However, the explicit calculation lead to

$$\left\langle Q^{(4)}(t)\right\rangle \equiv \left\langle Q^4(t)\right\rangle - 3\left\langle Q^2(t)\right\rangle^2 = 2(2t^2-1)(t-2)/n \propto t^3 \ (t >> 1)$$

(this easy reproducable calculation was performed ten years ago and sent to 1988 Conference on noise in physical systems in Vilnius, but, unfortunately, was not accepted either there or for journal publications).

This result proves the existence of infinitely long-living four-point correlations (cumulants) of the noise $u(t)$, and means that standard kinetical approaches can not present a qualitatively correct approximation (let being quantitatively good) of the noise. From the viewpoint of external observer, such 4-th cumulant looks as manifestation of low-frequency fluctuations of diffusivity, i.e. of instantaneous power spectrum of $u(t)$. In accordance with the above formulas, the corresponding spectrum is $\propto f^{-2}$. The factor $1/n$ is analogous to what takes place in the spectrum of relative resistance 1/f-noise in bulk conductors (example i) ).

## 5.4 NON-STATIONARY NOISE IN STATIONARY SYSTEM

Perhaply, this noise could not be qualified as what is due to forgetting the past, sooner the actual correlations carried in by random parameters $\sigma_j$ do work. Nevertheless, the obtained $1/f^2$-noise of noise is principally important by two reasons. First, it gives the example of completely calculatable long-living higher-order correlations in time-reversible dynamics. Secondly, it helps to understand why seemingly non-stationary fluctuations of kinetical values can occur even in equilibrium system.

With no doubts, the non-integrable spectrum $1/f^2$ belongs to stationary system . Indeed, the statistics of $Q(t)$ does not depend on the arbitrary initial time moment $t_0$. The spectrum $1/f^2$ reflects only the role of duration $t$ in the measurement of kinetical value (diffusivity) and means that the longer is the measurement the worse is its result. The similar strange property characterizes Levy flights with $\alpha_L < 1$ [12].

Of course, the guessing is that any kinetical value is delocalised in time. Therefore its bad behaviour does not contradict to ergodicity of visits of states. Moreover, we can expect that at fixed duration $t$ the four-point cumulant calculated with the help of ensemble averaging coinsides with that obtained from the time smoothing over $t_0$.

Besides, one should remember that non-stationarity in the sense of statistical moments coexists with stationarity of the most probable behavior. But because of non-stationarity the result of experiments can be sensitive to details of the measuring procedure [8,9].

## 5.5 1/F-NOISE OF FRICTION AND OF LIGHT SCATTERING IN PHONONIC SYSTEM

A quartz crystal is the system where thermodynamically equlibrium and nonequilibrium manifestations of one and the same 1/f-noise are observed by different methods. As it is known for a long time [14,15], when quartz is used as frequency stabilizator in electronic devices, its quality is restricted by 1/f-fluctuations of inner mechanical friction and consequently of dissipation.

Not long ago the 1/f-fluctuations of intensity of the laser light scattered by equilibrium quartz were discovered [16].

In [11] we showed that the statistical correlators which describe the fluctuations of dissipation can be generally connected with four-point equilibrium cumulants which describe thermal fluctuations of quantum probabilities of photon scattering by phonons. It was shown also that both the friction and light scattering 1/f-fluctuations can be reduced to thats of the relaxation rates (life-times) of phononic modes. In their turn, this fluctuations originate from the same phonon interactions what produce exponential instability, continuous spectra and the relaxation phenomenon itself.

The logical way to 1/f-noise looks as follows. Due to exponential instability, the interaction of given test phononic mode with the thermal bath consisting of other modes produces both its noisy disturbation and damping. In zero approximation, the friction is referred to linear response of the bath to test mode and is non-random, and correspondingly the noise is Gaussian. The linear response seems good approximation because the influence of the only test mode is $1/\sqrt{N}$ times smaller than summary interactions inside the bath, with $N$ being total number of modes, $N \sim 10^{20}$ .

Next, the equations which govern the linear response look as equations of coupled harmonic oscillators without friction but with fluctuating couplings. Even being Gaussian this fluctuations cause a stochastic instability of the response, in the sense of second and higher statistical momemts. Therefore in better approximation the friction becomes random with infinite variance and besides with infinitely long non-locality. As a consequence, the infinitely long-living four-point statistical correlations of phononic variables do occur.

In a more developed picture, one should consider the completely nonlinear response. But, as it was argued in [11], the long-living many-point correlations survive and correspond to giant low-frequency fluctuations of phase diffusivities of phonon modes and thus of their relaxation rates (relaxation times).

# 6 GENERALIZED RELATIONS BETWEEN FLUCTUATIONS AND DISSIPATION

## 6.1 PROBABILISTIC RELATIONS

In any theory concerning 1/f-noise in thermodynamical systems the exact generalized fluctuation-dissipation relations (FDR) can be useful which were considered in [32-34] (see also [7,9,11,38]). Except classical fluctuation-dissipation theorem, Kubo formulas and Onsager relations, FDR include infinitely many additional connections between nonlinear responces and higher-order equilibrium and nonequilibrium cumulants.

All the variety of FDR can be expressed with only relation what follows from the phase volume conservation and time reversibility of Hamiltonian dynamics. For instance, in case of initially equilibrium Hibbsian canonical ensemble later perturbed by some time-dependent variation of the Hamiltonian, the producing FDR can be written [32-34] as

$$\Pi\{Trajectory; Forces\}\exp(-Work/T) = \Pi\{\widetilde{Trajectory}; \widetilde{Forces}\}$$

There $\Pi\{Trajectory; Forces\}$ is the probability of realization of some process ($Trajectory$) under given external $Forces$=$Perturbations$+$Conditions$ (in the sense of probability density in functional space), $Work$ is the work produced by the perturbation during this process depending on $Trajectory$ and $Forces$ , $T$ is the temperature of initial Hibbsian probability distribution, and $\Pi\{\widetilde{Trajectory}; \widetilde{Forces}\}$ is the probability of realization of the time-reversed process under time-inverted perturbations and conditions. The operation $\tilde{}$ means the inversion of time direction and besides of signs of variables which have the odd time parity. From here, after integration over all possible processes, the remarkable equality $\langle\exp(-Work/T)\rangle = 1$ does follow what leads to inequality $\langle Work\rangle \geq 0$. Particularly, if the perturbation is described by Hamiltonian $H(q,p,t) = H_0(q,p) - \sum Q_j(q,p)x_j(t)$ , with $q$, $p$ and $x(t)$ being canonical coordinates and momentums and external forces, respectively, one has $Work = \int\sum J_j(t)x_j(t)dt$ , where $J_j(t) = \frac{d}{dt}Q_j(q(t),p(t))$ are the flows (velocities, currents, etc.) conjugated with the forces.

This FDR is valid also in quantum case, under certain definition of quantum characteristic and probability functionals [34]. Besides, this FDR can be extended to nonequlibrium initial ensembles [34,38], in other words, to a combination of dynamical perturbations (i.e. thats of Hamiltonian) and thermal perturbations (i.e. thats of probability distribution in phase space). The producung FDR looks quite similar besides that the factor $\exp(-Work/T)$ should be replaced with $\exp(-\Delta S)$ where $\Delta S\{t, Trajectory; Forces\}$ has the sense of increment of entropy (or similar thermodynamic potential) during $Trajectory$.

