Instability and Chaos in Spatially Homogeneous Field Theories

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Abstract

Spatially homogeneous field theories are studied in the framework of dynamical system theory. In particular we consider a model of inflationary cosmology and a Yang–Mills–Higgs system. We discuss also the role of quantum chaos and its application to field theories.

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I Introduction

Quantum field theory offers a wide variety of applications, in particular for condensed matter[1] and elementary particle physics[2]. Field theoretic ideas also reach for the cosmos through the development of the inflationary scenario – a speculative, but completely physical analysis of the early universe, which appears to be consistent with available observations[3].

In the last years there has been much interest in the chaotic behaviour of field theories[4–8]. In this paper we discuss and extend our recent results[12–17] on instability and chaos in classical and quantum field theory. In Section 2 we show how spatially homogeneous field theories can be studied by using the dynamical system theory and we introduce some basic definitions for the regular and chaotic dynamics of classical and quantum systems. In Section 3 we analyze the local stability of a inflationary scalar field minimally coupled to gravity and its point attractors in the phase space. The value of the scalar field in the vacuum is a bifurcation parameter and we discuss the existence of a stable limit cycle. Finally, in Section 4 we study the spatially homogenous SU(2) Yang–Mills–Higgs system. We show that for this system a classical order–chaos transition occurs both in classical and quantum mechanics.

II Field theories as dynamical systems

In this Section we introduce some basic ideas of the dynamical system theory. We clarify the concept of ergodic system giving a hierarchy of chaos.

Let us consider a classical relativistic scalar field theory with action

$$S[\phi] = \int d^4x \; L(\phi, \partial_\mu \phi) \; , \tag{1}$$

where L is the Lagrangian density of the system, $\partial_\mu = (\frac{\partial}{\partial t}, \nabla)$ is the covariant derivative and $\phi = \phi(x)$ is a real scalar field with $x_\mu = (t, \mathbf{x})$ the space–time position[2]. It is well known that by imposing the Hamilton's Least Action Principle

$$\delta S[\phi] = 0 \; , \tag{2}$$

we obtain the Euler–Lagrange equation of motion of the system

$$\frac{\partial L}{\partial \phi} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu \phi)} = 0 \; . \tag{3}$$

The homogenous space approximation means that we can neglect the spatial dependence of the field, thus we can perform the following substitution:

$$\phi(t, \mathbf{x}) \to \phi(t) \; , \tag{4}$$

and the resulting equation of motion is given by

$$\frac{\partial L}{\partial \phi} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} = 0 \; . \tag{5}$$

By introducing the momentum

$$\chi = \frac{\partial L}{\partial \dot{\phi}} \; , \tag{6}$$

and the Hamiltonian

$$H(\chi, \phi) = \dot{\phi}\chi - L(\phi, \dot{\phi}) \; , \tag{7}$$

the second order equation of motion can be written as a system of two first order Hamilton's equations

$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z}) \tag{8}$$

where $\mathbf{z} = (\phi, \chi)$ is a point in a two dimensional phase space and $\mathbf{f} = (f_1, f_2)$ is given by:

$$f_1(\phi, \chi) = \frac{\partial H}{\partial \phi} , \quad f_2(\phi, \chi) = -\frac{\partial H}{\partial \chi} . \tag{9}$$

This is a general result: any homogenous field theory can be written as a system of N first order differential equations, i.e. a dynamical system. In the next Sections we shall consider non–conservative and non–Abelian field theories.

II-A Dynamical System Theory

A dynamical system is defined by N first order differential equations

$$\dot{\mathbf{z}}(t) = \mathbf{f}(\mathbf{z}(t), t) , \tag{10}$$

where the variables $\mathbf{z} = (z_1, ..., z_N)$ are in the phase space Ω (the euclidean space R^N, unless otherwise specified). These equations describe the time evolution of the variables and the system they represent[9−11].

A solution of the dynamical system is a vector function $\mathbf{z}(\mathbf{z}_0, t)$, that satisfies (10) and the initial condition

$$\mathbf{z}(\mathbf{z}_0, 0) = \mathbf{z}_0 . \tag{11}$$

Usually one writes simply $\mathbf{z}(t)$ without the initial condition dependence.

The time evolution of $\mathbf{z} \in \Omega$ is obtained with the one parameter group of diffeomorfism g^t: $\Omega \to \Omega$, such that

$$\frac{d}{dt}(g^t \mathbf{z})|_{t=0} = \mathbf{f}(\mathbf{z}, 0) . \tag{12}$$

The group g^t is called phase flux and the solution is called orbit. The system is called Hamiltonian, if the dimension of Ω is even and there exists a function $H(\mathbf{z},t)$ given by

$$\mathbf{f}(\mathbf{z}(t),t) = J\nabla H(\mathbf{z},t) \ , \tag{13}$$

where:

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{14}$$

is the symplectic matrix and $H(\mathbf{z},t)$ is the Hamiltonian function.

