Lifshitz tail for Schrödinger operator with random magnetic field

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Abstract

We study the behavior of the density states at the lower edge of the spectrum for Schrödinger operators with random magnetic fields. We use a new estimate on magnetic Schrödinger operators, which is similar to the Avron-Herbst-Simon estimate but the bound is always nonnegative.

1 Introduction

In this paper, we consider the magnetic Schrödinger operator without scalar potential on $\mathbb{R}^d$ $(d \geq 2)$:

$$H = (p - A(x))^2 \qquad \text{on } L^2(\mathbb{R}^d),$$

where $p = -i\partial_x$ the momentum operator,

$$A(x) = (A_1(x), \dots, A_d(x)) \qquad x \in \mathbb{R}^d$$

is a vector potential. For the moment we suppose A is a C^1-class function. We identify $A(x)$ with the 1-form $A = \sum_{j=1}^d A_j(x)dx_j$. The magnetic field is the 2-form defined by

$$B(x) = dA = \sum_{i<j} B_{ij}(x) dx_i \wedge dx_j,$$

where $B_{ij}(x) = \partial_{x_i} A_j(x) - \partial_{x_j} A_i(x)$, and we identify B with the skew-symmetric matrix valued function $(B_{ij}(x))$ on $\mathbb{R}^d$. It is well-known that for any closed 2-form B, there is a 1-form A such that $B = dA$, and the corresponding Schrödinger operator H is uniquely determined by B modulo the gauge transformation group.

We suppose that $B = B^\omega$ is a metrically transitive random closed 2-form in the following sense: There exist a probability space $(\Omega, \Sigma, \mathbb{P})$ and $\mathbb{R}^d$ acts on Ω as a measure preserving transformation group. We denote the action by $\{T_x \mid x \in \mathbb{R}^d\}$. The action is supposed to be ergodic, i.e., if $E \subset \Omega$ is invariant under $\{T_x\}$ then $\mathbb{P}(E) = 0$ or 1. $B = B^\omega$ is assumed to satisfy

$$B^\omega(x - y) = B^{(T_x\omega)}(x), \qquad x, y \in \mathbb{R}^d,$$

and it is a continuous closed 2-form almost surely.

We denote

$$\Lambda_L = \{x \in \mathbb{R}^d \mid |x_j| \le L/2 \text{ for } j = 1, \dots, d\}, \qquad L > 0,$$

and let H_Λ^ω be the Schrödinger operator (with the same symbol as above) on $\Lambda \subset \mathbb{R}^d$ with, e.g., the Neumann boundary condition (cf. Section 3). We also denote the Lebesgue measure of Λ by $|\Lambda|$. Then it is well-known that the density of states

$$k(E) = \lim_{L\to\infty} \frac{1}{|\Lambda_L|} \#\{\text{eigenvalue of } H_{\Lambda_L}^\omega \le E\}$$

exists for almost all $E \in \mathbb{R}$ and $\omega \in \Omega$, and it is independent of ω (almost surely). Our main assumption is the following.

Assumption A. (i) B^ω is a metrically transitive random closed 2-form on $\mathbb{R}^d$ in the above sense.

(ii) There is a constant $M \in \mathbb{R}_+ = [0, \infty)$ such that

$$|B^\omega(x)| \le M \qquad \text{for any } x \in \mathbb{R}^d \text{ and almost all } \omega \in \Omega,$$

where $|B(x)| = (\sum_{i<j} |B_{ij}(x)|^2)^{1/2}$.

(iii) There exists a real-valued continuous function φ on $\mathbb{R}_+$ such that $\varphi(\lambda) \to 0$ as $\lambda \to \infty$, and it satisfies the following condition: Let $\Lambda_1, \Lambda_2 \in \Sigma$ and let $f \in L^1(\Omega)$, $g \in L^\infty(\Omega)$ such that f is Σ_{Λ_1}-measurable and g is Σ_{Λ_2}-measurable, where Σ_Λ denotes the σ-algebra generated by $\{B^\omega(x) \mid x \in \Lambda\}$. Then

$$|\mathbb{E}(fg) - \mathbb{E}(f)\mathbb{E}(g)| \le \varphi(\mathrm{dist}(\Lambda_1, \Lambda_2)) \|f\|_{L^1} \|g\|_{L^\infty},$$

where $\mathbb{E}(\cdot)$ denotes the expectation with respect to $\mathbb{P}$, and $\mathrm{dist}(\cdot, \cdot)$ is the Euclidean distance on $\mathbb{R}^d$.

