Qualitative Aspects of Classical Potential Scattering

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Abstract

We derive criteria for the existence of trapped orbits (orbits which are scattering in the past and bounded in the future). Such orbits exist if the boundary of Hill's region is non-empty and not homeomorphic to a sphere.

For non-trapping energies we introduce a topological degree which can be non-trivial for low energies, and for Coulombic and other singular potentials. A sum of non-trapping potentials of disjoint support is trapping iff at least two of them have non-trivial degree.

For $d \geq 2$ dimensions the potential vanishes if for any energy above the non-trapping threshold the classical differential cross section is a continuous function of the asymptotic directions.

1 Introduction

A large part of our knowledge concerning atoms and molecules comes from scattering experiments.

In the simplest case one scatters particles of definite initial velocity by a molecule and then observes the final distributions of their directions. This can be modeled by quantum potential scattering.

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The most prominent quantal phenomenon, namely the resonances of the differential cross section for the Schrödinger equation, is related to the classical phenomenon of *bounded orbits of positive energy*.

If a potential well of positive minimal height confines a bounded configuration space region (as it is the case for models of radioactive decay) then the classical orbits in that region are bounded for all times. As in this case the bounded orbits form a connected component of the energy shell, there need be no *trapped* orbits (orbits of positive energy which come from spatial infinity and are bounded in the future, or vice versa).

Quantum mechanically this then leads to so-called shape resonance poles in the complex energy plane. These come exponentially near to the real axis in Planck's constant $\hbar$ [2].

In this article we are interested in *semibounded* trapped orbits. Although these are necessarily of Liouville measure zero, they also give rise to quantal resonances (which, however, may have larger distance from the real axis and thus correspond to states with shorter life times [5]).

In **Sect. 2** we introduce some notation, give examples for trapping, and remark that trapped orbits exist iff there are bounded orbits in the unbounded component of the energy shell (Prop. 2.2).

Correspondingly, we derive in **Sect. 3** criteria for the existence for a special class of such bounded orbits. In Thm. 3.2 it is shown that such so-called brake orbits exist if some relative homotopy group of Hill's region w.r.t. its boundary is non-trivial. At least for the physical dimensions $d \leq 3$ this is the case iff that (non-empty) boundary is not homeomorphic to a sphere (Cor. 3.3).

After defining the differential cross section in **Sect. 4**, we introduce in **Sect. 5** for non-trapping energies a degree of the scattering map, which turns out to be non-trivial in many cases.

In **Sect. 6** we analyze potentials which can be decomposed into a sum of potentials with disjoint compact supports. If $n \geq 2$ of them have non-trivial degree, then the corresponding energy is trapping, and orbits, visiting these supports in any prescribed succession, can be found using symbolic dynamics (Thm. 6.1).

In the final **Sect. 7** we consider the differential cross section. Whereas it is smooth (up to the forward direction) for cases like the n-centre problem with a very complicated dynamics, it is never continuous for any large energy if $d \geq 2$ and for a smooth nonzero potential (Thm. 7.1).

2 Trapped Orbits

Let $V \in C^\infty(M,\mathbb{R})$ on configuration space $M := \mathbb{R}^d_{\vec{q}}$ be a smooth *short-range* potential, that is, for some $\alpha > 1$ the partial derivatives decay at infinity according to

$$\frac{\partial^n}{\partial q^n} V(\vec{q}) = \mathcal{O}\left(|\vec{q}|^{-|n|-\alpha}\right) \qquad (n \in \mathbb{N}_0^d), \tag{1}$$

with multi-index norm $|n| := \sum_{l=1}^d |n_l|$.

We denote the Hamiltonian flow generated by the restriction of

$$H : T^*M \to \mathbb{R} \quad , \quad H(\vec{p},\vec{q}) := \tfrac{1}{2}\vec{p}^{\,2} + V(\vec{q})$$

to the positive energy part $P := \{x \in T^*M \mid H(x) > 0\}$ of the phase space by

$$\Phi^t : P \to P \quad , \quad \Phi : \mathbb{R} \times P \to P \quad \text{or} \quad (\vec{p}(t,x_0), \vec{q}(t,x_0)) := \Phi^t(x_0),$$

and the energy shells $H^{-1}(E)$ by Σ_E. For arbitrary potentials V we set $V_{\max} := \sup_{\vec{q}} V(\vec{q})$. The phase space P is naturally partitioned into the invariant subsets

Definition 2.1

$$\begin{array}{rcll} b^\pm & := & \{x \in P \mid \vec{q}\,(\pm\mathbb{R}^+, x) \text{ is bounded }\} & , \quad b^\pm_E := b^\pm \cap \Sigma_E \\ b & := & b^+ \cap b^- \qquad (\text{the } \textit{bound} \text{ states}) & , \quad b_E := b \cap \Sigma_E \\ s^\pm & := & P \setminus b^\pm & , \quad s^\pm_E := s^\pm \cap \Sigma_E \\ s & := & s^+ \cap s^- \qquad (\text{the } \textit{scattering} \text{ states}) & , \quad s_E := s \cap \Sigma_E \\ t & := & P \setminus (b \cup s) \quad (\text{the } \textit{trapped} \text{ states}) & , \quad t_E := t \cap \Sigma_E. \end{array}$$

Time reversal $(\vec{p},\vec{q}) \mapsto (-\vec{p},\vec{q})$ interchanges b^+_E and b^-_E. It is known (see Hunziker [8]) that

$$\lim_{t\to\pm\infty} |\vec{q}(t,x_0)| = \infty \quad \text{iff} \quad x_0 \in s^\pm$$

so that these are indeed the $\pm$-scattering states.

By (1) for any $E > 0$ there exists a *virial radius* $R_{\mathrm{vir}}(E) > 0$ for which

$$|V(\vec{q})| < E/2 \quad \text{and} \quad |\langle \vec{q}, \nabla V(\vec{q})\rangle| < E/2 \qquad (\vec{q} \in M \setminus B(E)),$$

with the *interaction zone*

$$B(E) := \{\vec{q} \in M \mid |\vec{q}| \le R_{\mathrm{vir}}(E)\}. \tag{2}$$

The *virial identity*

$$\frac{d}{dt}\langle\vec{q}(t),\vec{p}(t)\rangle = 2(E - V(\vec{q}(t))) - \langle\vec{q}(t),\nabla V(\vec{q}(t))\rangle \tag{3}$$

holds true for any trajectory $t \mapsto (\vec{p}(t),\vec{q}(t)) = \Phi^t(x_0)$ with energy $E := H(x_0)$. In particular, a trajectory leaving the interaction zone cannot reenter it but goes to spatial infinity.

Although the Liouville measure λ_E of the trapped states t_E vanishes, they influence the neighbouring scattering orbits, which thus remain inside the interaction zone for a long time.

A compactness argument shows that $b_E^\pm \neq \emptyset$ if and only if $b_E \neq \emptyset$ (Prop. 2.1.2 in [3]).

We call $E > 0$ a *trapping energy* if

$$t_E \equiv (b_E^+ \cup b_E^-) \setminus b_E \equiv (b_E^+ \cap s_E^-) \cup (b_E^- \cap s_E^+) \neq \emptyset,$$

and denote the set of trapping energies by $\mathcal{TE}$. So for these energies there exist *trapped* trajectories, coming from infinity but bounded in the future, and vice versa. The complementary set

$$\mathcal{NT} := \mathbb{R}^+ \setminus \mathcal{TE}$$

of non-trapping energies is known to be open (see the proof of Prop. 2.4.1 of [3]).

Example. For $d = 1$ the set $\mathcal{TE}$ of trapping energies equals the set

$$\left\{ E > 0 \,\middle|\, \exists q \in \mathbb{R} : V(q) = E,\ DV(q) = 0,\ \sup_{q' \leq q} V(q') = E \text{ or } \sup_{q' \geq q} V(q') = E \right\}$$

of 'accessible' critical values, and Σ_E is not connected for $E \in \mathcal{NT}$.

For $d \geq 2$ and a *centrally symmetric* ($V(\vec{q}) = W(|\vec{q}|)$) potential each of the extrema of W at $q > 0$ in the trapping set $\mathcal{TE}_W$ of W gives rise to an interval $[W(q), u] \subset \mathcal{TE}_V$ in the trapping set $\mathcal{TE}_V$ of V, and $u > W(q)$ if the extremum at q is a non-degenerate maximum.

u is of the form $u = W_l(q')$ with $W_l'(q') = 0$ and $W_l''(q') = 0$, where

$$W_l(r) := W(r) + \frac{l^2}{2r^2} \tag{4}$$

is the *effective potential* .

For $E > 0$ *Hill's region*

$$\mathcal{R}_E := \{\vec{q} \in M \mid V(\vec{q}) \leq E\}$$

need not be connected (since there may be potential pits), but for $d \geq 2$ there is precisely one noncompact component $\mathcal{R}_E^u$ of this set, and the same is true for the energy shell Σ_E projecting to Hill's region. We denote this component by Σ_E^{u}.

It may well happen that

$$b_E^{\mathrm{u}} := b_E \cap \Sigma_E^{\mathrm{u}} \neq \emptyset. \tag{5}$$

Example. For centrally symmetric potentials the effective potential (4) has a positive local maximum at $r_{\max}$ for small values $l > 0$ of the angular momentum parameter, if $W < 0$ and $W(r) = \mathcal{O}(r^{-2-\varepsilon})$. This then leads to a non-empty set $b_E^{\mathrm{u}} = b_E$ of bound states for the energy $E = W_l(r_{\max}) > V_{\max} = 0$.

