Study of a Class of Regularizations of $1/|x|$ using Gaussian Integrals

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Abstract

This paper presents a comprehensive study of the functions $V_m^p(x) = \frac{pe^{x^p}}{\Gamma(m+1)} \int_x^\infty (t^p - x^p)^m e^{-t^p} dt$ for $x > 0$, $m > -1$ and $p > 0$. For large x these functions approximate x^{1-p}. The case $p = 2$ is of particular importance because the functions $V_m^2(x) \approx 1/x$ can be regarded as one-dimensional regularizations of the Coulomb potential $1/|x|$ which are finite at the origin for $m > -\frac{1}{2}$.

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The limiting behavior and monotonicity properties of these functions are discussed in terms of their dependence on m and p as well as x. Several classes of inequalities, some of which provide tight bounds, are established. Some differential equations and recursion relations satisfied by these functions are given. The recursion relations give rise to two classes of polynomials, one of which is related to confluent hypergeometric functions. Finally, it is shown that, for integer m, the function $1/V_m^2(x)$ is convex in x and this implies an analogue of the triangle inequality. Some comments are made about the range of p and m to which this convexity result can be extended and several open questions are raised.

Contents

1 Introduction

1.1 Definitions and background

In this paper we study the functions

$$\begin{aligned} V_m(x) &= \frac{2e^{x^2}}{\Gamma(m+1)} \int_x^\infty (t^2 - x^2)^m e^{-t^2} dt, \qquad m > -1 \qquad (1) \\ V_{-1}(x) &= \frac{1}{|x|} \end{aligned}$$

and their generalizations,

$$\begin{aligned} V_m^p(x) &= \frac{pe^{x^p}}{\Gamma(m+1)} \int_x^\infty (t^p - x^p)^m e^{-t^p} dt \qquad (2) \\ V_{-1}^p(x) &= x^{1-p}, \end{aligned}$$

for $0 < p < \infty$. These functions are well-defined for $x > 0$ and can be extended to complex m with $\Re(m) > -1$. For $\Re(m) > -\frac{1}{2}$ they are also well-defined for $x = 0$. Using symmetry or the equivalent forms (4) and (7) below, they can be extended to even functions on $\mathbf{R}$ or $\mathbf{R} \setminus \{0\}$. However, it suffices to consider only non-negative x and in this paper we restrict ourselves to that. We also restrict ourselves to real m.

Letting $p = 2$ in (2) yields (1). However, because this case is more important in applications we often drop the superscript and and simply write $V_m(x)$ for $V_m^2(x)$.

Our interest was motivated by studies of atoms in magnetic fields where these functions arise naturally for integer m. V_m can be regarded as a (two-dimensional) expectation of the (three-dimensional) Coulomb potential $1/|\mathbf{r}|$ with the state $\gamma_m(r,\theta) = \frac{1}{\sqrt{\pi m!}} e^{-im\theta} r^m e^{-r^2/2}$ (where we have used cylindrical coordinates $\mathbf{r} = (x, r, \theta)$ with the non-standard convention $r = \sqrt{y^2 + z^2}$ if $\mathbf{r} = (x, y, z)$ in rectangular coordinates). The state γ_m describes an electron in the lowest of the so-called "Landau levels" with angular momentum m in the direction of the field. In this context, it is natural to rewrite (1) in the form

$$\begin{aligned} V_m(x) &= \frac{2}{\Gamma(m+1)} \int_0^\infty \frac{r^{2m} e^{-r^2}}{\sqrt{x^2 + r^2}} r dr, \qquad (3) \\ &= \frac{1}{\Gamma(m+1)} \int_0^\infty \frac{u^m e^{-u}}{\sqrt{x^2 + u}} du \qquad (4) \end{aligned}$$

for $m > -1$. In this form, it is easy to see that $V_m(x) \approx 1/|x|$ for large x. The importance of V_m goes back at least to Schiff and Snyder [18] and played an essential role in the Avron, Herbst and Simon [2] study of the energy asymptotics of hydrogen in a strong magnetic field. More recently work in astrophysics and the work of Lieb, Solovej and Yngvason [13] on asymptotics of many-electron atoms in strong magnetic fields has renewed interest in this subject. Motivated by the LSY work, Brummelhuis and Ruskai [7, 8] have developed one-dimensional models of many-electron atoms in strong magnetic fields using the functions $V_m(x)$ as one-dimensional analogues of the Coulomb potential.

In the case of many-electron atoms, the anti-symmetry required by the Pauli exclusion principle suggests replacing the simple "one-electron" expectation above by an N-electron analogue in which the state γ_m is replaced by a Slater determinant of such states. This is discussed in detail in [8] where it is shown that, in the simple case corresponding to $m = 0 \ldots N-1$, the analogous one-dimensional potentials have the form

$$V_{\rm av}^N(x) = \frac{1}{N}\sum_{m=0}^{N-1} V_m(x). \tag{5}$$

In Section 3 we obtain recursion relations for V_m which, in addition to being of considerable interest in their own right, are extremely useful for studying potentials of the form (5).

For $m = 0$ the function

$$\frac{1}{\sqrt{2}} V_0\left(\frac{x}{\sqrt{2}}\right) = e^{x^2/2}\int_x^\infty e^{-t^2/2}dt \tag{6}$$

occurs in many other contexts and is sometimes called the "Mills ratio" [14]. Although it has been extensively studied, the class of inequalities we consider in Section 4.1 appears to be new (although some of our bounds coincide with known inequalities in other classes) and the realization that $1/V_0(x)$ is convex seems to be relatively recent [16, 17, 7].

The replacement of x^2 by x^p in (6) has been considered by Gautschi [10] and Mascioni [12], who (after seeing the preprint [15]) extended the results of Section 4.1 to this situation.

For the analysis of this generalization, it is useful to observe that (2) can be rewritten as

$$V_m^p(x) = \frac{1}{\Gamma(m+1)}\int_0^\infty \frac{u^m e^{-u}}{(x^p+u)^{\frac{p-1}{p}}}du \tag{7}$$

In this form, it is easy to verify that $V_m^p(x) \approx x^{p-1}$ for large x and that for $p = 1$, $V_m^1(x) = 1$ for all m .

Our first result shows that $V_m^p(x)$ is continuous in m and that our definition for $m = -1$ is natural.

Proposition 1 *For all $x > 0$, $\lim_{m \to -1^+} x^{p-1} V_m^p(x) = 1$.*

Proof: Note that $\dfrac{x^{p-1}}{(x^p + u)^{\frac{p-1}{p}}} = \dfrac{1}{\left(1 + \frac{u}{x^p}\right)^{\frac{p-1}{p}}}$ so that (7) implies

$$1 - x^{p-1} V_m^p(x) = \frac{1}{\Gamma(m+1)} \int_0^\infty u^m e^{-u} \left[1 - \left(1 + \frac{u}{x^p}\right)^{\frac{1-p}{p}} \right] du. \tag{8}$$

Since $\Gamma(m+1)$ becomes infinite as $m \to -1$, the desired result follows if the integral on the right above remains finite. To see that this is true, it is convenient to let $g(z) = \frac{1}{z}\left(1 - (1+z)^{\frac{1-p}{p}}\right)$ and note that (8) implies

$$\left|1 - x^{p-1} V_m^p(x)\right| \leq \frac{1}{\Gamma(m+1)} \int_0^\infty \frac{u^{m+1} e^{-u}}{x^p} \left| g\left(\frac{u}{x^p}\right) \right| du \tag{9}$$

It is easy to see that the large u portion of this integral causes no problems since $|g(z)|$ is bounded by a polynomial in z when $z > 1$. (For $p \geq 1$, it is bounded uniformly by 1; for $0 < p < 1$, it is bounded by a polynomial, namely $|g(z)| \leq 1 + (1+z)^k$, where $k \in \mathbf{N}$, $k \geq \frac{1}{p}$.) To see that it is also well-behaved for small u, we first note that for $p > 0$, $(1+z)^{\frac{1-p}{p}}$ is analytic for $\Re(z) > -1$. Then $g(z)$ has a removable singularity at $z = 0$ and can be extended to an analytic function on $\Re(z) > -1$. Thus, for small u, the integrand behaves like $x^{-p} u^{m+1} e^{-u}$ which ensures that the integral in (9) is finite for $m = -1$. **QED**

The rest of this paper is organized as follows. In the next two parts of the introduction we summarize the properties of V_m in the important case $p = 2$ and state some open questions. In Section 2 we state and prove the basic properties of V_m^p for general p. In Section 3 we derive recursion relations for V_m^p and study their consequences. Among these is a connection with confluent hypergeometric functions. In Section 4.1 we prove some optimal bounds for V_0. The optimal upper bound had been established earlier independently by Wirth [17] and by Szarek and Werner [16] who also showed that the upper bound is equivalent to the convexity of $1/V_0$. In Section 4.2 we discuss several classes of inequalities, beginning with optimal bounds on $V_0(x)$. We then consider optimal bounds on the ratio $R_m(x) = V_m(x)/V_{m-1}(x)$ and show that these have important consequences. In particular, we show that the upper bound is equivalent to the convexity (in x) of $1/V_m(x)$ and that the ratios increase with x. Proofs of the ratio bounds are then given in Section 5 where we also consider extensions to other p. Because the proof of the ratio bounds is via induction on m, the results of Sections 6 and 7 are only established for integer m. However, we believe that they hold for all $m > -1$.

1.2 Properties of $V_m(x)$

We now summarize some properties of $V_m(x)$ along with comments about the history and brief remarks about the proofs. Unless otherwise stated, these properties hold for $m > -1$ and $x > 0$.

a) $\dfrac{1}{\sqrt{x^2+m}} > V_m(x) > \dfrac{1}{\sqrt{x^2+m+1}}$

where the first inequality holds for $m > 0$ and the second for $m > -1$.

To prove the upper bound, which appears to be new, observe that $\mu = [u^{m-1}e^{-u}/\Gamma(m)]du$ is a probability measure on $(0, \infty)$. For fixed x, one can then apply Jensen's inequality to the concave function $f_x(u) = u(u+x^2)^{-1/2}$ to obtain

$$V_m(x) = \frac{1}{m}\int_0^\infty f_x(u)d\mu(u) \le \frac{1}{m} f_x\Big(\int_0^\infty \frac{u^m e^{-u} du}{\Gamma(m)}\Big)$$
$$= \frac{1}{m} f_x(m) = \frac{1}{\sqrt{x^2+m}}.$$

The lower bound was proved earlier (at least for integer m) by Avron, Herbst and Simon [2] who applied a similar argument to the probability measure $[u^m e^{-u}/\Gamma(m+1)]du$ and the convex function $f_x(u) = (u + x^2)^{-1/2}$.

b) $V_m(x)$ is decreasing in m. In particular, $V_{m+1}(x) < V_m(x) < \dfrac{1}{x}$.

The first inequality follows easily from property (a) which implies $V_m(x) < \frac{1}{\sqrt{x^2+m}} < V_{m-1}(x)$. Alternatively, one could use integration by parts on (4). The second inequality is easily verified from the integral representation (4). That $V_m(x)$ also decreases with m for non-integer jumps is more difficult, and the proof is postponed to Section 2 where it follows from the more general Theorem 6.

c) The expression $mV_m(x)$ is increasing in $m > -1$, $m \in \mathbf{R}$.
For integer jumps this holds for $m \geq -1$. Indeed, it is obvious that $-V_{-1} < 0 \cdot V_0 < V_1$. For integer jumps with $m \geq 1$, one can use property (a) to see that

$$mV_m(x) > \frac{m}{\sqrt{x^2+m+1}} > \frac{m-1}{\sqrt{x^2+m-1}} > (m-1)V_{m-1}(x). \quad (10)$$

The proof for general m is postponed to Theorem 6 in Section 2. The fact that $V_m(x)$ is decreasing in m, while $mV_m(x)$ is increasing gives an indication of the delicate behavior of V_m.

d) For $m > -1/2$, the definition of $V_m(x)$ can be extended to $x = 0$ and

$$V_m(0) = \frac{\Gamma(m+\frac{1}{2})}{\Gamma(m+1)}. \quad (11)$$

For integer m, this becomes

$$V_m(0) = \frac{(2m)!}{2^{2m}(m!)^2}\sqrt{\pi} = \frac{1 \cdot 3 \cdot 5 \ldots (2m-1)}{2 \cdot 4 \cdot 6 \ldots (2m)}\sqrt{\pi} \quad (12)$$

while for large m Stirling's formula implies

$$V_m(0) \approx \left(\frac{m-\frac{1}{2}}{m}\right)^m \left(\frac{e}{m}\right)^{1/2} \approx \frac{1}{\sqrt{m}} \quad (13)$$

which is consistent with property (a). Boyd [6, 14] has proved the more precise estimates

$$\frac{\sqrt{m+\frac{3}{4}+\frac{1}{32m+48}}}{m+\frac{1}{2}} < V_m(0) < \frac{1}{\sqrt{m+\frac{1}{4}+\frac{1}{32m+32}}} \quad (14)$$

e) For all $m \geq 0$, V_m satisfies the differential equation

$$V_m'(x) = 2x\left(V_m - V_{m-1}\right). \quad (15)$$

This can easily be verified using integration by parts in (4).

f) For each fixed $m \geq 0$, $V_m(x)$ is decreasing in x.

