Nontwist area preserving maps with reversing symmetry group

Emilia Petrisor

Department of Mathematics
"Politehnica" University of Timisoara
Str. M. Viteazul, nr. 1
1900 Timisoara, Romania

Abstract

The aim of this paper is to give a theoretical explanation of the rich phenomenology exhibited by nontwist mappings of the cylinder, in numerical experiments reported in [del-Castillo *et al*, 1996], [Howard & Humpherys, 1995], and to give new insights in the dynamics of nontwist standard–like maps. Our approach is based on the reversing symmetric properties of the nontwist standard–like systems. We relate the bifurcation of periodic orbits of such maps to their position with respect to a group–invariant circle, and to a critical circle, at whose points the twist property is violated. Moreover, we show that appearance of the meanders, i.e. homotopically nontrivial invariant circles of the cylinder, that are not graphs of real functions of angular variable, is a global bifurcation of a curve on the cylinder, with nodal–type singularities, and invariant with respect to the action of a reversing symmetry group.

1 Introduction

Area preserving diffeomorphisms of the annulus $\mathbb{S}^1 \times [a, b]$ or infinite cylinder $\mathbb{S}^1 \times \mathbb{R}$ were extensively studied since they represent the dynamics of Poincaré maps associated to two degree of freedom Hamiltonian systems. An area preserving map (a.p.m) $F : \mathbb{S}^1 \times \mathbb{R} \to \mathbb{S}^1 \times \mathbb{R}$ is a positive (negative) monotone twist map if one of its lifts $\tilde{F} : \mathbb{R}^2 \to \mathbb{R}^2$ is such that the function $y \to p_1 \tilde{F}(x, y)$ is strictly increasing (decreasing), for any $x \in \mathbb{S}^1$ ($p_1 : \mathbb{R}^2 \to \mathbb{R}$ is the projection $p_1(x, y) = x$). The basic properties of the monotone twist area preserving maps can be found in [Katok, A. & Hassenblatt B, 1995]. One of them is that a homotopically nontrivial F–invariant circle is the graph of a continuous function $\varphi : \mathbb{S}^1 \to \mathbb{R}$.

Last years physical phenomena whose evolution is described by the dynamics of nontwist area preserving diffeomorphisms were studied. Namely, nontwist maps are appropriate models for some processes in hydrodynamics, plasma physics, accelerator physics, celestial mechanics (see [del-Castillo *et al*, 1996], [Howard & Humpherys, 1995] and references therein).

The prototype for such a map is a two parameter family of area preserving nontwist diffeomorphisms $F_{\omega,k}$ of an annulus $\mathbb{S}^1 \times [a, b]$:

$$F(x, y) = \left(x + f_\omega \left(y - \frac{k}{2\pi} g(2\pi x) \right) \pmod 1,\ y - \frac{k}{2\pi} g(2\pi x) \right), \tag{1}$$

where $f_\omega : \mathbb{R} \to \mathbb{R}$, called *twist function*, has a unique critical point in (a, b), and it is an extremum, while g is a 2π–periodic function of zero average.

Numerical investigations, that revealed a wealth of phenomena in the dynamics of such maps, were done on the particular systems corresponding to $g(x) = \sin x$, and $f_\omega(y) = y - \omega y^2$ [Howard & Humpherys, 1995], repectively $f_\omega(y) = \omega(1 - y^2)$ [del-Castillo *et al*, 1996], [del-Castillo *et al*, 1997]. Mainly, one has detected collision and annihilation of periodic points, global bifurcations of some invariant circles, scaling properties.

A nontwist map corresponding to $g(x) = \sin x$, is nongeneric. In the generic case g has infinitely many harmonics. As an example we will take $g(x) = \sin(x - \arcsin(\epsilon \sin x))$, $\epsilon \in (0, 1)$.

As far as we know, there exist only few theoretical results concerning nontwist a.p.m. [Simó, 1998], [Delshams & de la Llave, 1998], [Franks & Le Calvez, 1999]. C. Simó [1998] proves the persistence of invariant circles in a generic perturbation of a nontwist area preserving map, and the existence of invariant circles that exhibit foldings in such a way that they are not graphs of functions defined on $\mathbb{S}^1$. These circles are called *meanders*. On the other hand, in [Delshams & de la Llave, 1998] it is proved the persistence of the invariant circles of a two-parameter family of nontwist a.p.m. having rotation number on the boundary of the range of the frequency, and it is given a partial justification of Greene's criterion [Greene, 1979] for nontwist maps (numerical results in [del-Castillo *et al*, 1997] were obtained on the basis of this criterion). The paper [Franks & Le Calvez, 1999] studies regions of instability for nontwist area preserving maps.

In the present work we give a theoretical explanation of the rich phenomenology displayed by the nontwist maps in numerical experiments reported in [del-Castillo *et al*, 1996], [Howard & Humpherys, 1995]. Our approach is based on the reversing symmetric properties of the analysed nontwist area preserving systems. A nontwist standard–like a.p.m. F has a reversing symmetry group $\mathcal{G}$, generated by a classical symmetry (equivariance), and two reversing symmetries: one orientation reversing, and another orientation preserving. The fixed points of the orientation preserving reversors record all global bifurcations underwent by the $\mathcal{G}$–invariant curve on the cylinder. We prove that appearance of the meanders is a global bifurcation of a curve with nodal singularities, invariant with respect to the action of the reversing symmetry group on the cylinder.

2 Properties of Nontwist Standard-like Maps

In the following, consider the circle $\mathbb{S}^1$ identified with the interval $[0, 2\pi)$. Let $F : \mathbb{S}^1 \times \mathbb{R} \to \mathbb{S}^1 \times \mathbb{R}$ be a nontwist a.p.m., $\Pi : \mathbb{R}^2 \to \mathbb{S}^1 \times \mathbb{R}$, the covering projection defined by $\Pi(X, Y) = (X \pmod{2\pi}, Y)$, and $\tilde{F} : \mathbb{R}^2 \to \mathbb{R}^2$ a lift of F to the covering space $\mathbb{R}^2$, i.e. $F\Pi = \Pi\tilde{F}$. Associate to a point $(x, y) \in \mathbb{S}^1 \times \mathbb{R}$, the sequence $(X_n, Y_n) = \tilde{F}^n(X_0, Y_0)$, where $n \in \mathbb{N}$, and (X_0, Y_0) is a point of the preimage $\Pi^{-1}(x, y) \subset \mathbb{R}^2$. If the limit:

$$\rho = \lim_{n \to \infty} \frac{X_n - X_0}{2n\pi} \tag{2}$$

exists, then it does not depend on the lift $\tilde{F}$, and it is called the *rotation number* of the orbit generated by $(x, y) \in \mathbb{S}^1 \times \mathbb{R}$.