## 6.2 FLUCTUATIONS OF DISSIPATION

Let $\langle J(t_1), ..., J(t_n)\rangle_q$ denotes the $n$-th order nonequilibrium cumulant corresponding to the time-cut modification of perturbing forces, $x(t) \Rightarrow x(t)\eta(t - \min(t_1, ..., t_n))$ , where $\eta(t) = 1$ at $t > 0$ and $\eta(t) = 0$ at $t \leq 0$ (thus the most early variable belongs to yet equilibrium state). Such correlators were termed in [34] quazi-equilibrium. Then the rigorous relation takes place [33,34,11],

$$\langle J_j(t)\rangle = \frac{1}{T}\int_{-\infty}^{t} \langle J_j(t), J_k(t')\rangle_q \, x_k(t')dt'$$

what extends Kubo formulas to arbitrary non-linear responce. As a consequence, the average value of the work (i.e. of the energy absorbed by the system), $\langle Work\rangle = \int_{-\infty}^{t} \langle J(t)\rangle \, x(t)dt$ , can be simply expressed via two-point quazi-equilibrium correlators.

The similar formulas for higher cumulants can be obtained [34] which lead to the formally exact expression for variance of the work [11] :

$$\left\langle Work^{(2)}\right\rangle = 2T\,\langle Work\rangle + \frac{2}{T}\int_{-\infty}^{t} x(1)x(2)\,\langle J(1), J(2), J(3)\rangle_q \, x(3)d1d2d3$$

where $\left\langle Work^{(2)}\right\rangle = \langle Work^2\rangle - \langle Work\rangle^2$, and for brevity the time variables (and indices) are denoted by integers and triple time integral is taken under condition 1>2>3 . Here the first term on the right-hand side describes shot noise accompanying the average energy influx, and the second term involves an excess noise. The latter enters by means of three-point cumulants, so one could not snatch at it in Gaussian approximation. Under infinitesimally small perturbation, $x \to 0$, the excess contribution is of order of $x^4$ and can be expressed as

$$\left\langle Work^{(2)}\right\rangle = 2T\,\langle Work\rangle +$$

$$+\frac{2}{T}\int_{-\infty}^{t} x(1)x(2)\,\langle J(1), \Gamma(2,3), J(4)\rangle_0 \, x(3)x(4)d1d2d3d4 + O(x^6)$$

where the dynamic linear differential responce function is introduced,

$$\Gamma_{jk}(t,t') \equiv [\delta J_j(t)/\delta x_k(t')]_{x=0}$$

the subscript 0 denotes averaging over equilibrium Hibbsian ensemble, and integrations are ordered by the conditions 1>4, 2>3, 2>4, 3>4 .

If low-frequency fluctuations of kinetical values responsible for energy dissipation take place then these fluctuations are reflected just the excess noise term. In a steady state, when $\langle Work \rangle \propto (t-t_0)x^2$ , with $t_0$ being the start time of perturbation, a $1/f^{\gamma}$-noise should result in $\propto (t-t_0)^{1+\gamma}x^4$ behaviour of the excess part of energy variance. Hence, nearly equilibrium fluctuations of dissipative values (resistance, friction, mobility, etc.) can be generally reduced to four-point equilibrium cumulants like $\langle J(1), \Gamma(2,3), J(4)\rangle_0$ which describe specifically non-Gaussian correlations between random responce and noisy flows. In their turn, these correlators can be connected with literally four-point cumulants $\langle J(1), J(2), J(3), J(4)\rangle_0$ , due to the four-point FDR [32,11], in part due to

$$\frac{1}{T}\langle J(1), J(2), J(3), J(4)\rangle_0 = \left[\frac{\delta}{\delta x(4)}\langle J(1), J(2), J(3)\rangle\right]_{x=0} =$$

$$= \langle J(1), \Gamma(2,4), J(3)\rangle_0 + \langle \Gamma(1,4), J(2), J(3)\rangle_0 + \langle J(1), J(2), \Gamma(3,4)\rangle_0$$

where the time moments are ordered as $1 > 2 > 3 > 4$. More rich information is given by the 7 independent four-point relations derived in [32], in particular, by the FDR

$$\frac{1}{T^2}\langle J(1), J(2), J(3), J(4)\rangle_0 = \left[\delta^2 \langle J(1), J(2)\rangle / \delta x(3)\delta x(4)\right]_{x=0} +$$

$$+ \left[\delta^2 \left\langle \widetilde{J(2)}, \widetilde{J(4)}\right\rangle / \delta \widetilde{x(3)} \delta \widetilde{x(1)}\right]_{x=0}$$

where the same time ordering is supposed, 1>2>3>4, and tilda means $\widetilde{x_j}(t_j) \equiv \varepsilon_j x_j(-t_j)$ , $\widetilde{J_j}(t_j) \equiv -\varepsilon_j J_j(-t_j)$ , with $\varepsilon_j$ being the time parities of forces.

## 6.3 CONNECTIONS BETWEEN EQUILIBRIUM AND NON-EQUILIBRIUM RANDOM WALKS

Let $Q(t) = Q(q(t), p(t))$ be coordinate of some random walk and $x(t)$ be generalized force conjugated with $Q$ in the above defined sense. In two of many possible examples $Q(t)$ is a coordinate of Brownian particle with $x(t)$ being the external force acting on it, or $Q(t)$ is electric charge transported through some conductor with $x(t)$ being applied voltage. Let $Q(t)$ be time-even variable, the force remains constant after its switching on ( $x = 0$ at $t < 0$ , $x = const \neq 0$ at $t > 0$) and $Q = 0$ at $t = 0$ . Then the general producing FDR results in simple relation [7,9,38],

$$W(Q,t)\exp(-\frac{Qx}{T}) = W(-Q,t)$$

$$W(Q,t) - W(-Q,t) = [W(Q,t) + W(-Q,t)]\tanh(\frac{Qx}{2T})$$

where $W(Q,t)$ is the probability density of $Q$ at $t > 0$. From here one gets $\langle Q(t)\rangle = \langle Q(t)\tanh[Q(t)x/2T]\rangle$ . The expansion of the latter relation over $x$ up to $x^3$ yields $\left[\frac{\partial}{\partial x}\langle Q(t)\rangle\right]_{x=0} = \langle Q^2(t)\rangle_0/2T$ , and [9]

$$\left[\frac{\partial^3}{\partial x^3}\langle Q(t)\rangle\right]_{x=0} = \frac{3}{2T}\left[\frac{\partial^2}{\partial x^2}\langle Q(t),Q(t)\rangle\right]_{x=0} - \frac{1}{4T^3}\langle Q(t),Q(t),Q(t),Q(t)\rangle_0$$

with $T$ being the initial temperature [32].

The first of these formulas is nothing but Einstein relation between low-field mobility and equilibrium diffusivity, or Nyquist relation, etc. The second formula asserts the connection between quadratic non-equilibrium addings to variance of $Q(t)$ (that is the excess noise), equilibrium fourth cumulant and cubic response. The latter could be due to either nonlinear dissipation mechanisms or to dependence on actual current temperature maintained by Joulean heating. But, in any case, the left-hand side can not grow in a more fast way than $\propto t$. On the contrary, in presence of $1/f^\gamma$ excess noise the first term on right-hand side should grow as $\propto t^{1+\gamma}$. Hence, at sufficiently long time intervals it must be compensated by the second term, and we come to relations between low-field excess nonequilibrium noise and four-point cumulant of equilibrium noise. The result can be written also as $K_m(\tau) = \frac{1}{T^2}K_D(\tau)$ [4,6,9], where $K_D(\tau)$ is the above defined correlator of diffusivity and $K_m(\tau)$ is the mobility correlation function as introducedd by the expansion $\langle Q(t),Q(t)\rangle = \langle Q(t),Q(t)\rangle_0 + x^2\int_0^t\int_0^t K_m(t'-t'')dt'dt'' + O(x^4)$. This is the extension of Einstein relation and similar relations to 1/f-fluctuations.

## 6.4 UNIFIED DESCRIPTION OF EQUILIBRIUM AND NONEQUILIBRIUM BROWNIAN MOTION

The producing FDR can be reformulated in terms of characteristic function (CF):

$$\left\langle \exp[(ik - \frac{x}{T})Q(t)] \right\rangle = \langle \exp[-ikQ(t)] \rangle$$

This is functional equation with respect to dependencies on $ik$ and $x$ . It helps to find the connections between the whole statistics of nonequilibrium random walk and that of equilibrium one, in presence of 1/f-noise, if we know something about statistics under neglecting 1/f-noise [4,6,8,38].