This follows directly from (b) and (e).

g) For $a > 0$, the expression $aV_m(ax)$ increases with a. Hence $aV_m(ax) > V(x)$ when $a > 1$ and $aV_m(ax) < V(x)$ when $a < 1$.

This property follows easily from the definition (3) or (4) and the observation that $\dfrac{a}{\sqrt{a^2x^2+u}} = \dfrac{1}{\sqrt{x^2+\frac{u}{a^2}}}$ is increasing in a. It is used in the proof of Theorem 6 and is important in the study of one-dimensional models for atoms in magnetic fields in which the electron-electron interaction takes the form of convex combinations of $\frac{1}{\sqrt{2}}V_m\left(\frac{|x_j-x_k|}{\sqrt{2}}\right)$.

h) $V_0(x)$ is convex in $x > 0$; however, $V_m(x)$ is *not* convex when $m > \frac{1}{2}$.

For $m = 0$, the differential equation (15) becomes $V_0'(x) = 2[xV_0 - 1]$. Since $xV_0 = \int_0^\infty \frac{e^{-u}}{\sqrt{1+u/x^2}}du$ is increasing for $x > 0$, it follows that $V_0(x)$ is convex. When $m > \frac{1}{2}$ it follows from (15) and (b) that $\lim_{x\to 0} V_m'(x) = 0$. Since V_m' is negative, V_m' must decrease on some small interval $(0, x_0)$. One can also show $\lim_{x\to\infty} V_m'(x) = 0$, so that one expects that there is an x_1 such that V_m is concave on $(0, x_1)$ and convex on (x_1, ∞). In Section 3 we will see that the convexity is recovered for the averaged potential $V_m^{\rm av}$.

i) For integer m, $1/V_m(x)$ is convex in $x > 0$.

This will be proved in Section 4.2 as Theorem 23. For large x, $1/V_m(x) \approx x$ so that the deviation from linearity is very small and the second derivative close to zero. This makes the proof quite delicate and lengthy.

The convexity of $1/V_m(x)$ can be rewritten as

$$\frac{1}{\frac{1}{2}V_m\left(\frac{x+y}{2}\right)} \leq \frac{1}{V_m(x)} + \frac{1}{V_m(y)}.$$

Using property (g) with $a = \frac{1}{2}$, one easily finds that the convexity of $1/V_m(x)$ implies

$$\frac{1}{V_m(x+y)} \leq \frac{1}{V_m(x)} + \frac{1}{V_m(y)}.$$

This subadditivity inequality plays the role of the triangle inequality in applications. (See, e.g. [7].)

j) Asymptotic estimates:

For large x, it follows from property (a) that

$$\frac{m}{2(x^2+m)^{3/2}} \leq \frac{1}{x} - V_m(x) < \frac{m+1}{2x^3} \tag{16}$$

The asymptotic expansion

$$V_m(x) = \frac{1}{x} - \frac{m+1}{2x^3} + \frac{3(m+2)(m+1)}{8x^5} + O\left(\frac{1}{x^7}\right) \tag{17}$$

can easily be obtained from (4). For details let $p = 2$ in the proof of Proposition 7 which gives a similar expansion for $p > 1$.

It follows from properties (a) and (c) that $V_m(x)$ decreases monotonically to zero for each fixed x as $m \to \infty$. In fact, since $V_m(x)$ is decreasing in x for all m, it suffices to show this for $x = 0$ which is easy since $V_m(0) < m^{-1/2}$.

1.3 Open questions

We now list some questions which remain open.

A) Do the ratio bounds and other results of Section 4.2 also hold for non-integer m? In particular, can the convexity of $1/V_m$ be established for non-integer m?

B) For what range of p is $1/V_m^p(x)$ convex (or concave) in $x > 0$?

For large x, $1/V_m^p(x) \approx x^{p-1}$ and x^{p-1} is concave for $1 < p < 2$. Hence we can not expect the convexity of $1/V_m$ to extend to $1/V_m^p$ in the range $(1, 2)$. The results of Section 7.4 provide evidence that $1/V_m^p(x)$ is convex for all $p \geq 2$. However, our method of proof in Section 7.4 gives this result only for a limited range of p.

C) Is $V_m(x)$ convex in m? In particular, is $2V_m(x) \leq V_{m+1}(x) + V_{m-1}(x)$ or, equivalently by (15), is $V_m'(x)$ increasing in m?

D) As discussed in property (i) above, one important consequence of the convexity of $1/V_m(x)$ is an analogue of the triangle inequality. For all values of p we have $[V_m^p(x)]^{\frac{1}{1-p}} \approx x$ for large x. For what range of p (or p and m) is the triangle inequality

$$[V_m^p(x+y)]^{\frac{1}{1-p}} \leq [V_m^p(x)]^{\frac{1}{1-p}} + [V_m^p(y)]^{\frac{1}{1-p}}$$

satisfied? Notice that for $p \geq 2$ the convexity of $1/V_m^p(x)$ implies only the weaker inequality

$$[V_m^p(x+y)]^{\frac{1}{1-p}} \leq 2^{\frac{p-2}{p-1}} \left([V_m^p(x)]^{\frac{1}{1-p}} + [V_m^p(y)]^{\frac{1}{1-p}} \right).$$

E) What can we say about the polynomials $Q_m^p(x)$?

In Section 3.3 we introduce two classes of polynomials $P_m^p(x)$ and $Q_m^p(x)$ which arise naturally from recursion relations for $V_m^p(x)$. We give an extensive discussion of the properties of P_m^p, but have much less information about $Q_m^p(x)$.

2 General p

We now study the basic properties of V_m^p in detail. As one would expect from $V_m^p(x) \approx x^{1-p}$, the behavior of V_m^p is often quite different for $p > 1$ and $p < 1$. At the boundary, $p = 1$, $V_m^1(x) = 1$ for all x. Proposition 2 describes the monotonicity and limiting behavior of $V_m^p(x)$ as p varies with m and x fixed. Proposition 3 gives a simple expression for V_m^p in the special case that $1/p$ is an integer.

The next four results generalize properties of Section 1.2 to general p. Proposition 4 generalizes the inequalities from property (a); Proposition 5 generalizes properties (d), (e), (f), and (g); and, Theorem 6 extends the monotonicity properties (b) and (c). Moreover, the proof of monotonicity for non-integer jumps is provided here. Finally, Proposition 7 gives the asymptotic behavior of $V_m^p(x)$ for large x when $p > 1$.

Proposition 2 *Let $m > -1$ and $x > 0$ be fixed. Then*

(i) $\lim_{p\to 0} V_m^p(x) = \infty$.

(ii) *For all $x \geq 1$, V_m^p is decreasing in p. Moreover,*

$$\begin{aligned} \text{if } x > 1, \quad \lim_{p\to\infty} V_m^p(x) &= 0, \quad \text{and} \\ \text{if } x = 1, \quad \lim_{p\to\infty} V_m^p(1) &= \frac{1}{\Gamma(m+1)} \int_0^\infty \frac{u^m e^{-u} du}{1+u}. \end{aligned}$$

(iii) *For all* $0 < x < 1$ *and* $m > 0$, $\lim_{p\to\infty} V_m^p(x) = \frac{1}{m}$.

Proof: We use the expression (7) for V_m^p.

(i) Since $\lim_{p\to 0}(x^p + u)^{1/p} = \infty$ for $x > 1$,

$$\lim_{p\to 0} V_m^p(x) \geq \lim_{p\to 0} \frac{1}{\Gamma(m+1)} \int_1^3 \frac{u^m e^{-u}(x^p+u)^{\frac{1}{p}} du}{x^p + u} = \infty.$$

(ii) Differentiating (7) yields

$$\frac{d}{dp} V_m^p(x) = \frac{1}{\Gamma(m+1)} \int_0^\infty \frac{u^m e^{-u}}{p^2 (x^p+u)^{\frac{2p-1}{p}}} \times \left[(1-p) x^p \ln(x^p) - (x^p+u)\ln(x^p+u) \right] du. \tag{18}$$

For $x = 1$ or $p = 1$, the first term in square brackets above is zero leaving a quantity which is clearly negative. When both $x > 1$ and $p > 1$ both terms in (18) are clearly negative. When $x > 1$ and $0 < p < 1$, the quantity in square brackets in (18) is negative since

$$\begin{aligned}&(1-p)x^p \ln(x^p) - (x^p+u)\ln(x^p+u) \\ &\quad\leq\ x^p \ln(x^p) - (x^p+u)\ln(x^p+u) < 0.\end{aligned}$$

The last inequality follows from the fact that the function $f(w) = w \ln w$ is increasing for $w > 1$. Thus, $\frac{d}{dp} V_m^p(x) \leq 0$ for all $x \geq 1$ and for all $p \in (0, \infty)$.

(iii) Since $\lim_{p\to\infty}(x^p + u)^{1/p} = 1$ for $x < 1$,

$$\lim_{p\to\infty} V_m^p(x) = \frac{1}{\Gamma(m+1)} \int_0^\infty u^{m-1} e^{-u} du = \frac{1}{m}.\textbf{QED}$$

Remark: The behavior for $x < 1$ depends upon m. For "small" m, there is a p_0 such that V_m^p decreases (below 1) on $(0, p_0)$ and increases on (p_0, ∞) to $\frac{1}{m}$. For "big" m , V_m^p simply decreases to $\frac{1}{m}$ as p increases.

The next result shows that in the special case that $1/p$ is an integer V_m^p reduces to a polynomial in $x^p = x^{1/n}$ of degree $n - 1$.

Proposition 3 *For* $n \in \mathbf{N}$, $n \geq 2$,

$$V_m^{1/n}(x) = \frac{1}{\Gamma(m+1)} \sum_{k=0}^{n-1} \binom{n-1}{k} \Gamma(m+n-k)\ x^{k/n}$$

for all $x \geq 0$ *and* $m > -1$.

Proof: It follows from (7) that for $p = \frac{1}{n}$

$$V_m^{1/n}(x) = \frac{1}{\Gamma(m+1)} \int_0^\infty \left(x^{1/n} + u\right)^{n-1} u^m e^{-u} du.$$

When n is an integer ≥ 2, the result then follows easily from the binomial expansion applied to $(x^{\frac{1}{n}} + u)^{n-1}$ and the definition of the Γ-function.

Proposition 4 *For all $x > 0$*

$$\frac{1}{(x^p + m + 1)^{\frac{p-1}{p}}} \leq V_m^p(x) \leq \frac{1}{(x^p + m)^{\frac{p-1}{p}}} \quad \text{for } p > 1,$$

where the first inequality holds for $m > -1$ and the second for $m \geq 0$.

$$(x^p + m + 1)^{\frac{1-p}{p}} \geq V_m^p(x) \geq (x^p + m)^{\frac{1-p}{p}} \quad \text{for } \frac{1}{2} \leq p < 1,$$

where the first inequality holds for $m > -1$ and the second for $m \geq 0$.

$$V_m^p(x) \geq (x^p + m + 1)^{\frac{1-p}{p}} \quad \text{for } 0 < p \leq \frac{1}{2},$$

and the inequality holds for $m > -1$.

Proof: The proofs are done using Jensen's inequality as in property (a) of Section 1.2.

Proposition 5 *For all $x > 0$*

(i) *For $m > -\frac{1}{p}$, $V_m^p(0)$ is defined and*

$$V_m^p(0) = \frac{\Gamma(m + \frac{1}{p})}{\Gamma(m+1)}.$$

(ii) *For all $m \geq 0$, $x > 0$, V_m^p satisfies the differential equation*

$$\frac{d}{dx} V_m^p(x) = p x^{p-1} \left(V_m^p(x) - V_{m-1}^p(x)\right). \tag{19}$$

(iii) *For all $m > -1$, V_m^p is decreasing in x, if $p > 1$, identically equal to 1 for all x, if $p = 1$, increasing in x, if $p < 1$.*

(iv) *Let $m > -1$ and $x > 0$. For $a > 0$, the expression $a^{p-1}V_m^p(ax)$ increases in a, if $p > 1$ and decreases in a if $p < 1$.*

Proof: The proofs are straightforward extensions of those given in Section 1.2. In (iii), one can verify that V_m^p is also increasing for $0 < p < \frac{1}{2}$ by computing the derivative directly.