Proposition 2.2 *An energy $E > 0$ is non-trapping if and only if $b_E^{\mathrm{u}} = \emptyset$.*

Proof. $b_E^+ \setminus b_E$ lies in Σ_E^{u}. So if the closed, Φ^t-invariant set $b_E^+ \cap \Sigma_E^{\mathrm{u}}$ is non-empty, then the set of its ω-limit points lying in the compact region of the energy shell over $B(E)$ is non-empty, too (see also Prop. 2.1.2 of [3]). Thus $b_E^{\mathrm{u}} \neq \emptyset$.

To show the inverse implication, we assume that $E > 0$ is non-trapping, so that $t_E = \emptyset$. Then

$$\Sigma_E^{\mathrm{u}} = s_E \,\dot{\cup}\, b_E^{\mathrm{u}}$$

so that for $b_E^{\mathrm{u}} \neq \emptyset$ there would be a sequence $x_i \in s_E$ of points on scattering orbits converging to $x := \lim_{i\to\infty} x_i \in b_E^{\mathrm{u}}$. Then there exist unique times t_i such that $y_i \equiv (\vec{p}_i, \vec{q}_i) := \Phi^{t_i}(x_i)$ enter the interaction zone, i.e. meet $|\vec{q}_i| = R_{\mathrm{vir}}(E)$ and $\langle \vec{p}_i, \vec{q}_i \rangle \leq c < 0$.

By compactness there exists an accumulation point $y \equiv (\vec{p}, \vec{q})$ of the y_i. Since $|\vec{q}| = R_{\mathrm{vir}}(E)$ and $\langle \vec{p}, \vec{q} \rangle \leq c$, it is backward scattering ($y \in s_E^-$). But the times $t_i \nearrow \infty$, so that $y \in b_E^+$, too. Thus y belongs to a trapped orbit. □

The virial identity (3) implies that the motion is *non-trapping* above some (optimal) energy threshold $E_{\mathcal{NT}} \geq V_{\max}$, i.e.

$$]E_{\mathcal{NT}}, \infty[\, \subset \mathcal{NT} \quad \text{and} \quad E_{\mathcal{NT}} \in \mathcal{TE} \text{ or } E_{\mathcal{NT}} = 0,$$

since for E large $\frac{d}{dt}\langle \vec{q}(t), \vec{p}(t) \rangle > E$ for all $\vec{q} \in \mathbb{R}^d_{\vec{q}}$. This implies a unique minimum of $t \mapsto |\vec{q}(t, x_0)|$ at, say $t = 0$, and the estimate

$$\vec{q}^{\,2}(t, x_0) \geq \vec{q}_0^{\,2} + Et^2 \qquad (t \in \mathbb{R}). \tag{6}$$

Remark 2.3 Without a smoothness assumption for the potential V this need not be true even if $V < 0$. Namely, for the physically important *n-centre potentials* of the form

$$V(\vec{q}) = \sum_{l=1}^{n} \frac{-Z_l}{|\vec{q} - \vec{s}_l|}, \tag{7}$$

one has for $n \geq 2$ in $d = 2$ dimensions $\mathcal{TE} = \mathbb{R}^+$, at least if all charges $Z_l > 0$, see [9]. For $n \geq 2$, $d = 3$ and arbitrary $Z_l \neq 0$, the set $\mathcal{TE}$ of trapping energies contains an interval $[E_{\mathrm{th}}, \infty[$, see [12].

However, $\mathcal{NT} \neq \emptyset$, too if all charges Z_l are negative, since then the radial component of the force $-\nabla V$ is positive outside a ball containing all $\vec{s}_l$, and since for small $E > 0$ Hill's region does not contain that ball.

As this example shows, non-trapping energies can lie below, not only above, trapping energies.

Example. In the smooth case for $d = 1$ the threshold energy is $E_{\mathcal{NT}} = V_{\max}$, and for $E > E_{\mathcal{NT}}$ all scattering is in the forward direction.

3 Brake Orbits

We saw in Prop. 2.2 that $E \in \mathcal{NT}$ iff $b_E^u = \emptyset$. Here we derive a criterion for the existence of bound states b_E^u for energies $E \leq V_{\max}$.

The set $\mathcal{TE}$ of trapping energies contains all critical values E of V with critical points $\vec{q} \in \partial\mathcal{R}_E^u$, since then the phase space point $(\vec{0}, \vec{q}) \in \Sigma_E^u$ belongs to the set b_E^u defined in (5).

So we may ask ourselves whether a regular value $E < V_{\max}$ of V (or rather of $V\restriction_{\mathcal{R}_E^u}$) is a trapping energy. This is certainly the case if there exists a periodic orbit in Σ_E^u.

Definition 3.1 *A Φ^t–orbit on Σ_E is called* **brake orbit** *if its configuration space projection touches the boundary $\partial\mathcal{R}_E$ of Hill's region.*

Theorem 3.2 *If for some $k \leq d$ the relative homotopy group $\pi_k(\mathcal{R}_E^u, \partial\mathcal{R}_E^u)$ of Hill's region w.r.t. its boundary is non-trivial, then there exists a periodic brake orbit in Σ_E^u, and thus E is a trapping energy.*

Proof. We may assume that E is a regular value of $V\restriction_{\mathcal{R}_E^u}$, so that $\mathcal{R}_E^u$ is a smooth d-manifold with boundary. That boundary $\partial\mathcal{R}_E^u \subset M$ lies in the interaction zone $B(E)$ around the origin, see (2). The *Jacobi metric*

$$g_E(\vec{q}) := (E - V(\vec{q})) \cdot g(\vec{q}) \qquad (\vec{q} \in \mathcal{R}_E^u)$$

(with g being the Euclidean metric) is only degenerate on $\partial\mathcal{R}_E^u$.

All trajectories $t \mapsto \vec{q}(t, x_0)$, $x_0 \in \Sigma_E$, in the interior of $\mathcal{R}_E^u$ coincide, up to time parametrization, with the geodesics of g_E.

By the virial identity (3) the trajectories touching $\partial B(E)$ (that is, $\vec{q}_0 \in \partial B(E)$ and $\vec{p}_0 \perp \vec{q}_0$) meet the inequality $\partial^2 \vec{q}^{\,2}(t)/\partial t^2 > 0$, so that $\partial B(E)$ is convex in the Jacobi metric.

Now for the first $k \geq 1$ with nontrivial $\pi_k(\mathcal{R}_E^u, \partial\mathcal{R}_E^u)$ we consider an essential map $f_0 : (B^k, \partial B^k) \mapsto (\mathcal{R}_E^u, \partial\mathcal{R}_E^u)$.

We then apply to f_0 a curve shortening process, originally devised by Seifert in [13] and used by Gluck and Ziller in [6]. Here one considers f_0 as a $(k-1)$-parameter family of curves whose ends lie in $\partial\mathcal{R}_E^u$.

This is possible since $\partial\mathcal{R}_E^u$ is compact, $\nabla V \neq \vec{0}$ on $\partial\mathcal{R}_E^u$, and so one may apply to $\partial\mathcal{R}_E^u$ the metric surgery described in [6].

Although $\mathcal{R}_E^u$ is not compact, a Palais-Smale condition holds: by convexity of $\partial B(E)$ w.r.t. g_E the curve shortening leads to curves still lying inside $B(E)$. Alternatively one may choose the radius $R_{\mathrm{vir}}(E)$ of $B(E)$ so large that the g_E-distance between $\partial\mathcal{R}_E^u$ and $\partial B(E)$ is larger than the maximal length of a curve in the family f_0.

So the shortening process leads to a non-trivial geodesic segment with two end points in $\partial\mathcal{R}_E^u$. This corresponds to a periodic brake orbit of energy E. $\Box$

In fact this criterion is often met:

Corollary 3.3 *If for $d \leq 3$ the boundary $\partial\mathcal{R}_E^u$ of Hill's region is not empty or homeomorphic to S^{d-1}, then there exists a periodic brake orbit in Σ_E^{u}, and thus $E \in \mathcal{TE}$.*

Proof. By a remark at the beginning of this section we may again assume that E is a regular value of $V\restriction_{\mathcal{R}_E^u}$, so that $\mathcal{R}_E^u$ is a smooth d-manifold with boundary. For $d = 1$ then one only has the alternatives $\partial\mathcal{R}_E^u = \emptyset$ or $\partial\mathcal{R}_E^u \cong S^0$. So assume $d \geq 2$, and denote by $\overline{\mathcal{R}_E^u}$ the compact manifold which arises from $\mathcal{R}_E^u \subset M = \mathbb{R}^d$ by the one-point compactification of $\mathbb{R}^d$. Now $\overline{\mathcal{R}_E^u}$ is a compact manifold with boundary $\partial\overline{\mathcal{R}^u_E} = \partial\mathcal{R}^u_E$. Thus not all relative *homology* groups $H_k(\overline{\mathcal{R}^u_E}, \partial\overline{\mathcal{R}^u_E})$, $k = 1, \ldots, n$, are trivial (cf. Spanier [15], Chapter 4).

For $d \geq 2$ Hill's region $\mathcal{R}^u_E$ and thus also $\overline{\mathcal{R}^u_E}$ is connected. We may assume that $\partial\overline{\mathcal{R}^u_E} = \partial\mathcal{R}^u_E$ is connected, too, since otherwise $\pi_1(\mathcal{R}^u_E, \partial\mathcal{R}^u_E)$ is non-trivial and we can apply Theorem 3.2. This already shows our claim for $d = 2$, since the only closed connected (non-empty) 1-manifold is S^1.