An orbit $\mathcal{O}(x, y) = \{(x_n, y_n) = F^n(x, y), n \in \mathbb{Z}\}$ of a nontwist a.p.m of the cylinder is called *rotational orbit* if there is a homotopically nontrivial circle on the cylinder, that interpolates the points of the orbit (here a circle is meant in the topological sense).

Let $(x, y) = \Pi(X, Y)$ be a q–periodic point of the map F, i.e. $F^q(x, y) = (x, y)$. Denoting by $(X_q, Y_q) = \tilde{F}^q(X, Y)$, we have $\Pi(X_q, Y_q) = \Pi(X, Y)$. Hence $X_q - X \in 2\pi\mathbb{Z}$, $Y_q = Y$, and thus a q-periodic point of F is the projection of a point $(X, Y) \in \mathbb{R}^2$ such that

$$\begin{aligned} X_q &= X + 2p\pi, \quad p \in \mathbb{Z} \\ Y_q &= Y \end{aligned} \tag{3}$$

The periodic point $(x, y) = \Pi(X, Y)$ is called (p, q)-periodic point or periodic point of type (p, q).

The behaviour of the nontwist maps, investigated in [del-Castillo *et al*, 1996], [del-Castillo *et al*, 1997] and [Howard & Humpherys, 1995], is captured by the *model map* defined on the infinite cylinder $\mathbb{S}^1 \times \mathbb{R}$, $F_{\omega,k}(x, y) = (\overline{x}, \overline{y})$:

$$F_{\omega,k}(x, y) = (x + f_\omega(\overline{y}) \pmod{2\pi}, y + k \sin x), \tag{4}$$

where $f_\omega(y) = 2\pi\omega + y^2$. This choice for the form of the map, instead of that of type Eq. (1) is subjective (it allows faster guesses for particular initial conditions in different simulations). All figures in this paper, excepting Fig. 4, display different features of the system defined by Eq. (4).

For $k = 0$, the model map is an integrable system:

$$\begin{aligned} \overline{x} &= x + 2\pi\omega + y^2 \pmod{2\pi} \\ \overline{y} &= y \end{aligned} \tag{5}$$

The twist condition is violated along the circle $\mathbb{S}^1 \times \{0\}$, i.e. $\dfrac{\partial \tilde{F}_1}{\partial y}(x, 0) = 0$, $\forall x \in \mathbb{R}$. Each point on the invariant circle $\mathbb{S}^1 \times \{y\}$ has the rotation number equal to $\omega + y^2/2\pi$. Hence the critical circle is the circle of minimum rotation number among all circles that foliate the annulus (if we chose $f_\omega(y) = 2\pi\omega - y^2$, then this would be the circle of maximum rotation number).

Observe that the map f_ω in all families of nontwist a.p.m. mentioned here, is symmetric with respect to a line $y = y^*$ (i.e. $f(2y^* - y) = f(y)$, $\forall y \in \mathbb{R}$), and it is either a strictly convex or a strictly concave function.

In the following, consider the family of diffeomorphisms $F_{\omega,k}$ of the cylinder defined by a strictly convex twist function f_ω, which is symmetric with respect to the line $y = y^*$, (y^* is the point of extremum for f_ω), and $g(x) = \sin x$. Call such a map *nontwist standard-like map*. The systems corresponding to strictly concave twist functions exhibit similar behaviour.

Next, we drop the indices ω, k and (mod 2π) for the first component of each mapping of the cylinder. If $\tilde{F}$ is a lift of the nontwist standard–like map F, then $\dfrac{\partial \tilde{F}_1}{\partial y} = 0$ along the curve $C : y + k \sin x = y^*$. This curve is called the *nonmonotone curve* or *circle* of the nontwist map F. It is not a F–invariant circle.

Definition 2.1 *An orbit $\{(x_n, y_n), n \in \mathbb{Z}\}$, of a nontwist standard–like map F is a pseudomonotone orbit if there exist $i, j \in \mathbb{Z}$, such that $f'(y_i + k\sin x_i) > 0$, and $f'(y_j + k \sin x_j) < 0$. A q–periodic orbit of F, $\{z_0, z_1, \ldots, z_{q-1}\}$, $z_i = (x_i, y_i)$, for which there exists $j = \overline{0, q-1}$, such that $z_j \in C$, is called nonmonotone periodic orbit.*

A nontwist standard-like map is symmetric with respect to the transformation S of the cylinder, defined by

$$S(x, y) = (x + \pi, 2y^* - y)\,, \tag{6}$$

i.e. $SF = FS$. Observe that $S^2 = \mathrm{id}$, that is S is an involution of the cylinder. On the other hand, F is reversible with respect to the involution R defined by:

$$R(x, y) = (-x, y + k \sin x)\,, \tag{7}$$

i.e. $F^{-1} = RFR$. R is called *involutive reversor* of the map F. F decomposes as a product of involutions $F = IR$, where

$$I(x, y) = (-x + f(y), y)\,. \tag{8}$$

Associate to these involutions the basic symmetry lines denoted $\mathcal{L}_0, \mathcal{L}_1$, $\mathcal{L}_0 := \mathrm{Fix}(R)$, and $\mathcal{L}_1 := \mathrm{Fix}(I)$. Each symetry line has two components:

$$\begin{aligned} &\mathcal{L}_0 \left\{ \begin{array}{l} \Gamma_0 : x = 0 \\ \Gamma_0' : x = \pi \end{array} \right. \\ &\mathcal{L}_1 \left\{ \begin{array}{l} \Gamma_1 : x = f(y)/2 \pmod{2\pi} \\ \Gamma_1' : x = f(y)/2 + \pi \pmod{2\pi} \end{array} \right. \end{aligned} \tag{9}$$

Observe that

$$S(\Gamma_i) = \Gamma_i', \ i = 0, 1 \tag{10}$$

Next we recall some properties of periodic orbits of a reversible system.