One of possible solutions on the functional equation is $\langle \exp[ikQ(t)] \rangle = \Xi(ikx/T - k^2, t)$ , with only two independent arguments. It is easy to see that both the CF of quazi-Gaussian Brownian motion and CF of asymmetric one-directed random walk with Cauchy statistics can be unified into just such form [6,8],

$$\langle \exp\{ik\Delta Q\} \rangle \approx \Xi(ikx/T - k^2, t) \ , \ \Xi(\xi, t) \equiv \exp\{-D_0 \xi t A(t) \ln(\frac{\tau_0}{t} - \xi q_0^2)\}$$

Here $x = 0$ corresponds to quazi-Gaussian case, while the change $ikx/T - k^2 \Rightarrow ikx/T$ , under the condition $|k|\, x/T >> k^2$, leads to the second case. The condition means that only the drift component of the walk is observed which dominates at large time scales. Comparing the CFs we find that the unification needs in relations $U_0 = D_0 x/T$ and $Q_0 = q_0^2 x/T$ (or $V = D' x/T$ ). These is nothing but Einstein formula (or Nyquist formula, etc.) and its generalization for 1/f-fluctuations.

Besides, we obtain the statistical model of nonequilibrium Brownian motion, with taking into account both the drift and diffusional displacements and common 1/f-fluctuations of diffusuvity and mobility (in [6] FDR were used even in order to derive the non-equilibrium Cauchy statistics from quazi-Gaussian statistics obtained in [4,5]). The additional generalization of the model allows for arbitrary dependencies $D_0 = D_0(x)$ and $D' = D'(x)$.

# 7 CURRENT STATE AND PARADOXES OF GAS KINETICS BBGKY HIERARHY

## 7.1 BBGKY HIERARCHY

The probabilistic formulation of the Hamiltonian dynamics of classical gas is given by Bogolyubov-Born-Green-Kirkwood-Yvon (BBGKY) equations

$$\left[\frac{\partial}{\partial t} + v_1 \frac{\partial}{\partial r_1}\right] F_1 = \rho \iint \Lambda_{12} F_2 dp_2 dr_2 \equiv I_{12}(F_2)$$

$$\left[\frac{\partial}{\partial t} + v_1 \frac{\partial}{\partial r_1} + v_2 \frac{\partial}{\partial r_2} + \Lambda_{21}\right] F_2 = \rho \iint (\Lambda_{13}+\Lambda_{23}) F_3 dp_3 dr_3 \equiv I_{13}(F_3)+I_{23}(F_3)$$

$$\left[\frac{\partial}{\partial t} + L^{(n)}\right] F_n = \rho \iint \sum_j \Lambda_{jn+1} F_{n+1} dp_{n+1} dr_{n+1} \equiv \sum_j I_{jn+1}(F_{n+1})$$

$$L^{(n)} \equiv \sum_j v_j \frac{\partial}{\partial r_j} + \sum_{j>k} \Lambda_{jk} \ , \ \Lambda_{jk} \equiv f(r_j - r_k)(\frac{\partial}{\partial p_j} - \frac{\partial}{\partial p_k})$$

Here $L^{(n)}$ is the $n$-particle Liouville operator, $F_n = F_n(t, r_1..r_n, p_1..p_n)$ is the $n$-particle probability distribution function (DF), normalized to whole (formally infinite) gas volume $\Omega$, $\int F_n dp_n dr_n/\Omega = F_{n-1}$, the operators $\Lambda_{jk}$ decribe pair interactions, with $f(r)$ being the interaction force, and $v_j = p_j/m$ with $m$ being the mass of particles.

## 7.2 BOLTZMANN EQUATION

More than 100 years ago Boltzmann suggested the phenomenological kinetical equation to describe the evolution of rarefied gas with a short-range repulsive interactions between particles. The Boltzmann equation (BE) can be written as

$$\left[\frac{\partial}{\partial t} + v_1 \frac{\partial}{\partial R}\right] F_1 =$$

$$= \rho \int dp_2 \int d^2 B \, |v_1 - v_2| \, \{F_2^{in}(t, R, p_1', p_2') - F_2^{in}(t, R, p_1, p_2)\} \equiv S_{12}(F_2^{in}) \tag{1}$$

where the ansatz about molecular chaos (the famous Stosshalansatz) is introduced,

$$F_2^{in}(t, R, p_1, p_2) = F_1(t, r_1 \approx R, p_1) F_1(t, r_2 \approx R, p_2)$$

Here $F_2^{in}(t, R, p_1, p_2)$ is DF of two particles which come into collision at a point $R$ from the preceeding in-state, $B$ is the two-dimensional impact parameter vector perpendicular to the relative velocity $v = v_1 - v_2$ ( $B = r - v(v, r)/\left|v\right|^2$ with $r = r_1 - r_2$ ), and $p'_1, p'_2$ are the momentums what should be attributed to initial in-state in order to transform into the momentums $p_1, p_2$ at final out-state after collision. In accordance with Stosshalansatz, the join two-particle DF is factored into the product of one-particle DFs.

## 7.3 IS BOLTZMANN EQUATION DERIVABLE FROM BBGKY HIERARCHY ?

With no doubts, BE is the well grounded kinetical model. Thus it is the striking surprise that it still is not derived from BBGKY equations.

The widely spread opinion is that the derivation was performed by Bogolyubov. However, it was based on two ansatzes. First, the evolution of statistical ensemble governed by BBGKY equations tends to the stage when all the DFs become time-local functionals of one-particle DF. Secondly, the molecular chaos (mistakenly identified wih the weakening of correlations of far spaced particles).

Both these assumptions seem quite reasonable. But if it is really so, both should be deducable from BBGKY formalism. The possibility of rigirous derivation of BE from BBGKY equations was carefully analysed by O.Lanford, under a suitable definition of the norm in DF's functional space in the Boltzmann-Grad limit [35,36] (see also [30,37]). It was shown that the divergency of the series expansion of solution of BBGKY hierarchy to solution of BE takes place, but only during a short time after start smaller than the free flight time.

The attempts to extend this result to larger time intervals run into likely principal difficulties. The matter is that the singularities (discontinuities) of binary DF and higher DFs arise at hypersurfaces in the phase space which correspond to collisional configurations (for instance, when the vector $v_2 - v_1$ is parallel to $r_2 - r_1$ ). Though these singularities are concentrated in regions whose Lebesgue measure tends to zero in Boltzmann-Grad limit, thats are just the regions which determine the evolution of DFs of colliding particles and reliability of molecular chaos.

## 7.4 OBJECTIONS TO MOLECULAR CHAOS AND BOLTZMANN EQUATION

M.Kac [31] had doubts that BE is derivable from the BBGKY theory in general spatially non-uniform case. He treated BE sooner as the good model than as formal approximation of exact theory. According to Kac, in the uniform situation one could come to BE by means of spatial averaging in addition to ensemble averaging. In this case DFs contain no information about previous motion of gas particles, therefore the statistical ensemble makes no difference between two actually colliding particles and two arbitrary particles, and molecular chaos looks quite inevitable. But in general one losses the help of spatial averaging.

This criticizm was developed in [9]. According to BE, a life of a gas particle is prescribed by quite certain portions of collision of various sorts (by the portioning what corresponds to uniform distribution of the impact vector $B$ ). But in reality the distribution over sorts is out of both dynamical and thermodynamical control of the system. Even if it happened that some portion was much larger than its expected value, this incident will not influence the future. There are no mechanism what would be able to remember the history of preceding collisions and use it in order to compensate previous occasions and keep a certain distribution (histogram) of collisions over sorts.