(iv) Let C_0 be the unit cube in $\mathbb{R}^d$. Then

$$\mathbb{E}\Big(\int_{C_0} |B^\omega(x)| dx\Big) > 0.$$

Theorem 1. *Suppose Assumption A. Then*

$$\limsup_{E\downarrow 0} \log(-\log k(E))/\log E \le -d/2.$$

The Lifshitz tail (or the Lifshitz singularity) of the density of states for Schrödinger operator with random potential have been studied by many authors (see, e.g., the textbook of Carmona and Lacroix [2] or the lecture note of Kirsch [3]), but very few results have been obtained on the Lifshitz tail for Schrödinger operator with random magnetic field so far. The author is aware of only a work by Ueki [7] where the Lifshitz tail is studied for Schrödinger operators with a class of Gaussian random magnetic field, and a work by the author on 2D discrete Schrödinger operator with Anderson type random magnetic field [5].

The strategy of the proof is similar to [5]. We first prove that magnetic Schrödinger operator is bounded from below by a nonnegative function, which does not vanish where B does not (Section 2). This estimate is similar to the well-known AHS (Avron-Herbst-Simon) estimate:

$$H \geq \pm B_{ij}(x), \qquad x \in \mathbb{R}^d, i \neq j.$$

The right hand side of the AHS estimate is not necessarily positive. In order to obtain global lower bounds of the operator, we have to use a partition of unity, and thus introduce (not necessarily positive) error terms. Our estimate (Theorem 2) is in fact weaker than the AHS estimate in many situations, but the right hand side of the estimate is always nonnegative, and hence easier to apply in our context. In Section 3, we discuss Neumann decoupling and prove a simple a priori estimate on the density of states for magnetic Schrödinger operators. We then combine them with a result on the Lifshitz tail of Schrödinger operator with random potential to conclude the assertion (cf. Kirsch-Martinelli [4], Kirsch [3]).

Notation: We denote the definition domain of an operator A by $\mathfrak{D}(A)$. The quadratic form domain of A is denoted by $\mathcal{Q}(A)$. The inner product of a Hilbert space $\mathcal{H}$ is denoted by $\langle\varphi|\psi\rangle$ ($\varphi, \psi \in \mathcal{H}$), which is linear in the second entry. We denote the space of the Schwartz functions on $\mathbb{R}^d$ by $\mathcal{S}(\mathbb{R}^d)$.

2 Energy estimates for magnetic Schrödinger operators

We consider a deterministic Schrödinger operator $H = (p - A(x))^2$ with the magnetic field $B = dA$. We fix a constant $r > 0$. For $x \in \mathbb{R}^d$ and $1 \leq i < j \leq d$, we denote

$$\begin{aligned} D_{ij}(x) &= \big\{ y \in \mathbb{R}^d \bigm| y_k = x_k \text{ for } k \neq i, j, \\ &\qquad\qquad (x_i - y_i)^2 + (x_j - y_j)^2 \leq r^2 \big\}, \\ \gamma_{ij}(x) &= \partial D_{ij}(x). \end{aligned}$$

We define $b_{ij}(x) \in \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ by

$$b_{ij}(x) \equiv \int_{D_{ij}(x)} B_{ij}(y) dy_i dy_j \pmod{2\pi\mathbb{Z}}.$$

We identify $\mathbb{T}$ with $[-\pi, \pi)$ and set

$$W(x) = \frac{1}{(d-1)4\pi^3 r^3} \sum_{i<j} \int_{\gamma_{ij}(x)} |b_{ij}(y)|^2 dy.$$