Theorem 6 *For each fixed $x > 0$, and for m in the region $m > -1$,*

(i) *$V_m^p(x)$ is strictly decreasing in m for $p > 1$ and strictly increasing in m for $p < 1$.*

(ii) *$mV_m^p(x)$ is strictly increasing in m for $p > 1$ and strictly decreasing in m for $p < 1$.*

Proof: To prove (i) we differentiate (7) to get

$$\frac{d}{dm}V_m^p(x) = \frac{1}{\Gamma(m+1)}\int_0^\infty \frac{u^m \ln u \ e^{-u}}{(x^p+u)^{1-\frac{1}{p}}}du - V_m^p(x)\frac{\Gamma'(m+1)}{\Gamma(m+1)} \tag{20}$$

Using the same procedure as that used (see, e.g. [1, 11]) to obtain the standard integral representation

$$\psi(z) \equiv \frac{\Gamma'(z)}{\Gamma(z)} = \int_0^\infty \left[\frac{e^{-s}}{s} - \frac{1}{s(1+s)^z}\right] ds \tag{21}$$

one finds

$$\begin{aligned}
&\frac{1}{\Gamma(m+1)}\int_0^\infty \frac{u^m \ln u \ e^{-u}}{(x^p+u)^{1-\frac{1}{p}}}du \\
&= \frac{1}{\Gamma(m+1)}\int_{s=0}^\infty \frac{ds}{s}\int_0^\infty [e^{-s} - e^{-su}]\frac{e^{-u}u^m}{(x^p+u)^{1-\frac{1}{p}}}du \\
&= V_m^p(x)\int_0^\infty \frac{e^{-s}}{s}ds - \frac{1}{\Gamma(m+1)}\int_0^\infty \frac{ds}{s(s+1)^{m+\frac{1}{p}}}\int_0^\infty \frac{e^{-w}w^m dw}{[x^p(s+1)+w]^{1-\frac{1}{p}}} \\
&= V_m^p(x)\int_0^\infty \frac{e^{-s}}{s}ds - \int_0^\infty V_m^p[x(s+1)]^{\frac{1}{p}})\frac{ds}{s(s+1)^{m+\frac{1}{p}}},
\end{aligned}$$

where we made the change of variable $w = (s+1)u$ to obtain $V_m((s+1)^{\frac{1}{p}}x)$. Now we use Proposition 4 (iv) with $a = (s+1)^{\frac{1}{p}} > 1$ to obtain

$$\begin{aligned}
\frac{1}{\Gamma(m+1)}\int_0^\infty \frac{u^m \ln u \ e^{-u}}{(x^p+u)^{1-\frac{1}{p}}}du &\le \int_0^\infty V_m^p(x)\left(\frac{e^{-s}}{s} - \frac{1}{s(s+1)^{m+1}}\right)ds \\
&= V_m^p(x)\ \psi(m+1)
\end{aligned}$$

when $p > 1$. For $p < 1$, Proposition 4 (iv) gives the inequality in the opposite direction. Hence inserting the result in (20) yields

$$\frac{d}{dm}V_m(x)\begin{cases} < 0 & \text{if } p > 1 \\ > 0 & \text{if } p > 1. \end{cases}$$

To prove (ii) it is slightly more convenient to consider the logarithmic derivative $\frac{d}{dm}\ln\left[mV_m^p(x)\right]$ and show that it is positive for $p > 1$ and negative for $p < 1$. Proceeding as above, we find for $p > 1$

$$\begin{aligned}
\frac{d}{dm}\ln\left[mV_m^p(x)\right] &= \frac{1}{m} + \frac{\frac{d}{dm}\left[V_m^p(x)\right]}{V_m^p(x)} \\
&= \frac{1}{m} + \int_0^\infty \frac{e^{-s}}{s}ds - \frac{1}{V_m^p(x)}\int_0^\infty \frac{V_m^p\left[x(s+1)^{1/p}\right]}{s\ (s+1)^{m+\frac{1}{p}}}ds - \psi(m+1) \\
&< \frac{1}{m} + \int_0^\infty \frac{e^{-s}}{s}ds - \int_0^\infty \frac{1}{s(s+1)^m}ds - \psi(m+1) \\
&= \frac{1}{m} + \psi(m) - \psi(m+1) \\
&= \frac{1}{m} - \int_0^\infty \frac{1}{(s+1)^{m+1}} = 0
\end{aligned}$$

where we have used (21) and the following inequality with $a = (s+1)^{1/p}$.

$$V_m^p(ax)\left\{\begin{array}{c} < \\ > \end{array}\right\} aV_m^p(x) \quad \text{for} \quad \left\{\begin{array}{c} p > 1 \\ p < 1 \end{array}\right\}. \tag{22}$$

for all $a \geq 1$. This is easily verified and implies that the inequality proved above for $\frac{d}{dm}\ln\left[mV_m^p(x)\right]$ is reversed when $p < 1$. **QED**

The following result gives the asymptotic behavior of $V_m^p(x)$ for large x.

Proposition 7 *For $p > 1$, $V_m^p(x)$ has the asymptotic expansion*

$$\frac{1}{x^{p-1}} - \frac{(p-1)(m+1)}{p\ x^{2p-1}} + \frac{(2p^2-3p+1)(m^2+3m+2)}{2p^2\ x^{3p-1}} + O\left(\frac{1}{x^{4p-1}}\right)$$

Proof: This follows from (7) since

$$\begin{aligned}
V_m^p(x) &= \frac{1}{\Gamma(m+1)\ x^{p-1}}\int_0^\infty \frac{u^m e^{-u}}{\left(1+\frac{u}{x^p}\right)^{\frac{p-1}{p}}}du \\
&= \frac{1}{\Gamma(m+1)\ x^{p-1}}\int_0^\infty u^m e^{-u}\left[1 - \frac{(p-1)\ u}{p\ x^p} + \frac{(p-1)(2p-1)\ u^2}{2p^2\ x^{2p}} + \ldots\right]du \\
&= \frac{1}{x^{p-1}}\left[1 - \frac{(p-1)\ \Gamma(m+2)}{p\ \Gamma(m+1)\ x^p} + \frac{(2p^2-3p+1)\ \Gamma(m+3)}{(2p^2)\ \Gamma(m+1)\ x^{2p}} + O\left(\frac{1}{x^{3p}}\right)\right]
\end{aligned}$$

3 Recursion Relations and their Consequences

3.1 Recursion relations for V_m^p

Although the case $p = 2$ is of primary interest in applications, we continue to study general p is this section, as the proofs for general p are identical to those for $p = 2$. In these recursions, our convention that $V_{-1}^p(x) = x^{1-p}$ plays an important role.

Proposition 8 *For all $m \in \mathbf{R}$, $m \geq 1$, for all $x > 0$,*

$$V_m^p(x) = \frac{1}{m}\left[(m - 1 + \frac{1}{p} - x^p)V_{m-1}^p(x) + x^p V_{m-2}^p(x)\right]. \tag{23}$$

Proof: For $m = 1$, one gets that

$$V_1^p(x) = pe^{x^p}\left[(\frac{1}{p} - x^p)\int_x^\infty e^{-t^p}dt + x\right] = (\frac{1}{p} - x^p)V_0^p(x) + x^{p-1}V_{-1}^p(x).$$

For $m > 1$, using (2) and integration by parts, we find

$$\begin{aligned}
V_m^p(x) &= \frac{pe^{x^p}}{\Gamma(m+1)}\int_x^\infty (t^p - x^p)^{m-1}(t^p - x^p)e^{-t^p}dt \\
&= \frac{pe^{x^p}}{m\Gamma(m)}\left[(-x^p\int_x^\infty e^{-t^p}(t^p - x^p)^{m-1}dt\right. \\
&\quad \left. + \frac{1}{p}\int_x^\infty e^{-t^p}((t^p - x^p)^{m-1} + (m-1)pt^p(t^p - x^p)^{m-2}t^2)dt\right] \\
&= \frac{pe^{x^p}}{m\Gamma(m)}\left[(m - 1 + \frac{1}{p} - x^p)\int_x^\infty e^{-t^p}(t^p - x^p)^{m-1}dt\right. \\
&\quad \left. +\ (m-1)x^p\int_x^\infty (t^p - x^p)^{m-2}e^{-t^2}dt\right] \\
&= \frac{1}{m}\left[(m - 1 + \frac{1}{p} - x^p)V_{m-1}^p(x) + +x^p V_{m-2}^p(x)\right]
\end{aligned}$$

Repeated application of (23) gives a useful corollary. For $m \in \mathbf{R}$, let $\lfloor m \rfloor$ denote the "floor" of m, i.e., the largest natural number less than or equal to m.

Corollary 9 *Let $m \in \mathbf{R}$, $m \geq 1$ and let $n \in \mathbf{N}$ such that $n \leq \lfloor m \rfloor$. Then*

$$\begin{aligned}
V_m^p(x) &= \frac{1}{2m}\left[(1 - px^p)V_{m-1}^p(x) + V_{m-2}^p(x) + \ldots\right. \\
&\qquad \left. + \ldots [p(m-n) + 1]V_{m-n}^p(x) + px^p V_{m-n-1}^p(x)\right].
\end{aligned} \tag{24}$$

In particular, if m is a positive integer, then

$$V_m^p(x) = \frac{1}{2p}\left[(1-px^p)V_{m-1}^p(x) + \sum_{k=0}^{m-2} V_k^p(x) + px^{p-1}V_{-1}^p(x)\right]. \tag{25}$$

The expression (25) is well-defined for $x = 0$. Putting $x = 0$ and using Proposition 5 (i), we obtain the (presumably well-known) identity

$$\frac{\Gamma(m+\frac{1}{p})}{\Gamma(m+1)} = \frac{1}{pm}\sum_{k=0}^{m-1}\frac{\Gamma(k+\frac{1}{p})}{\Gamma(k+1)} \tag{26}$$

3.2 Averaged potentials

These recursion relations are quite useful for studying the average of the first N of the V_m. For N a positive integer, we extend (5) to

$$V_{\mathrm{av}}^{p,N}(x) = \frac{1}{N}\sum_{m=0}^{N-1} V_m^p(x) \tag{27}$$

Note that for $p = 1$, $V_{\mathrm{av}}^{1,N}(x) = 1$, for all $x \geq 0$.

The next result follows immediately from (25).

Corollary 10 $V_{\mathrm{av}}^{p,N}(x) = pV_N^p(x) - \frac{px^p}{N}\left[V_{-1}^p(x) - V_{N-1}^p(x)\right].$

For the important case $p = 2$, this reduces to

$$V_{\mathrm{av}}^N(x) = 2V_N(x) - \frac{2x^2}{N}\left[V_{-1}(x) - V_{N-1}(x)\right] \tag{28}$$

The function $V_0(|x|)$ is convex on $(0,\infty)$ but has a cusp at $x = 0$. However, as discussed in property (f), for higher m both the convexity and cusp are lost. Thus, for higher m, the V_m are somewhat smoother than one might want for one-dimensional approximations to the Coulomb potential. The next result, although straightforward, is important because it implies that the averaged potentials $V_{\mathrm{av}}^N(x)$ retain the cusp and convexity properties of V_0 near the origin.

Proposition 11 *The function $V_{\mathrm{av}}^N(x)$ is convex for all $x > 0$ and*

$$\lim_{x\to 0+}\frac{d}{dx}V_{\mathrm{av}}^N(x) = -\frac{2}{N}.$$

Proof: Using (5) and (15) one finds

$$\frac{d}{dx}NV_{\rm av}^{N}(x)=\sum_{m=0}^{N-1}2x[V_m(x)-V_{m-1}]=2x[V_{N-1}-V_{-1}]=2xV_{N-1}-2.$$

Therefore, to show that $V_{\rm av}^{N}$ is convex, we need to show that

$$xV_{N-1}=\frac{1}{\Gamma(N)}\int_0^\infty\frac{u^{N-1}e^{-u}du}{(1+\frac{u}{x^2})^{\frac{1}{2}}}$$

is increasing. This holds as, for $x>0$, the function $h_u(x)=[1+u/x^2]^{-1/2}$ is increasing.