Consequently, in accordance with the above phenomenology, the number of collisions (of any given sort) per unit time may have no predictable limit. In other words, the rate and the efficiency (the effective cross-section) of the flow of collisions of a gas particle undergo scaleless flicker fluctuations. The effect of these fluctuations on the evolution of $F_1$ could be diminished only by spatial averaging in the uniform gas, but in no way in the framework of non-uniform ensemble, when the spatial dependencies of DFs carry an information about history of Brownian motion of gas particles [9].

Indeed, the rate (the diffusivity) of the Brownian motion is closely correlated with fluctuations in the number and efficiency of collisions, hence, with the currently possible collision too. Therefore, the present collision has long-living statistical correlations with the past motion of colliding particles. In view of these correlations, the binary DF for particles going into collision acquires the specific statistical meanig as the conditional probability distribution, under the condition that the joint collision is realized in a fixed space-time point. Hence, this DF can not be factored into product of DFs $F_1$.

The generally possible factoration has the form

$$F_2^{in}(t,R,p_1,p_2) = F_1'(t,R,p_1)F_1''(t,R,p_2)$$

with conditional marginal distributions different from the conditionless one-particle DF $F_1$, $F_1' \neq F_1$, $F_1'' \neq F_1$ , what corresponds to mutual independency of particles in the momentum space but not in the real configurational space. Thus, the molecular chaos fails, at least because of spatial statistical correlation of colliding particles.

Such the ensemble can not be reduced to BE. It is amusing that the violation of molecular chaos takes place only between colliding particles, that is just where the molecular chaos is especially believed. This picture is obviously incompatible also with the ansatzes used by Bogolyubov.

Therefore, we must introduce a different ansatz, with the same purpose to build a kinetical model and firstly to narrow a variety of kinetically describable solutions of BBGKY equations.

# 8 CORRECT REFORMULATION OF BOLTZMANNIAN KINETICS

## 8.1 PRIMARY ANSATZ OF GAS KINETICS

The essence of coarsened kinetical language is that all the set of successive dynamical stages of a two-particle interaction process is treated as the unique momentary collision event. This event looks as a "black box" (compressed into point under Boltzmann-Grad limit) which instantly transforms the input velocities into output ones.

Clearly, this collision event should conserve probabilities, else a model would effectively allow for creation or annihilation of particles inside the black box. In other words, in any kinetical statistical ensemble, the flow of particles entering the collision should currently coinside with the flow of particles what leave it.

In our opinion, just this trivial requirement should serve as the basic ansatz when deriving a kinetical model from BBGKY equations. In order to provide with the probability conservation, the exclusion of internal dynamics of collision from consideration must be compensated by definite bondary conditions at the black box borders. Concretely, one should keep the probability measure of any final output state at the border to be currently equal to the probability of corresponding initial input state.

This condition is formally similar to the one for DFs at $|r_j - r_k| = d$ in the rigid sphere gas (with $d$ being the diameter of spheres). In general, a choise of the borders of collision box is rather arbitrary, at least in the framework of Boltzmann-Grad limit. Therefore the above mentioned ansatz is equivalent to that probabilities of all the inner stages of any fixed sort of collision equal to one and the same quantity. If this condition was not satisfied, it would be impossible to use the concept of momentary collision event at all!

In such the way we narrow the set of solutions of BBGKY hierarchy to be under kinetical consideration: the distributions somehow modulated along a path inside collision box are excluded (although arbitrary modulation from one path to another is permissible).

The full characterization of a sort of binary collision consists of its location defined by three space coordinates, the two-dimensional impact parameter vector and two velocities of particles, that is totally by 3+2+6=11 scalars instead of total 12 phase space variables of the pair. The rest twelfth variable, let be denoted as $\Theta$ , just enumerates inner dynamical stages of the collision in the black box interior. This variable should be excluded from a kinetical picture. Thus, with respect to $F_2$ , the primary ansatz means that

$$\frac{\partial}{\partial\Theta} F_2 = 0 \ , \ r_2 - r_1 \in collision\, box \tag{2}$$

One always can choose $\Theta$ as the inner time of the collision, for instance, as the time necessary to achieve its given stage from its most proximity stage. Then the differentiation by $\Theta$ looks as

$$\frac{\partial}{\partial\Theta} = v\frac{\partial}{\partial r} + f(r)(\frac{\partial}{\partial p_2} - \frac{\partial}{\partial p_1}) = v\frac{\partial}{\partial r} + \Lambda_{21} \equiv L^{(2)}_{rel} \tag{3}$$

with $r = r_2 - r_1$ and $v = v_2 - v_1$ being the relative distance and velocity of particles, respectively. The operator $L^{(2)}_{rel}$ is nothing but the part of full two-particle Liouville operator $L^{(2)}$ responsible for the inner dynamics of collision.

Of course, none of rigorous solutions on BBGKY equations exactly satisfies (2), as well as the ansatzes introduced by Boltzmann and Bogolyubov. But if the kinetic approximation is actually available, we may hope that at a more or less late stage (kinetical stage) of the ensemble evolution the ansatz

(2) is quantitatively satisfied to the extent sufficient to regard it as formal identity.

## 8.2 COLLISION INTEGRAL

In fact, one inevitably needs in the primary ansatz in order to construct the Boltzmannian collision integral, even under the molecular chaos postulate, and thus to reduce the explicite dynamics to the momentary collisions representation. Indeed, let us rewrite the right-hand side of first BBGKY equation as

$$I_{12}(F_2) = \rho \iint \Lambda_{12} F_2 dp_2 dr_2 =$$

$$= \rho \int \int_{CB} (v_2 - v_1) \frac{\partial}{\partial r} F_2 dr dp_2 - \rho \int \int_{CB} \frac{\partial}{\partial \Theta} F_2 dr dp_2$$

where $CB$ marks the spatial integration over the collision black box. After transformation the volume integral in the first right-hand term into surface integral $\oint$ over the collision box board and after separation of in-states and out-states, regarding of sign of the scalar product $(vr)$ , one obtains

$$I_{12}(F_2) = \rho \int \left[ \oint_{vr>0} |v| \, F_2^{out} d^2 B' - \oint_{vr<0} |v| \, F_2^{in} d^2 B - \int_{CB} \frac{\partial F_2}{\partial \Theta} dr \right] dp_2$$

Here in both the surface integrals one and the same velocity arguments are present while the impact parameter vectors $B'$ and $B$ relate to final and initial state, respectively.

This expression can be reduced to conventional collision integral only if the probabilities of the final states constantly coinside with thats of preceding incoming states,

$$F_2^{out}(t, R, p_1, p_2, B') = F_2^{in}(t, R, p_1', p_2', B)$$

Here $R$ is the location of collision, and in view of Boltzmann-Grad limit we do not make difference between spatial locations of in-states and out-states.

The validity of this relation is ensured just by ansatz (2). Besides, due to (2) the last integral, what formally acts like a bulk source of particles, turns into zero. As the result, we come to the kinetic equation (1), but, as it will be seen, the molecular chaos hypothesis fails.

## 8.3 VIOLATION OF MOLECULAR CHAOS

According to primary ansatz, in the interior of collision box the second BBGKY equation reduces to

$$\left[\frac{\partial}{\partial t} + U_2 \frac{\partial}{\partial R} + \frac{\partial}{\partial \Theta}\right] F_2 \Rightarrow \left[\frac{\partial}{\partial t} + U_2 \frac{\partial}{\partial R}\right] F_2 = I_{13}(F_3) + I_{23}(F_3) \qquad (4)$$

with $R \equiv (r_1 + r_2)/2$ and $U_2 \equiv (v_1 + v_2)/2$ being the position and velocity of the mass center of colliding pair, i.e. the position and velocity of the collision as a whole.

Hence, the spatial drift of the DF of colliding or closely spaced particles is governed by the mass center velocity. This consequence of (2) has very simple physical meaning. First, if the relative motion of particles is included into the collision interior, then quite naturally it should be automatically excluded from the transport of the collision as a whole. Secondly, the DF related to collision box represents the conditional probability under the condition that the particles are partners in one and the same encounter. It is natural that this specified DF is ruled by an equation different from what describes the less informative unconditional DF. Evidently, the conditional DF $F_2$ in (4) can be interpreted as the probability measure of that a collision occures in the point $R$ , i.e. as the measure of ensemble-averaged density of collisions.