Definition 2.2 *A R–invariant orbit $\mathcal{O}(x, y) = \{f^j(x, y), j \in \mathbb{Z}\}$ of the R-reversible system F, i.e. $R(\mathcal{O}(x, y)) = \mathcal{O}(x, y)$, is called R-symmetric orbit or orbit symmetric with respect to R.*

Denote by $\mathcal{L}_{2n} = F^n(\mathcal{L}_0)$ and $\mathcal{L}_{2n+1} = F^n(\mathcal{L}_1)$, and call them symmetry lines of order $2n$, respectively, $2n + 1$. It is known that periodic orbits of a reversible system, which are R–symmetric, can be found as the intersections of the symmetry lines $\mathcal{L}_i, \mathcal{L}_j$. Namely, a point belonging to $\mathcal{L}_i \cap \mathcal{L}_j$ is periodic and its period q divides $|i - j|$.

R–symmetric periodic orbits have a special position on the basic symmetry lines [de Vogelaere, 1958]:

Proposition 2.1 *A R–symmetric q-periodic orbit $\mathcal{O}(x, y)$ of the revesible diffeomorphism F, $F = IR$, has the property:*

i) if q is odd, the orbit has a unique point on the symmetry line $\mathcal{L}_0$, and a unique point on $\mathcal{L}_1$. If $(x, y) \in \mathcal{L}_0$ then $F^{(q+1)/2}(x, y) \in \mathcal{L}_1$.

ii) if q is even, the orbit does not intesect both symmetry lines $\mathcal{L}_0$, $\mathcal{L}_1$, but it intersects twice one of them. If $(x, y) \in \mathcal{L}_j$, then $F^{q/2}(x, y) \in \mathcal{L}_j, j = 0, 1$.

Because the symmetry S is an involution that commutes with R and I, it is straightforward to see that SR and SI are involutions, too, and F decomposes as $F = (SI)(SR)$. Denote by $\mathcal{G}$ the group generated by S, R, I and call it the *reversing symmetry group* of the nontwist standard-like map F. For a more general reversing symmetry group of a map see [Lamb, 1992]. The name of reversing symmetry group is due to the fact that it contains symmetries of the map F, and reversing symmetries i.e. involutive reversors of F. The composition of two reversors is a symmetry (F is a symmetry for itself), and the composition of a symmetry with an involutive reversor is an involutive reversor (in the case of a nontwist standard–like map, not in general).

While the reversors R and I are orientation reversing, SR and SI are orientation preserving. Hence their fixed point set is not a curve, but two points: $\mathrm{Fix}(RS) = \{\mathbf{p}_0, \mathbf{p}_0'\}$, where

$$\mathbf{p}_0\left(\frac{\pi}{2}, y^* - \frac{k}{2}\right), \ \mathbf{p}_0'\left(\frac{3\pi}{2}, y^* + \frac{k}{2}\right), \tag{11}$$

and $\mathrm{Fix}(IS) = \{\mathbf{p}_1, \mathbf{p}_1'\}$,

$$\mathbf{p}_1\left(\left(\frac{1}{2}f(y^*) + \frac{\pi}{2}\right) \pmod{2\pi}, y^*\right), \ \mathbf{p}_1'\left(\left(\frac{1}{2}f(y^*) + \frac{3\pi}{2}\right) \pmod{2\pi}, y^*\right) \tag{12}$$

Observe that

$$S(\mathbf{p}_0) = \mathbf{p}_0', \ R(\mathbf{p}_0) = \mathbf{p}_0' \tag{13}$$

and

$$S(\mathbf{p}_1) = \mathbf{p}_1', \;\; I(\mathbf{p}_1) = \mathbf{p}_1' \tag{14}$$

Our purpose is to analyse how the symmetry S and the reversors R and SR manifest themselves in the dynamics of the map F.

The hypothesis that the twist function f of nontwist standard–like map F is a strictly convex function (i.e. $f''(y) > 0$, $\forall\, y \in \mathbb{R}$), ensures the existence of nonmonotone $\mathcal{G}$–invariant annuli (a nonmonotone annulus is one having as the complement, two S–symmetric unbounded components, $U^+, U^- \subset \mathbb{S}^1 \times \mathbb{R}$, and the restriction $F_{|U^+}$, respectively $F_{|U^-}$, is a positive, respectively, a negative monotone twist map). We will show that interesting behaviour, and bifurcations occurs within such an annulus. Denote by A, the largest annulus, whose all rotational orbits are pseudomonotone or monotone.

The dynamics in monotone twist regions is well known (see [Katok & Hassenblatt, 1995]). One of the basic results, for an exact area preserving monotone twist map of the infinite cylinder, asserts that for any $p \in \mathbb{Z}$, $q \in \mathbb{N} \setminus \{0\}$ relative prime numbers, there exist two periodic orbits of the same period q, and rotation number $\frac{p}{q}$, one elliptic and another hyperbolic. If the system is close to an integrable one, the two orbits form a chain of islands, called Poincaré–Birkhoff chain. Elliptic points are surrounded by invariant circles, and hyperbolic points are connected by heteroclinic orbits.

Because of the S–symmetry of a nontwist standard–like map, Poincaré–Birkhoff chains exist in pairs. S maps the points (x_n, y_n) of an orbit to $(x_n + \pi, 2y^* - y_n)$, $\forall n \in \mathbb{Z}$, hence two S–symmetric orbits have the same rotation number. Denote by C_u^q, C_l^q the upper, respectively, the lower Poincaré–Birkhoff chain of period q, $C_l^q = S(C_u^q)$. Next we investigate the relative position on the basic symmetry lines of elliptic and hyperbolic points of the S–symmetric pair (C_u^q, C_l^q).