Theorem 2. *$H \geq W$ in the operator sense, i.e.,*

$$\langle \psi | H\psi \rangle \geq \int_{\mathbb{R}^d} W(x)|\psi(x)|^2 dx, \qquad \psi \in \mathfrak{D}(H).$$

Remark . Since $|b_{ij}(x)| \leq \pi$, we have

$$0 \leq W(x) \leq \frac{1}{(d-1)4\pi^3 r^3} \cdot \frac{d(d-1)}{2} 2\pi r \cdot \pi^2 = \frac{d}{4r^2}, \qquad x \in \mathbb{R}^d.$$

Proof. We denote the velocity operator by

$$v_j = p_j - A_j(x), \qquad j = 1, \dots, d.$$

We parameterize $\gamma_{ij}(x)$ as follows: For $t \in [0, 2\pi r)$, we set

$$y_k(x;t) = \begin{cases} x_k & \text{if } k \neq i, j, \\ x_i + r\cos(t/r) & \text{if } k = i, \\ x_j + r\sin(t/r) & \text{if } k = j. \end{cases}$$

For $\psi \in \mathcal{S}(\mathbb{R}^d)$, we compute

$$E_x(\psi) = \int_{\gamma_{ij}(x)} |\dot{y} \cdot (v\psi)|^2 dy, \tag{2.1}$$

where $\dot{y}(x;t) = \frac{d}{dt} y(x;t)$. If we denote

$$\tilde{\psi}(t) = \psi(y(x;t)), \qquad \tilde{A}(t) = \dot{y}(x;t) \cdot A(y(x;t)),$$

then the right hand side of (2.1) is

$$\int_0^{2\pi r} \left|(-i\partial_t - \tilde{A}(t))\tilde{\psi}(t)\right|^2 dt = \langle \tilde{\psi} | h\tilde{\psi} \rangle$$

where h is an operator on $L^2([0, 2\pi r))$ defined by

$$h\varphi(t) = (-i\partial_t - \tilde{A}(t))^2 \varphi(t), \qquad \varphi \in \mathfrak{D}(h) = H^2(\mathbb{R}/(2\pi r\mathbb{Z})),$$

i.e., $\varphi \in \mathfrak{D}(h)$ satisfies the periodic boundary condition. Then using the gauge transform $G(t) = \exp(i\int_0^t \tilde{A}(s)ds)$, we can easily observe that h is unitarily equivalent to $\tilde{h}$ defined by

$$\begin{aligned} &\tilde{h}\varphi(t) = -\partial_t^2 \varphi(t), \\ &\varphi \in \mathfrak{D}(\tilde{h}) = \left\{\varphi \in H^2((0, 2\pi r)) \;\middle|\; \varphi(2\pi r) = e^{ib}\varphi(0)\right\}, \end{aligned}$$

where $b \equiv \int_0^{2\pi r} \tilde{A}(t)dt \pmod{2\pi\mathbb{Z}}$. Now we note

$$\begin{aligned} \int_0^{2\pi r} \tilde{A}(t)dt &= \int_{\gamma_{ij}(x)} \dot{y} \cdot A(y(x;t))dt \\ &= \int_{D_{ij}(x)} B_{ij}(y)dy_i dy_j \equiv b_{ij}(x) \pmod{2\pi\mathbb{Z}} \end{aligned}$$

by the Stokes formula. Thus $b = b_{ij}(x)$. The spectrum of $\tilde{h}$ is easily computed, and we learn

$$\sigma(\tilde{h}) = \left\{ \left(\frac{b}{2\pi r} + \frac{n}{r} \right)^2 \;\middle|\; n \in \mathbb{Z} \right\}.$$

In particular,

$$\inf \sigma(h) = \inf \sigma(\tilde{h}) = \left(\frac{b_{ij}(x)}{2\pi r} \right)^2 = \frac{1}{4\pi^2 r^2} |b_{ij}(x)|^2.$$