On the phase space Ω one usually defines a probability measure $\mu : \Omega \to \Omega$, such that $\mu(\Omega) = 1$. If we choose a subspace A of Ω, the system is measure preserving if

$$\mu(g^t A) = \mu(A) \ . \tag{15}$$

We observe that for measure preserving dynamical systems one gets $\mathrm{div}(\mathbf{f}) = 0$. It is well known that Hamiltonian systems preserve their measure: the Liouville measure. Dynamical systems which do not preserve their measure are called dissipative, and usually have a measure contraction in time evolution.

The dynamics of a system is called regular if the orbits are stable to infinitesimal variations of initial conditions. It is called chaotic if the orbits are unstable to infinitesimal variations of initial conditions. Useful quantities to calculate this behaviour are the Lyapunov exponents, which give the stability of a single orbit.

A vector of the tangent space $T\Omega_{\mathbf{z}}$ to the phase space Ω in the position $\mathbf{z}$ is given by

$$\omega(\mathbf{z}) = \lim_{s\to 0} \frac{\mathbf{r}(s) - \mathbf{r}(0)}{s} \ , \tag{16}$$

where $\mathbf{r}(0) = \mathbf{z}$ and $\mathbf{r}(s) \in \Omega$. The tangent space vectors are the velocity vectors of the curves on M; there are obviously N independent vectors.

Now we can define the Lyapunov exponent

$$\lambda(\mathbf{z}) = \lim_{t\to\infty} \frac{1}{t} \ln |\omega(t)|, \tag{17}$$

where $\omega(t)$ is a tangent vector to $\mathbf{z}(t)$ with the condition that $|\omega(0)| = 1$.

It can be demonstrated that the limit given by the previous equation exists for a compact phase space, and that it is metric independent. Fixing an orbit in the N dimensional phase space, there are N distinct exponents $\lambda_1, ..., \lambda_N$, called first order Lyapunov exponents. If the orbit has positive Lyapunov exponents, it is chaotic.

To characterize globally the chaoticity of a system, we can introduce the Kolmogorov–Sinai entropy, which is given by

$$h_{KS}(\mu) = \int_A d\mu(\mathbf{z}) \sum_{\lambda_i > 0} \lambda_i(\mathbf{z}) \ , \tag{18}$$

with A subspace of Ω and λ_i Lyapunov exponents. The Kolmogorov–Sinai entropy is a very useful tool for showing chaotic behaviour in the region A.

A system is called ergodic if the time average is equal to phase space average

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t dt f(g^t \mathbf{z}(t)) = \int_\Omega d\mu(\mathbf{z}) f(\mathbf{z}) \ . \tag{19}$$

Incidentally, as is well known, Boltzmann started from the "ergodic hypothesis" to obtain statistical mechanics of equilibrium. But ergodicity is not sufficient to reach an equilibrium state: one must consider mixing systems.

In a mixing system, every finite element of the phase space occupies for $t \to \infty$ the entire phase space Ω; more precisely: $\forall A, B \subset \Omega$ with $\mu(A)$ and

$\mu(B) \neq 0$,

$$\lim_{t\to\infty} \frac{\mu(B \cap g^t A)}{\mu(B)} = \mu(A) \ . \tag{20}$$

To have quantitative information of orbit separations, we must introduce K–systems (Kolmogorov), which are mixing systems with a positive metric entropy, i.e. $h_{KS} > 0$. Such systems are typical chaotic systems.

Among the K–systems, the most unpredictable ones are the B–systems (Bernoulli), which have the Kolmogorov–Sinai entropy equal to the entropy of every partition, i.e. $h_{KS} = h(A_i(0), \mu)$, $\forall A_i(0)$.

II-B Hamiltonian dynamics

Let us consider an Hamiltonian system with n degrees of freedom described by the Hamiltonian function $H(\mathbf{z})$, where $\mathbf{z} = (q_1, ...q_n, p_1, ..., p_n)$ so that the phase space is $N = 2n$ dimensional. The Hamiltonian system is called integrable if there are N functions $F_i = F_i(\mathbf{z})$ defined on Ω in involution:

$$[F_i, F_j]_{PB} = \sum_{k=1}^{n} \frac{\partial F_i}{\partial q_k}\frac{\partial F_j}{\partial p_k} - \frac{\partial F_j}{\partial q_k}\frac{\partial F_i}{\partial p_k} = 0, \quad \forall i, j \tag{21}$$

and linearly independent. $[\ ,\ ]_{PB}$ are the Poisson brackets.

For conservative systems we have $F_1 = H(\mathbf{z})$ and also

$$\frac{dF_i}{dt} = [H, F_i]_{PB} = 0 \ . \tag{22}$$

Because there are n constants of motion, every orbit can explore only the n dimensional manifold $\Omega_f = \{\mathbf{z} : F_i(\mathbf{z}) = f_i, i = 1, ..., n\}$. If Ω_f is compact and connected, it is equivalent to a n dimensional torus $T^n = \{(Q_1, ..., Q_n) \quad mod \quad 2\pi\}$. There are n irreducible and independent circuits

γ_i on Ω_f and there exists a canonical transformation $(\mathbf{p}, \mathbf{q}) \to (\mathbf{P}, \mathbf{Q})$, generated by the function $S(\mathbf{q}, \mathbf{P})$, such that

$$\begin{aligned} P_i &= \oint_{\gamma_i} d\mathbf{q} \cdot \mathbf{p} \\ Q_i &= \frac{\partial S}{\partial P_i} \, . \end{aligned} \tag{23}$$

The P_i are called action variables and the Q_i are called angle variables. The moments $\mathbf{p}$ and coordinates $\mathbf{q}$ are periodic functions of $\mathbf{Q}$ with period 2π. The Hamiltonian depends only on action variables, i.e. $H = H(\mathbf{P})$.