It is easy to see that in a spatially non-uniform ensemble, when $\partial F_2/\partial R \neq 0$ , the Eq.4 forbids the factoration of this DF into product of one-particles DFs, $F_2 \neq F_1(t, p_1, R) * F_1(t, p_2, R)$ , regardless of contribution of the right-hand side. Therefore the molecular chaos inevitably fails, and just with respect to colliding particles! As a consequence, in general the density of collisions is not simply proportional to [density of particles]$^2$ .

## 8.4 THREE-PARTICLE PROCESSES AND EVOLUTION OF DENSITY OF COLLISIONS

Let us take in mind the Boltzmann-Grad limit (BGL),

$$\rho \to \infty,\ d \to 0,\ \lambda = 1/\rho d^2 = const$$

where $\lambda$ is the mean free path and $d$ is characteristic radius of interaction.

The right-hand side of the Eq.4 describes a change of density of collisions due to three-particle processes. Thats are not factual triple collisions, which are of no importance under BGL, but two successive pair collisions separated by a distance comparable with $d$ but perhaply much greater than $d$ . One of the collisions involves a third external particle representing the rest gas. Under BGL, a spatial region taken by this process contracts into point if measured by $\lambda$ scale although can grow up to infinity in $d$ units.

In order to express the right-hand side of Eq.4 in kinetic fashion, in terms of collision integrals, we need in the obvious three-partricle extention of the ansatz (2),

$$\partial F_3/\partial\Theta = L^{(3)}_{rel} F_3 \Rightarrow 0 \ , \ L^{(3)}_{rel} = L^{(3)} - U_3 \partial/\partial R$$

where $\Theta$ is a suitably choosen inner time of the process, and $R$ and $U_3$ are the position and velocity of mass center of the close three-particle configuration under consideration. Eventually, the Eq.4 is transformed into the kinetical equation

$$\left[\frac{\partial}{\partial t} + U_2 \frac{\partial}{\partial R}\right] F_2 = S_{13}(F_3^{in}) + S_{23}(F_3^{in}) \tag{5}$$

where the superscript $in$ means that the external third particle is at in-state with respect to 1-st and 2-nd particles (i.e. is coming to them from the infinity, in $d$ units). This equation describes the evolution of density of binary encounters and collisions as changed by collisions of the given two particles with the rest gas.

## 8.5 NO FAR CORRELATIONS ALTHOUGH NO MOLECULAR CHAOS

In contrary to the conditional binary DF subjected to Eq.4, the general binary DF for arbitrary configurations placed outside the collision box is subjected to the usual equation

$$\left[\frac{\partial}{\partial t} + v_1 \frac{\partial}{\partial r_1} + v_2 \frac{\partial}{\partial r_2}\right] F_2 = S_{13}(F_3^{in}) + S_{23}(F_3^{in}) \tag{6}$$

Evidently, this equation allows for the factored solutions, $F_2 = F_1(t, r_1, p_1) * F_1(t, r_2, p_2)$ , with $F_3^{in} = F_2^{in}(t, r_1, p_1, p_3, B) * F_1(t, r_2, p_2)$ in $S_{13}(F_3^{in})$ and $F_3^{in} = F_2^{in}(t, r_2, p_2, p_3, B) * F_1(t, r_1, p_1)$ in $S_{23}(F_3^{in})$ , where each of $F_1$ multipliers is the solution on the Eq.1.

Thus the vanishing of correlations of far distanced particles coexists with invalidity of molecular chaos with respect to closely spaced particles (in particular, colliding ones). If measure the inter-particle distance with the free flight path $\lambda$ then, under BGL, the weakening of correlations, $F_2 = F_1(t, r_1, p_1) * F_1(t, r_2, p_2)$ , can be rigorously satisfied anywhere except the hypersurface $r_2 - r_1 = 0$ . In view of this circumstance, the singularities of DFs at collisional hypersurfaces met in [35,36] confirm the necessity to specially consider the specific DFs for encountering configurations with $r_1 = .. = r_n$ .

## 8.6 MANY-PARTICLE ENCOUNTERS AND CHAIN OF KINETICAL EQUATIONS

In the framework of BGL, one should continue the above mentioned procedure up to infinity, with extention of primary ansatz to many-particle events, namely, to sequences of pair collisions and close many-particle encounters, step by step building the kinetical model of spatially non-uniform gas. The extention looks as $\partial F_n/\partial\Theta = 0$ , with the equality to be satisfied inside a collision (encounter) black box alloted for the event and $\Theta$ being its inner time. This is the inevitable payment for decision to reformulate the ensemble evolution in terms of collision integrals.

Because of identities $\partial/\partial\Theta = L^{(n)}_{rel}$ , $L^{(n)} = U_n\partial/\partial R + L^{(n)}_{rel}$ , where $L^{(n)}_{rel}$ is the part of $n$-particle Liouville operator responsible for relative motion inside the event, the result is the infinite chain of coupled kinetic equations [9]

$$\left[\frac{\partial}{\partial t} + U_n \frac{\partial}{\partial R}\right] F_n = \sum_{j=1}^{n} S_{jn+1}(F^{in}_{n+1}) \tag{7}$$

Here $R = (r_1 + .. + r_n)/n$ , $U_n = (v_1 + .. + v_n)/n$ .

The DFs on the left represent the probability measure of occurence of $n$-particle encounters, in particular, constituted by $(n-1)$ successive collisions. The right-hand DFs relate to $(n+1)$-particles events, with an external $(n+1)$-th particle being at precollisional in-state with respect to the left $n$-particle configuration. In general, any of these $F_n$ depends on time, the location of the encounter $R$ , $n$ momentums, $(n-1)$ impact parameter vectors and, besides, on $(n-2)$ distancies between binary encounters if $n > 2$ . Again, the relative motion turns into the hidden inner property of the event and so does not influence the drift of the probability of the event as a whole.

In spatially uniform case, when $\partial/\partial R \to 0$ , the solution on the Eqs.7 can be sought in the conventional molecular chaos form, $F_n = \prod_j F_1(t, r_j, p_j)$ , with no factual dependence on impact parameters, and the model reduces to conventional BE.

## 8.7 MORE ABOUT PRIMARY ANSATZ

To formulate the primary ansatz specially for BGL, let us write the interaction force in the form $f(r) \Rightarrow f(r/d)/d$ and measure the position of a cluster of close particles in $\lambda$ units (and time in free flight time units) while interparticle distancies in $d$ units. Then BBGKY equations turn into

$$\left[\frac{\partial}{\partial t} + U_n \frac{\partial}{\partial R} + \frac{1}{\mu} L_{rel}^{(n)}\right] F_n^{(\mu)} = \iint \sum_j \Lambda_{jn+1} F_{n+1}^{(\mu)} dp_{n+1} dr_{n+1}$$

where $\mu = d/\lambda = \rho d^3 \to 0$ is small gas parameter, and DFs dependence on $\mu$ is marked. Clearly, if a definite limit of all the $F_n^{(\mu)}$ does exist at $\mu \to 0$ , then without fail the relation $\partial F_n^{(0)}/\partial\Theta = L_{rel}^{(n)} F_n^{(0)} = 0$ is satisfied. Hence, the ansatz is equivalent to the assumption that $L_{rel}^{(n)} F_n^{(\mu)} = o(\mu)$ , at least for configurations corresponding to chains of far spaced binary collisions (i.e excluding actually triple or multiple collisions).

## 8.8 COLLISIONS, ENCOUNTERS AND COARSENED KINETIC EQUATIONS

In principle, under a suitable geometrical parametrization of the events the left-hand DFs in Eqs.7 can be identified with the right-hand ones. However, a full classification of configurations would lead to an extremely complicated theory.