This implies

$$E_x(\psi) = \langle \tilde{\psi} | h \tilde{\psi} \rangle \geq \inf \sigma(h) \|\tilde{\psi}\|^2 = \frac{1}{4\pi^2 r^2} |b_{ij}(x)|^2 \|\tilde{\psi}\|^2.$$

Integrating this inequality over $x \in \mathbb{R}^d$, we have

$$\begin{aligned} \int_{\mathbb{R}^d} E_x(\psi) dx &\geq \frac{1}{4\pi^2 r^2} \int_{\mathbb{R}^d} \int_0^{2\pi r} |b_{ij}(x)|^2 |\psi(y(x;t))|^2 dt dx \\ &= \frac{1}{4\pi^2 r^2} \int_{\mathbb{R}^d} \int_0^{2\pi r} |b_{ij}(y(x';t'))|^2 |\psi(x')|^2 dt' dx' \qquad (2.2) \end{aligned}$$

where $x' = y(x,t)$ and $t' \equiv t + \pi r \pmod{2\pi\mathbb{Z}}$.

Now we estimate the left hand side of (2.2). We use the same change of variables as above to learn

$$\begin{aligned} \int_{\mathbb{R}^d} E_x(\psi) dx &= \int_{\mathbb{R}^d} \int_0^{2\pi r} \left| \dot{y}(x;t) \cdot \psi(y(x;t)) \right|^2 dt dx \\ &= \int_{\mathbb{R}^d} \int_0^{2\pi r} \left\{ \sin^2(t/r) |v_i \psi(x)|^2 + \cos^2(t/r) |v_j \psi(x)|^2 \right\} dt dx \\ &= \pi r \int_{\mathbb{R}^d} \left\{ |v_i \psi(x)|^2 + |v_j \psi(x)|^2 \right\} dx. \end{aligned}$$

Combining this with (2.2), we obtain

$$\begin{aligned} &\int_{\mathbb{R}^d} \left\{ |v_i \psi(x)|^2 + |v_j \psi(x)|^2 \right\} dx \\ &\qquad\qquad \geq \frac{1}{4\pi^3 r^3} \int_{\mathbb{R}^d} \int_0^{2\pi r} |b_{ij}(y(x;t))|^2 |\psi(x)|^2 dt dx. \end{aligned}$$

Then we sum up this estimate over $1 \leq i < j \leq d$ to conclude

$$(d-1) \sum_{i=1}^{d} |v_i \psi(x)|^2 \geq \frac{1}{4\pi^3 r^3} \int_{\mathbb{R}^d} \sum_{i<j} |b_{ij}(y(x;t))|^2 |\psi(x)|^2 dt dx.$$

and this implies the assertion of Theorem 2. □

Example 1. Let H be the Schrödinger operator on $\mathbb{R}^2$ with constant magnetic field with $B > 0$. We choose $r > 0$ so that $b = b_{12} = \pi$, i.e., $r = B^{-1/2}$. Then we have $W(x) = B/2$, and hence $H \geq B/2$. It is well-known, however, that $\inf \sigma(H) = B$, and the estimate is not optimal.

The following estimate (or, strictly speaking, its generalization) will be useful in the proof of Theorem 1.

Corollary 3. *Let W as above, and let $H_0 = p^2$ be the free Schrödinger operator. Then*

$$\inf \sigma(H) \geq \frac{1}{2} \inf \sigma(H_0 + W).$$

Proof. By Kato's inequality ([6] Theorem X.33), we have

$$\langle \psi | H \psi \rangle \geq \langle |\psi| \, | H_0 |\psi| \rangle, \qquad \psi \in \mathcal{S}(\mathbb{R}^d).$$

Combining this with Theorem 1, we learn

$$\langle \psi | H \psi \rangle \geq \frac{1}{2} \langle |\psi| \, |(H_0 + W) |\psi| \rangle,$$

and hence

$$\begin{aligned}
\inf \sigma(H) &= \inf_{\psi \neq 0} \frac{\langle \psi | H \psi \rangle}{\|\psi\|^2} \\
&\geq \frac{1}{2} \inf_{\psi \neq 0} \frac{\langle |\psi| \, |(H_0 + W)|\psi| \rangle}{\|\psi\|^2} \geq \frac{1}{2} \inf \sigma(H_0 + W).
\end{aligned}$$