For $d \geq 3$ we can apply the relative Hurewitz isomorphism theorem ([15], Chapter 7.5) to show that there exists a nontrivial relative homotopy group $\pi_k(\overline{\mathcal{R}^u_E}, \partial\overline{\mathcal{R}^u_E})$. Let k be the smallest such integer. If $k < d$, we are finished, since by a transversality argument

$$\pi_k(\mathcal{R}^u_E, \partial\mathcal{R}^u_E) \supset \pi_k(\overline{\mathcal{R}^u_E}, \partial\overline{\mathcal{R}^u_E}) \qquad (k < d).$$

So we assume that $\pi_k(\overline{\mathcal{R}^u_E}, \overline{\partial\mathcal{R}^u_E})$ is non-trivial iff $k = d$, and consider the remaining case $d = 3$. We assume $\partial\mathcal{R}^u_E \neq \emptyset$. Then $\overline{\mathcal{R}^u_E}$ can be considered as a true subset of $S^3 = M \cup \{\infty\}$, not containing the complement $\mathbb{R}^d \setminus \mathcal{R}^u_E \neq \emptyset$ of Hill's region.

Thus $\pi_3(\overline{\mathcal{R}^u_E})$ is trivial, and the exact sequence

$$\pi_3(\overline{\mathcal{R}^u_E}) \to \pi_3(\overline{\mathcal{R}^u_E}, \partial\overline{\mathcal{R}^u_E}) \to \pi_2(\partial\overline{\mathcal{R}^u_E})$$

then implies that $\pi_2(\partial\overline{\mathcal{R}^u_E})$ is non-trivial, too, so that the orientable closed connected surface $\partial\mathcal{R}^u_E = \partial\overline{\mathcal{R}^u_E}$ is homeomorphic to S^2. $\square$

4 Definition of the Differential Cross Section

Under the decay assumption (1), away from the interaction zone the flow Φ^t becomes similar to the free flow

$$\Phi^t_\infty : P_\infty \to P_\infty, \qquad (\vec{p}, \vec{q}) \mapsto (\vec{p}, \vec{q} + t\vec{p})$$

generated by the Hamiltonian function

$$H_\infty(\vec{p}, \vec{q}) := \tfrac{1}{2}\vec{p}^2 \quad \text{on} \quad P_\infty := \{x \in T^*M \mid H_\infty(x) > 0\}.$$

More precisely, the *Møller transformations*

$$\Omega^\pm := \lim_{t \to \pm\infty} \Phi^{-t} \circ \Phi^t_\infty$$

exist (pointwisely) on P_∞, and are symplectic diffeomorphisms onto their images $s^\pm$, see [14].

In particular the *asymptotic momentum*

$$\vec{p}^\pm : s^\pm \to \mathbb{R}^d \quad , \quad \vec{p}^\pm(x_0) := \lim_{t\to\pm\infty} \vec{p}(t, x_0),$$

the *asymptotic direction*

$$\hat{p}^\pm : s^\pm \to S^{d-1} \quad , \quad \hat{p}^\pm(x) := \frac{\vec{p}^\pm(x)}{|\vec{p}^\pm(x)|}$$

and the *impact parameter*

$$\vec{q}_\perp^{\,\pm} : s^\pm \to \mathbb{R}^d, \quad \vec{q}_\perp^{\,\pm}(x_0) := \lim_{t\to\pm\infty} \big(\vec{q}(t, x_0) - \big\langle \vec{q}(t, x_0), \hat{p}^\pm(x_0)\big\rangle \, \hat{p}^\pm(x_0)\big)$$

are smooth Φ^t-invariant functions.

The impact parameter is orthogonal to the asymptotic direction, and for $E > 0$

$$A_E^\pm : s_E^\pm/\mathbb{R} \to I_E^\pm \subset T^*S^{d-1} \quad , \quad x \mapsto (\vec{q}_\perp^{\,\pm}(x), \hat{p}^\pm(x))$$

is a homeomorphism onto its (open and dense) image $I_E^\pm$. Note in comparison to the inverse Møller transformations $\Omega_*^\pm : s^\pm \to P_\infty$, that the energy now appears as a parameter, and that orbits are mapped to points, so that we disregard time delay etc.

For $\hat{I}_E^\pm := A_E^\pm(s_E/\mathbb{R})$ the energy E *scattering map*

$$(\vec{Q}_E, \hat{P}_E) : \hat{I}_E^- \to \hat{I}_E^+ \quad , \quad (\vec{q}_\perp^{\,-}, \hat{p}^-) \mapsto A_E^+ \circ (A_E^-)^{-1}(\vec{q}_\perp^{\,-}, \hat{p}^-) \tag{8}$$

from the initial to the final asymptotic data is a symplectic diffeomorphism w.r.t. the canonical symplectic form ω_N on the cotangent bundle

$$N := T^*S^{d-1}$$

of the sphere of directions. In particular it preserves the Liouville measure

$$\lambda_N := \frac{\omega_N \wedge \ldots \wedge \omega_N}{(d-1)!}$$

on N.

The differential cross section $\frac{d\sigma}{d\hat{\theta}^+}(E, \hat{\theta}^-, \hat{\theta}^+)$ is the (density of the) number of particles per second scattered in the final direction $\hat{\theta}^+ \in S^{d-1}$, assuming a uniform flux of one particle per second and unit area of incoming particles of energy E and initial direction $\hat{\theta}^- \in S^{d-1}$.

So we consider the restriction

$$\hat{P}_{E,\hat{\theta}^-} := \hat{P}_E \restriction_{\hat{I}^-_{E,\hat{\theta}^-}} \tag{9}$$

of the *final direction map* $\hat{P}_E$ to the intersection

$$\hat{I}^-_{E,\hat{\theta}^-} := \hat{I}^-_E \cap T^*_{\hat{\theta}^-} S^{d-1}$$

of its domain with the cotangent space of the sphere at $\hat{\theta}^-$.

Definition 4.1 *For $E > 0$ and $\hat{\theta}^- \in S^{d-1}$ the* **cross section measure** $\sigma(E, \hat{\theta}^-)$ *on S^{d-1} is the image measure*

$$\sigma(E, \hat{\theta}^-) := \left(\hat{P}_{E,\hat{\theta}^-}\right)^{-1} (\lambda_{\hat{\theta}^-}), \tag{10}$$

$\lambda_{\hat{\theta}^-}$ being Lebesgue measure on the cotangent plane $T^*_{\hat{\theta}^-} S^{d-1}$.

If $\sigma(E, \hat{\theta}^-)$ on $S^{d-1} \setminus \{\hat{\theta}^-\}$ is absolutely continuous w.r.t. Lebesgue measure $\lambda_{S^{d-1}}$, the Radon-Nikodym derivative $\frac{d\sigma}{d\hat{\theta}^+}(E, \hat{\theta}^-, \hat{\theta}^+)$ is called the **differential cross section**.

If the set $\mathcal{IP} := \hat{P}^{-1}_{E,\hat{\theta}^-}(\hat{\theta}^+)$ of initial impact parameters is countable, we may thus write the differential cross section as the sum

$$\frac{d\sigma}{d\hat{\theta}^+}(E, \hat{\theta}^-, \hat{\theta}^+) = \sum_{\vec{q}^{\,-}_\perp \in \mathcal{IP}} \left| D\hat{P}_{E,\hat{\theta}^-}(\vec{q}^{\,-}_\perp) \right|^{-1}.$$

Example. For the Coulomb potential $V(\vec{q}) = Z/|\vec{q}|$, $Z \neq 0$ on $\mathbb{R}^d \setminus \{0\}$ one has the so-called *Rutherford* differential cross section

$$\frac{d\sigma}{d\hat{\theta}^+}(E, \hat{\theta}^-, \hat{\theta}^+) = \left(\frac{|Z|}{4E \sin^2(\frac{1}{2}\sphericalangle(\hat{\theta}^+, \hat{\theta}^-))} \right)^{d-1}. \tag{11}$$

5 The Degree of the Scattering Map

For non-trapping energies $E \in \mathcal{NT}$ the scattering map (8) is a symplectic diffeomorphism

$$(\vec{Q}_E, \hat{P}_E) : N \to N$$

of (N, ω_N), and for each $\hat{\theta}^- \in S^{d-1}$ the restriction (9)

$$\hat{P}_{E,\hat{\theta}^-} : T^*_{\hat{\theta}^-} S^{d-1} \to S^{d-1}$$

of the final direction map is smooth. For $d \geq 2$

$$\lim_{\vec{q}_\perp^- \to \infty} \hat{P}_{E,\hat{\theta}^-}(\vec{q}_\perp^-) = \hat{\theta}^-.$$

Thus we may extend it uniquely to a continuous map

$$\hat{\mathbf{P}}_{E,\hat{\theta}^-} : \left(T^*_{\hat{\theta}^-} S^{d-1} \cup \{\infty\}\right) \cong S^{d-1} \to S^{d-1}. \tag{12}$$

The choice of an orientation on the sphere fixes an orientation of the cotangent space $T^*_{\hat{\theta}^-} S^{d-1}$, too, and we denote by

$$\deg(E) := \deg(\hat{\mathbf{P}}_{E,\hat{\theta}^-})$$

the topological degree of this map (see, e.g., Hirsch [7]). That degree is independent of the choice of orientation. By continuity of the final direction map $\hat{P}_E$ it is independent of the choice of initial direction $\hat{\theta}^-$.

Furthermore, $\hat{P}_E$ depends continuously on $E \in \mathcal{NT}$, so that the *non-trapping degree*

$$\deg : \mathcal{NT} \to \mathbb{Z}$$

is locally constant on the (open) set of non-trapping energies.