Proposition 2.2 *Let C_u^q, C_l^q be two S– symmetric Poincaré–Birkhoff chains of odd period q. The two chains have points of opposite stability type on the same component of a basic symmetry line.*

Proof: Let (x_0, y_0) be an elliptic point of the chain C_u^q and (x_0', y_0') a hyperbolic one. If, for instance, $\mathcal{O}(x_0, y_0)$ intersects Γ_i, and $\mathcal{O}(x_0', y_0')$ intersects Γ_i', $i \in \{0, 1\}$, then because an odd periodic orbit intersects both basic symmetry lines $\mathcal{L}_0$, $\mathcal{L}_1$, we can suppose that $\mathcal{O}(x_0, y_0)$ intersects Γ_j' too, while $\mathcal{O}(x_0', y_0')$ has a point on Γ_j, $j \in \{0, 1\}$, $j \neq i$. But $S(\Gamma_i) = \Gamma_i'$, hence $S(\mathcal{O}(x_0, y_0))$ intersects Γ_i' and Γ_j, while $S(\mathcal{O}(x_0', y_0'))$ intersects Γ_i and Γ_j'. □

Because on the same component of a basic symmetry line there exists two points of opposite stability type, belonging to different chains that are S–symmetric (see Fig.1), when the parameter k is fixed and ω increases, these points approach each other, and can collide.

Proposition 2.3 *Let $\mathcal{O}(x_0, y_0)$ be a periodic orbit of period $2q$ and type t (elliptic or hyperbolic), intersecting one of the two components of the symmetry line $\mathcal{L}_i$, $i = 0, 1$.*

The orbit $S(\mathcal{O}(x_0, y_0))$ has the same type t, and intersects the same component of the line $\mathcal{L}_i$.

Proof: To make a choice suppose that $(x_m, y_m) = F^m(x_0, y_0) \in \Gamma_i$, $i \in \{0, 1\}$. Then, by Eq. (10), $S(x_m, y_m)$ lies on Γ_i'. Because an orbit of even period intersects twice the same symmetry line $\mathcal{L}_i$ we have that there is an $n \in \{0, 1, \dots, 2q-1\} \setminus \{m\}$ such that $(x_n, y_n) \in \Gamma_i$. Because equivariances preserve the stability type of periodic orbits, $\mathcal{O}(x_0, y_0)$ and $S(\mathcal{O}(x_0, y_0))$ are both, in the nondegenerate case, either hyperbolic or elliptic.

□

Remark 2.1 *This result shows that even periodic orbits exist in pairs of the same stability type on a component of a basic symmetry line (Fig. 1). In this case a hyperbolic–hyperbolic and elliptic–elliptic collision is possible.*

Fig. 1: Two pairs of Poincaré-Birkhoff chains of period 3, respectively 4, outside the nonmonotone annulus of the model map Eq. (4), corresponding to $k = 0.3$, $\omega = 0.231$; The figure also displays the components of the basic symmetry lines, the nonmonotone circle C and the $\mathcal{G}$–invariant curve γ_s (the curve containing four marked points.

Besides Poincaré–Birkhoff chains, invariant circles of the same rotation number exist in S–symmetric pairs, too.

The amount of (reversing) symmetry of a subset Λ of the cylinder is measured by its isotropy subgroup $\Sigma_\Lambda \subseteq \mathcal{G}$,

$$\Sigma_\Lambda = \{\sigma \in \mathcal{G} |\ \sigma\Lambda = \Lambda\} \tag{15}$$

We are interested in determining the isotropy group of an orbit of the system, or of a F–invariant homotopically nontrivial circle on the cylinder. For example, the union of two S–symmetric invariant circles has $\mathcal{G}$ as isotropy group. This set has two connected components. Next, we are interested whether there exists a connected $\mathcal{G}$-invariant subset Λ of the cylinder or not.

Proposition 2.4 *A nontwist standard–like map has at most one $\mathcal{G}$-invariant homotopically nontrivial circle.*

Proof: Let SR, F, S be the generators of the group $\mathcal{G}$, and $\mathbf{P}_0(\frac{\pi}{2}, y^* - \frac{k}{2})$, $\mathbf{P}_1(\frac{1}{2}f(y^*) + \frac{\pi}{2}, y^*) \in \mathbb{R}^2$. Consider the lift U of SR such that $U(\mathbf{P}_0) = \mathbf{P}_0$, and the lift V of SI, with $V(\mathbf{P}_1) = \mathbf{P}_1$.

U is a nonlinear symmetry with respect to the point $\mathbf{P}_0$, i.e. the differential $DU(\mathbf{P}_0)$ has the matrix

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{16}$$

Hence there exists an infinity of U–invariant curves containing the point $\mathbf{P}_0$. But at most one of these curves has a F-invariant projection onto $\mathbb{S}^1 \times \mathbb{R}$, i.e. for all points (X, Y) lying on this curve, and for any $n \in \mathbb{Z}$, there exist m and $l \in \mathbb{Z}$ such that $U\tilde{F}^n(X, Y) = \tilde{F}^m(X, Y) + (2\pi\, l, 0)$ ($\tilde{F} = UV$). If such a curve exists, it contains the point $\mathbf{P}'_0(\frac{3\pi}{2}, y^* + \frac{k}{2})$. Indeed, let n be an arbitrary integer. Then, $U\tilde{F}^n(\mathbf{P}'_0) = \tilde{F}^{-n}U(\mathbf{P}'_0) + (2\pi\, l, 0)$ for some $l \in \mathbb{Z}$ (here for chosen U, V, $l = 0$). Hence $U\tilde{F}^n(\mathbf{P}'_0) = \tilde{F}^{-n}(\mathbf{P}'_0 + (-2\pi, 0)) = \tilde{F}^{-n}(\mathbf{P}'_0) + (-2\pi, 0)$. The projection onto the cylinder of this curve is a homotopically nontrivial circle, because it contains the arc of ends $U(\mathbf{P}'_0)$, $\mathbf{P}'_0$. Denote this circle by γ_s. Obviously it is a connected set.