It seems reasonable to simplify the model with the help of spatial averaging of DFs over $r_j$ inside the collision (encounter) regions, under fixed mass center position $R$ . Of course, then a dependence on impact parameters will be lost, except the only parameter on the right-hand sides what relates to the incoming $(n+1)$-th external particle. As a consequence, one will be forced to distinguish left-hand DFs $F_n$ and right-hand DFs $F_n^{in}$ .

Besides, for more simplicity, in the framework of BGL we may choose the dimensions of the collision (encounter) regions (boxes) much larger than interaction diameter $d$ . Then the most part of configurations under averaging will correspond to encounters, when particles are separated by a distance comparable with $d$ but without factual interaction (and so out-states coinside with in-states).

We may hope that in spite of such the crucial simplification the essential features of statistics of collisions will not be lost. Indeed, in view of incident origin of close $n$-particle clusters, there are no principal difference between efficient collisions and merely encounters. The latters can be regarded as collisions with negligibly weak scattering, and fluctuations in the flow of collisions should be similar to thats in the flow of encounters.

## 8.9 WEAKENED MOLECULAR CHAOS

It is interesting that the chain (7) is still formally time-reversible if all the geometry of binary and many-particle events is kept under control. Indeed, the combined inversion of time and velocities and rearrangement of in-states and out-states does not change these equations. However, this does not prevent existence of seemingly irreversible solutions of the equations, all the more if a part of the complete characterization of many-particle configurations is lost.

The DFs smoothed over collision regions have the clear physical meaning: thats represent ensemble-averaged densities of $n$-particle encounters, each depending on time, location $R$ and $n$ velocities. But in order to close the Eqs.7 one needs in some connections between $F^{in}_{n+1}$ and the left-side functions. This is just the point where one more ansatz like molecular chaos should be attracted.

It is natural to assume that the velocity distribution of $(n+1)$-th right-hand external particle is statistically independent on velocities of the left-hand cluster particles:

$$F^{in}_{n+1}(t,R,p^{(n+1)}) = G_n(t,R,p^{(n)})F_1(t,R,p_{n+1})$$

where the notation $p^{(n)} \equiv \{p_1,..,p_n\}$ is introduced. In general this factoration is not identical to absolute statistical independence of particles, since it allows for spatial correlations whose manifestation is that functions $G_n$ differ from $F_n$ .

Clearly, one should connect $F^{in}_{n+1}$and thus $G_n$ with $F_{n+1}$ , not with $F_n$ . The matter is that the spatial dependencies of both $F^{in}_{n+1}$and $F_{n+1}$ reflect the probability density of the same $(n+1)$-particle encounter events. Therefore the degree of specifically spatial correlations must be one and the same in both $F^{in}_{n+1}$and $F_{n+1}$ (this assertion follows also merely from the proximity of $(n+1)$-th particle to the left-side cluster in $\lambda$ units). This requirement implies

$$\int F_{n+1} dp^{(n+1)} = \int F^{in}_{n+1} dp^{(n+1)} \Rightarrow \int G_n(t,R,p^{(n)}) dp^{(n)} \int F_1(t,R,p') dp'$$

One can say also that this relation expresses the conservation of particles and probabilities during encounters.

Besides, take into account that the degree of correlations inside the left-hand cluster must be the same in $F^{in}_{n+1}$and in $F_{n+1}$. Then the only reasonable form of $G_n$ is $G_n = \int F_{n+1} dp_{n+1} / \int F_1 dp_1$, and we finally come to the connection [9]

$$F^{in}_{n+1}(t,R,p^{(n+1)}) = \int F_{n+1}(t,R,p^{(n)},p') dp' \frac{F_1(t,R,p_{n+1})}{\int F_1(t,R,p') dp'} \qquad (8)$$

This is the weakened form of molecular chaos which establishes the statistical independence of the external particle only in the velocity space, but not in the configurational space.

## 8.10 SPATIALLY NON-UNIFORM BUT EQUILIBRIUM GAS AND MODIFIED BOLTZMANN-LORENTZ MODEL

One naturally is forced to consider some spatially non-uniform solutions on BBGKY hierachy even in thermodynamically equilibrium case, if wants to analyse the statistics of Brownian motion of a test gas particle. If the position of even a single test particle is known at initial time moment, or even merely its distribution differs from the uniform one, then one should deal with spatially varying DFs.

Let DF $F_1(t,R,p_1)$ be related to the test particle. Then the higher order DFs $F_{n+1}$ subjected to Eqs.7 and 8 relate to the clusters consisting of the test particle plus $n$ arbitrary close particles. All the DFs together encode the information about statistics of encounters, collisions and random displacements of test particle.

But, of course, now the DF of test particle, $F_1(t,R,p_1)$ , and the one-particle DF $F_1(t,R,p_{n+1})$ in (8) are different things. The latter relates to arbitrary gas particle and so should coinside with the Maxwellian distribution in equilibrium gas:

$$F_1(t,R,p_{n+1}) = F^{eq}(p_{n+1}) \ , \ \ F^{eq}(p) \equiv (2\pi Tm)^{-3/2} \exp(-p^2/2mT)$$

The weakened molecular chaos condition transforms into

$$F^{in}_{n+1}(t,R,p^{(n)},p_{n+1}) = F^{eq}(p_{n+1}) \int F_{n+1}(t,R,p^{(n)},p')dp' \qquad (9)$$

The sets of equations (7) and (9) represent the analogy of the well known Boltzmann-Lorentz model but correctly modified for a spatially non-uniform ensemble.

# 9 SELF-DIFFUSION AND DIFFUSIVITY 1/F-NOISE

## 9.1 KINETICAL EQUATIONS FOR BROWNIAN MOTION OF TEST PARTICLE

Let $r(t)$ and $u(t)$ be coordinate and velocity of any particle and $Q(t) = \int_0^t u(t')dt'$ be its displacement during time interval $t$ (more accurately, let $u$ and $Q(t)$ be the projections of velocity and displacement vector onto a fixed direction). The statistics of self-diffusion (Brownian motion) of gas particle is described by the characteristic function (CF), $\langle \exp[ikQ(t)] \rangle$ , but, as we know, the most significant information is contained in the first two terms of expansion of $\ln \langle \exp[ikQ(t)] \rangle$ over $k^2$ , which reflect the mean diffusivity and the correlation function of diffusivity fluctuations.

The Laplace transform of CF always can be represented as

$$\int_0^\infty \langle \exp[ikQ(t)] \rangle \exp(-zt)dt = [z + kD(z,ik)k]^{-1}$$

with $D(z,ik)$ being the frequency dependent diffusivity kernel. From the other hand, with the help of generalized Green-Kubo formulas, one can connect $D(z,ik)$ and the response of probability distribution of a test particle to a weak external force $f_{ext}(r)$ , $f_{ext} \to 0$ , which influence it after switching on at $t = 0$ (being parallel to the choosen direction). Concretely [9],

$$\int_0^\infty dt \exp(-zt) \int dr \exp(-ikr) \int vF_1(t,r,p)dp =$$

$$= \frac{D(z,ik)}{T[z+kD(z,ik)k]}\int \exp(-ikr)f_{ext}(r)dr$$

where $v = p/m$ , the DF $F_1(t,r,p)$ describes the test particle and equals to $F^{eq}(p)$ at $t < 0$ .

Hence, the long-time statistics of equilibrium self-diffusion can be obtained from the spatially non-uniform evolution (formally nonequilibrium but in fact equilibrium in the macroscopic sense) of ensemble disturbed by a force what acts on the only test particle. In the framework of the kinetical theory constructed, the problem reduces to solution of Eqs.7 and 9 , after adding the terms $f_{ext}(R)\partial F_n/\partial p_1$ to all left sides of Eqs.7.