(Note that the last inequality is in fact equality, since the ground state of Schrödinger operator is nonnegative.) □

3 Proof of Theorem 1

As in the last section, we fix $A(x)$ and consider a deterministic magnetic Schrödinger operator $H = (p - A)^2$. Let $\Lambda \subset \mathbb{R}^d$ be an open set. Then the Neumann Hamiltonian H_Λ on $L^2(\Lambda)$ is defined by

$$\begin{aligned}
&H_\Lambda \psi(x) = (p - A(x))^2 \psi(x), \qquad \psi \in \mathfrak{D}(H_\Lambda), \\
&\mathcal{Q}(H_\Lambda) = \mathfrak{D}((H_\Lambda)^{1/2}) \\
&\qquad = \big\{ \psi \in H^1_{loc}(\Lambda) \,\big|\, (p_j - A_j(x))\psi \in L^2(\Lambda), j = 1, \dots, d \big\}.
\end{aligned}$$

Note that $\mathcal{Q}(H_\Lambda)$ is not necessarily the same as $H^1(\Lambda)$ if Λ is unbounded. If $\tilde{\Lambda} \subset \Lambda$ are open sets in $\mathbb{R}^d$ such that $|\Lambda \setminus \tilde{\Lambda}| = 0$, then we can regard $L^2(\Lambda) = L^2(\tilde{\Lambda})$, and H_Λ and $H_{\tilde{\Lambda}}$ may be considered as operators on the same Hilbert space. Since $\mathcal{Q}(H_{\tilde{\Lambda}}) \supset \mathcal{Q}(H_\Lambda)$ and the symbols are the same, we learn

$$H_{\tilde{\Lambda}} \leq H_\Lambda \qquad \text{on } L^2(\Lambda).$$

Hence, in particular, by the min-max principle ([6] Theorem XIII-1) we have

$$\frac{1}{|\Lambda|} \#\{\text{eigenvalues of } H_\Lambda \leq E\} \leq \frac{1}{|\Lambda|} \#\{\text{eigenvalues of } H_{\tilde{\Lambda}} \leq E\}$$

for $E \in \mathbb{R}$ if $|\Lambda|$ is finite. We denote

$$C(L; x) = \big\{ y \in \mathbb{R}^d \,\big|\, x_j < y_j < x_j + L, j = 1, \dots, d \big\},$$

for $L > 0$, $x \in \mathbb{R}^d$. In particular, we write $\Lambda_L = C(L, 0)$. We will need the following a priori bound on the number of eigenvalues.

Lemma 4. *Suppose that $|B(x)| \le M$ with $M < \infty$. Then there exists a constant $C > 0$ depending only on d and M such that*

$$\frac{1}{|\Lambda_L|}\#\{\text{eigenvalues of } H_{\Lambda_L} \le E\} \le C(1+E^{d/2})$$

for any $E \ge 0$ and for any integer $L > 0$.

Proof. Let $\Lambda = \Lambda_L$, and let

$$\tilde{\Lambda} = \bigcup\{C(1,n) \mid n \in \mathbb{Z}^d, 0 \le n_j < L, j = 1,\dots,d\}.$$

Then $\tilde{\Lambda} \subset \Lambda$ and $|\Lambda \setminus \tilde{\Lambda}| = 0$, and hence it suffices to estimate

$$\frac{1}{|\Lambda_L|}\#\{\text{e.v.'s of } H_{\tilde{\Lambda}} \le E\} = \Lambda^{-d} \sum_{\substack{0 \le n_j < L \\ j=1,\dots,d}} \#\{\text{e.v.'s of } H_{C(1,n)} \le E\} \tag{3.1}$$