Finally we have to prove that γ_s is S-invariant. $S\gamma_s = S(F(\gamma_s)) = F(S\gamma_s)$. Hence $S\gamma_s$ is F–invariant. On the other hand, $S\gamma_s = S(SR\gamma_s) = SR(S\gamma_s)$, i.e. $S\gamma_s$ is also SR–invariant. Because there is at most one F, and SR–invariant circle, we can conclude that $S\gamma_s = \gamma_s$. □

The $\mathcal{G}$–invariant circle is in fact the *shearless curve* defined in [del-Castillo *et al*, 1996] using approximation by periodic orbits. The figures in this paper display the $\mathcal{G}$–invariant circle of the model map, as the circle containing four marked points, as in:

Proposition 2.5 *The fixed points $\mathbf{p}_0, \mathbf{p}'_0$, $\mathbf{p}_1, \mathbf{p}'_1$ of the reversors SR, IR, belong to the $\mathcal{G}$–invariant curve.*

Proof: It is clear that $\mathbf{p}_0$ and $\mathbf{p}'_0$ belong to γ_s. Let us show that the orbit of $\mathbf{p}_1$ is SR–invariant. We prove that for any $n \in \mathbb{Z}$ there exists an $m \in \mathbb{Z}$ such that $SRF^n(\mathbf{p}_1) = F^m(\mathbf{p}_1)$. $SRF^n(\mathbf{p}_1) = F^{-n}(SR(\mathbf{p}_1)) = F^{-n}SRIR(\mathbf{p}_1) = F^{-n}F^{-1}\mathbf{p}_1 = F^{-n-1}(\mathbf{p}_1)$. $S(\mathbf{p}_1) = \mathbf{p}'_1$ implies that $\mathcal{O}(\mathbf{p}'_1) \in \gamma_s$. □

The orbits of these particular points are hence related by: $\mathcal{O}(\mathbf{p}_i) = SR(\mathcal{O}(\mathbf{p}'_i))$, $i = 0, 1$. Next we derive necessary and sufficient conditions that the orbits of these points to be S–invariant. First, recall that as in the case of orientation reversing reversors, if $\ell_0 = \mathrm{Fix}(SR)$, and $\ell_1 = \mathrm{Fix}(SI)$, then, denoting by $\ell_{2n} = F^n(\ell_0)$, $\ell_{2n+1} = F^n(\ell_1)$, we have the following result: If $\ell_i \cap \ell_0 \neq \emptyset$, $i \neq 0$, or $\ell_i \cap \ell_1 \neq \emptyset$, $i \neq 1$, then the points of intersection are periodic points of period i, respectively $|i - 1|$.

Proposition 2.6 *The orbit of a point in $\ell_0 \cup \ell_1$ has full isotropy group iff it is a periodic point of even period.*

Proof: First we prove that a point in $\ell_0 \cup \ell_1$, say $\mathbf{p}_0$, has a S–invariant orbit iff it is a periodic point of even period. Indeed, $\mathbf{p}_0$ has a S–symmetric orbit iff for any $n \in \mathbb{Z}$, there is a $m \in \mathbb{Z}$, such that $SF^n(\mathbf{p}_0) = F^m(\mathbf{p}_0)$. By the S–symmetry of the map F, and the fact that $S(\mathbf{p}_0) = \mathbf{p}'_0$, this relation is equivalent to $F^n(\mathbf{p}'_0) = F^m(\mathbf{p}_0)$ or $F^{n-m}(\mathbf{p}'_0) = \mathbf{p}_0$. But this means that $\ell_{2(n-m)} \cap \ell_0 \ni \mathbf{p}_0$, i.e. $\mathbf{p}_0$ is a $2(n - m)$ periodic point.

If one of the points of ℓ_0 (ℓ_1) has an even periodic orbit, then the second one belongs to its orbit. Namely, if $\mathbf{p}_0$ ($\mathbf{p}_1$) is $2q$–periodic then $F^q(\mathbf{p}_0) = \mathbf{p}'_0$ ($F^q(\mathbf{p}_1) = \mathbf{p}'_1$). Hence, because of Eqs. (13)-(14) it is easy to see that even periodic orbits of the points $\mathbf{p}_i$, $i = 0, 1$, have R–symmetric orbits. □

Remark 2.2 *Eq. (10) ensures that a periodic orbit of period $2q$ of one of the points $\mathbf{p}_i$, $i = 0, 1$, contains a point of tangential intersection between one of the lines of $\mathcal{L}_{2q}$ and one of $\mathcal{L}_0$ or one of the lines of $\mathcal{L}_{2q+1}$ and one of $\mathcal{L}_1$.*

3 Description of the Route to Collision of Periodic Points

3.1 Collision of odd periodic orbits

Collision of q–periodic points of a nontwist standard–like map F was detected numerically in [del-Castillo *et al*, 1996], [Howard & Humpherys, 1995]. The route to collision/annihilation is described in [Simó, 1998] using the interpolating Hamiltonian of a small perturbation of an integrable nontwist a.p.m. The approximating Hamiltonian captures the behaviour of the map F^q, corresponding to a fixed small value of k and to a range of the parameter ω, for which the fixed points of the map F^q lie within the annulus A of nonmonotone dynamics.

Next we extract more information on the route to collision, derived from reversing symmetry properties of the interpolating Hamiltonian of a nontwist standard–like map. Exploiting symmetry properties of the nontwist standard–like map F, we are also able to characterize the position of the points of collision with respect to the $\mathcal{G}$–invariant curve, and in some particular cases with respect to the nonmonotone circle C.

the circle

Let us discuss the scenario of elliptic–hyperbolic collision–annihilation of odd periodic points.

In [Simó, 1998] it is derived the interpolating Hamiltonian for a generic perturbation of the integrable nontwist area preserving map

$$F_0(x, y) = (x + \omega + y^2 + O(y^3), y) \tag{17}$$

Incidentally, the chosen truncation:

$$H(x, y) = \mu y + \frac{1}{3}y^3 + \cos x \tag{18}$$

of the Hamiltonian function defines a vector field

$$X_H = \left(-\frac{\partial H}{\partial y}, \frac{\partial H}{\partial x}\right), \tag{19}$$

having a reversing symmetry group $\mathcal{G}$ generated by the reversor $R(x, y) = (-x, y)$, and the symmetry $S(x, y) = (x + \pi, -y)$, i.e. $RX_H = -X_H R$, and $DS \circ X_H = X_H \circ S$. It is known that Hamiltonian normal forms are almost always reversible [Lamb & Roberts, 1998]. The symmetry S, however, induces a feature of the elliptic–hyperbolic collision that is not exhibited by the generic perturbations of the integrable systems, as we shall see next.