Similarly, one can show that for $p>1$, $V_{\rm av}^{p,N}$ is convex on $(0,\infty)$ and

$$\lim_{x\to 0+}\frac{d}{dx}V_{\rm av}^{p,N}(x)=-\frac{p}{N}.$$

For $p<1$, the derviative becomes infinite at the origin; however, concavity of $V_{\rm av}^{p,N}$ on $(0,\infty)$ still holds.

3.3 Polynomials defined by recursion

We now observe that by repeatedly using (23) to eliminate the V_m^p with the largest value of m from (24) allows us to write V_m^p in terms of the two "lowest" functions (e.g., V_0^p and V_{-1}^p in the case of integer m) and that the coefficients in such expressions define two classes of polynomials related to confluent hypergeometric functions. We discuss the properties of these polynomials in some detail. First, we make the statement above explicit.

Corollary 12 *For $m\geq 1$ there are polynomials $P_m^p(y)$ and $Q_m^p(y)$ of degree $\lfloor m\rfloor$ such that*

$$V_m^p(x) \;=\; P_m^p(x^p)V_{m-\lfloor m\rfloor}^p(x)+x^pQ_{m-1}^p(x^p)V_{m-\lfloor m\rfloor-1}^p(x). \tag{29}$$

In the case of integer m (29) becomes

$$\begin{aligned}V_m^p(x) &= P_m^p(x^p)V_0^p(x)+x^pQ_{m-1}^p(x^p)V_{-1}^p(x)\\ &= P_m^p(x^p)V_0^p(x)+xQ_{m-1}^p(x^p)\end{aligned} \tag{30}$$

where the second expression follows from our convention $V_{-1}^p(x)=x^{1-p}$. We define $P_m^p(y)=1$ for $m\in[0,1)$ and $Q_m^p(y)=0$ for $m\in[-1,0)$. Then (29) holds trivially for $m\in[0,1)$.

Proof: The desired polynomials are defined recursively. First let

$$P^p_m(y) \;=\; \frac{1}{m}\left(m-1+\frac{1}{p}-y\right) \quad \text{for} \quad m\in[1,2), \quad \text{and} \tag{31}$$

$$Q^p_m(y) \;=\; \frac{1}{m+1} \quad \text{for} \quad m\in[0,1). \tag{32}$$

Then (29) holds because it is equivalent to (23) for $m \in [1,2)$. For $m \geq 2$ define $P^p_m(y)$ by

$$P^p_m(y) = \frac{1}{m}\left[\left(m-1+\frac{1}{p}-y\right)P^p_{m-1}(y) + y\; P^p_{m-2}(y)\right] \tag{33}$$

and for $m \geq 1$ define $Q^p_m(y)$ by

$$Q^p_m(y) \;=\; \frac{1}{m+1}\left[\left(m+\frac{1}{p}-y\right)Q^p_{m-1}(y) + y\; Q^p_{m-2}(y)\right]. \tag{34}$$

It is straightforward to use induction to check that (23) yields (29). **QED**

We now restrict ourselves to $m \in \mathbf{N}$ and study these polynomials in more detail. The first few polynomials are given in the following Table.

m	P^p_m	Q^p_m
0	1	1
1	$\frac{1}{p}-y$	$\frac{1}{2}(1+\frac{1}{p}-y)$
2	$\frac{1}{2}\left[\left(y-\frac{1}{p}\right)^2+\frac{1}{p}\right]$	$\frac{1}{3}\left[y+\frac{1}{2}(1+\frac{1}{p}-y)(2+\frac{1}{p}-y)\right]$

The following useful results, which hold for $m \geq 1$, are easily checked by induction. $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ is the Beta function.

$$P^p_m(0) \;=\; \frac{\Gamma(m+\frac{1}{p})}{\Gamma(m+1)\Gamma(\frac{1}{p})} = \frac{1}{m\; B(m,\frac{1}{p})}$$

$$Q^p_m(0) \;=\; \frac{\Gamma(m+1+\frac{1}{p})}{\Gamma(m+2)\Gamma(\frac{1}{p})} = \frac{1}{(m+1)\; B(m+1,\frac{1}{p})}$$

$$P_m^p(y) = \frac{1}{m}\left[\frac{1}{p}\sum_{j=0}^{m-1} P_j^p(y) - y\; P_{m-1}^p(y)\right], \quad \text{and} \tag{35}$$

$$Q_m^p(y) = \frac{1}{m+1}\left[\frac{1}{p}\sum_{j=0}^{m-1} Q_j^p(y) - y\; Q_{m-1}^p(y) + 1]\right]. \tag{36}$$

We now obtain two expressions for $\frac{d}{dx}V_m^p(x)$. First, observe that using (30) in (19) yields

$$\frac{d}{dx}V_m^p(x) = px^{p-1}\left(\left[P_m^p(x^p) - P_{m-1}^p(x^p)\right]V_0^p(x) + x\left[Q_{m-1}^p(x^p) - Q_{m-2}^p(x^p)\right]\right).$$

Differentiating (30) yields after some simplifications

$$\frac{d}{dx}V_m^p(x) = px^{p-1}\Big[\left(P_m^p\right)'(x^p)V_0^p(x) + P_m^p(x^p)[V_0^p(x) - V_{-1}^p(x)] + x^p\left(Q_{m-1}^p\right)'(x^p)V_{-1}^p(x) + \tfrac{1}{p}Q_{m-1}^p(x^p)V_{-1}^p(x)\Big]$$

where $\left(P_m^p\right)'(y)$ denotes $\frac{d}{dy}P_m^p(y)$. Equating these expressions yields

$$-\left[\left(P_m^p\right)'(x^p) + P_{m-1}^p(x^p)\right]V_0^p(x) = \left[x^p\left(Q_{m-1}^p\right)'(x^p) - P_m^p(x^p) + (\tfrac{1}{p} - x^p)Q_{m-1}^p(x^p) + x^pQ_{m-2}^p(x^p)\right]V_{-1}^p(x) \tag{37}$$

This provides motivation for the following

Lemma 13 *For $m \in \mathbf{N}$, $m \geq 1$,*

$$\frac{d}{dy}P_m^p(y) = -P_{m-1}^p(y), \quad \textit{and} \tag{38}$$

$$y\;\frac{d}{dy}Q_{m-1}^p(y) = P_m^p(y) - (m+1)Q_m^p(y) + mQ_{m-1}^p(y) \tag{39}$$

Proof: We first prove (38) by induction. It can be verified for $m = 1, 2$ using the Table above. Then using (33), we find

$$\begin{aligned}
&m\frac{d}{dy}P_m^p(y)\\
&= \left(m - 1 + \tfrac{1}{p} - y\right)\frac{d}{dy}P_{m-1}^p(y) - P_{m-1}^p(y) + y\frac{d}{dy}P_{m-2}^p(y) + P_{m-2}^p(y)\\
&= -P_{m-1}^p(y) - \left(m - 1 + \tfrac{1}{p} - y\right)P_{m-2}^p(y) - y\;P_{m-3}^p(y) + P_{m-2}^p(y)\\
&= -P_{m-1}^p(y) - \left(m - 2 + \tfrac{1}{p} - y\right)P_{m-2}^p(y) - y\;P_{m-3}^p(y)\\
&= -mP_{m-1}^p(y).
\end{aligned}$$

This implies that the coefficient of V_0^p in (37) is identically zero. Therefore the coefficient of V_{-1}^p must also be identically zero. Substituting $y = x^p$ gives

$$\begin{aligned} y\,\frac{d}{dy}Q^p_{m-1}(y) &= P^p_m(y) + \left(y - \frac{1}{p}\right) Q^p_{m-1}(y) - yQ^p_{m-2}(y) \\ &= P^p_m(y) - (m+1)Q^p_m(y) + mQ^p_{m-1}(y) \end{aligned}$$

where we used (34). **QED**

Note that since the left side of (39) is a polynomial of degree $m-1$, this implies the coefficients of the y^m terms in P^p_m and $(m+1)Q^p_m(y)$ are identical. In fact, one can use (33) and (34) to see that the leading terms of P^p_m is $(-1)^m y^m/m!$ and that of Q^p_m is $(-1)^m y^m/(m+1)!$

A set of polynomials $\{p_n(x)\}$belongs to the class known as Appell polynomials [4] if they satisfy $\frac{d}{dx}p_n(x) = p_{n-1}(x)$. Therefore, (38) implies that for each fixed p, the set $(-)^m P^p_m(y)$ forms a family of Appell polynomials.

One can use (38) in (33) to replace P^p_{m-1} and P^p_{m-2} by derivatives of P^p_m and obtain a second order differential equation satisfied by P^p_m. This allows us to obtain a relationship between the polynomials P^p_m and confluent hypergeometric functions, which we denote ${}_1F_1(\alpha, \gamma, y)$.

Theorem 14 *For $m \in \mathbf{N}$, $m \geq 1$, $P^p_m(y)$ satisfies the differential equation*

$$y\phi''(y) - \left(m - 1 + \frac{1}{p} - y\right)\phi'(y) - m\phi(y) = 0. \tag{40}$$

Standard techniques show that (40) has a polynomial solution of the form $\phi(y) = \sum_{k=0}^m b_k y^k$ with $b_k = -\frac{m+1-k}{k(m+\frac{1}{p}-k)}b_{k-1}$, $k \geq 1$, $b_0 \neq 0$ arbitrary, and a second solution of the form $\phi(y) = \sum_{k=0}^\infty c_k y^{k+m+1/p}$ with $c_k = -\frac{k-1+1/p}{k(m+\frac{1}{p}+k)}c_{k-1}$, $k \geq 1$, $c_0 \neq 0$ arbitrary. Since $P^p_m(0) = \frac{1}{m\ B(m,\frac{1}{p})}$, we conclude that $b_k = \frac{(-1)^k}{k!\ (m-k)B(m-k,\frac{1}{p})}$ and

$$P^p_m(y) = \sum_{k=0}^m \frac{(-1)^k \Gamma(m+\frac{1}{p}-k)}{\Gamma(k+1)\Gamma(m+1-k)\Gamma(\frac{1}{p})}\ y^k.$$

The restriction that $1/p$ be non-integer in the next result is neither serious, nor unexpected, in view of Proposition 3.

Corollary 15 *Let $p \neq \frac{1}{n}$ for $n \in \mathbf{N}$. Then*

$$P_m^p(y) = \frac{1}{m\ B(m, \frac{1}{p})}\ e^{-y}\ {}_1F_1\left(1 - \tfrac{1}{p}, 1 - \tfrac{1}{p} - m, y\right) \tag{41}$$

Proof: Write $\phi(y) = e^{-y}\widehat{\phi}(y)$. Then it follows from (40) that $\widehat{\phi}$ satisfies

$$y\widehat{\phi}''(y) - \left(m - 1 + \frac{1}{p} + y\right)\widehat{\phi}'(y) - \left(1 - \frac{1}{p}\right)\widehat{\phi}(y) = 0. \tag{42}$$

which has the form of the differential equation satisfied by the confluent hypergeometric function. Comparing the behavior of $P_m^p(y)$ near $y = 0$ with that of the well-known solutions to (42) suffices to complete the proof. **QED**

It is well-known [3] that for real α and γ, ${}_1F_1(\alpha, \gamma, y)$ has at most finitely many zeros on the real line. Hence, the same holds for P_m^p. In fact, we can show that P_m^p has no zeros when m is even and exactly one when m is odd.

To show this, it is convenient to introduce the new variable $z = \frac{1}{p} - y$ and write $P_m^p(y) = \tilde{P}_m^p\left(\frac{1}{p} - y\right)$. The first few of these polynomials are $\tilde{P}_0^p(z) = 1$, $\tilde{P}_1^p(z) = z$, and $\tilde{P}_2^p(z) = \frac{1}{2}\left(z^2 + \frac{1}{p}\right)$.

Lemma 16 *For $m \in \mathbf{N}$, $m \geq 2$, the polynomials $\tilde{P}_m^p(z)$ satisfy*

(i) $\tilde{P}_m^p(z) = \dfrac{1}{m}\left[(m - 1 + z)\tilde{P}_{m-1}^p(z) + \left(\dfrac{1}{p} - z\right)\tilde{P}_{m-2}^p(z)\right]$,

(ii) $\tilde{P}_m^p(z) = \dfrac{1}{m}\left[\dfrac{1}{p}\displaystyle\sum_{j=0}^{m-2}\tilde{P}_j^p(z) + z\tilde{P}_{m-1}^p(z)\right]$, *and*

(iii) $\frac{d}{dz}\tilde{P}_m^p(z) = \tilde{P}_{m-1}^p(z)$.