Adding a small perturbation $V(\mathbf{P}, \mathbf{Q})$ to an integrable Hamiltonian $H_0(\mathbf{P})$, the total Hamiltonian can be written:

$$H(\mathbf{P}, \mathbf{Q}) = H_0(\mathbf{P}) + gV(\mathbf{P}, \mathbf{Q}) \, , \tag{24}$$

and, generically, the integrability is destroyed. As a consequence, parts of phase space become filled with chaotic orbits, while in other parts the toroidal surfaces of the integrable system are deformed but not destroyed; thus we have a quasi–integrable system. By growing g, chaotic motion develops near the regions of phase space where all the frequencies on the torus $\omega_i = \frac{\partial H(\mathbf{P})}{\partial P_i}$ are commensurate. Conversely, tori of the integrable system, on which the ω_i are incommensurate, are deformed, but not destroyed immediately (Kolmogorov–Arnold–Moser (KAM) theorem)[9,18]. As g increases, the phase space generically develops a highly complex structure, with islands of regular motion (filled with quasi–periodic orbits) interspersed in regions of chaotic motion, but containing in turn more regions of chaos. As g grows further, the fraction of phase space filled with chaotic orbits grows until it reaches unity as the last KAM surface is destroyed. Then the motion is completely chaotic everywhere, except possibly for isolated periodic orbits[9,18].

It is very useful to plot a $2n-1$ surface of section $\mathcal{P} \subset \Omega$, called Poincaré section. For an integrable system with two degrees of freedom, the $q_1 = 0$ Poincaré section of a rational (resonant) torus is a finite number of points along a closed curve, while the section of an irrational (non resonant) torus is a continuous closed curve. Adding a perturbation, the section presents closed curves (KAM tori), whose points are stable (elliptic), and also curves formed by substructures, residua of resonant tori, whose points are unstable (hyperbolic). As the perturbation parameter increases, the closed curves are distorted and reduced in number.

II-C Quantum Chaos

We use the term quantum chaotic system in the precise and restricted sense of a quantum system whose classical analogue is chaotic. In particular we concentrate on energy levels of quantum systems (see, for example, Ref. 18 and 19).

Let us consider a classical regular Hamiltonian system. The short–range properties of the corresponding quantal spectrum tend to resemble those of a spectrum of randomly distributed numbers. This is because regular classical motion is associated with integrability or separability of the classical equations of motion. In quantum mechanics the separability corresponds to a number of independent conserved quantities (such as angular momentum), and each energy level can be characterised by the associated quantum numbers. Superimposing the terms arising from the various quantum numbers, a spectrum is generated like that of random numbers, at least over short intervals. In particular, the distribution $P(s)$ of nearest–neighbour spacings $s_i = (\epsilon_{i+1} - \epsilon_i)/d$, where d is the mean level spacing, is expected to follow the Poisson limit, i.e. $P(s) = \exp(-s)$.

Instead, when the classical dynamics of a physical system is chaotic, the system cannot be integrable and there must be fewer constants of motion than degrees of freedom. Quantum mechanically this means that once all good quantum numbers due to obvious symmetries etc. are accounted for, the energy levels cannot simply be labelled by quantum numbers associated with certain constants of motion. The short–range properties of the energy spectrum then tend to resemble those of eigenvalue spectra of matrices with randomly chosen elements and one gets a result very close to $P(s) = (\pi/2)s\exp(-\frac{\pi}{4}s^2)$, which is the so–called Wigner distribution.

The distribution $P(s)$ is the best spectral statistics to analyze shorter series of energy levels and the intermediate regions between order and chaos. This distribution can be compared to the Brody distribution

$$P(s,\omega) = \alpha(\omega+1)s^{\omega}\exp(-\alpha s^{\omega+1}) \ , \tag{25}$$

with

$$\alpha = (\Gamma[\frac{\omega+2}{\omega+1}])^{\omega+1} \ . \tag{26}$$

The Brody distribution interpolates between the Poisson distribution ($\omega = 0$) of integrable systems and the Wigner distribution ($\omega = 1$) of chaotic ones, and thus the parameter ω can be used as a simple quantitative measure of the degree of chaoticity.

III A model for inflationary cosmology

In this Section we study the stability of a scalar inflaton field[12,13] and analyze its bifurcation properties in the framework of the dynamical system theory.

It is generally believed that the universe, at a very early stage after the big bang, exhibited a short period of exponential expansion, the so–called inflationary phase. In fact the assumption of an inflationary universe solves three major cosmological problems: the flatness problem, the homogeneity problem, and the formation of structure problem[3].

The Friedmann–Robertson–Walker metric of a homogeneous and isotropic expanding universe is given by

$$ds^2 = dt^2 - a^2(t)[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)], \tag{27}$$

where $k = 1, -1$, or 0 for a closed, open, or flat universe, and $a(t)$ is the scale factor of the universe.