Starting with the integrable (unperturbed) system $F_0(x, y) = (x + \omega + y^2 + O(y^3), y))$, which is S–symmetric with respect to the transformation $S(x, y) = (x + \pi, -y)$ one can use the same method as in [Simó, 1998], with some additional assumptions on the perturbation of the generating function of F_0, that ensures S–symmetry for the associated perturbed map F_ε, to construct the interpolating Hamiltonian H for F_ε. We are not interested here in the method of construction of the symmetric interpolating Hamiltonian, but in the further information given by its reversing symmetry properties. Its truncation is now Eq. (18), and it describes the behaviour of the map F^q, q–odd, where F is a nontwist standard–like map corresponding to a fixed small value of k, and a range of ω for which q–periodic orbits are in the nonmonotone annulus A.

For $\mu < 0$ the Hamiltonian system has two pairs of equilibrium points, one above the energy curve $H(x, y) = 0$, and another bellow it (Fig. 2a) :

a) b) c) d)

Fig. 2: The route to collision of equilibrium points of the Hamiltonian system defined by Eq. (18) Discussion in text.

$\mathbf{u}_h(0, \sqrt{-\mu})$, $\mathbf{u}_e(\pi, \sqrt{-\mu})$, respectively $\mathbf{d}_e(0, -\sqrt{-\mu})$, $\mathbf{d}_h(\pi, -\sqrt{-\mu})$, where $\mathbf{u}$ ($\mathbf{d}$) stands for up (down), and the indices e, h for elliptic, respectively, hyperbolic. As μ increases, the two pairs of equilibrium points approach each other on the symmetry lines $x = 0, x = \pi$ of the reversor R, and at $\mu = \mu_c = -(9/4)^{1/3}$ the upper and lower hyperbolic points cross the $\mathcal{G}$–invariant energy curve $H(x, y) = 0$ (Fig. 2b and Fig. 3a). Because SR is a reversor, we have the following relation between the stable and unstable manifolds W^s, W^u, of the corresponding hyperbolic points:

$$SR(W^{s,u}(\mathbf{u}_h)) = W^{u,s}(SR(\mathbf{u}_h)) = W^{u,s}(\mathbf{d}_h) \tag{20}$$

Hence along with hyperbolic points, the union of their F–invariant manifolds is SR–invariant, too, and as a consequece the $\mathcal{G}$–invariant curve $H(x, y) = 0$ is the union of saddles and their invariant manifolds. This curve contains in fact three rotational circles (Fig. 3a): one containing distinct saddles $\mathbf{u}_h, \mathbf{d}_h$ and heteroclinic arcs connecting them. These arcs contain the fixed points $(\pi/2, 0)$, $(3\pi/2, 0)$ of the reversor SR. The other two rotational circles are homoclinic manifolds to $\mathbf{u}_h$, respectively, $\mathbf{d}_h$. These last two rotational circles are not S–invariant, only their union is S–invariant.

In order to find out the sense of flowing on the three circles, evaluate the vector field X_H at the fixed points $(\pi/2, 0)$, $(3\pi/2, 0)$ of the reversor SR. We get that at $\mu = \mu_c$, the sense of flowing on the heteroclinic arcs is anti–clockwise. Hence on the homoclinic arc to $\mathbf{u}_h$ the sense is from the right to left, and the same sense for its S–symmetric arc, homoclinic to $\mathbf{d}_h$.

$\mu = \mu_c$ is a point of global bifurcation of the $\mathcal{G}$–invariant curve $H(x, y) = 0$. At $\mu = \mu_c$ it is a curve with singularities, having a topological type different from that of a circle (Fig 3a). For $\mu < \mu_c$, by implicit function theorem, it is a graph of a differentiable function defined on $\mathbf{S}^1 \equiv [0, 2\pi)$ (Fig 2a), while for $\mu > \mu_c$ the $\mathcal{G}$–invariant curve has foldings in such a way that it is no longer a graph (Fig.2c).

On the other hand, when μ crosses μ_c, the hyperbolic points leave their elliptic partner in the Poincaré–Birkhoff chain, and homoclinic loops to these hyperbolic points encircle elliptic points, lying on the same component of the symmetry line, of the S–symmetric chain (Fig. 2c). A global bifurcation from heteroclinic to homoclinic connections occurs.

For $\mu \leq 0$ not only the $\mathcal{G}$–invariant curve, but also its neighbors $H(x, y) = h$, $h \in (H(\mathbf{d}_h), H(\mathbf{u}_h))$, fold (Fig. 2c). That is why, in [Simó, 1998], they are called *meanders*. Note that as μ increases from μ_c to 0 the interval $(H(\mathbf{d}_h), H(\mathbf{u}_h)) = (\frac{2}{3}(\sqrt{-\mu})^3 - 1, -\frac{2}{3}(\sqrt{-\mu})^3 + 1)$ gets larger, i.e. the strip of meanders enlarges, because new rotational orbits are born (see Fig. 3).

a) b) c) d)

Fig. 3: The birth of orbits as μ increases in $[\mu_c, 0)$. The first picture shows the $\mathcal{G}$–invariant curve, $H(x, y) = 0$, at $\mu = \mu_c$, while the next, the new born S–symmetric meanders.

Remark 3.1 *According to C. Simó [1998] the period of the $\mathcal{G}$–invariant curve decreases as μ increases from μ_c to 0. As a consequence, for each $\mu \in (\mu_c, 0)$ the new–born orbit is the $\mathcal{G}$–invariant circle, while the former has generated two S–symmetric circles. Hence, we have a "central" source of orbits.*

As μ approaches 0, the homoclinic loops shrink to the elliptic points, and at $\mu = 0$ the hyperbollic–elliptic collision occurs (Fig. 2d), and a rotational circle $H(x, y) = H(\mathbf{u}_e) = H(\mathbf{d}_e)$, having cusps at points of collision, is born. For $\mu > 0$ the system has no equilibrium points and no meanders. The energy curves $H(x, y) = h$ are again graphs.

After the above analysis, we can conclude that a nontwist standard–like mapping F exhibits a rich phenomenology in the ranges of ω for which odd periodic points lie within the nonmonotone annulus A of the cylinder. For each fixed small parameter k in the definion of the map F, denote by $\omega_1(p/q)$ the value of the parameter ω for which the S–symmetric hyperbolic odd–periodic orbits of rotation number p/q, enter the $\mathcal{G}$–invariant curve, and by $\omega_2(p/q)$ the collision threshold of the odd periodic orbits having the same rotation number p/q. Note that hyperbolic–elliptic collision occurs in a strip including the $\mathcal{G}$–invariant circle, but not exactly on this circle.