We may avoid a part of non-principal mathematical difficulties if replace the Boltzmannian collision integrals on right-hand sides of Eqs.7 with the Fokker-Planck approximation of the Boltzmann-Lorentz collision operator,

$$S_{jn+1}(\int F_{n+1}(t,R,p^{(n)},p')dp' * F^{eq}(p_{n+1})) \Rightarrow \gamma\frac{\partial}{\partial p_j}(p_j + mT\frac{\partial}{\partial p_j})\int F_{n+1}dp_{n+1}$$

with $\gamma$ being the inverse free flight time. Then we have to solve the equations [9]

$$\left[\frac{\partial}{\partial t} + U_n\frac{\partial}{\partial R} + f_{ext}(R)\frac{\partial}{\partial p_1}\right] F_n = \sum_{j=1}^{n}\gamma\frac{\partial}{\partial p_j}(p_j + mT\frac{\partial}{\partial p_j})\int F_{n+1}dp_{n+1} \quad (10)$$

with $U_n = (v_1 + .. + v_n)/n$ and equilibrium initial conditions $F_n \mid_{t=0} = \prod_j F^{eq}(p_j)$ .

## 9.2 NON-LOCAL DIFFUSIVITY KERNEL AND LONG-LIVING STATISTICAL CORRELATIONS

But even the analysis of these simplific equations is rather hard, and only the lowest terms of the expansion

$$D(z,ik) = D_0(z) + (ik)^2 D_1(z) + ...$$

were found in [9]. Nevertheless, these terms always allow to check the existence of low-frequency duffusivity fluctuations and estimate their variance.

In accordance with the $D(z,ik)$ definition, one can verify that $D_0(z)$ is the Laplace transform of the two-point equilibrium velocity correlator, $K_u(t) \equiv \langle u(t)u(0)\rangle$ , and

$$D_1(z) = \int_0^\infty \{K_D(t) + \, K_u(t) \int_0^t (t-t')K_u(t')dt'\} \exp(-zt)dt$$

Here $K_D(t)$ is the above discussed diffusivity correlation function as expressed via four-point equilibrium cumulant of $u(t)$ . Therefore, at low frequencies, $|z| << \gamma$ , $D_1(z)$ is nothing but the Laplace transform of $K_D(t)$ , that is $Re[D_1(2\pi if+0)]$ is responsible for the power spectrum of diffusivity fluctuations. From the other hand, one can connect $D_1(z)$ strictly with the fourth statistical moment of the displacement:

$$\int_0^\infty \left\langle Q^4(t)\right\rangle \exp(-zt)dt \ = \frac{24}{z^2}[D_1(z) + D_0^2(z)/z]$$

The latter function was obtained in [9] in the form of double series via $\gamma/z$ which can be exactly transformed into an integral. Here we omit these manipulations. At $|z| << \gamma$ , the final result reeds as

$$D_1(z) \approx \frac{1}{2z} D_0^2(z)[\ln^2\frac{\gamma}{z} + const]$$

with $D_0(z) = \frac{T}{m}(\gamma+z)^{-1}$ and $D_0 = D_0(0) = T/m\gamma$ being the usual diffusivity. In the temporal reprezentation, these results mean that $\langle Q^2(t)\rangle = 2D_0t$ and

$$\left\langle Q^{(4)}(t)\right\rangle = \left\langle Q^4(t)\right\rangle - 3\left\langle Q^2(t)\right\rangle^2 \approx 6D_0^2t^2[\ln^2\gamma t + c'\ln\gamma t + c'']$$

at $\gamma t >> 1$ , where $c', c''$ are numerical constants. In view of the above consideration, this is the clear evidence of existence of 1/f type self-diffusivity fluctuations.

Formally, these results are produced by the peculiar drift terms $U_n\frac{\partial}{\partial R}$ on right-hand sides of the Eqs.10. But our experience showed the remarkable fact that the concrete form $U_n = (v_1 + .. + v_n)/n$ is of no principal importance. Almost any sequence of $U_n = \sum c_{nj}v_j$ with different $c_{nj}$ results in an excess contribution to the fourth cumulant $\left\langle Q^{(4)}(t)\right\rangle$ which grows faster than proportionally to time and thus corresponds to a low-frequency diffusivity noise. In particular, if the mass of the test particle $M$ differs from masses of gas particles, and so $U_n = (Mv_1 + mv_2 + ...)/(M+nm)$ , then the Eqs.10 lead to $\left\langle Q^{(4)}(t)\right\rangle \sim t^{1+\gamma}$ , $\gamma > 0$ , where $\gamma$ differs from unit and depends on the ratio $M/m$ [9].

## 9.3 1/F-FLUCTUATIONS OF DIFFUSIVITY

The power spectrum of the relative diffusivity fluctuations resulting from the obtained $D_1(z)$ for sufficiently low frequences, $f << \gamma$ , is

$$S_{\delta D}(f) \approx \frac{1}{2f} \ln \frac{\gamma}{f}$$

Formally this is non-integrable spectrum like that of a slightly non-stationary noise. But we already emphasized that it is fictituous non-stationarity what refers only to the duration of observation, not to the time moment when observation takes beginning. Besides, it is necessary to underline that the spectrum is insensitive to spatial dimensionality of gas, giving the illustration of so-called zero dimensionality of 1/f-noise.

# 10 RESUME

We reviewed the explanation of 1/f-noise based on the idea that its sources are not some specifically slow processes but, in opposite, the same processes which realize forgetting the past, relaxation and related white noise. If a system producing randomness constantly forgets its past and keeps in its memory scope only a limited amount of recently happened events, then it is unable to definitely distinguish a proper part of events and deviation from it. As a consequence, the fluctuations in the amount of incidently emerging events grow similarly to its average value, resulting in just 1/f-fluctuations of the rate of random flow of events.

In order to comprehend such a random behaviour, one should make difference between long-living statistical correlations and long-living causal correlations.

We noted that 1/f-noises in different systems can be interpreted in such manner. The corresponding phenomenological models of purely random flow of events and of Brownian motion were considered, whose important feature is violation of the central limit theorem with respect to long-range statistics and close connections with Cauchy statistics.

The attempts to develope an adequate philosophy of 1/f-noise were made. The quantitative estimates of 1/f-noise in terms of only short-range (microscopic) characteristic scales were presented, showing that the wider is the memory scope of the system the lower is 1/f-noise level.

For thermodynamical systems, the general connections between 1/f-noise and four-point equilibrium cumulants were analysed. The existence of long-living four-point correlations equivalent to flicker noise in exactly solvable Kac's ring model was found.

The connections with Hamiltonian models of relaxation and irreversible processes were discussed, with hopes that if any realistic time-reversible many-particles dynamics produces noise and irreversibility then it also produces 1/f-fluctuations of their rates. This expectation can be testified in such principally important cases as slightly unharmonic crystal lattice (phononic system) and classical gas with short repulsive interactions (in the Boltzmann-Grad limit).

We pointed out that collisions of any gas particle with other particles form just such a random flow of events whose rate is being constantly forgotten and free of a thermodynamical control by the system. We argued that consequently 1/f fluctuations of the rate (number of collisions per unit time) and of related quantities (effective cross-section, self-diffusuvity, mobility) should occur and manifest themselves as the spatial statistical correlations of colliding particles in a spatially non-uniform statistical ensemble, resulting in violation of both the molecular chaos ansatz and the ansatz used by Bogolyubov to deduce Boltzmann equation from BBGKY hierarchy.

The euristic reasonings were confirmed by critical analysis of kinetical theory of gas. It was argued that reformulation of the gas ensemble evolution in terms of coarsened description with Boltzmannian collision integrals inevitably needs in another ansatz whose meaning is the conservation of particles and constancy of probabilities on the path what connects in-state and out-state through the collision box (i.e. the space-time region to be excluded from kinetical description).