Now we will work out a series of examples.

Proposition 5.1 *For a smooth short-range potential V*

$$\deg(E) = 0 \qquad (E > E_{\mathcal{NT}}).$$

Proof. This is obvious for large energies E, since then the map $\hat{\mathbf{P}}_{E,\hat{\theta}^-}$ is not onto S^{d-1}: The curvature k of the trajectory, that is, the inverse radius of the osculating circle, can be considered as a phase space function, and equals

$$k : P \setminus \Big(\{\vec{0}\} \times M\Big) \to [0, \infty[\quad , \quad k(\vec{p}, \vec{q}) := \frac{|(\mathbb{1} - \Pi_{\vec{p}})\ddot{q}|}{|\dot{q}|^2},$$

where $\Pi_{\vec{p}}$ denotes the orthogonal projection in the direction of $\vec{p}$. Inserting Hamilton's equation, we see that

$$k(\vec{p}, \vec{q}) = \frac{|(\mathbb{1} - \Pi_{\vec{p}})\nabla V(\vec{q})|}{2(E - V(\vec{q}))} \leq \frac{|\nabla V(\vec{q})|}{2(E - V(\vec{q}))}. \tag{13}$$

For large E by (1) the integral of (13) is integrable and, using (6), is seen to be uniformly of order

$$\int_{\mathbb{R}} k \circ \Phi^t(x_0)\, |d\vec{q}(t, x_0)/dt|\, dt = \mathcal{O}(H(x_0)^{-1}).$$

This implies absence of back-scattering for large E and thus $\deg(E) = 0$. As the degree is locally constant, the result follows for all $E \in]E_{\mathcal{NT}}, \infty[$. $\Box$

The following proposition generalizes the case of the Kepler potential (which corresponds to $n = 1$).

Proposition 5.2 *For $d > 1$ let $\hat{M} := \mathbb{R}^d \setminus \{\vec{0}\}$. Then for $n \in \mathbb{N}$, the flow generated by the potential*

$$V(\vec{q}) := -|\vec{q}|^{-2n/(n+1)} \qquad (\vec{q} \in \hat{M}) \tag{14}$$

can be regularized, all positive energies are non-trapping ($\mathcal{NT} = \mathbb{R}^+$), and the degree of the scattering map equals

$$\deg(E) = \begin{cases} -n & d \text{ even} \\ \frac{1}{2}(1 - (-1)^n) & d \text{ odd} \end{cases} \qquad (E > 0). \tag{15}$$

Proof. Due to the singularity at the origin the Hamiltonian flow in the phase space $T^*\hat{M}$ is incomplete. We will show, however, that this flow can be completed in an essentially unique way.

To that aim we calculate the total deflection angle $\Delta\varphi(E, l)$ of a trajectory with energy E and modulus l of the angular momentum. Considering for a moment an arbitrary centrally symmetric potential $V(\vec{q}) = W(|\vec{q}|)$ and for $l > 0$ its effective potential W_l (see (4)), we have (see Chapter 2.8 of Arnold [1])

$$\Delta\varphi(E,l) = 2\int_{r_{\min}}^{\infty} \frac{\dot{\varphi}}{\dot{r}}\,dr - \pi = 2\int_{r_{\min}}^{\infty} \frac{l/r^2}{\sqrt{2(E - W_l(r))}}\,dr - \pi, \tag{16}$$

where the pericentral radius $r_{\min}$ is the largest $r > 0$ with $W_l(r) = E$.

Setting $W(r) := -r^{-\alpha}$ with $0 < \alpha < 2$, and substituting

$$v := \frac{(l/\sqrt{2})^{1/(1-\alpha/2)}}{r},$$

we obtain

$$\Delta\varphi(E,l) = 2\int_0^{v_{\max}} \frac{dv}{\sqrt{2El^{\frac{\alpha}{1-\alpha/2}}2^{\frac{1-\alpha}{2-\alpha}} + v^\alpha - v^2}} - \pi$$

with $2El^{\frac{\alpha}{1-\alpha/2}}2^{\frac{1-\alpha}{2-\alpha}} + v_{\max}^\alpha - v_{\max}^2 = 0$. Since $\alpha < 2$, in the collision limit of vanishing angular momentum the first term in the square root vanishes, and we are left with

$$\Delta\varphi := \lim_{l\to 0}\Delta\varphi(E,l) = 2\int_0^1 \frac{dv}{\sqrt{v^\alpha - v^2}} - \pi = \frac{2\pi}{2-\alpha} - \pi, \tag{17}$$

which equals $n\pi$ if $\alpha = 2n/(n+1)$. Thus precisely for the exponents appearing in (14) we can continuously regularize the collision orbits with $l = 0$, since then the ($(d-2)$-dimensional) family of orbits with given E, $\hat{\theta}^-$ and l converges to the same orbit as $l \searrow 0$.

That collision orbit can thus be parametrized by its energy $E \in \mathbb{R}$ and, say, initial direction $\hat{\theta}^- \in S^{d-1}$. So by setting

$$P := T^*\hat{M} \,\dot{\cup}\, (\mathbb{R} \times S^{d-1}),$$

we may thus regularize the motion on this new phase space and obtain a complete, continuous flow

$$\Phi^t : P \to P \qquad (t \in \mathbb{R}).$$

In fact, P can be made a smooth symplectic manifold and Φ^t a smooth Hamiltonian flow. See [9] and [12] for details of the construction in the representative case $n = 1$ of the Kepler potential.

Since there are no bounded orbits of positive energy, $\mathcal{NT} = \mathbb{R}^+$. In the case of $d = 2$ dimensions the outgoing angle

$$\hat{\mathbf{P}}_{E,\hat{\theta}^-}(q_\perp) = \hat{\theta}^- - \Delta\varphi(E, \sqrt{2E}q_\perp) \qquad (q_\perp \geq 0)$$

is continuous decreasing in $q_\perp$. So in this case it follows from (17) that

$$\int_{-\infty}^{\infty} \frac{d}{dq_\perp}\hat{\mathbf{P}}_{E,\hat{\theta}^-}(q_\perp)dq_\perp = -2\Delta\varphi = -2\pi n,$$

proving

$$\deg(E) = -n \qquad (E > 0).$$

For $d > 2$ we consider a family of trajectories with fixed E and $\hat{\theta}^-$, whose impact parameter $\vec{q}_\perp$ varies on a one-dimensional subspace $L \subset T^*_{\hat{\theta}^-}S^{d-1}$.

$\hat{\theta}^-$ and this subspace span a 2–plane in $\mathbb{R}^d$, and $\hat{\theta}^+$ lies in that plane. To avoid degeneracies we choose a $\hat{\theta}^+$ which is linear independent from $\hat{\theta}^-$. Then there are exactly n impact parameters $\vec{q}^{\,1}_\perp, \ldots, \vec{q}^{\,n}_\perp \in L$ with $\hat{\mathbf{P}}_{E,\hat{\theta}^-}(\vec{q}^{\,i}_\perp) = \hat{\theta}^+$.

$[n/2]$ of them have a scalar product $\left\langle \vec{q}^{\,i}_\perp, \hat{\theta}^+ \right\rangle > 0$, and $\left\langle \vec{q}^{\,i}_\perp, \hat{\theta}^+ \right\rangle < 0$ for the rest. For the first group the restriction of the linearization of the final angle map to the subspace $\{\vec{v} \in T^*_{\hat{\theta}^-}S^{d-1} \mid \vec{v} \perp L\}$ gives a positive sub-determinant, whereas for the second group the sign equals $(-1)^{d-2}$. So

$$\deg(E) = -\left([n/2] + (-1)^{d-2}(n - [n/2])\right),$$

proving (15). $\square$

Proposition 5.3 *For a centrally symmetric short-range potential V*

$$\deg(E) = +1 \quad \textit{if} \quad E \in \mathcal{NT} \cap]0, V_{\max}[.$$

Proof. When we substitute $v := r_{\min}/r$ in the formula (16) for the deflection angle, we get

$$\Delta\varphi(E, l) = 2\int_0^1 \frac{dv}{\sqrt{2r_{\min}^2(E - V(r_{\min}/v))/l^2 - v^2}} - \pi.$$

For $d = 2$ the degree equals

$$\begin{aligned}
\deg(E) &= -\frac{2}{\pi}\int_0^\infty \frac{\partial}{\partial l}\Delta\varphi(E,l)dl \qquad (18)\\
&= -\frac{2}{\pi}\int_0^1 \frac{dv}{\sqrt{2(r_{\min}/l)^2(E - V(r_{\min}/v)) - v^2}}\Bigg|_{l=0}^{l=\infty}\\
&= \frac{2}{\pi}\int_0^1 \frac{dv}{\sqrt{1-v^2}} = 1,
\end{aligned}$$

since $\lim_{l\to\infty} r_{\min}(E,l)/l = 1/\sqrt{2E}$ and $\lim_{l\to 0} r_{\min}(E,l)/l > 0$, using the assumption $E < V_{\max}$. The exchange of integration and differentiation in (18) is justified by the non-trapping assumption, which is equivalent to the assumption $W_l'(r_{\min}) < 0$.

The case of higher dimension $d > 2$ is treated similar as in Prop. 5.2. $\square$

I conjecture that for the above energy range the degree equals one, even if the potential is not centrally symmetric.