The evolution of the scale factor $a(t)$ is given by the Einstein equations

$$\ddot{a} = -\frac{4\pi}{3}G(\rho + 3p)a,$$

$$(\frac{\dot{a}}{a})^2 + \frac{k}{a^2} = \frac{8\pi}{3}G\rho, \tag{28}$$

where ρ is the energy density of matter in the universe, and p its pressure. The gravitational constant $G = M_p^{-2}$ (with $\hbar = c = 1$), where $M_p = 1.2 \cdot 10^{19}$ GeV is the Plank mass, and $H_u = \dot{a}/a$ is the Hubble "constant", which in general is a function of time.

The inflationary models postulate the existence of a scalar field ϕ, the so–called inflation field, with Lagrangian

$$L = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) \tag{29}$$

where the potential $V(\phi)$ depends on the type of inflation model considered. The scalar field, if minimally coupled to gravity, satisfies the equation

$$\Box\phi = \ddot{\phi} + 3(\frac{\dot{a}}{a})\dot{\phi} - \frac{1}{a^2}\nabla^2\phi = -\frac{\partial V}{\partial\phi}, \tag{30}$$

where $\Box$ is the covariant d'Alembertian operator. The Hubble "constant" H_u is related to the energy density of the field by

$$H_u^2 + \frac{k}{a^2} = (\frac{\dot{a}}{a})^2 + \frac{k}{a^2} = \frac{8\pi G}{3}[\frac{\dot{\phi}^2}{2} + \frac{(\nabla\phi)^2}{2} + V(\phi)]. \tag{31}$$

In a flat universe $k = 0$ and, if the inflaton field is sufficiently uniform (i.e. $\dot{\phi}^2$, $(\nabla\phi)^2 << V(\phi)$), we obtain an homogenous field theory in one dimension

$$\ddot{\phi} + 3H_u(\phi)\dot{\phi} + \frac{\partial V}{\partial\phi} = 0, \tag{32}$$

where the Hubble "constant" H_u is an explicit function of ϕ:

$$H_u^2 = \frac{8\pi G}{3}V(\phi). \tag{33}$$

III-A Local instability for the inflationary self–energy

The second order equation of motion of our cosmological model can be written as a system of two first order differential equations

$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z}) \tag{34}$$

where $\mathbf{z} = (\phi, \chi)$ is a point in the two dimensional phase space and $\mathbf{f} = (f_1, f_2)$ is given by

$$f_1(\phi, \chi) = \chi, \quad f_2(\phi, \chi) = -3H_u(\phi)\chi - \frac{\partial V(\phi)}{\partial \phi}. \tag{35}$$

The system is non–conservative because the function

$$\mathrm{div}(\mathbf{f}) = \frac{\partial g_1}{\partial \phi} + \frac{\partial g_2}{\partial \chi} = -3H_u(\phi) \tag{36}$$

is not identically zero. The fixed points of the system are those for which $f_1(\phi, \chi) = 0$ and $f_2(\phi, \chi) = 0$, i.e

$$\chi = 0, \quad \frac{\partial V(\phi)}{\partial \phi} = 0. \tag{37}$$

The deviation $\delta\mathbf{z}(t) = \hat{\mathbf{z}}(t) - \mathbf{z}(t)$ from the two initially neighboring trajectories $\mathbf{x}$ and $\hat{\mathbf{x}}$ in the phase space satisfies the linearized equations of motion

$$\frac{d}{dt}\delta\mathbf{z}(t) = \Gamma(t)\delta\mathbf{z} \tag{38}$$

where $\Gamma(t)$ is the stability matrix

$$\Gamma(t) = \begin{pmatrix} 0 & 1 \\ -\frac{\partial^2 V}{\partial \phi^2} - 3\chi\frac{\partial H}{\partial \phi} & -3H_u(\phi) \end{pmatrix}. \tag{39}$$

At least if an eigenvalue of $\Gamma(t)$ is real the separation of the trajectories grows exponentially and the motion is unstable. Imaginary eigenvalues correspond to stable motion. In the limit of time that goes to infinity, from the eigenvalues of the stability matrix we can obtain the Lyapunov exponents. For two–dimensional dynamical system the Lyapunov exponents can not be positive[9] and so the system is not chaotic, i.e. there is not global instability.

However, we can be assured that the universe is crowded with many interacting fields of which the inflaton is but one. The nonlinear nature of these interactions can result in a complex chaotic evolution of the universe and the local instability of the inflaton field is a precursor phenomenon of chaotic motion.

The eigenvalues of the stability matrix are given by

$$\sigma_{1,2} = -\frac{3}{2}H_u(\phi) \pm \frac{1}{2}\sqrt{9H_u^2(\phi) - 4\frac{\partial^2 V}{\partial \phi^2} - 12\chi\frac{\partial H_u}{\partial \phi}}. \tag{40}$$

The pair of eigenvalues become real and there is exponential separation of neighboring trajectories, i.e. unstable motion, if

$$\frac{\partial^2 V}{\partial \phi^2} + 3\chi\frac{\partial H_u}{\partial \phi} < 0. \tag{41}$$

Particularly when $\chi = 0$, e.g. the fixed points, we obtain local instability when

$$\frac{\partial^2 V}{\partial \phi^2} < 0, \tag{42}$$

i.e. for negative curvature of the potential energy. The fixed points are stable if they are point of local minimum of $V(\phi)$ and unstable if are points of local maximum.