We showed that this requirement leads to the infinite set of specific kinetical equations for the probability distributions on the hypersurfaces which correspond to binary and many-particle encounters and collisions (the sequences of binary collisions, not actually multiple ones). In a spatially non-uiform ensemble these kinetical equations forbid conventional molecular chaos, because of producing statistical correlations of close particles in the configuration space, and can be reduced to standard Boltzmann equation only in the limit uniform case.

In our opinion, this is in agreement with the results by O.Lanford who noted the appearance of singularities of distribution functions at the above mentioned hypersurfaces in the formal series solution of BBGKY equations when investigating the possibility of reduction of BBGKY hierarchy to Boltzmann equation. The singularities just indicate that one needs in special consideration of probabilities of colliding and close configurations.

The obtained chain of kinetical equations, with addition Green-Kubo formulas and higher-order fluctuation-dissipation relations, was applied to analyse the statistics of self-diffusion (Brownian motion of a test particle) in equilibrium gas. The four-order cumulant of Brownian displacement was calculated whose time behaviour proves the existence of 1/f-fluctuations of diffusivities (and thus of mobilities) of gas particles.

Hence, we have at least one example of principally correct formulation of kinetical approximation which does not loss 1/f-noise, and the example of dynamical system by no means producing slow processes but in spite of this producing 1/f-noise.

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**REFERENCES**

1. P.Dutta, P.M.Horn, "Low-frequency fluctuations in solids: 1/f noise", Rev.Mod.Phys., 53(3), 497 (1981).

2. F.N.Hooge, T.G.M.Kleinpenning, L.K.J.Vandamme, "Experimental studies of 1/f noise", Rep.Prog.Phys., 44, 481 (1981).

3. N.S.Krylov, "Works on the foundations of statistical mechanics", Princeton U.P., Princeton, 1979 (the original books published in Russian in 1950).

4. Yu.E.Kuzovlev and G.N.Bochkov. "On nature and statistics of 1/f-noise". Preprint No.157, NIRFI (Scientific-Research Radiophysics Institute), Gorkii, USSR, 1982 (in Russian).

5. Yu.E.Kuzovlev and G.N.Bochkov. "On the origin and statistical characteristics of 1/f-noise". Izvestiya VUZov.-Radiofizika, 26, 310 (1983). Transl. in English in "Radiophysics and Quantum Electronics" (RPQEAC, USA), No 3 (1983).

6. G.N.Bochkov and Yu.E.Kuzovlev. "On the probabilistic characteristics of 1/f-noise". Izvestiya VUZov.-Radiofizika, 27, 1151 (1984). Transl. in English in "Radiophysics and Quantum Electronics" (RPQEAC, USA), No

9 (1984).
7. G.N.Bochkov and Yu.E.Kuzovlev. "New in 1/f-noise studies". Uspekhi fizicheskikh nauk, 141, 151 (1983). English translation: Sov.Phys.-Usp., 26, 829 (1983).
8. G.N.Bochkov and Yu.E.Kuzovlev. "On the theory of 1/f-noise". Preprint No.195, NIRFI (Scientific-Research Radiophysics Institute), Gorkii, USSR, 1985 (in Russian).
9. Yu.E.Kuzovlev. "Bogolyubov-Born-Green-Kirkwood-Yvon equations, self-diffusion and 1/f-noise in a slightly nonideal gas". Zh. Eksp. Teor. Fiz, 94, No.12, 140 (1988). English translation: Sov.Phys.JETP, 67(12), 2469 (1988).
10. Yu.E.Kuzovlev, "Purely random time partitions and 1/f-noise". Phys.Lett., A 194, 285 (1994).
11. Yu.E.Kuzovlev. "Relaxation and 1/f-noise in phonon systems". Zh. Eksp. Teor. Fiz, 111, No.6, 2086 (1997). English translation: JETP, 84(6), 1138 (1997)
12. V.Feller, "Introduction to probability theory and its applications", Mir, Moscow, 1967.
13. D.S.Ornstein, "Ergodic theory, randomness and dynamical systems", New Haven, London, Yale University Press, 1974.
14. P.H.Handel, "Nature of 1/f frequency fluctuations in quartz crystal resonators", Sol.-St. El., 22, 875 (1979).
15.Y.Nogushi, Y.Teramashi and T.Musha, "1/f frequency fluctuations of a quartz oscillator", Appl.Phys.Lett., 40, 872 (1982).
16. T.Musha, B.Gabor and M.Shoji, "1/f photon-number fluctuations in quartz observed by light scattering", Phys.Rev.Lett., 64(20), 2394 (1990).
17. R.F.Voss, "Evolution of long-range fractal correlations and 1/f noise in DNA base sequences", Phys.Rev.Lett., 68, 3805 (1992).
18. W.Li, T.G.Marr and K.Kaneko, "Understanding long-range correlations in DNA sequences", Physica, D 75, 392 (1994).
19. R.F.Voss and J.Clarke, Phys.Rev., B 13, 556, 1976.
20. M.Nelkin and A.M.S.Tremblay, "Deviation of 1/f voltage fluctuations from scale-similar Gaussian behaviour", J.Stat.Phys., 25, 253 (1981).
21. J.L.Tandon and H.P.Bilger, "1/f-noise as a non-stationary process: evidences and some analytical conditions", J.Appl.Phys., 47, 1697 (1976).
22. J.J.Brophy, "Statistics of 1/f-noise", Phys.Rev., 166(3), 827 (1968).
23. J.J.Brophy, "Low-frequency variance noise", J.Appl.Phys., 41, 2913 (1970).

24. A.A.Aleksandrov, G.N.Bochkov, A.A.Dubkov and A.I.Chikin, "Measuring probability distribution of 1/f-noise", Izvestiya VUZov.-Radiofizika, 29, 980 (1986). Transl. in English in "Radiophysics and Quantum Electronics" (USA).

25. A.Sadbery, "Quantum mechanics and the particles of nature", Cambridge Univ. Press, 1986.

26. I.Prigogine, "From being to becoming", W.H.Freeman and Co., San Francisco, 1980.

27. R.D.Hughes, E.W.Montroll and M.F.Shlesinger, J.Stat.Phys., 28, 111 (1982).

28. I.Procaccia and H.Schuster, "Functional renormalization-group theory of universal 1/f noise in dynamical systems", Phys.Rev., A 28, 1210 (1983).

29. D.V.Anosov et al., "Dynamical systems with hyperbolic behaviour", in "Modern problems of mathematics. Fundamentals. Vol. 66", ed. R.V.Gamkrelidze, VINITI, Moscow, 1991 (in Russian).

30. G.Gallavotti, "Statistical mechanics", 1998.

31. M.Kac, "Probability and related topics in physical sciences", Intersci. Publ., London, N.-Y., 1957.

32. G.N.Bochkov and Yu.E.Kuzovlev, "On general theory of thermal fluctuations in nonlinear systems", Sov.Phys.-JETP, 45, 125 (1977).

33. G.N.Bochkov and Yu.E.Kuzovlev, "Fluctuation-dissipation relations for nonequilibrium processes in open systems", Sov.Phys.-JETP, 49, 543 (1979).

34. G.N.Bochkov and Yu.E.Kuzovlev, "Non-linear fluctuation-dissipation relations and stochastic models in nonequilibrium thermodynamics", Physica, A 106, 443 (1981).

35. O.E.Lanford, "Time evolution of large classical systems", in "Dynamical systems, theory and applications", ed. J.Moser, Lectures Notes in Physics, vol.38, 1975.

36. O.E.Lanford, "On a derivation of Boltzmann equation", in "Nonlinear phenomena. 1: The Boltzmann equation", eds. J.L.Lebowitz, E.W.Montroll, N.-H., Amsterdam, 1983.

37. H.Spohn, "Theory of fluctuations and Boltzmann equation", ibid.

38. G.N.Bochkov and Yu.E.Kuzovlev, "Fluctuation-dissipation theory of nonlinear viscosity", Zh. Eksp. Teor. Fiz., 79, 2239, (1980) (transl. in English in Sov.Phys.-JETP, No 12, 1980).