Proof: Follows immediately from substitution in (33), (35) and (38).

Corollary 17 *For $m \in \mathbf{N}$, $m \geq 0$, all coefficients in the polynomials $\tilde{P}_m^p(z)$ are positive.*

Proof: This follows immediately from the explicit expressions above for $\tilde{P}_m^p$ when $m = 0, 1$ and part (ii) of Lemma 16.

Proposition 18 *Let $m \in \mathbf{N}$, $m \geq 1$*

If m is even, $\tilde{P}^p_m(z) \geq 0$.

If m is odd, $\tilde{P}^p_m(z)$ has exactly one root z_m.

Moreover, the roots form a strictly decreasing sequence with $-m+1 \leq z_m \leq 0$.

Proof: First note that Corollary 17 implies that $\tilde{P}^p_m(z) \geq 0$ for all $z \geq 0$ and $\lim_{z\to\infty} \tilde{P}^p_m(z) = \infty$. Now we claim that

$$\lim_{z\to-\infty} \tilde{P}^p_m(z) = \begin{cases} -\infty & \text{if m is odd} \\ \infty & \text{if m is even} \end{cases}$$

This can easily be verified by induction using part(i) of Lemma 16 (i) above.

For m odd, $\tilde{P}^p_m(z)$ has at least one root z_m. We now prove by induction that z_m is the only root if m is odd and that $\tilde{P}^p_m(z) \geq 0$ for all z if m is even. The induction hypothesis is easily seen to hold for $m = 0, 1$. Suppose it is true up to $m-1$ and consider $\tilde{P}^p_m$, with m odd. Then, by Lemma 16 (iii), $\frac{d}{dz}\tilde{P}^p_m(z) = \tilde{P}^p_{m-1}(z)$. Since $m-1$ is even, by the induction hypothesis $\tilde{P}^p_{m-1}(z) \geq 0$. Thus $\tilde{P}^p_m(z)$ is increasing for all $z \in \mathbf{R}$, which implies that for m odd $\tilde{P}^p_m(z)$ has only one root. Since $\tilde{P}^p_m(z) > 0$ when $z > 0$ that one root must satisfy $z_m \leq 0$.

If m is even, then $\frac{d}{dz}\tilde{P}^p_m(z) = \tilde{P}^p_{m-1}(z)$ which, by the induction hypothesis, has exactly one root $z_{m-1} \leq 0$. Therefore $\tilde{P}^p_m(z)$ has a local extremum at z_{m-1}. Since $\frac{d^2}{(dz)^2}\tilde{P}^p_m(z) = \tilde{P}^p_{m-2}(z)$ and $m-2$ is even, $\tilde{P}^p_{m-2}(z) \geq 0$ by the induction hypothesis and z_{m-1} is a local minimum for $\tilde{P}^p_m(z)$. By Lemma 16 (i) we have

$$m\tilde{P}^p_m(z_{m-1}) = \left(\tfrac{1}{p} - z_{m-1}\right)\tilde{P}^p_{m-2}(z_{m-1})$$

since $\tilde{P}^p_{m-1}(z_{m-1}) = 0$. But by the induction hypothesis $z_{m-1} \leq 0$ and $\tilde{P}^p_{m-2}(z_{m-1}) > 0$ so that $\tilde{P}^p_m(z_{m-1}) > 0$ as required.

It remains to be shown that the roots are decreasing and bounded below by $-(m-1)$. Both can be easily checked for $m = 1, 3$ and then proved by induction using Lemma 16 (i). We now let m be odd. Since $\tilde{P}^p_m(z)$ is increasing to show that $z_m > -m+1$, it suffices to show that $\tilde{P}^p_m(-m+1) < 0$. For $z = -m+1$ the recursion relation reduces to

$$m\tilde{P}^p_m(-m+1) = \left(\tfrac{1}{p} + m - 1\right)\tilde{P}^p_{m-2}(-m+1)$$

which is negative by the induction assumption hypothesis that $z_{m-2} \geq -m+3$. To show that $z_m < z_{m-2}$ it suffices to show that $\tilde{P}^p_m(z_{m-2}) > 0$. But

$$m\tilde{P}^p_m(z_{m-2}) = (m-1+z_{m-2})P^p_{m-1}(z_{m-2}) \geq 0$$

since $P^p_{m-2}(z_{m-2}) = 0$, $z_{m-2} > -m+3 > -m+1$, and $P^p_{m-1}(z)$ is positive.
QED

We now restate the results above in terms of the behavior of the original polynomials $P^p_m(y)$.

Corollary 19 *Let $m \in \mathbf{N}$, $m \geq 1$. Then*

(i) *If m is even, then $P^p_m(y) \geq 0$ for all $y \in \mathbf{R}$.*

(ii) *If m is odd, then $P^p_m(y)$ has exactly one root $y_m \geq 0$.*

(iii) *For all $m \geq 1$,*

$$\begin{aligned}
\lim_{y\to-\infty} P^p_m(y) &= \infty \\
\lim_{y\to\infty} P^p_m(y) &= \begin{cases} -\infty & \text{if } m \text{ is odd} \\ \infty & \text{if } m \text{ is even} \end{cases}
\end{aligned}$$

4 Inequalities and Convexity

4.1 Inequalities for $V_0(x)$

We first illustrate our strategy by proving a special class of inequalities for V_0. The convexity of $1/V_0(x)$ follows directly from the optimal upper bound in this class as given in Theorem 20 below. Although, as discussed at the end of this section, these inequalities generalize to V_m the resulting upper bound is not sufficient to establish the convexity of $1/V_m$. For this we need a bound on the ration $V_m(x)/V_{m-1}(x)$. Nevertheless, these simple inequalities for V_0, which can also be interpreted as ratio bounds, are of some interest in their own right in a variety of applications. Because the geometric strategy is also used in our more complex proofs of ratio bounds, we think there is some merit in presenting it first here.

We now define

$$g_k(x) = \frac{k}{(k-1)x + \sqrt{x^2+k}}. \tag{43}$$

Theorem 20 *For $x \geq 0$*

$$g_\pi(x) \leq V_0(x) < g_4(x) \tag{44}$$

and these inequalities are optimal for functions of the form (43) with equality only at $g_\pi(0) = V_0(0) = \sqrt{\pi}$.

Proof: It is easy to see that the family of functions $g_k(x)$ is increasing in k and that $0 < g_k(x) < 1/x$. In order to prove that the upper bound is optimal, we first observe that $g_k'(x) = -k[g_k(x)]^2[x + (k-1)\sqrt{x^2+k}\,]/\sqrt{x^2+k}$ and $xg_k(x) - 1 = -kg_k(x)/[x + \sqrt{x^2+k}\,]$. Then one can verify that

$$\begin{aligned} & g_k'(x) > 2[xg_k(x) - 1] \\ & \qquad \Longleftrightarrow \quad \frac{k}{\sqrt{x^2+k}} \, \frac{(k-1)\sqrt{x^2+k} + x}{(k-1)x + \sqrt{x^2+k}} < \frac{2k}{x + \sqrt{x^2+k}} \\ & \qquad \Longleftrightarrow \quad (k-2)x^2 + k(k-3) < (k-2)x\sqrt{x^2+k} \\ & \qquad \Longleftrightarrow \quad x^2(k-2)(k-4) + k(k-3)^2 < 0 \end{aligned} \tag{45}$$

when $k > 3$. We now restrict attention to $3 \le k \le 4$ and let $h_k(x) = g_k(x) - V_0(x)$. For $k = 4$, the expression (45) implies $g_4'(x) < 2[xg_k(x) - 1]$ so that $h_4'(x) < 2xh_4(x)$ for all $x \ge 0$; whereas for $k < 4$ this holds only for $x < a_k = \sqrt{\frac{k(k-3)^2}{(k-2)(4-k)}}$. Since both $V_0(x)$ and $g_k(x)$ are positive and bounded above by $1/x$, their difference also satisfies $|h_k(x)| < 1/x \to 0$.

For $k = 4$, if $h_4(x) \le 0$ for some $x > 0$, then $h_4'(x) < 2xh_4(x)$ is negative and thus h_4 is negative and strictly decreasing from a certain x on, which contradicts $\lim_{x\to\infty} h_4(x) = 0$. Thus $h_4(x) > 0$ so that $g_4(x) > V_0(x)$, for all x. Now suppose that for some $k < 4$, g_k is an upper bound, i.e. $h_k(x) \ge 0$ for all $x \ge 0$. In particular, $h_k(x) \ge 0$ for all $x > a_k$. For $k < 4$, we find however that $h_k'(x) > 2xh_k(x)$ holds for $x > a_k$. Thus we get $h_k(x) \ge 0$ and strictly increasing for all $x > a_k$ which contradicts $\lim_{x\to\infty} h_k(x) = 0$. Thus the upper bound can *not* hold when $x > a_k$ and $k < 4$. The lower bound also fails for $k > \pi$ since then $h_k(0) = g_k(0) - V_0(0) = \sqrt{k} - \sqrt{\pi} > 0$.

To establish the improved lower bound $g_\pi \le V_0(x)$ we note that the argument above implies that $h_k(x)$ is negative for $x > a_k$ and $3 < k \le \pi$. However for $k < \pi$ we have $h_k(0) < 0$ so that $h_k(x)$ is also negative for very small x. If $h_k(x)$ is ever non-negative, we can let b denote the first place $h_k(x)$ touches or crosses the x-axis, i.e., $h_k(b) = 0$ and $h_k(x) < 0$ for $x < b$. Then h_k must be increasing on some interval of the form (x_0, b). However, by the remarks above, $h_k(b) = 0$ implies $b \le a_k$ so that $h_k'(x) < 2xh_k(x) < 0$ on (x_0, b). Since this contradicts h_k increasing on (x_0, b), we must have $h_k(x) < 0$ for all $x \ge 0$ if $k < \pi$.

Thus we have proved the lower bound $g_k(x) < V_0(x)$ on $[0, \infty)$ for $k < \pi$. Since g_k is continuous and increasing in k, it follows that $g_\pi(x) \le V_0(x)$. To show that this inequality is strict except at $x = 0$, note that the right derivative of h_k at 0 satisfies $h_k'(0) = 2 - k$ so that $h_\pi'(0) < 0$ and $h_\pi(x)$

is negative at least on some small interval $(0, x_1)$. Then we can repeat the argument above to show that $h_\pi(x) < 0$ if $x > 0$. **QED**

As discussed in [7, 16, 17] the upper bound implies the convexity of $1/V_0(x)$ on $(0, \infty)$; in fact, it is not hard to use the fact that (15) reduces to $\frac{d}{dx}V_0(x) = 2(xV_0 - 1)$ to see that the upper bound is equivalent to convexity. It was established independently by Wirth [17] and by Szarek and Werner [16]. (The latter actually proved slightly more by using (1) to define an asymmetric extension of $V_0(x)$ to negative x. They showed in [16] that this extension is convex for $x > -\frac{1}{\sqrt{2}}$.)

Both bounds in (44) are sharper than the inequalities of Komatsu [9, 14]. The weaker lower bound $g_3(x) < V_0(x)$ was used in [7] to show that the function $\left[1/V_0(x) - x\right]^2 / V_0(x)$ is decreasing for $x \geq 0$. The lower bound $g_\pi(x) \leq V_0(x)$ was established earlier by Boyd [5] as the optimal bound in a different class of inequalities. There is an extensive literature (see. e.g. [14]) on bounds for $V_0(x)$; however, the class of inequalities obtained using functions of the form $g_k(x)$ does not seem to have been considered before so that the optimality of bounds of this type for $k = \pi$ and $k = 4$ seems new.

Mascioni [12] generalized the upper bound to $p \geq 2$ for which he showed

$$V_0^p(x) < \frac{4p}{3px^{p-1} + \sqrt{p^2 x^{2p-2} + 8p(p-1)x^{p-2}}}$$

and also showed that this implies convexity of $1/V_0^p(x)$ for $p \geq 2$.

In view of Property (a) of Section 1.2, it would seem natural to try to generalize (20) using functions of the form

$$g_k^m(x) = \frac{k}{(k-1)x + \sqrt{x^2 + m + k}}. \tag{46}$$

Note that the functions g_k^m are increasing in k and that $\lim_{k\to\infty} g_k^m(x) = \frac{1}{x}$. Therefore Property (a) implies that

$$g_1^m(x) \leq V_m(x) < \lim_{k\to\infty} g_k^m(x).$$

As g_k^m is continuous in k, there must exist i_m and j_m such that

$$g_{i_m}^m(x) \leq V_m(x) < g_{j_m}^m(x). \tag{47}$$

However, we have not obtained explicit expressions for i_m and j_m. One might expect that the optimal lower bound occurs when i_m is chosen to satisfy $g_{i_m}^m(0) = V_m(0)$. However, numerical evidence shows that this is false; in fact, this choice for i_m does not even yield an inequality.