The potential $V(\phi)$ depends on the type of inflation model considered, and it is usually some kind of double–well potential. We choose a symmetric double–well potential

$$V(\phi) = \frac{\lambda}{4}(\phi^2 - v^2)^2, \tag{43}$$

where $\pm v$ are the values of the inflaton field in the vacuum, i.e. the points of minimal energy of the system.

We observe that the inflaton field value in the vacuum v is a bifurcation parameter. Bifurcation is used to indicate a qualitative change in the features of the system under the variation of one or more parameters on which the system depends. First of all we consider the case $v = 0$, i.e. $V(\phi) = (\lambda/4)\phi^4$. In this situation there is only one fixed point $(\phi^* = 0, \chi^* = 0)$ which is a stable one being

$$\frac{\partial^2 V}{\partial \phi^2} = 3\lambda\phi^2 \geq 0. \tag{44}$$

The fixed point $(\phi^* = 0, \chi^* = 0)$ is a point attractor.

Instead for $v \neq 0$ there are three fixed points

$$(\phi^* = 0, \chi^* = 0), \quad (\phi^* = v, \chi^* = 0), \quad (\phi^* = -v, \chi^* = 0), \tag{45}$$

and the condition for the instability becomes

$$-\frac{v}{\sqrt{3}} < \phi < \frac{v}{\sqrt{3}}. \tag{46}$$

Obviously $(\phi^* = 0, \chi^* = 0)$ is an unstable fixed point, and in particular a saddle point because the stability matrix has real and opposite eigenvalues. On the other hand $(\phi^* = \pm v, \chi^* = 0)$ are stable fixed points.

There are four possible functions for the Hubble "constant"

$$H_u(\phi) = \pm\gamma|\phi^2 - v^2|, \tag{47}$$

but also

$$H_u(\phi) = \pm\gamma(\phi^2 - v^2), \tag{48}$$

where $\gamma = \sqrt{2\pi G\lambda/3}$ is the dissipation parameter. The choice of the Hubble function is crucial for the dynamical evolution of the system.

In certain non–conservative systems we could find closed trajectories or limit cycles toward which the neighboring trajectories spiral on both sides. It is sometime possible to know that no limit cycle exist and the Bendixson criterion[22], which establishes a condition for the non–existence of closed trajectories, is useful in some cases. Bendixson criterion is as follows: if div($\mathbf{f}$) is not zero and does not change its sign within a domain D of the phase space, no closed trajectories can exist in that domain. In our case we have div($\mathbf{f}$) $= -3H_u(\phi)$, and so the presence of periodic orbit is related to the sign of $H_u(\phi)$.

If $H_u(\phi) = \gamma|\phi^2 - v^2|$ we do not find periodic orbits and the inflaton field goes to one of its two stable fixed points, which are points attractors (see Figure 1). The vacuum is degenerate but if we choose an initial condition around the saddle point there is a dynamical symmetry breaking towards the positive v or negative $-v$ value of the inflaton field in the vacuum. This symmetry breaking is unstable because neighbour initial conditions can go in different point attractors.

Instead, if we choose $H_u(\phi) = \gamma(\phi^2 - v^2)$ the numerical calculations of Figure 2 show that exists a limit cycle, the two stable fixed points are not point attractors, and the inflaton field oscillates forever. Obviously more large is v more large is the limit cycle.

III-B A limit cycle in the cosmological model

The equation of motion of the inflaton field with $H_u(\phi) = \gamma(\phi^2 - v^2)$ reads

$$\ddot{\phi} + 3\gamma(\phi^2 - v^2)\dot{\phi} + \lambda\phi(\phi^2 - v^2) = 0 \; . \tag{49}$$

This equation can be written as

$$\frac{d}{dt}[\dot{\phi} + 3\gamma \int_0^{\phi} (u^2 - v^2) du] + \lambda \phi (\phi^2 - v^2) = 0 \; , \tag{50}$$

and if we put

$$F(\phi) = 3 \int_0^{\phi} (u^2 - v^2) du = \phi(\phi^2 - 3v^2) \; , \quad G(\phi) = \phi(\phi^2 - v^2) \; , \tag{51}$$

and also $\omega = \dot{\phi} + \gamma F(\phi)$, we obtain the system

$$\begin{aligned} \dot{\phi} &= \omega - \gamma F(\phi) \\ \dot{\omega} &= -\lambda G(\phi) \; . \end{aligned} \tag{52}$$

For systems of this kind the Lienard theorem[23] states that there is an unique and stable limit cycle if the following conditions are satisfied: $F(\phi)$ is an odd function and $F(\phi) = 0$ only at $\phi = 0$ and $\phi = \pm\alpha$; $F(\phi) < 0$ for $0 < \phi < \alpha$, $F(\phi) > 0$ and is increasing for $\phi > \alpha$; $G(\phi)$ is an odd function and $\phi G(\phi) > 0$ for all $\phi > \alpha$. It is easy to check that the functions $F(\phi)$ and $G(\phi)$ satisfy all the conditions of the Lienard theorem with $\alpha = v$. The cubic force $G(\phi)$ tends to reduce any displacement for large $|\phi|$, whereas the damping $F(\phi)$ is negative at small $|\phi|$ and positive at large $|\phi|$. Since small oscillations are pumped up and large oscillations are damped down, it is not surprising that the system tends to seattle into a self–sustained oscillation of some intermediate amplitude.