Because of the S–symmetry of the system, collision of the two pairs of SR–symmetric odd periodic points occurs simultaneously. In the generic case, when such a symmetry is not present this collision occurs in stages, as we can see in Fig. 4d, where upper three periodic orbits have already annihilated, while the lower still persist within a homoclinic chain.

a) b)
c) d)

Fig. 4: The phase space of the nontwist a.p.m. $F(x,y) = (\overline{x},\overline{y})$, $\overline{x} = x + f_\omega(\overline{y})$, $\overline{y} = y + k\,\sin(x - \arcsin(\varepsilon \sin(x))$, $k = 0.2$, $\varepsilon = 0.38$, displaying a pair of three periodic chains in a nonmonotone annulus, whose periodic points collide in stages; ω is respectively, a) 0.3285 b) 0.3297655, c) 0.3318, d) 0.332.

In the generic case, the corresponding nontwist map has merely reversing symmetries, and no symmetries (equivariances). More precisely, the system is double weakly reversible [Roberts & Quispel, 1992]. Exploiting this property we will explain, in a forthcoming paper, the collision in stages of periodic points of opposite stability type and common rotation number, observed in the dynamics of Hénon map, too [Van der Weele & Valkering, 1990].

The birth of meanders of nontwist standard-like mappings is a consequence of a global bifurcation of a $\mathcal{G}$–invariant homotopically nontrivial circle on the cylinder, where $\mathcal{G}$ is a group acting on $\mathbb{S}^1 \times \mathbb{R}$:

Proposition 3.1 *Let $\mathcal{G}$ be the group acting on the cylinder $\mathbb{S}^1 \times \mathbb{R}$, generated by the transformations R, S, $R(x,y) = (-x,y)$, $S(x,y) = (x+\pi,-y)$, $\mathbf{S}^1 \equiv [0,2\pi)$. A family c_μ: $\Phi(x,y,\mu) = 0$ of $\mathcal{G}$– invariant analytic curves on a domain $\{(x,y) |\, Im(x) < M_1, |y| < M_2\}$, such that $\Phi(x,y,\mu_0) = 0$ has nodal singular points, undergoes at $\mu = \mu_0$ a global bifurcation: for $\mu < \mu_0$ the curves are graphs of real functions defined on $\mathbb{S}^1$, while for $\mu > \mu_0$ they are not graphs any more.*

Proof: Let Φ be an analytic function on a domain, as above, and its level curves $L_h : \Phi(x,y) = h$. Such a curve is $\mathcal{G}$–invariant if along with a point $(x,y) \in L_h$, $R(x,y)$ and $S(x,y)$ belong to L_h, too. Zero level set of such a function satisfying:

$$\begin{aligned} \Phi(-x,y) &= \Phi(x,y) && (21)\\ \Phi(x+\pi,-y) &= -\Phi(x,y) && (22) \end{aligned}$$

is the only $\mathcal{G}$–invariant curve on the cylinder. In this case Φ is periodic and even in x. Its Fourier expansion is:

$$\Phi(x,y) = a_0(y) + \sum_{n=0}^{\infty} a_n(y) \cos x \tag{23}$$

The S–invariance of the zero level set implies that a_0 is an odd function, the sum is taken over odd natural numbers, $n = 2k+1$, and a_{2k+1} are even analytical functions, for all $k \geq 1$. Hence:

$$\Phi(x,y) = a_0(y) + \sum_{k=0}^{\infty} a_{2k+1}(y) \cos(2k+1)x \tag{24}$$

Truncating $a_0(y)$ at $ay + b\,y^3$, and the sum at $\cos\, x$, let us investigate when the curve $ay{+}b\,y^3{+}\cos\, x = 0$ can have singular points. A simple computation gives that for $ab < 0$ the curve has critical points. Taking $b > 0$ and rescaling, we get $\mu\, y + \frac{1}{3}y^3 + \cos x = 0$, where $\mu = a < 0$. For $\mu_0 = -(9/4)^{1/3}$ our curve has nodal–type singular points. Implicit function theorem ensures that for $\mu < \mu_0$ the curve is a graph, while for $\mu > \mu_0$ it has foldings in such a way that is not a graph.

$\square$

This result explains the appearance of invariant circles that are meanders, in the dynamics of nontwist standard–like maps.

Because of this global bifurcation, and the continuity of the system, the rotational circles, near the $\mathcal{G}$–invariant one, are meanders, too, for $\mu \in (\mu_0, 0)$ (Figs. 3b, 3c, 3d).

The above described route to elliptic/hyperbolic collision of odd periodic orbits of nontwist standard–like mapings does not give any information on the position of periodic points with respect to the nonmonotone circle. Next we analyse this aspect.

Consider a nontwist standard–like map F, whose twist function f has y^* as the extremum point. Hence the nonmonotone circle C has the equation: $y + k\sin x = y^*$. Denote by $\mathbf{n}_0 = (0, y^*)$, $\mathbf{n}'_0 = (\pi, y^*)$ the points of the intersection $C \cap \mathcal{L}_0$, and $\mathbf{n}_1, \mathbf{n}'_1 \in C \cap \mathcal{L}_1$. Their orbits are R–symmetric nonmonotone orbits. Hence if these particular points have rotational orbits, then the $\mathcal{G}$–invariant curve lies "between" the orbits $\mathcal{O}(\mathbf{n}_0)$ $\mathcal{O}(\mathbf{n}'_0)$, respectively $\mathcal{O}(\mathbf{n}_1)$, $\mathcal{O}(\mathbf{n}'_1)$. As a consequence, for k small (i.e. the system is close to an integrable one) if γ_s is a meander, its height, defined as the difference, along the meander, of the maximal and minimal values of y, is less then the minimum of the Hausdorff distances between the pairs of invariant circles or S and R–symmetric nonmonotone chains containing these special points.