4.2 Ratio Bounds

One of our main goals is to show that the function $\frac{1}{V_m(x)}$ is convex for integer $m \geq 1$. The key to this is the realization that (44) can also be rewritten to give bounds on the ratio $V_0(x)/V_{-1}(x) = xV_0(x)$. We now let

$$G_k^m(y) = \frac{ky}{(k-1)y - m + \sqrt{(y+m)^2 + ky}} \tag{48}$$

and note that $xg_k(x) = G_k^0(x^2)$ so that (44) is equivalent to

$$G_\pi^0(x^2) \leq xV_0(x) = \frac{V_0(x)}{V_{-1}(x)} < G_4^0(x^2).$$

For integer $m > 0$, convexity of $\frac{1}{V_m(x)}$ can be shown to be equivalent to

$$R_m(x) \equiv \frac{V_m(x)}{V_{m-1}(x)} < G_4^m(x^2).$$

In addition to this upper bound, we can show the following

Theorem 21 *Let $m \in \mathbf{N}$, $m \geq 0$. Then the inequalities*

$$G_8^{m-1}(x^2) < R_m(x) < G_4^m(x^2) \tag{49}$$

hold and are optimal in k for the class of functions of the form $G_k^m(x^2)$.

The upper bound is optimal in k for all m. The lower bound is optimal in the sense that 8 is the largest integer for which the lower bound in (49) holds for all m. However, as we discuss at the end of Section 5.3, for fixed m one can find $k(m)$ such that $G_{k(m)}^{m-1}(x^2) < R_m(x)$ holds with $k(m) > 8$.

Since $G_k^m(y)$ is increasing in both m and k, its behavior at zero and infinity allows us to also draw some conclusions about the optimality in m of (49). $R_m(0) = 1 - \frac{1}{2m}$ and $G_k^m(0) = 1 - \frac{1}{1+2m}$ for all k. Therefore, $G_k^\nu(0) < R_m(0) < G_{k'}^\mu(0)$ implies $\nu \leq m - \frac{1}{2}$ and $\mu \geq m - \frac{1}{2}$ for all k, k'. Thus, if we insist that m be integer, there is no choice of k which allows $m-1$ to be replaced by m in the lower bound when $m > 0$, or m by $m-1$ in the upper bound. This argument does not, however, rule out the possibility of bounds of the form $G_k^{m-\frac{1}{2}}(x^2) < R_m(x) < G_{k'}^{m-\frac{1}{2}}(x^2)$.

To examine the behavior at infinity, note that

$$\begin{aligned} G_k^m(y) &= 1 - \frac{1}{2y} + \frac{4m+k+2}{8y^2} + O\left(\frac{1}{y^3}\right), \quad \text{and} \\ R_m(\sqrt{y}) &= 1 - \frac{1}{2y} + \frac{4m+6}{8y^2} + O\left(\frac{1}{y^3}\right) \end{aligned}$$

where the asymptotic expansion for R_m follows from Proposition 7. It then follows that $R_m(\sqrt{y}) < G_{k'}^{\mu}(y)$ implies $\mu > m + 1 - \frac{k}{4}$. Thus m is optimal for the upper bound if $k \leq 4$ and any attempt to decrease m would require an increase in k. Furthermore, $\mu = m$ implies $k \geq 4$ so that the upper bound in (49) is optimal in k.

We postpone the proof of Theorem 21, which requires a lengthy computation even for the case $p = 2$, to the next section. Our proof uses induction on m. Therefore, we are able to establish (49) and the theorems in the next section only for m a positive integer. We believe that they are also true for non-integer m. However, a proof would require either a different method or independent verification of the upper bound for an initial range, such as $-1 < m < 0$.

The ratio $R_m(x)$ is of interest in its own right, and our results are sufficient to establish that it is increasing in x on $(0, \infty)$. This is proved in the next section after Theorem 23, which uses a similar argument.

Theorem 22 *For $m \in \mathbf{N}$, the ratio $R_{m+1}(x) = \frac{V_{m+1}(x)}{V_m(x)}$ is increasing in x.*

4.3 Convexity of $1/V_m$

We now prove some important consequences of Theorem 21. The first is

Theorem 23 *For all $m \in \mathbf{N}$, the function $1/V_m(x)$ is convex on $[0, \infty)$.*

Proof: We need to show that

$$\left(\frac{1}{V_m(x)}\right)'' = \frac{2[V_m(x)']^2 - V_m(x)(V_m(x))''}{V_m(x)^3} > 0. \tag{50}$$

It follows from the differential equation (15) and the recursion relation (23) that

$$V_m(x)'' = 2\left[V_m(x)(1 + 2m + 2x^2) - 2V_{m-1}(x)(x^2 + m)\right]$$

so that (50) holds if and only if

$$[V_m(x)]^2(1+2m-2x^2)+2V_{m-1}(x)V_m(x)(3x^2-m)-4x^2[V_{m-1}(x)]^2\leq 0.$$

After division by $[V_{m-1}(x)]^2$ this can be rewritten as $P[R_m(x)]\leq 0$ where

$$P(z)=z^2(1+2m-2x^2)+2z(3x^2-m)-4x^2.$$

Writing the roots of $P(z)=Az+2Bz+C$ in the non-standard form $\frac{-C}{B\pm\sqrt{B^2-AC}}$, we find that $G_4^m(x^2)$ is either the smaller of two positive roots (when $x^2>m+\frac{1}{2}$) or the only positive root (when $x^2<m+\frac{1}{2}$). Since $P(0)<0$, in both cases, we can conclude that

$$z<G_4^m(x^2)\quad\text{implies}\quad P(z)<0.$$

Therefore, it follows from the upper bound in Theorem 21 that $P[R_m(x)]<0$, hence (50) holds. **QED**

Proof of Theorem 22: Using (15) one finds that

$$\frac{d}{dx}R_{m+1}(x)=2x\left[\frac{R_{m+1}(x)}{R_m(x)}-1\right].$$

After rewriting this in terms of V_m and then using the recursion relation (23) with $p=2$ to eliminate V_{m+1}, one finds that $R'_{m+1}(x)\geq 0$ if and only if

$$2(m+1)[R_m(x)]^2-(2m+1-2x^2)R_m(x)-2x^2\leq 0.$$

The polynomial $P(z)=2(m+1)z^2-(2m+1-2x^2)z-2x^2$ has one positive and one negative root, and $R'_{m+1}(x)\geq 0$ if and only if $R_m(x)$ lies between these two roots. Since $1\geq R_m(x)>0$, it follows that $R_{m+1}(x)$ is increasing if and only if $R_m(x)$ is less than the larger root, i.e.,

$$R_m(x)\leq\frac{4x^2}{\sqrt{4(x^2+m)^2+1+4m+12x^2}+2x^2-2m-1}.$$

where, we have again written the root in the non-standard form $\frac{C}{-B+\sqrt{B^2-AC}}$. Then using the upper bound of Theorem 21, we see that it suffices to show that

$$\begin{aligned}R_m(x)&\leq\frac{4x^2}{\sqrt{(x^2+m)^2+4x^2}+3x^2-m}\\&\leq\frac{4x^2}{\sqrt{4(x^2+m)^2+1+4m+12x^2}+2x^2-2m-1}\end{aligned}$$

or, equivalently that

$$\begin{aligned}&\sqrt{(x^2+m)^2+4x^2}+3x^2-m\\ &\qquad\geq\ \sqrt{4(x^2+m)^2+1+4m+12x^2}+2x^2-2m-1\end{aligned}$$

which is easily checked.

5 Proof of Ratio Bounds

The proofs in this section, although elementary, are quite long and tedious. The details were checked using Mathematica.

5.1 Differential inequality

In order to prove Theorem 21, it suffices to establish the following

Lemma 24 *Let G_k^m be given by (48). Then*

(i) *For $m \geq 1$,* $\quad \dfrac{d}{dx}G_4^m(x^2) \leq 2x(\dfrac{G_4^m(x^2)}{G_4^{m-1}(x^2)}-1).$

(ii) *For $m \geq 4$,* $\quad \dfrac{d}{dx}G_8^m(x^2) \geq 2x(\dfrac{G_8^m(x^2)}{G_8^{m-1}(x^2)}-1),$

but the inequality (ii) does not hold for $m < 4$.

Proof: For simplicity, we put $x^2 = y$ and assume $y \geq 0$. Then (i) is equivalent to

$$E_m(y) \equiv \left(\frac{G_4^m(y)}{G_4^{m-1}(y)}-1\right)-\frac{d}{dy}G_4^m(y) \geq 0. \tag{51}$$

Let $B_m = (m^2+y^2+4y+2my)^{\frac{1}{2}}$. Then

$$G_4^m(y) = \frac{4y}{B_m+3y-m}.$$

and

$$E_m(y) = \frac{B_m\,[4m+(B_m+3y-m)(B_{m-1}-B_m+1)]-(4m^2+8y+4my)}{B_m\ (B_m+3y-m)^2}.$$

Thus $E_m(y) \geq 0$ if and only if

$$\begin{aligned} & B_m\Big(3m + B_{m-1}[B_m + 3y - m]\Big) + m^3 + 2my \\ \geq\ & 3m^2 + 4y + 11y^2 + m^2y + 5my^2 + 3y^3 + B_m\Big((y+m)^2 + y\Big). \end{aligned}$$

We put

$$\begin{aligned} s &= s(y,m) = 3m^2 + 4y + 11y^2 + m^2y + 5my^2 + 3y^3, \\ t &= t(y,m) = 2my + m^3, \quad \text{and} \\ h &= h(y,m) = (y+m)^2 + y - 3m. \end{aligned}$$

Then $E_m(y) \geq 0$ if and only if

$$B_m B_{m-1}(B_m + 3y - m) \geq B_m h + s - t. \tag{52}$$

Notice that both sides of (52) are positive. For the left side this follows immediately from $B_m > m$. For the right side note that $B_m h(0) + s(0) - t(0) = 0$ and

$$\begin{aligned} & \frac{d}{dy}\left[B_m h(y) + s(y)t(y)\right] \\ =\ & \frac{1}{B_m}\left[6y + 12my + 9m^2y + 12y^2 + 9my^2 + 3y^3\right. \\ & \left. +\ B_m(22y + 10my + 9y^2) + 4B_m - 2mB_m + 3m^3 - 6m + m^2B_m\right]. \end{aligned}$$

Now observe that

$$\begin{aligned} & 4B_m - 2mB_m + 3m^3 - 6m + m^2B_m \\ & \qquad = 3m(m^2-2) + B_m(m^2 - 2m + 4) \geq 3m(m^2-1) \geq 0. \end{aligned}$$

since $m \geq 1$ and $B_m \geq m$. Hence $B_m h + s - t$ is increasing in y and the right side of (52) is also positive. Therefore we can square both sides of (52) to conclude that it is equivalent to

$$F(y) = B_m f_1(y) - f_2(y) \geq 0, \tag{53}$$

where

$$f_1(y) = (m+y)(y^3 + my^2 + 3y^2 - m^2y - 3my + 2y - m^3 + 2m^2)$$

and

$$f_2(y) = y^5 + 3my^4 + 5y^4 + 2m^2y^3 + 5my^3 + 2y^3 \\ - \; 2m^3y^2 - 3m^2y^2 - 3m^4y - m^3y + 6m^2y - m^5 + 2m^4.$$

Note that $F(0) = 0$. Therefore, to prove (53) it is enough to show that $\frac{d}{dy}F(y) = B_m f_1'(y) + \frac{f_1(y)(2+y+m)}{B_m} - f_2'(y) \geq 0$, or equivalently

$$D(y) \equiv d_1(y) - B_m d_2(y) \geq 0, \tag{54}$$

where

$$\begin{aligned} d_1(y) &= B_m^2 f_1'(y) + f_1(y)(2+y+m) \\ &= 6m^3 - m^4 - 3m^5 + 12my + 4m^2y - 13m^3y - 7m^4y + 20y^2 \\ &\quad + 14my^2 + 7m^2y^2 + 2m^3y^2 + 48y^3 + 49my^3 + 18m^2y^3 + 30y^4 \\ &\quad + 17my^4 + 5y^5, \quad \text{and} \\ d_2(y) &= f_2'(y) = 6m^2 - m^3 - 3m^4 - 6m^2y - \; 4m^3y \\ &\qquad + \; 6y^2 + 15my^2 + 6m^2y^2 + 20y^3 + 12my^3 + 5y^4. \end{aligned}$$