Let us consider a typical trajectory of the system. After the scaling $\psi = \lambda\omega$ we obtain

$$\begin{aligned} \dot{\phi} &= \lambda[\psi - \tfrac{\gamma}{\lambda} F(\phi)] \; , \\ \dot{\psi} &= -G(\phi) \; . \end{aligned} \tag{53}$$

The cubic nullcline $\psi = (\gamma/\lambda)F(\phi)$ is the key to understand the motion. Suppose that $\lambda >> 1$ and the initial condition is far from the cubic nullcline, then we have $|\dot{\phi}| \sim O(\lambda) >> 1$; hence the velocity is enormous in the horizontal direction and tiny in the vertical direction, so trajectories move practically horizontally. If the initial condition is above the nullcline then $\dot{\phi} > 0$, thus the trajectory moves sideways toward the nullcline. However, once the trajectory gets so close that $\psi \simeq (\lambda/\gamma)F(\phi)$ then the trajectory crosses the nullcline vertically and moves slowing along the backside of the branch until it reaches the knee and can jump sideways again. The period T of the limit cycle is essentially the time required to travel along the two slow branches, since the time spent in the jumps is negligible for large λ. By symmetry, the time spent on each branch is the same so we have

$$T \simeq 2 \int_{t_A}^{t_B} dt \tag{54}$$

where A and B are the initial and final points on the positive slow branch. To derive an expression for dt we note that on the slow branches with a good approximation $\psi \simeq (\gamma/\lambda)F(\phi)$ and thus

$$\frac{d\psi}{dt} \simeq \frac{\gamma}{\lambda}F'(\phi)\frac{d\phi}{dt} = 3\frac{\gamma}{\lambda}(\phi^2 - v^2)\frac{d\phi}{dt}\ . \tag{55}$$

Since $d\psi/dt = -\phi(\phi^2 - v^2)$, we obtain $dt \simeq -3\frac{\gamma}{\lambda}\frac{d\phi}{\phi}$, on the slow branches. The slow positive branch begins at $\phi_A = 2\gamma v/\lambda$ and ends at $\phi_B = \gamma v/\lambda$. Because $\gamma = \sqrt{2\pi G\lambda/3}$ we get $T \simeq 2\ln 2\sqrt{\frac{6\pi G}{\lambda}}$.

IV The homogenous SU(2) YMH system

Now we study the suppression of classical chaos in the spatially homogenous SU(2) Yang–Mills–Higgs (YMH) system induced by the Higgs field[4,14,15]. We analyze also the energy fluctuation properties of the system, which give a clear quantum signature of the classical chaos–order transition of the system.

The SU(2) YMH system describes the interaction between a scalar Higgs field ϕ and three non–Abelian Yang–Mills fields A_{μ}^{a}, $a = 1, 2, 3$. The Lagrangian density of the YMH system is given by

$$L = \frac{1}{2}(D_{\mu}\phi)^{+}(D^{\mu}\phi) - V(\phi) - \frac{1}{4}F_{\mu\nu}^{a}F^{\mu\nu a} , \tag{56}$$

where

$$(D_{\mu}\phi) = \partial_{\mu}\phi - igA_{\mu}^{b}T^{b}\phi , \tag{57}$$

$$F_{\mu\nu}^{a} = \partial_{\mu}A_{\nu}^{a} - \partial_{\nu}A_{\mu}^{a} + g\epsilon^{abc}A_{\mu}^{b}A_{\nu}^{c} , \tag{58}$$

with $T^{b} = \sigma^{b}/2$, $b = 1, 2, 3$, generators of the SU(2) algebra, and where the potential of the scalar field (the Higgs field) is

$$V(\phi) = \mu^{2}|\phi|^{2} + \lambda|\phi|^{4} . \tag{59}$$

We work in the (2+1)–dimensional Minkowski space ($\mu = 0, 1, 2$) and choose spatially homogeneous Yang–Mills and the Higgs fields

$$\partial_{i}A_{\mu}^{a} = \partial_{i}\phi = 0 , \quad i = 1, 2 \tag{60}$$

i.e. we consider the system in the region in which space fluctuations of fields are negligible compared to their time fluctuations.

In the gauge $A_0^a = 0$ and using the real triplet representation for the Higgs field we obtain

$$L = \dot{\phi}^2 + \frac{1}{2}(\dot{A}_1^2 + \dot{A}_2^2) - g^2[\frac{1}{2}A_1^2 A_2^2 - \frac{1}{2}(A_1 \cdot A_2)^2 +$$

$$+(A_1^2 + A_2^2)\phi^2 - (A_1 \cdot \phi)^2 - (A_2 \cdot \phi)^2] - V(\phi) \ , \qquad (61)$$

where $\phi = (\phi^1, \phi^2, \phi^3)$, $A_1 = (A_1^1, A_1^2, A_1^3)$ and $A_2 = (A_2^1, A_2^2, A_2^3)$.