6 Multiple Scattering

We now consider potentials $V \in C_0^\infty(M,\mathbb{R})$, $d \geq 2$, whose support is contained in the union of n disjoint balls

$$B_l := \{\vec{q} \in M \mid |\vec{q} - \vec{s}_l| \leq r_l\} \qquad (l = 1, \ldots, n),$$

and represent V in the form $V = \sum_{l=1}^n V_l$ with $\mathrm{supp}(V_l) \subset B_l$.

Our aim is to compare the flow Φ^t generated by H with the flows Φ_l^t generated by the Hamiltonian functions $H_l : P \to \mathbb{R}$, where $H_l(\vec{p},\vec{q}) := \frac{1}{2}\vec{p}^2 + V_l(\vec{q})$. In general objects corresponding to V_l will carry a subindex l. For $E > 0$ we have

$$\mathcal{R}_E^u = \bigcap_{l=1}^n \mathcal{R}_{l,E}^u \quad \text{and} \quad b_E^u \supset \bigcup_{l=1}^n b_{l,E}^u,$$

since $d \geq 2$ and the supports of the V_l are disjoint. So by Prop. 2.2 the set $\mathcal{NT}$ of non-trapping energies of H meets

$$\mathcal{NT} \subset \bigcap_{l=1}^n \mathcal{NT}_l.$$

We now assume that V is *non-shadowing*, by which we mean that every straight line in M meets at most two balls B_l. Moreover, we only consider scattering from and to directions in which the balls do not shadow each other. We thus exclude the cones of angles

$$\alpha_{k,l} := \arcsin\left(\frac{r_k + r_l}{d_{k,l}}\right) \quad \text{with} \quad d_{k,l} := |\vec{s}_k - \vec{s}_l| \tag{19}$$

around the axes $\hat{s}_{k,l} := (\vec{s}_k - \vec{s}_l)/d_{k,l}$, and restrict the initial and final directions $\hat{p}^{\pm}$ to the subset

$$\tilde{S}^{d-1} := \{\hat{x} \in S^{d-1} \mid \sphericalangle(\hat{x}, \hat{s}_{k,l}) > \alpha_{k,l},\ 1 \le k \ne l \le n\} \tag{20}$$

of the sphere not contained in any such cone.

In order to use symbolic dynamics, we introduce *symbol sequences*

$$\underline{k} = (k_i)_{i \in I} \in \mathcal{S}^I \quad \text{over the } \textit{alphabet} \quad \mathcal{S} := \{1, \ldots, n\},$$

where

$$I \equiv I_l^r := \{i \in \mathbb{Z} \mid l \le i \le r\}$$

for $l, r \in \mathbb{Z} \cup \{\pm\infty\}$ is a (finite, half-infinite or bi-infinite) *interval*.

$\underline{k}$ is called *admissible* if $k_i \ne k_{i+1}$ for all $\{i, i+1\} \subset I$, and

$$\mathbf{X}_l^r := \{\underline{k} \in \mathcal{S}^I \mid \underline{k} \text{ admissible}\}.$$

Theorem 6.1 *Let $n \ge 2$, E be non-trapping for the individual potentials V_l ($E \in \cap_{l=1}^n \mathcal{NT}_l$) and $\deg_l(E) \ne 0$, $1 \le l \le n$.*

Then for every interval I_l^r, $\underline{k} \in \mathbf{X}_l^r$ and $\hat{p}^{\pm} \in \tilde{S}^{d-1}$ there is a trajectory in Σ_E meeting exactly the balls B_{k_i}, $i \in I_l^r$, in succession.

- *If $l \ne -\infty$, then this trajectory in s_E^- has initial direction $\hat{p}^-$. Otherwise it belongs to b_E^-.*
- *If $r \ne \infty$, then this trajectory in s_E^+ has final direction $\hat{p}^+$. Otherwise it belongs to b_E^+.*

In particular E is a trapping energy for V ($E \in \mathcal{TE}$).

Proof. We only need to consider the case $l = 1 < r < \infty$ of a scattering orbit, as the other cases follow from this by limit arguments.
1) We decompose the boundary of the region

$$\mathcal{D}_k := \{(\vec{p}, \vec{q}) \in \Sigma_E \mid \vec{q} \in B_k\} \qquad (k = 1, \ldots, n)$$

of the energy shell into the disjoint union

$$\partial\mathcal{D}_k = \overset{\circ}{V}(k) \,\dot{\cup}\, \partial\mathcal{D}_k^0 \,\dot{\cup}\, \overset{\circ}{U}(k),$$

with

$$V(k) := \{(\vec{p}, \vec{q}) \in \partial\mathcal{D}_k \mid \langle \vec{p}, \vec{q} - \vec{s}_k \rangle \le 0\},$$

$$U(k) := \{(\vec{p}, \vec{q}) \in \partial\mathcal{D}_k \mid \langle \vec{p}, \vec{q} - \vec{s}_k \rangle \ge 0\}$$

and $\partial\mathcal{D}_k^0 := V(k) \cap U(k)$.
2) Setting

$$V := \bigcup_{k=1}^{n} V(k) \quad \text{and} \quad U := \bigcup_{k=1}^{n} U(k),$$

the hypersurfaces $\overset{\circ}{V}$ and $\overset{\circ}{U}$ of Σ_E are transversal to the flow Φ^t, and by assumption $E \in \cap_{k=1}^{n} \mathcal{NT}_k$. So the *interior return time* $T^i : V \to \mathbb{R}$ given by $T^i\restriction_{V \cap U} := 0$ and

$$T^i(x) := \inf\{t > 0 \mid \Phi^t(x) \in U\} \qquad (x \in \overset{\circ}{V})$$

is finite, and smooth on $\overset{\circ}{V}$. The *interior Poincaré map*

$$\mathcal{P}^i : V \to U \quad , \quad x \mapsto \Phi(T^i(x), x)$$

is a diffeomorphism:
By transversality its restriction to $\overset{\circ}{V}$ is a diffeomorphism, and its restriction to $V \cap U$ equals the identity. Finally, $\mathcal{P}^i$ is also smooth at the boundary of its domain. Namely, by enlarging the balls B_k a bit (without loosing the non-shadowing property), we may assume that $\mathrm{supp}(V_k) \cap \partial B_k = \emptyset$, so that the dynamics near the boundary is the free dynamics. Thus near the component $\partial\mathcal{D}_k^0$ of $V \cap U$ the interior Poincaré map acquires the smooth form

$$\mathcal{P}^i(\vec{p}, \vec{q}) = (\vec{p}, \vec{q} - 2\,\langle \vec{q} - \vec{s}_k, \hat{p} \rangle \cdot \hat{p}), \quad \text{with} \quad \hat{p} := \vec{p}/|\vec{p}|.$$

On $V(k)$ and on $U(k)$ we use the smooth coordinates

$$(\vec{q}_k^{\perp}, \hat{p}) \quad \text{with} \quad \vec{q}_k^{\perp} := (\mathbb{1} - \Pi_{\vec{p}})(\vec{q} - \vec{s}_k)$$

($\Pi_{\vec{p}}$ being the $\vec{p}$–projection), which map $V(k)$ resp. $U(k)$ homeomorphically onto the disk bundle

$$B^k S^{d-1} := \left\{ (\vec{v}, \hat{\theta}) \in T^* S^{d-1} \mid |\vec{v}| \leq r_k \right\},$$

and $\overset{\circ}{V}(k)$ resp. $\overset{\circ}{U}(k)$ diffeomorphically onto the interior.

When we write $\mathcal{P}^i(k) = (\vec{Q}_k^{\perp}, \hat{P}_k)$, then for a given incoming direction $\hat{p} \in S^{d-1}$ the map

$$B_{\hat{p}}^k S^{d-1} \ni \vec{q}^{\perp} \mapsto \hat{P}_k(\vec{q}^{\perp}, \hat{p}) \in S^{d-1} \tag{21}$$

sends the points $\vec{q}^{\perp}$ of modulus r_k onto $\hat{p}$ and thus can be considered as a map

$$\hat{\mathbf{P}}_{k,\hat{p}} : S^{d-1} \to S^{d-1} \quad \text{from the } (d-1)\text{-sphere} \quad B_{\hat{p}}^k S^{d-1} / \sim \; \cong S^{d-1} \tag{22}$$

to the $(d-1)$–sphere of outgoing directions. Here $\sim$ identifies the points $\vec{q}^{\perp} \in B_{\hat{p}}^k S^{d-1}$ of modulus r_k.

The trajectories of Φ_k^t which do not meet B_k are straight lines. So the degree of the continuous map $\hat{\mathbf{P}}_{k,\hat{p}}$ equals the degree $\deg_k(E)$ which is non-zero by assumption. In particular we see that for $r = 1$ there is a trajectory with initial resp. final directions $\hat{p}^-, \hat{p}^+$ meeting only B_{k_1}. So assume from now on $r \geq 2$.