We note that the number of eigenvalues are independent of the choice of gauge. Hence we may choose $A(x)$ on each $C(1,n)$ so that $|A_j(x)| \le M$ $(j = 1,\dots,d)$. Then

$$\begin{aligned} H_{C(1,n)} &= p^2 - A\cdot p - p\cdot A + A^2 \\ &= \frac{1}{2}p^2 + \frac{1}{2}(p-2A)^2 - A^2 \\ &\ge \frac{1}{2}p^2 - dM^2. \end{aligned}$$

Hence we learn

$$\begin{aligned} \#\{\text{e.v.'s of } H_{C(1,n)} \le E\} &\le \#\{\text{e.v.'s of } (\tfrac{1}{2}p^2 - dM^2) \le E\} \\ &\le c_1(E+dM)^2 \le c_2(1+E^{d/2}) \end{aligned} \tag{3.2}$$

with some $c_1, c_2 > 0$. The lemma now follows from (3.1) and (3.2). □

We also need a generalization of the results of the last section for the Neumann Hamiltonian. Let $r > 0$ and $W(x)$ as in the last section. For $\Lambda \subset \mathbb{R}^d$, we set

$$W_\Lambda(x) = \begin{cases} W(x) & \text{if } \operatorname{dist}(x,\Lambda^c) > r, \\ 0 & \text{if } \operatorname{dist}(x,\Lambda^c) \le r. \end{cases}$$

Then, by exactly the same argument as in the last section, we can prove the following estimates:

Lemma 5. *$H_\Lambda \ge W_\Lambda$ as operators on $L^2(\Lambda)$. Hence, in particular,*

$$\inf\sigma(H_\Lambda) \ge \frac{1}{2}\inf\sigma(H_{0,\Lambda} + W_\Lambda),$$

where $H_{0,\Lambda}$ denotes the free Schrödinger operator with the Neumann boundary condition on Λ.

Proof of Theorem 1. Here we suppose that $B = B^\omega$ satisfies Assumption A. Let $L > 0$ and we set

$$k_L(E) = \frac{1}{|\Lambda_L|}\mathbb{E}(\#\{\text{e.v.'s of } H_L^\omega \le E\}), \qquad E \in \mathbb{R},$$

where H_L^ω is the Neumann Hamiltonian on Λ_L. Then we can show

$$k(E) \le k_L(E), \qquad \text{for any } E \text{ and } L, \tag{3.3}$$

using exactly same argument as in the case of Schrödinger operator with random potential (cf. [3] Chapter 7, Corollary 2). Now we apply the argument of Kirsch-Martinelli [4]. It follows from Assumption A-(iii) that $W(x)$ satisfies Assumption B-ii) of [4]. By Assumption A-(iv), we have

$$\mathbb{E}\Big(\int_{C_0} |W(x)|dx\Big) > 0$$

and hence

$$\mathbb{E}(|\{x \in C_0 \mid W(x) = 0\}|) < 1.$$

Then we learn from the argument of Section 3 of [4] that there are c_1, $c_2 > 0$ such that if $L \sim c_1 E^{-1/2}$ then

$$\mathbb{P}\big(\inf \sigma(H_{0,\Lambda_L} + W_{\Lambda_L}) \le E\big) \le \exp(-c_2 E^{-d/2}), \qquad E \sim 0.$$

We note that the support of $W - W_{\Lambda_L}$ is contained in a neighborhood of $\partial\Lambda_L$ with the width r, and the volume is $O(L^{d-1})$. Hence the difference of W and W_{Λ_L} does not affect the large deviation argument. This and Lemma 5 imply

$$\mathbb{P}\big(\inf \sigma(H_L^\omega) \le E\big) \le \exp(-c_3 E^{-d/2})$$

with $c_3 > 0$. Combining this with Lemma 4, we have

$$\begin{aligned} k_L(E) &\le C(1 + E^{d/2}) \cdot \mathbb{P}\big(\inf \sigma(H_L) \le E\big) \\ &\le \exp(-c_4 E^{-d/2}) \end{aligned}$$

with $c_4 > 0$ if E is sufficiently small. Theorem 1 follows from this and (3.3). $\square$

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