When $\mu^2 > 0$ the potential V has a minimum at $|\phi| = 0$, but for $\mu^2 < 0$ the minimum is at

$$|\phi_0| = \sqrt{\frac{-\mu^2}{4\lambda}} = v \ ,$$

which is the non zero Higgs vacuum. This vacuum is degenerate and after spontaneous symmetry breaking the physical vacuum can be chosen $\phi_0 = (0, 0, v)$. If $A_1^1 = q_1$, $A_2^2 = q_2$ and the other components of the Yang–Mills fields are zero, in the Higgs vacuum the Hamiltonian of the system reads

$$H = \frac{1}{2}(p_1^2 + p_2^2) + g^2 v^2 (q_1^2 + q_2^2) + \frac{1}{2} g^2 q_1^2 q_2^2 \ , \qquad (62)$$

where $p_1 = \dot{q}_1$ and $p_2 = \dot{q}_2$. Here $w^2 = 2g^2v^2$ is the mass term of the Yang–Mills fields. This YMH Hamiltonian is a toy model for classical non–linear dynamics, with the attractive feature that the model emerges from particle physics.

IV-A From chaos to order in the YMH system

The chaotic behaviour of the YMH system can be studied by using the Toda criterion of the Gaussian curvature of the potential energy[20,21]. For our YMH system the potential energy is given by

$$V(q_1, q_2) = g^2 v^2 (q_1^2 + q_2^2) + \frac{1}{2} g^2 q_1^2 q_2^2 \; . \tag{63}$$

At low energy, the motion near the minimum of the potential, where the Gaussian curvature is positive, is periodic or quasi–periodic and is separated from the instability region by a line of zero curvature; if the energy is increased, the system will be, for some initial conditions, in a region of negative curvature, where the motion is chaotic. According to this scenario, the energy E_c of chaos–order transition is equal to the minimum value of the line of zero Gaussian curvature $K_G(q_1, q_2)$ on the potential–energy surface. For our potential the gaussian curvature vanishes at the points that satisfy the equation

$$(2g^2v^2 + g^2q_2^2)(2g^2v^2 + g^2q_1^2) - 4g^4q_1^2q_2^2 = 0 \; . \tag{64}$$

It is easy to show that the minimal energy on the zero–curvature line is given by:

$$E_c = V_{min}(K_G = 0, \bar{q_1}) = 6g^2v^4 \; , \tag{65}$$

and by inverting this equation we obtain $v_c = (E/6g^2)^{1/4}$. Thus the curvature criterion suggest that there is a order–chaos transition by increasing the energy E of the system and a chaos–order transition by increasing the value v of the Higgs field in the vacuum. Thus, there is only one transition regulated by the unique parameter $E/(g^2v^4)$.

It is important to stress that the Toda criterion is not a fully reliable indicator of chaos[21]. In fact, the local instability of the Toda Criterion does not necessarily imply the global one and the idea of an order–chaos transition with a critical energy is not strictly correct. The Toda curvature criterion should therefore be combined with the Poincarè sections, which are shown in Figure 3. The numerical results confirm the analytical predictions of the curvature criterion: with $E=10$ and $g=1$ we get the critical value of the onset of chaos $v_c=(E/6g^2)^{1/4}\simeq 1.14$.

IV-B Spectral statistics of the YMH system

In quantum mechanics the generalized coordinates of the YMH system satisfy the usual commutation rules $[\hat{q}_k,\hat{p}_l]=i\delta_{kl}$, with $k,l=1,2$. Introducing the creation and destruction operators

$$\hat{a}_k=\sqrt{\frac{\omega}{2}}\hat{q}_k+i\sqrt{\frac{1}{2\omega}}\hat{p}_k\ ,\quad \hat{a}_k^+=\sqrt{\frac{\omega}{2}}\hat{q}_k-i\sqrt{\frac{1}{2\omega}}\hat{p}_k\ , \tag{66}$$

the quantum YMH Hamiltonian can be written

$$\hat{H}=\hat{H}_0+\frac{1}{2}g^2\hat{V}\ , \tag{67}$$

where

$$\hat{H}_0=\omega(\hat{a}_1^+\hat{a}_1+\hat{a}_2^+\hat{a}_2+1)\ , \tag{68}$$

$$\hat{V}=\frac{1}{4\omega^2}(\hat{a}_1+\hat{a}_1^+)^2(\hat{a}_2+\hat{a}_2^+)^2\ , \tag{69}$$

with $\omega^2=2g^2v^2$ and $[\hat{a}_k,\hat{a}_l^+]=\delta_{kl}$, $k,l=1,2$.