Note that $D(0) = 0$. Therefore, to prove (54) it is enough to show that $\frac{d}{dy}D(y) = d_1'(y) - B_m d_2'(y) - \frac{d_2(y)(2+y+m)}{B_m} \geq 0$, or equivalently,

$$G(y) \equiv B_m g_1(y) - g_2(y) \geq 0, \tag{55}$$

where

$$\begin{aligned} g_1(y) &= d_1'(y) \\ &= 12m + 4m^2 - 13m^3 - 7m^4 + 40y + 28my + 14m^2y + 4m^3y \\ &\qquad + \; 144y^2 + 147my^2 + 54m^2y^2 + 120y^3 + 68my^3 + 25y^4, \quad \text{and} \\ g_2(y) &= B_m^2 d_2'(y) + d_2(y)(2+y+m) \\ &= 12m^2 + 4m^3 - 13m^4 - 7m^5 - 18m^2y - 13m^3y - 3m^4y \\ &\quad + 60y^2 + 180my^2 + 183m^2y^2 + 58m^3y^2 + 298y^3 \\ &\quad + 353my^3 + 122m^2y^3 + 170y^4 + 93my^4 + 25y^5. \end{aligned}$$

Note that $G(0) = 0$. Therefore, to prove (55) it is enough to show that $\frac{d}{dy}G(y) = B_m g_1'(y) + \frac{g_1(y)(2+y+m)}{B_m} - g_2'(y) \geq 0$, or equivalently

$$H(y) = h_1(y) - B_m h_2(y) \geq 0, \tag{56}$$

where

$$\begin{aligned}
h_1(y) &= B_m^2 g_1'(y) + g_1(y)(2+y+m) \\
&= 24m + 60m^2 + 6m^3 - 13m^4 - 3m^5 + 240y + 300my + 460m^2y \\
&\quad + 347m^3y + 113m^4y + 1520y^2 + 2246my^2 + 1663m^2y^2 + 482m^3y^2 \\
&\quad + 2112y^3 + 2233my^3 + 738m^2y^3 + 930y^4 + 497my^4 + 125y^5, \quad \text{and} \\
h_2(y) &= g_2'(y) \\
&= -18m^2 - 13m^3 - 3m^4 + 120y + 360my + 366m^2y + 116m^3y \\
&\quad + 894y^2 + 1059my^2 + 366m^2y^2 + 680y^3 + 372my^3 + 125y^4.
\end{aligned}$$

Note that $H(0) = 12m(2 + 5m + 2m^2) > 0$. Therefore, to prove (56) it is enough to show that $\frac{d}{dy}H(y) = h_1'(y) - B_m h_2'(y) - \frac{h_2(y)(2+y+m)}{B_m} \geq 0$, or equivalently

$$l_1(y)B_m - l_2(y) \geq 0, \tag{57}$$

where

$$\begin{aligned}
l_1(y) &= h_1'(y) \\
&= 240 + 300m + 460m^2 + 347m^3 + 113m^4 + 3040y + 4492my \\
&\quad + 3326m^2y + 964m^3y + 6336y^2 + 6699my^2 + 2214m^2y^2 + \\
&\quad + 3720y^3 + 1988my^3 + 625y^4, \quad \text{and} \\
l_2(y) &= B_m^2 h_2'(y) + h_2(y)(2+y+m) \\
&= 84m^2 + 316m^3 + 347m^4 + 113m^5 + 720y + 2520my + 5046m^2y \\
&\quad + 3899m^3y + 1077m^4y + 9180y^2 + 15780my^2 + 11727m^2y^2 \\
&\quad + 3178m^3y^2 + 12202y^3 + 13145my^3 + 4202m^2y^3 + 4970y^4 \\
&\quad + 2613my^4 + 625y^5.
\end{aligned}$$

Note that $l_1(y) \geq 0$ and $l_2(y) \geq 0$ for all $y \geq 0$. Therefore (57) holds, if and only if $L(y) = B_m^2(l_1(y))^2 - (l_2(y))^2 \geq 0$, which follows immediately from the fact that all the coefficients are positive in

$$\begin{aligned}
L(y) &= 4(14400m^2 + 36000m^3 + 75936m^4 + 97368m^5 + 78972m^6 + 37188m^7 \\
&\quad + 8136m^8 + 57600y + 172800my + 717360m^2y + 1373400m^3y \\
&\quad + 1732428m^4y + 1360314m^5y + 599454m^6y + 111444m^7y + 1344000y^2 \\
&\quad + 3838560my^2 + 8437260m^2y^2 + 11062920m^3y^2 + 8495031m^4y^2 \\
&\quad + 3499083m^5y^2 + 595986m^6y^2 + 9342880y^3 + 24217360my^3
\end{aligned}$$

$$+ 32546720m^2y^3 + 24561680m^3y^3 + 9950080m^4y^3 + 1694280m^5y^3$$
$$+ 17918380y^4 + 37038224my^4 + 34271234m^2y^4 + 15627870m^3y^4$$
$$+ 2862630m^4y^4 + 15343236y^5 + 23700982my^5 + 13930330m^2y^5$$
$$+ 2982516m^3y^5 + 6445963y^6 + 6618363my^6 + 1887294m^2y^6$$
$$+ 1302640y^7 + 667056my^7 + 101250y^8).$$

To prove (ii) we proceed similarly, but now let $B_m(y) = \sqrt{(y+m)^2 + 8y}$ and $E_m(y) = \frac{d}{dy}G_8^m(y) - (\frac{G_8^m(y)}{G_8^{m-1}(y)} - 1)$, noting that

$$G_8^m(y) = \frac{8y}{B_m + 7y - m}.$$

We now need to show that $E_m(y) \geq 0$ for all $m \geq 4$. As the argument is similar to that above, we omit the details except to indicate the steps leading to the condition $m \geq 4$. Observe that $E_m(y) \geq 0$ if and only if

$$\begin{aligned} &B_m((y+m)^2 + y - 7m) \\ &\quad +7m^2 - m^3 + 24y - 2my + 5m^2y + 55y^2 + 13my^2 + 7y^3 \\ &\qquad \geq B_m B_{m-1}(7y - m + B_m). \end{aligned} \tag{58}$$

Again, both sides of the inequality are positive. Hence we can square both sides of the inequality and, as above get that (58) is equivalent to

$$F(y) = f_1(y) - B_m f_2(y) \geq 0, \tag{59}$$

with the appropriate f_1 and f_2. Again $F(0) = 0$. Therefore, in order to prove (59), it is enough to show that $\frac{d}{dy}F(y) \geq 0$, or equivalently, after rewriting,

$$D(y) = B_m d_1(y) - d_2(y) \geq 0, \tag{60}$$

with the appropriate d_1 and d_2. And again $D(0) = 0$. We repeat the procedure: To prove (60), it is enough to show that $\frac{d}{dy}D(y) \geq 0$, or equivalently,

$$e_1(y) - B_m e_2(y) \geq 0,$$

with the appropriate e_1 and e_2. e_1 and e_2 turn out to be both positive for $y \geq 0$. Therefore (60) holds if

$$L(y) = (e_1(y))^2 - (B_m e_2(y))^2 \geq 0.$$

$L(0) = 0$ and

$$L'(0) = 192m^2(m-4)(1+2m)(480 + 64m + 90m^2 + 33m^3).$$

Thus $L'(0) \geq 0$ if and only if $m \geq 4$. For all $m \geq 4$, $L''(y) \geq 0$ for all $y \geq 0$. This finishes (ii). **QED**

5.2 Proof of Theorem 21

We will prove by induction that $R_m(x) < G_4^m(x^2)$ for $m = 0, 1, 2, 3, \ldots$ As observed earlier, this inequality holds for $m = 0$, since it is then equivalent to the upper bound in (44). Let

$$H_m(x) = G_4^m(x^2) - R_m(x).$$

Then the upper bound in Theorem 21 is equivalent to $H_m(x) \geq 0$. This can be verified using the strategy of Section 4.1 if the following conditions hold

(i) $H_m(0) > 0$.

(ii) $\lim_{x\to\infty} H_m(x) = 0$.

(iii) $H'_m(x) \leq F_{\text{pos}}(x) H_m(x)$ for some strictly positive function $F_{\text{pos}}(x) > 0$.

Conditions (i) and (ii) hold. Indeed,

$$H_m(0) = \frac{2m}{1+2m} - \frac{\Gamma(m)\Gamma(m+\frac{1}{2})}{\Gamma(m+1)\Gamma(m-\frac{1}{2})} = \frac{1}{2m(1+2m)},$$

and

$$\lim_{x\to\infty} H_m(x) = 0,$$

since $\lim_{x\to\infty} R_m(x) = 1$, and for all $k \geq 1$, $\lim_{x\to\infty} G_k^m(x) = 1$.

We check now condition (iii). It follows from Lemma 24 (i) and (51) that

$$\begin{aligned}
H'_m(x) &\leq 2x \left[\frac{G_4^m(x^2)}{G_4^{m-1}(x^2)} - \frac{R_m(x)}{R_{m-1}(x)} \right] \\
&= \frac{2x}{G_4^{m-1}(x^2) R_{m-1}(x)} \left[G_4^m(x^2) R_{m-1}(x) - G_4^{m-1}(x^2) R_m(x) \right] \\
&\leq \frac{2x}{R^{m-1}(x)} \left[G_4^m(x^2) - R_m(x) \right] \\
&= \frac{2x}{R^{m-1}(x)} H_m(x)
\end{aligned}$$

where the inequality follows from the induction hypothesis $R_{m-1}(x) < G_4^{m-1}(x^2)$. Thus (iii) holds with $F_{\text{pos}}(x) = 2x/R_{m-1}(x)$.

For $m \geq 4$, the lower bound is proved similarly. One considers $H_m(x) = R_m(x) - G_8^m(x^2)$ instead and uses Lemma 24(ii). The cases R_1, R_2 and R_3

have to be verified directly as Lemma 24 (ii) only covers the cases R_m for $m \geq 4$.

Using (23), $R_1 \geq G_8^0$ is equivalent to showing that

$$\frac{x}{V_0(x)} \geq \frac{9x + 14x^3 + (2x^2 - 1)\sqrt{8 + x^2}}{2(7x + \sqrt{8 + x^2})}.$$

As V_0 is always positive, this inequality holds trivially for those x, for which the right hand side is negative or zero. Therefore we only need to prove the inequality on the interval $[x_0, \infty)$, $x_0 \simeq 0.2511$, where

$$9x + 14x^3 + (2x^2 - 1)\sqrt{8 + x^2} \geq 0.$$

Hence we need to show that for all $x \in [x_0, \infty)$,

$$\begin{aligned} V_0(x) &\leq 2x \frac{7x + \sqrt{8 + x^2}}{9x + 14x^3 + (2x^2 - 1)\sqrt{8 + x^2}} \\ &= 2x \frac{6x^2 - 1}{1 + 6x^2 + 12x^4 - 2x\sqrt{8 + x^2}}. \end{aligned} \tag{61}$$

Put $h_1(x) = 2x \frac{6x^2-1}{1+6x^2+12x^4-2x\sqrt{8+x^2}}$. By Theorem 20 of Section 4.1, inequality (61) is true for all $x \in [x_0, \infty)$, for which

$$g_4(x) \leq h_1(x).$$

This last inequality holds only on an interval $[x_0, x_1]$, $x_1 \simeq 1.399$. For all $x \geq x_1$, we show that

$$h_1(x) < \frac{1}{x}$$

and

$$h_1' \leq 2(xh_1 - 1).$$

Then (61) follows as in Section 4.1.