3) Since the motion outside the balls B_k is free, the *exterior return time*

$$T^e : U \to \mathbb{R} \cup \{\infty\} \quad , \quad T^e(x) := \inf \left\{ t > 0 \,\middle|\, \Phi^t(x) \in V \right\}, \tag{23}$$

is bounded below by the minimal distance between the balls, divided by the speed $\sqrt{2E}$. Due to our non-shadowing assumption, on $U' := \{x \in U \mid T^e(x) < \infty\}$ the *exterior Poincaré map*

$$\mathcal{P}^e : U' \to W' := \mathcal{P}^e(U') \quad , \quad \mathcal{P}^e(x) := \Phi(T^e(x), x)$$

is continuous, and smooth on $\overset{\circ}{U}{}'$. By composing it with the interior map, we obtain

$$\mathcal{P} := \mathcal{P}^e \circ \mathcal{P}^i \restriction_{V'} : V' \to W', \quad \text{where} \quad V' := (\mathcal{P}^i)^{-1}(U').$$

By recursion in the length $r \in \mathbb{N}$ of the symbol sequence we define the *iterated Poincaré maps*

$$\mathcal{P}(\underline{k}) : V(\underline{k}) \to W(\underline{k}) := \mathcal{P}(\underline{k})(V(\underline{k})) \quad , \quad (\underline{k} \in \mathbf{X}_1^r)$$

on

$$V(\underline{k}) := V(k_1) \cap \mathcal{P}^{-1}(V(k_2, \ldots, k_r)) \quad \text{by} \quad \mathcal{P}(\underline{k}) := \mathcal{P}^{r-1}\restriction_{V(\underline{k})}.$$

So in particular $\mathcal{P}(k)$ is the identity map on $V(k) = W(k)$, $k = 1, \ldots, n$.

The $\mathcal{P}(\underline{k})$ are diffeomorphisms, but a priori some $V(\underline{k})$ may be empty. Our next task is to show the converse, using the non-vanishing of the degrees $\deg_l(E)$, and the non-shadowing assumption. We start by observing that this assumption implies

$$V(k, l) \neq \emptyset \qquad \text{iff } k \neq l.$$

4) To that aim we consider a point $(\vec{q}_l^{\perp}, \hat{p}_l) \in W(k, l)$, $1 \leq k \neq l \leq n$ and set

$$c := \langle \hat{s}, \hat{p}_l \rangle \quad \text{for} \quad \hat{s} := \hat{s}_{k,l}.$$

Then $c \geq \cos(\alpha_{k,l}) > 0$, see (19), and the rotation $\mathcal{M} \equiv \mathcal{M}(\hat{p}_l, \hat{s}) \in \mathrm{SO}(d)$ in the plane spanned by $\hat{s}$ and $\hat{p}_l$, given by

$$\mathcal{M}(\vec{v}) := \vec{v} + \frac{\hat{s}((1 + 2c)\langle \hat{p}_l, \vec{v} \rangle - \langle \hat{s}, \vec{v} \rangle) - \hat{p}_l \langle \hat{s} + \hat{p}_l, \vec{v} \rangle}{1 + c} \qquad (\vec{v} \in \mathbb{R}^d)$$

is well defined. $\mathcal{M}$ maps $\hat{p}_l$ to $\hat{s}$.

The one-parameter family of rotations $\mathcal{M}_t$ on $\mathbb{R}^d$

$$\mathcal{M}_t := \exp(t \log(\mathcal{M}_1)) \qquad (t \in [0, 1])$$

is well-defined and smooth in t and $\hat{p}_l$, since $\mathcal{M} = \mathcal{M}_1$ rotates by an angle $< \pi$.

$\mathcal{M}_t(\hat{p}_l)$ acts on $W(k, l)$ by $\mathcal{N}_t \equiv \mathcal{N}_t(k, l) : W(k, l) \to B^l S^{d-1}$, $t \in [0, 1]$,

$$\mathcal{N}_t(\vec{q}_l^{\perp}, \hat{p}_l) := \left(\mathcal{M}_t(\hat{p}_l, \hat{s})(\vec{q}_l^{\perp}), \mathcal{M}_t(\hat{p}_l, \hat{s})(\hat{p}_l) \right), \tag{24}$$

since the length of these two vectors are preserved as well as the right angle between them. Furthermore

$$\mathcal{N}_1(\vec{q}_l^{\perp}, \hat{p}_l) \in B_{\hat{s}}^l S^{d-1}.$$

5) We now return to the situation considered in the theorem. For $\hat{p}^- \in S^{d-1}$ and $r \geq 2$ we let

$$V_{\hat{p}^-}(\underline{k}) := \{(\vec{q}^\perp, \hat{p}) \in V(\underline{k}) \mid \hat{p} = \hat{p}^-\} \quad \text{and} \quad W_{\hat{p}^-}(\underline{k}) := \mathcal{P}(\underline{k})(V_{\hat{p}^-}(\underline{k})).$$

Now if $\hat{p}^- \in \tilde{S}^{d-1}$ (with $\tilde{S}^{d-1}$ defined in (20)), we notice that

$$\mathcal{P}(\underline{k})(\partial V_{\hat{p}^-}(\underline{k})) \subset \partial B^{k_r} S^{d-1}, \tag{25}$$

that is, boundaries are mapped into boundaries by the iterated Poincaré map, and that

$$\mathcal{P}(k_1, \ldots, k_l)(\partial V_{\hat{p}^-}(\underline{k})) \cap \partial B^{k_l} S^{d-1} = \emptyset, \qquad (l = 1, \ldots, r-1).$$

This follows from our non-shadowing assumption and the form of the domain $\tilde{S}^{d-1}$ of initial directions $\hat{p}^-$: trajectories coming from the k_{i-1}–st ball (resp. having direction $\hat{p}^-$ if $i = 1$), and meet the ball B_{k_i} tangentially, do not hit a further ball but go to spatial infinity.

For the same reason, setting $U_{\hat{p}^+}(l) := \{(\vec{q}^\perp, \hat{p}) \in U(l) \mid \hat{p} = \hat{p}^+\}$,

$$\mathcal{P}^i \circ \mathcal{P}(\underline{k})(\partial V_{\hat{p}^-}(\underline{k})) \cap U_{\hat{p}^+}(k_r) = \emptyset \quad \text{if} \quad \hat{p}^+ \in \tilde{S}^{d-1}.$$

By (25) and Def. (24) $\mathcal{N}_1 \circ \mathcal{P}(\underline{k})$ maps $\partial V_{\hat{p}^-}(\underline{k})$ into the $(d-2)$–sphere $\partial B_{\hat{s}}^{k_r} S^{d-1}$ of radius r_{k_r}, where now $\hat{s} := \hat{s}_{k_{r-1},k_r}$.

Using the identification (22), we may regard

$$\mathcal{Q}_{\hat{p}^-} := \mathcal{N}_1 \circ \mathcal{P}(\underline{k}) \restriction_{V_{\hat{p}^-}(\underline{k})} : V_{\hat{p}^-}(\underline{k}) \to B_{\hat{s}}^{k_r} S^{d-1} \tag{26}$$

as a continuous map on the pair

$$(V_{\hat{p}^-}(\underline{k}), \partial V_{\hat{p}^-}(\underline{k})) \to \left(S^{d-1}, \{*\}\right)$$

(with $* \in S^{d-1}$). Although $\mathcal{Q}_{\hat{p}^-}$ is not a mapping between closed $(d-1)$-manifolds, its degree $\deg(\mathcal{Q}_{\hat{p}^-}, y)$ at $y \in S^{d-1}$ is independent of $y \neq *$.

We claim that, with the same identification (22)

$$\deg(\mathcal{Q}_{\hat{p}^-}) = \prod_{i=1}^{r-1} \deg_{k_i}(E). \tag{27}$$

6) Assuming (27) for a moment, we remark that it implies

$$\deg(\hat{P}^+_{\hat{p}^-}(\underline{k})) = \prod_{i=1}^{r} \deg_{k_i}(E). \tag{28}$$

with

$$\hat{P}^{+}_{\hat{p}^-}(\underline{k}) := \hat{P}_{k_r} \circ \mathcal{P}(\underline{k})\restriction_{V_{\hat{p}^-}(\underline{k})} \tag{29}$$

This follows, using interpolation with

$$\hat{P}_{k_r} \circ \mathcal{N}_t \circ \mathcal{P}(\underline{k})\restriction_{V^t_{\hat{p}^-}(\underline{k})} \qquad (t \in [0,1]), \tag{30}$$

$V^t_{\hat{p}^-}(\underline{k})$ consisting of the preimages $(\vec{q}^{\perp}, \hat{p})$ with $\hat{p} = \hat{p}^-$. For $t = 0$ the map (30) coincides with (29).

For $t = 1$ by (26) we may take (21) with $\hat{p} := \hat{s}_{k_{r-1},k_r}$. So using multiplication of degrees for composed maps, for $t = 1$ the degree of (30) equals (28), implying that $\hat{P}^{+}_{\hat{p}^-}(\underline{k})$ has the same degree.

7) Finally (27) follows by induction in the number r of symbols, using an interpolation argument for the shifted ball

$$B^t_{k_r} := \{\vec{q} \in M \mid |\vec{q} - \vec{s}_{k_r} + t\hat{s}_{k_{r-1},k_r}| = r_{k_r}\} \qquad (t \geq 0).$$

We leave the details to the reader. □

Remarks 6.2 **1)** In a generic situation and $l, r \in \mathbb{Z}$ one finds at least $\prod_{i=l}^{r} |\deg_{k_i}(E)|$ such scattering orbits.

2) Instead of assuming that *all* the degrees $\deg_k(E)$, $k \in \{1, \ldots, n\}$, of the local potentials V_k are non-zero, it suffices to assume this for at least two of them. Then a corresponding statement holds true for the symbolic dynamics built on the corresponding subset of symbols.

3) We did not use the assumption that the interior Poincaré maps were due to the Hamiltonian dynamics generated by the smooth local potentials V_k. Instead one could use, e.g., the singular potentials of Prop. 5.2, localized by a smooth cutoff function. More generally we could take any Hamiltonian dynamics inside the balls B_k leading to a smooth interior Poincaré map of non-trivial degree.