We compute the energy levels of the YMH system with a numerical diagonalization of the truncated matrix of the quantum YMH Hamiltonian in the basis of the harmonic oscillators (see also Ref. 16 and 17). If $|n_1 n_2>$ is the basis of the occupation numbers of the two harmonic oscillators, the matrix elements are

$$< n_1^{'} n_2^{'} | \hat{H}_0 | n_1 n_2 >= \omega (n_1 + n_2 + 1) \delta_{n_1^{'} n_1} \delta_{n_2^{'} n_2} \; , \tag{70}$$

and

$$< n_1^{'} n_2^{'} | \hat{V} | n_1 n_2 >= \frac{1}{4\omega^2} [\sqrt{n_1(n_1-1)} \delta_{n_1^{'} n_1 - 2} + \sqrt{(n_1+1)(n_1+2)} \delta_{n_1^{'} n_1 + 2} +$$
$$+ (2n_1+1) \delta_{n_1^{'} n_1}] \times [\sqrt{n_2(n_2-1)} \delta_{n_2^{'} n_2 - 2} + \sqrt{(n_2+1)(n_2+2)} \delta_{n_2^{'} n_2 + 2} + (2n_2+1) \delta_{n_2^{'} n_2}] \; . \tag{71}$$

The symmetry of the potential enables us to split the Hamiltonian matrix into 4 sub–matrices reducing the computer storage required. These sub–matrices are related to the parity of the two occupation numbers n_1 and n_2: even–even, odd–odd, even–odd, odd–even. The numerical energy levels depend on the dimension of the truncated matrix: we compute the numerical levels in double precision increasing the matrix dimension until the first 100 levels converge within 8 digits (matrix dimension 1156×1156).

We have seen previously that the most used quantity to study the local fluctuations of the energy levels is the distribution $P(s)$ of nearest–neighbour spacings s_i of the energy levels. It is obtained by accumulating the number of spacings that lie within the bin $(s, s+\Delta s)$ and then normalizing $P(s)$ to unit.

We use the first 100 energy levels of the 4 sub–matrices to calculate the $P(s)$ distribution. In order to remove the secular variation of the level density as a function of the energy E, for each value of the coupling constant the corresponding spectrum is mapped into one which has a constant level density.

The Figure 4 shows the $P(s)$ distribution of Brody for three different values of the Higgs vacuum v. The best fit Brody parameter ω is obtained by using the nearest–neighbour spacings of the first 100 unfolded energy levels of the YMH system. There is Wigner–Poisson transition by increasing the value v of the Higgs field in the vacuum. Thus, by using the P(s) distribution, it is possible to give a quantitative measure of the degree of quantal chaoticity of the system. Our numerical calculations show clearly the quantum chaos–order transition and its correspondence to the classical one.

V Conclusions

We have seen that spatially homogeneous field theories can be studied as dynamical systems. After a brief review of the dynamical system theory, we have discussed two schematic models of field theory.

First, we have considered the stability of a non–conservative scalar inflaton field. The value of the inflaton field in the vacuum is a bifurcation parameter which changes dramatically the phase space structure. The main point is that for some functional solutions of the Hubble "constant" the system goes to a limit cycle, i.e. to a periodic orbit. The inflaton field is not chaotic but its local instability can give rise to a complex chaotic evolution of the universe due to its nonlinear interactions with other fields. In the future it will be very interesting to study these effects which can perhaps lead to some observable implications like a fractal pattern in the spectrum of density fluctuations.

We have then analyzed the non–Abelian SU(2) Yang–Mills–Higgs system.

We have given an analytical estimation (confirmed by numerical results of Poincarè sections) of the classical chaos–order transition as a function of the Higgs vacuum, the Yang–Mills coupling constant and the energy of the system. A quantum signature of a chaos–order transition has been obtained by using the distribution $P(s)$ of nearest–neighbour spacings. The Wigner–Poisson transition of the $P(s)$ distribution follows very well the classical results of the Poincarè sections.

To conclude, we observe that there are yet many open problems about chaos in field theory. We make a list of some of them: i) spatial chaos and space–time chaos; ii) classical and quantum chaos in more realistic systems, for example in QCD (some results can be found in Ref. 7 and 8); iii) connection between chaos and critical phenomena (finite temperature field theory).

Figure Captions

Figure 1: The Hubble function *vs* time (top) and the phase space trajectory of the inflaton field (bottom); for $H_u(\phi) = \gamma|\phi^2 - v^2|$ with $\gamma = 1/2$, $\lambda = 3$ and $v = 1$. Initial conditions: $\phi = 0$ and $\dot{\phi} = 1/2$.

Figure 2: The Hubble function *vs* time (top) and the phase space trajectory of the inflaton field (bottom); for $H_u(\phi) = \gamma(\phi^2 - v^2)$ with $\gamma = 1/2$, $\lambda = 3$ and $v = 1$. Initial conditions: $\phi = -1/2$ and $\dot{\phi} = 0$.

Figure 3: The Poincarè sections of the YMH system. From the top: $v = 1$, $v = 1.1$ and $v = 1.2$. Energy $E = 10$ and interaction $g = 1$.

Figure 4: $P(s)$ distribution of Brody of the YMH system. First 100 energy levels and $g = 1$. The best fit Brody parameter is given by: $\omega = 0.92$ for $v = 1.0$, $\omega = 0.34$ for $v = 1.1$ and $\omega = 0.01$ for $v = 1.2$.

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