Next, we find that $R_2 \geq G_8^1$ is equivalent to $V_0(x) \geq h_2(x)$, where

$$h_2(x) = 2x \frac{3 + 9x^2 + 14x^4 + (2x^2 - 3)(8x^2 + (1 + x^2)^2))^{\frac{1}{2}}}{-3 - 7x^2 + 32x^4 + 28x^6 + (3 - 4x^2 + 4x^4)(8x^2 + (1 + x^2)^2))^{\frac{1}{2}}}$$

and $R_3 \geq G_8^2$ is equivalent to $V_0(x) \leq h_3(x)$ where

$$h_3(x) = \frac{2x}{N(x)} \left[-30 - 23x^2 + 32x^4 + 28x^6 + \sqrt{8x^2 + (2 + x^2)^2}\,(15 - 8x^2 + 4x^4) \right],$$

with

$$N(x) = 30 + 3x^2 - 42x^4 + 92x^6 + 56x^8$$
$$+(8x^2 + (2 + x^2)^2))^{\frac{1}{2}}(-15 + 18x^2 - 12x^4 + 8x^6).$$

Again, we have to check these inequalities for V_0 only for those x, for which the right hand sides are positive. We then proceed as for R_1 and show that $g_\pi \geq h_2$ up to a certain x_2 and that $h_2 < \frac{1}{x}$, $h_2' \geq 2(xh_1 - 1)$ on $[x_2, \infty)$. Similarly, we show that $g_4 \leq h_3$ up to a certain x_3 and that $h_3 < \frac{1}{x}$, $h_3' \leq 2(xh_1 - 1)$ on $[x_3, \infty)$. **QED**

Note that these arguments also show that on the interval $[x_1, \infty)$ the function h_1 is a better upper bound for V_0 than g_4; on $[x_2, \infty)$ the function h_2 is a better lower bound for V_0 than g_π; and on $[x_3, \infty)$ the function h_3 is a better upper bound for V_0 than g_4. In fact, $h_3 \leq h_1 \leq g_4$ for $x > x_3$.

5.3 Optimality of bounds

We still need to consider optimality of the lower bound in upper bound in (49 in the parameter k. We continue the strategy above using similar notation so that now $B_m = \sqrt{(y+m)^2 + ky}$ and $E_m(y) = \left[\frac{G_k^m(y)}{G_k^{m-1}(y)} - 1\right] - \frac{d}{dy}G_k^m(y)$ with

$$G_k^m(y) = \frac{ky}{B_m + (k-1)y - m}.$$

Then $E_m(y) \leq 0$ if and only if,

$$\begin{aligned} &2B_m B_{m-1}(B_m + (k-1)y - m) \\ &\quad \leq \ 2B_m(y + (y+m)^2 - (k-1)m) + P, \end{aligned} \tag{62}$$

where

$$\begin{aligned} P = &-2m^2 + 2km^2 - 2m^3 - 2ky + k^2y - 4my - 6m^2y + 2km^2y \\ &- \ 2y^2 - 2ky^2 + 2k^2y^2 - 6my^2 + 4kmy^2 - 2y^3 + 2ky^3. \end{aligned}$$

For $m \geq 1$ and $k \geq 2$ both sides of the inequality are positive. Therefore we can square both sides and get that $E_m(y) \leq 0$ if and only if

$$F(y) = f_1(y) - B_m f_2(y) \geq 0,$$

where

$$\begin{aligned}
f_1(y) &= -16km^4 + 8k^2m^4 + 32m^5 - 16km^5 - 20k^2m^2y + 8k^3m^2y + 8km^3y \\
&- 4k^2m^3y + 160m^4y - 96km^4y + 8k^2m^4y - 4k^3y^2 + k^4y^2 \\
&+ 120km^2y^2 - 84k^2m^2y^2 + 12k^3m^2y^2 + 320m^3y^2 - 224km^3y^2 \\
&+ 32k^2m^3y^2 + 20k^2y^3 - 20k^3y^3 + 4k^4y^3 + 152kmy^3 - 124k^2my^3 \\
&+ 24k^3my^3 + 320m^2y^3 - 256km^2y^3 + 48k^2m^2y^3 + 56ky^4 - 52k^2y^4 \\
&+ 12k^3y^4 + 160my^4 - 144kmy^4 + 32k^2my^4 + 32y^5 - 32ky^5 \\
&+ 8k^2y^5 \qquad \text{and} \\
f_2(y) &= 4\left(-4km^3 + 2k^2m^3 + 8m^4 - 4km^4 - 3k^2my + k^3my + 2km^2y\right. \\
&- k^2m^2y + 32m^3y - 20km^3y + 2k^2m^3y + k^2y^2 + 16kmy^2 - 12k^2my^2 \\
&+ 2k^3my^2 + 48m^2y^2 - 36km^2y^2 + 6k^2m^2y^2 + 10ky^3 - 9k^2y^3 + 2k^3y^3 \\
&+ \left. 32my^3 - 28kmy^3 + 6k^2my^3 + 8y^4 - 8ky^4 + 2k^2y^4\right).
\end{aligned}$$

Then $F(0) = 0$ and in order that $F \geq 0$, we have to have at least that $\frac{d}{dy}F(0) \geq 0$. Computing $\frac{d}{dy}F(y)$, we find that $\frac{d}{dy}F(0) = 0$. We apply the same procedure as in the proof of Lemma 24, compute the successive derivatives and evaluate them at 0. Evaluating the derivative at 0, in the fourth step of the procedure gives the value

$$24k^3m(1 + 2m)(km - 6m - k).$$

Therefore, in order that (62) (which is the condition for the lower bound) holds for all $m \geq 2$, we have to have at least that $k \geq \lim_{m\to\infty} \frac{6m}{m-1} = 6$. Thus for $m = 2$, $k \geq 12$ will do, for $m = 3$, $k \geq 9$, for $m = 4$, $k \geq 8$ and so fourth. Therefore, as G_k^m is increasing in k, it seems a natural choice to pick $k = 12$ or bigger for the lower bound. And indeed, one can check that G_k^m satisfies the lower bound condition of Lemma 24 for $k \geq 12$ and $m \geq 2$. However, it is not true that for $k > 8$, G_k^{m-1} is a lower bound for R_m, for *all* $m \geq 1$. It is a lower bound for all $m \geq m(k)$, from a certain $m(k)$ on. Thus the induction in the proof of Theorem 21 cannot start at $m = 0$ or $m = 1$. For $m < m(k)$, there exists x_m such that $R_m - G_k^{m-1} \geq 0$ on $[0, x_m]$ and $R_m - G_k^{m-1} < 0$ on (x_m, ∞).

5.4 Extensions to general p

For $p = 1$, all the functions involved are identically equal to 1 and hence trivially convex. For large x, $1/V_m^p(x) \approx x^{p-1}$ and x^{p-1} is concave for $1 <$

$p < 2$. Hence we can not expect convexity of $1/V_m^p$ on $(0,\infty)$ for p is in $(1,2)$. It was shown in [12] that $\frac{1}{V_0^p}$ is not convex on $\mathbf{R}^+$ for $0 < p < 1$. Therefore, we study only generalizations to $p > 2$. Our method of proof yields verification of the convexity of $\frac{1}{V_m^p}$ for all $m \geq 1$ up to at least $p = 4$. However this method breaks down for larger p.

We generalize our previous notation to $R_m^p(x) = \frac{V_m^p(x)}{V_{m-1}^p(x)}$, and observe that

$$\frac{d}{dx} R_m^p(x) = p x^{p-1} \left[\frac{R_m^p(x)}{R_{m-1}^p(x)} - 1 \right]. \tag{63}$$

For $k \geq 1$, $m \geq 0$, $p \geq 1$ we generalize G_k^m to

$$G_k^{m,p}(x^p) = \frac{kpx^p}{p[(k-1)x^p - m] + \sqrt{p^2(x^p+m)^2 + 2kp(p-1)x^p}}.$$

The proofs of Theorems 21, 22 and 23 can be extended provided that the analogue of Lemma 24 holds. This is not the case for large p. However, although the generalization of the upper bound in Theorem 21 is a necessary and sufficient condition for convexity of $1/V_m^p(x)$, Lemma 24 is only a sufficient condition for Theorem 21. Indeed, we were able to establish the lower bound in Theorem 21 for $m = 1, 2, 3$ even though part (ii) of Lemma 24 does not hold for $m < 4$. Hence, the fact that Lemma 24 breaks down for large p does not preclude convexity of $1/V_m^p(x)$. On the contrary, numerical evidence suggests that $\frac{1}{V_m^p}$ is convex for all $p \geq 2$.

Lemma 25 *For all $4 \geq p \geq 2$, $m \geq 1$,*

$$\frac{d}{dx} G_4^{m,p}(x^p) \leq p x^{p-1} \left[\frac{G_4^{m,p}(x^p)}{G_4^{m-1,p}(x^p)} - 1 \right]$$

This is equivalent to

$$E_m^p = p x^{p-1} \left[\frac{G_4^{m,p}(x^p)}{G_4^{m-1,p}(x^p)} - 1 \right] - \frac{d}{dx} G_4^{m,p}(x^p) \geq 0$$

which allows us to make some remarks about the range of validity. Although Lemma 25 can probably be extended to some higher p, it does *not* hold for all p, m. On the contrary, for all $m \geq 1$ there exists $p(m)$ and an interval $(x_1^{p(m)}, x_2^{p(m)})$ such that $E_m^p < 0$ on that interval for all $p \geq p(m)$. For example, numerical results show that for $m = 1$, an interval on which $E_m < 0$ exists when $p \geq 10$; for $m = 2$ when $p \geq 14$ and for $m = 3$ when $p \geq 18$.

For simplicity, we only sketch the proof of Lemma 25 and give the final expressions for $p = 3$. Similar expressions can be given for $p = 4$. We have checked the details for general p using Mathematica, but omit the long expressions. As the expressions involved are monotone in p, this suffices for the entire interval $2 \leq p \leq 4$.

Proof: Let $B_m = \sqrt{p^2(y+m)^2 + 8p(p-1)y}$. Then $E_m^p \geq 0$ is equivalent to

$$\begin{aligned}
&B_m B_{m-1}(B_m + 3py - pm) \\
&\quad \geq \; pB_m\left[p(y+m)^2 + y(5p-8) - 3pm\right] + p^2\left\{3m^2p - m^3p\right. \\
&\qquad + \; \left. y(-8 + 8m + 8p - 6mp + m^2p) + y^2(-24 + 23p + 5mp) + 3py^3\right\}.
\end{aligned}$$

Going through the same procedure as in the proof of Lemma 24 (i) and with the analogous notation we arrive at a function H such that $H(0) > 0$. Thus to show $E_m^p \geq 0$ if would suffice to show that

$$l_1(y)B_m - l_2(y) \geq 0, \tag{64}$$

where, for $p = 3$,

$$\begin{aligned}
l_1(y) &= 3(5120 + 2880m + 13800m^2 + 11034m^3 + 3051m^4 + 169600y \\
&\quad + 199632my + 116820m^2y + 26028m^3y + 298296y^2 + 239706my^2 \\
&\quad + 59778m^2y^2 + 133920y^3 + 53676my^3 + 16875y^4), \qquad \text{and} \\
l_2(y) &= 27(-2048m + 192m^2 + 3448m^3 + 3678m^4 + 1017m^5 25600y \\
&\quad + 49920my + 76152m^2y + 45330m^3y + 9693m^4y + 200000y^2 \\
&\quad + 250152my^2 + 139266m^2y^2 + 28602m^3y^2 + 195880y^3 \\
&\quad + 157254my^3 + 37818m^2y^3 + 59640y^4 + 23517my^4 + 5625y^5).
\end{aligned}$$

Both $l_1(y)$ and $l_2(y)$ in (64) are positive, and hence (64) is equivalent to

$$L = l_1(y)^2 B_m^2 - l_2(y)^2 \geq 0.$$

where

$$\begin{aligned}
L &= 1728\left[-540672m^2 + 1714176m^3 + 12955392m^4 + 14819328m^5\right. \\
&\quad + 9516528m^6 + 4234032m^7 + 988524m^8 + 6553600y + 54067200my \\
&\quad + 203688960m^2y + 252786816m^3y + 260067456m^4y + 195994728m^5y \\
&\quad + 82262304m^6y + 13540446m^7y + 158924800y^2 + 186808320my^2 \\
&\quad + 1242179136m^2y^2 + 2017181664m^3y^2 + 1490651748m^4y^2
\end{aligned}$$

$$+\,526526568m^5y^2 + 72412299m^6y^2 + 3716085760y^3 + 7607804160my^3$$
$$+\,8146902528m^2y^3 + 4860751248m^3y^3 + 1546464960m^4y^3$$
$$+\,205855020m^5y^3 + 6023830080y^4 + 9658917216my^4 + 6950005956m^2y^4$$
$$+\,2457752544m^3y^4 + 347809545m^4y^4 + 4108955904y^5 + 4877383680my^5$$
$$+\,2204096832m^2y^5 + 362375694m^3y^5 + 1340628192y^6 + 1052098632my^6$$
$$+\,229306221m^2y^6 + 207787680y^7 + 81047304my^7 + 12301875y^8\big]\,.$$

The positivity of this expression is easy to verify for $m \geq 1$. **QED**

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