7 Discontinuous Cross Section

The Rutherford differential cross section (11) for the Kepler potential is smooth on $\left(S^{d-1} \times S^{d-1} \setminus \mathrm{Diag}\right)$ for all $E > 0$. As shown in [9], the same is true for the

n-centre potential (7) in $d = 2$ dimensions, although this Hamiltonian system is non-integrable for $n \geq 3$. See [12] for similar results in $d = 3$ dimensions.

On the other hand, the differential cross section is smooth (again, up to the forward direction) for many smooth potentials V and energies below $V_{\max}$.

So the next result may be unexpected.

Theorem 7.1 *Let $d \geq 2$ and V a smooth short-range potential of decay rate $\alpha = 2(d-1)$ in (1). If the differential cross section*

$$(\hat{\theta}^-, \hat{\theta}^+) \mapsto \frac{d\sigma}{d\hat{\theta}^+}(E, \hat{\theta}^-, \hat{\theta}^+)$$

is continuous on $\left(S^{d-1} \times S^{d-1} \setminus \mathrm{Diag}\right)$ for any non-trapping energy $E > E_{\mathcal{NT}}$, then $V \equiv 0$.

Proof. For $E > V_{\max}$ the configuration space trajectories $t \mapsto \vec{q}(t, x_0)$ with initial conditions $x_0 \in \Sigma_E = H^{-1}(E)$ coincide, up to time parametrization, with the geodesics in the Jacobi metric on $M = \mathbb{R}^d_{\vec{q}}$

$$g_E(\vec{q}) := (E - V(\vec{q})) \cdot g(\vec{q}) \qquad (\vec{q} \in M), \tag{31}$$

which is conformally equivalent to the Euclidean metric g.

For $\hat{\theta} \in S^{d-1}$ we consider the Lagrange submanifolds

$$L_{\hat{\theta}} := \{x \in s_E \mid \hat{p}^-(x) = \hat{\theta}\}.$$

If the potential V is constant, then the particle has constant momentum. In that case, every energy shell Σ_E, $E > V$, has the form of a principal bundle $\pi : \Sigma_E \to B \cong S^{d-1}$ with base space B diffeomorphic to the $(d-1)$-dimensional sphere of directions. Furthermore, every invariant Lagrange submanifold $L_{\hat{\theta}} = \pi^{-1}(\hat{\theta}) \subset \Sigma_E$, $\hat{\theta} \in B$, projects diffeomorphically to the configuration space M under the restriction $\tau_{\hat{\theta}}$ of $\tau : \Sigma_E \to M$ to $L_{\hat{\theta}}$.

Let us now assume that for some potential V and some energy $E > E_{\mathcal{NT}}$ all Lagrange submanifolds of the energy shell Σ_E project diffeomorphically to M. Then we prove that $V \equiv 0$, contradicting the assumption of the theorem.

The metric g_E defines a connection and thus a canonical decomposition of $T(TM)$ (the space of phase space vectors) into a horizontal and a vertical subspace:

$$T_X TM = T_{X,h} TM \oplus T_{X,v} TM$$

for each phase space point $X = (\dot{\vec{q}}, \vec{q}) \in TM$. Both $T_{X,h}TM$ and $T_{X,v}TM$ are canonically isomorphic to the n-dimensional space T_qM. A vector in $T_{X,v}TM$ varies the velocity of the particle keeping its position fixed, whereas the horizontal space $T_{X,h}TM$ describes the direction of parallel transport.

Thus any vector $w \in T_XTM$ can be decomposed into its horizontal and vertical component: $w = w_h + w_v$. The symplectic two-form ω is described by the formula

$$\omega(w^1, w^2) = \langle w_h^1, w_v^2 \rangle - \langle w_h^2, w_v^1 \rangle \tag{32}$$

(Prop. 3.1.14 of [10]).

Let λ be a Lagrangian subspace of T_XTM which is transversal to the vertical subspace $T_{X,v}TM$, i.e. $\lambda \cap T_{X,v}TM = \{0\}$. Then there exists an operator

$$S : T_{X,h}TM \to T_{X,v}TM \tag{33}$$

such that the vertical and horizontal component of any vector $w = w_h + w_v \in \lambda$ obey the relation

$$w_v = Sw_h. \tag{34}$$

The symplectic two-form ω vanishes on λ. Therefore by (32),

$$0 = \omega(w^1, w^2) = \langle w_h^1, w_v^2 \rangle - \langle w_h^2, w_v^1 \rangle = \langle w_h^1, Sw_h^2 \rangle - \langle w_h^2, Sw_h^1 \rangle,$$

i.e., the operator S describing the Lagrangian space λ is symmetric.

By assumption no Lagrangian tangent space $\lambda(x)$, $x \in \Sigma_E$, turns vertical. Hence, using eq. (33), we can describe $\lambda(x)$ by a symmetric operator $S(x)$. Let $\Psi_t : T_1M \to T_1M$ denote the geodesic flow in the unit tangent bundle T_1M of (M, g_E) and let η be any vector in the tangent space T_XT_1M at the point $X = (q, \dot{q}) \in T_1M$ of this energy shell. Then after time t, X has moved to $X_t := \Psi_t(X)$, and the vector η has moved to $\eta_t := T\Psi_t(\eta)$ The horizontal part $\eta_{t,h} = Y(t)$ equals a Jacobi field along the curve $q(t) = \tau\Psi_t(q, \dot{q})$, whose covariant derivative $\nabla Y(t) = \eta_{t,v}$ equals the vertical part of η (Lemma 3.1.17 of [10]). By definition, a Jacobi field $Y(t)$ satisfies the so-called *Jacobi equation*

$$\nabla^2 Y(t) + R_{X_t} Y(t) = 0$$

for the *curvature operator*

$$R_V : T_qM \to T_qM, \qquad W \mapsto R(W, V)V, \tag{35}$$

R being the Riemann curvature tensor.

Thus we know that

$$\nabla^2 Y(t) = \nabla(SY(t)) = \left(\nabla S + S^2\right) Y(t) = -R_{X_t} Y(t)$$

for all Jacobi fields $Y(t)$. Hence the operator S satisfies the Riccati equation

$$\nabla S + S^2 + R_X = 0. \tag{36}$$

By Lemma 7.3 below we may integrate the trace of this equation over the unit tangent bundle $T_1 M$. The integral of the covariant derivative ∇S vanishes, and the integral of trace(S^2) is positive. Hence

$$\int_{T_1 M} \text{trace}(R_X) do\, dm \leq 0, \tag{37}$$

where we denote by $dm = \sqrt{\det g^J(q)} dq_1 \wedge \ldots \wedge dq_n$ the measure on M and by do the measure on the unit sphere ($\int_{S^{d-1}} do = \text{vol}(S^{d-1})$). But

$$\int_{T_1 M} \text{trace}(R_X) do\, dm = \frac{\text{vol}(S^{d-1})}{d} \int_M \mathcal{R}(\vec{q}) dm, \tag{38}$$

with the scalar curvature $\mathcal{R}$ of the Jacobi metric.

If the particle moves on a plane $M = \mathbb{R}^2_{\vec{q}}$, then $\int_M \mathcal{R}(\vec{q}) dm = 0$ as a consequence of the Gauss-Bonnet formula. For $d \geq 3$, that equality is wrong in general. But in our case the Jacobi metric (31) is conformally flat. Defining the positive function

$$u : M \to \mathbb{R}^+ \quad \text{by} \quad u := (E - V)^{(d-2)/4}, \tag{39}$$

the measure dm on M equals $dm = u^{\frac{2d}{d-2}} dq_1 \wedge \ldots \wedge dq_n$. The scalar curvature $\mathcal{R}$ equals

$$\mathcal{R} = 4 \frac{1-d}{d-2} u^{-\frac{d+2}{d-2}} \Delta u \tag{40}$$

(with the Euclidean Laplacian $\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial q_i^2}$ on M).

Therefore

$$\int_M \mathcal{R}(\vec{q})dm = -4\frac{d-1}{d-2}\int_M u^{-\frac{d+2}{d-2}}(\Delta u)u^{\frac{2d}{d-2}}dq_1\wedge\ldots\wedge dq_d$$
$$= -4\frac{d-1}{d-2}\int_M u(\Delta u)dq_1\wedge\ldots\wedge dq_d$$
$$= +4\frac{d-1}{d-2}\int_M (\nabla u)(\nabla u)dq_1\wedge\ldots\wedge dq_d \geq 0. \qquad (41)$$

Eqs. (38) and (41) are compatible with (36) only if the potential V is constant, and thus equal to zero.

As we assumed that V is non-vanishing, not all Lagrange manifolds project diffeomorphically to the configuration space M.

But since the Hamiltonian function is a positive quadratic form, the folds of the Lagrange manifolds over M extend to spatial infinity, see Duistermaat [4].

This implies a divergence of the differential cross section. □

Remark 7.2 The above theorem only implies that if *all* scattering is in the forward direction, then the potential vanishes. In fact, there exist non-zero potentials which give rise to pure forward scattering in *some* directions.

In the next lemma we show decay estimates used in the proof of Theorem 7.1.

Lemma 7.3 *For a short range potential $V \in C^\infty(M,\mathbb{R})$ with $\alpha = 2(d-1)$ in (1), an energy $E > V_{\max}$ and a smooth field of symmetric operators*

$$T_1M \ni X \mapsto S(X) \in L(T_{X,h}TM, T_{X,v}TM)$$

meeting the Riccati equation (36) along any geodesic flow line,

$$\int_M |\mathcal{R}(\vec{q})|\, dm < \infty \qquad (42)$$

for the scalar curvature $\mathcal{R}$, and

$$\int_{T_1M} \left\|S^2(X)\right\| do\, dm < \infty \qquad (43)$$

Proof. For $E > V_{\max}$ the Jacobi metric g_E is well-defined, and converges at infinity to a Euclidean metric. In the case $d > 2$, the function u defined in (39) is bounded away from zero. The expression (40) for the scalar curvature, together with (1) implies $\mathcal{R}(\vec{q}) = \mathcal{O}(\langle\vec{q}\rangle^{-\alpha-2})$ with $\langle\vec{q}\rangle := \sqrt{1+\vec{q}^2}$, so that (42) holds true.

The $(d=2)$-dimensional case gives the same decay estimate, since there

$$\mathcal{R}(\vec{q}) = \frac{(E-V(\vec{q}))\Delta V(\vec{q}) - (\nabla V(\vec{q}))^2}{2(E-V(\vec{q}))^3}.$$

Estimating the norm of S is more complicated. The idea is to exploit that S is finite everywhere, and to prove the estimate

$$\int_{S_{\vec{q}}^{d-1}} \left\|S^2(\vec{v},\vec{q})\right\| d\vec{v} = \mathcal{O}\left(\langle\vec{q}\rangle^{-d-\varepsilon}\right) \tag{44}$$

on the unit sphere over $\vec{q}$, which then implies (43).

We first note that the Riccati equation (36) tends to develop singularities. More precisely, if for some $T>0$ and $X \in T_1M$

$$\|S(X)\| > 2/T \quad \text{and} \quad \|R_{\psi^t(X)}\| \le 2/T^2 \qquad (t \in [-T,T]), \tag{45}$$

then $S(\psi^t(X))$ meeting (36) cannot be finite in the whole interval $[-T,T]$.

Namely let $Y(0)$ be a norm one eigenvector of $S(X)$ with eigenvalue $s(0) = \pm\|S(X)\|$ (such an eigenvector exists since $S(X)$ is symmetric). Then by using time inversion, if necessary, we may assume that $s(0) = -\|S(X)\|$.

We set $Y(t) = y(t)\hat{Y}(t)$ where the unit vector $\hat{Y}(t)$ is the parallel transport of $Y(0)$ along the flow line. Then by assumption (45) $s(t) := \dot{y}(t)/y(t)$ meets the scalar inequality $\dot{s}(t) \le -s^2(t) + 2/T^2$, or $u(t) := 1/s(t)$ meets

$$\dot{u}(t) \ge 1 - 2u^2(t)/T^2 \quad \text{, and} \quad -T/2 < u(0) < 0.$$

Thus $\dot{u}(t) \ge \frac{1}{2}$ as long as $u(t) \le 0$. This implies $u(t_0) = 0$ for some $t_0 \in]0,T]$, that is, divergence of $s(t)$ at t_0, contradicting our assumption. Thus

$$\|S(X)\| \le 2/T \quad \text{if} \quad \|R_{\psi^t(X)}\| \le 2/T^2 \quad \text{for} \quad |t| \le T \tag{46}$$

Such an assumption on the curvature is *uniformly* satisfied on the unit sphere $S_{\vec{q}}^{d-1}$ over $\vec{q}$ only for relatively small times $|T| = \mathcal{O}(|\vec{q}|)$, since otherwise one has trajectories of lengths T connecting $\vec{q} \in M$ with the the interaction zone, where the curvature may be large.

Thus we partition $S_{\vec{q}_0}^{d-1}$ into the union

$$R(\vec{q}) := \left\{\vec{v} \in S_{\vec{q}_0}^{d-1} \mid \sphericalangle(\vec{v}, \pm\vec{q}) \le \langle\vec{q}\rangle^{-\beta}\right\}$$

of two cones, and its complement. For $1 > \beta > 1 - 1/(d-1)$ and $T = c \cdot |\vec{q}|$ in (46) we get the contribution

$$\int_{R(\vec{q}_0)} \left\| S^2(\vec{v}_0, \vec{q}_0) \right\| d\vec{v}_0 = \mathcal{O}\left(\langle \vec{q}_0 \rangle^{-\beta(d-1)-2} \right) = \mathcal{O}\left(\langle \vec{q}_0 \rangle^{-d-\varepsilon} \right) \tag{47}$$

of $R(\vec{q}_0)$ to (44).

So let $\vec{v}_0 \in S^{d-1}_{\vec{q}_0} \setminus R(\vec{q}_0)$ be the initial direction of $X_0 := (\vec{v}_0, \vec{q}_0) \in T_1 M$ and $\theta_0 := \sphericalangle(\vec{v}_0, -\vec{q}_0)$ so that $\langle \vec{q}_0 \rangle^{-\beta} < \theta_0 < \pi - \langle \vec{q}_0 \rangle^{-\beta}$.

We claim that for $\langle \vec{q}_0 \rangle$ large

$$\inf_{t \in \mathbb{R}} |\vec{q}(t, X_0)| \geq \tfrac{1}{2} |\vec{q}_0| |\sin(\theta_0)|. \tag{48}$$

For vanishing potential $V \equiv 0$, we would have motion on straight lines and thus $\inf_{t \in \mathbb{R}} |\vec{q}(t, X_0)| = |\vec{q}_0| |\sin(\theta_0)|$.

Since the flow is reversible, we may assume w.l.o.g. that $\theta_0 \leq \pi/2$. We prove (48) by a self-consistent estimate for the double cone

$$\mathcal{C}(X_0) := \left\{ \vec{q} \in M \;\middle|\; \vec{q} = \vec{q}_0 \text{ or } \min_{\vec{v}_0' = \pm \vec{v}_0} \sphericalangle(\vec{q} - \vec{q}_0, \vec{v}_0') < \tfrac{1}{2} \langle \vec{q}_0 \rangle^{-\beta} \right\}$$

in configuration space M with vertex $\vec{q}_0$ and axis $\vec{v}_0$. Note that for $\langle \vec{q}_0 \rangle$ large

$$\operatorname{dist}(\mathcal{C}(X_0), \vec{0}) = |\vec{q}_0| \sin(\theta_0 - \tfrac{1}{2} \langle \vec{q}_0 \rangle^{-\beta}) \geq |\vec{q}_0|^{1-\varepsilon} \theta_0, \tag{49}$$

since $\theta_0 > \langle \vec{q}_0 \rangle^{-\beta}$.

We claim that the trajectory stays in the cone for all times in the sense that

$$\vec{q}(t, X_0) \in \mathcal{C}(X_0) \quad \text{and} \quad \sphericalangle\left(\dot{\vec{q}}(t, X_0), \vec{v}_0 \right) < \tfrac{1}{2} \langle \vec{q}_0 \rangle^{-\beta} \qquad (t \in \mathbb{R}) \tag{50}$$

It suffices to prove the second inequality, since the first follows from the second and the definition of $\mathcal{C}(X_0)$.

The geodesic curvature of the trajectory $t \mapsto \vec{q}(t, X_0)$ in the Euclidean metric on $M = \mathbb{R}^d_{\vec{q}}$ is given by $k(\psi^t(X_0))$, where the phase space function k is defined in (13).

For $\langle \vec{q}_0 \rangle$ large and as long as $\vec{q}(t, X_0) \in \mathcal{C}(X_0)$, this is bounded above by

$$\begin{aligned} \frac{|\nabla V(\vec{q}(t, X_0))|}{E} &= \mathcal{O}\left(|\vec{q}(t, X_0)|^{-\alpha-1} \right) \\ &= \mathcal{O}\left((|\vec{q}_0|^{1-\varepsilon} \theta_0)^{-\alpha-1} \right) = \mathcal{O}\left(|\vec{q}_0|^{-2-\varepsilon} \right) \end{aligned}$$

for $\beta - (1 - 1/(d-1)) > 0$ small, using (1) and (49).

Integrating this curvature along a segment of length $T := 2|\vec{q}_0|$, we see that within that segment the angle between the initial and the actual direction is of the order

$$\sphericalangle \left(\dot{\vec{q}}(t, X_0), \vec{v}_0 \right) = \mathcal{O}(\langle \vec{q}_0 \rangle^{-1-\varepsilon})$$

which implies (50) for the segment. Moreover at time T the trajectory already passed its (unique) pericentral point of minimal distance $|\vec{q}(t, X_0)|$ from the origin.

An estimate analogous to (6) then allows for a similar statement for all times $t \geq T$ and $t \leq 0$, proving (50).

So we may conclude from (1) and (49) that (35) meets the estimate

$$\|R_{\psi^t(X_0)}\| \leq \left(\mathrm{dist}(\mathcal{C}(X_0), \vec{0}) \right)^{-\alpha-2} = \mathcal{O}\left(\langle \vec{q}_0 \rangle^{-2d(1-\varepsilon)} \theta_0^{-2d} \right).$$

By (46) $\|S^2(X_0)\| = \|S(X_0)\|^2$ is of the same order so that

$$\begin{aligned} \int_{S_{\vec{q}_0}^{d-1} \setminus R(\vec{q}_0)} \left\| S^2(\vec{v}_0, \vec{q}_0) \right\| d\vec{v}_0 &= \mathcal{O}\left(\langle \vec{q}_0 \rangle^{-2d(1-\varepsilon)} \int_{\langle \vec{q}_0 \rangle^{-\beta}}^{\pi/2} \theta^{-2d} \, \theta^{d-2} d\theta \right) \\ &= \mathcal{O}\left(\langle \vec{q} \rangle^{-d - \frac{2}{d-1} + \varepsilon} \right). \end{aligned}$$

Together with the similar estimate (47) for $R(\vec{q}_0)$ we have thus shown the decay estimate (44). $\square$

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