For each fixed small parameter k of a nontwist standard–like map, the meander γ_s, corresponding to $\omega \in (\omega_1(p/1), \omega_2(p/1)]$, $p \in \mathbb{Z}$ (i.e. the meander associated to fixed points), seems to have the largest height among all $\mathcal{G}$–invariant meanders associated to any other odd periodic orbit. This property is due to the fact that in this range of the parameter ω, the R–symmetric nonmonotone points have non–rotational orbits, allowing the meanders to exceed the amplitude of the nonmonotone circle C. To illustrate this property, let us analyse the route to collision of the fixed points of the model map Eq. (4) corresponding to $\omega > 0$ (the case $\omega \leq 0$ implies some sign changes). For $n - 1 < \omega \leq n$, F has the fixed points $(0, \pm\sqrt{2n\pi - 2\pi\omega})$, $(\pi, \pm\sqrt{2n\pi - 2\pi\omega})$, of rotation number $p/q = n/1$.

points are points of intersection of the basic symmetry lines $\mathcal{L}_0, \mathcal{L}_1$. Hence their collision takes place when one of the lines of $\mathcal{L}_0$ has tangential contact to one of $\mathcal{L}_1$, i.e. when $\omega = n \in \mathbb{N}$.

The route to collision of fixed points corresponds to the variation of the parameter ω in intervals of the form $(n-1, n]$, $n \in \mathbb{N}$. The points of collision are $(0, 0)$ on Γ_0 and $(\pi, 0)$ on Γ'_0, i.e. collision occurs on the nonmonotone circle C. differential DF, at a p'_0, p_1 have the same shearless curve is not a graph.

Next we "follow" the fixed points in their way towards collision. In this way three topological changes in the phase space occur:

1. For ω close to $n-1$, the points $(0,0)$, $(\pi,0)$ have rotational orbits (see Fig. 5a). As ω increases the fixed points approach each other, hence the homoclinic manifolds to hyperbolic points, that are nearer to C, get closer and closer. At a threshold $\omega_0(n/1)$, the point $(0,0)$ (hence $(\pi,0)$) crosses the homoclinic manifold (Fig. 5b) and a global bifurcation occurs: the rotational orbits of points $(0,0)$, $(\pi,0)$ become orbits on invariant circles that are contractible to elliptic fixed points (Fig. 5c).

a) b)
c) d)

Fig. 5: $k=0.2$; Orbits of the points $(0,0)$, $(\pi,0)$ (these points are marked with $\diamond$, while the points $p_i, p_i' \in \gamma_s$, $i=0,1$, with crosses) for different values of ω; a) $\omega=0.88$: the points $(0,0)$, $(\pi,0)$ have rotational orbits; b) $\omega=0.888345$: orbits homoclinic to hyperbolic fixed points; c) $\omega=0.9$: orbits surrounding elliptic fixed points; d) $\omega=0.955$: the meander γ_s having larger height than the amplitude of the nonmonotone circle C.

2. The second global bifurcation was already discussed above: at $\omega=\omega_1(n/1)$, γ_s bifurcates from a graph to a meander.

3. At the collision threshold $\omega=\omega_2(n/1)$, the points $(0,0)$, $(\pi,0)$ have again rotational orbits. Hence a reconnection of a homotopically nontrivial circle occured. At the reconnection threshold, these points are cusps.

Next we investigate whether the orbits of the special points $\mathbf{n}_0$, $\mathbf{n}_0'$, and $\mathbf{n}_1, \mathbf{n}_1'$ can be included in γ_s or not. Note that the nonmonotone circle C is S–invariant, but not R or F–invariant, and along with $\mathbf{n}_0, \mathbf{n}_0'$, the points $F^{-1}(\mathbf{n}_0)$, $F^{-1}(\mathbf{n}_0')$ belong to C, too.

Proposition 3.2 *Let F be a nontwist standard–like map, having the twist function f_ω. For each small fixed k, the orbits of the R–symmetric nonmonotone points $\mathbf{n}_0(0,y^*)$, $\mathbf{n}_0'(\pi,y^*)$, are included in the graph γ_s, but $\mathcal{O}(\mathbf{n}_1)\not\subset\gamma_s$, iff ω is a solution of the equation $f_\omega(y^*)=(2m+1)\pi$, $m\in\mathbb{Z}$.*

Proof: Suppose that γ_s is the projection of the graph(φ), where $\varphi:\mathbf{R}\to\mathbf{R}$ is a continuous 2π–periodic function. As a consequence of S, and R–invariance of γ_s, φ satisfies the conditions:

$$\begin{aligned}\varphi(x)+\varphi(x+\pi)&=2y^*\\ \varphi(x)-\varphi(-x)&=-k\sin x\end{aligned}\tag{25}$$

The additional information that γ_s contains the points $\mathbf{n}_0(0,y^*)$ and $\mathbf{p}_0(\frac{\pi}{2},y^*-\frac{k}{2})$ leads to $\varphi(x)=-\frac{k}{2}\sin x+y^*$. Taking into account that

$$F^{-1}(0,y^*)=(-f(y^*),y^*+k\sin(f(y^*)))\in C\tag{26}$$

we get that $F^{-1}(0, y^*) \in \gamma_s$, too. This implies $f_\omega(y^*) = n\pi$. For n even, $\mathcal{L}_1$ has tangential contact to $\mathcal{L}_0$, i.e. the fixed points collide. ¿From the analysis of the approximating Hamiltonian, we know that at the collision threshold the circle γ_s is still a meander. For ω that makes $f_\omega(y^*) = (2m+1)\pi$, $m \in \mathbb{Z}$, $\mathcal{L}_2$ has tangential contact to $\mathcal{L}_0$, thus in this case collision of two 2–periodic orbits of the same stability type occurs. Hence for such a value of the parameter ω the orbit of the R–symmetric nonmonotone point $(0, y^*)$ and its S–symmetric point (π, y^*), is included in γ_s which is a graph (see below the collision of even periodic orbits) .

The proof for the converse statement is straightforward. $\square$

Remark 3.2 *Depending on the twist function f_ω, the equation $f_\omega(y^*) = (2m+1)\pi$, can have or not a solution ω. For the model map, and the nontwist standard like–mapings investigated in [del-Castillo et al, 1996], [Howard & Humpherys, 1995], this equation has solutions.*

Proposition 3.3 *For a nontwist standard–like map, the orbit of the point $\mathbf{n}_0(0, y^*) \in C$ is periodic of period three, and included in the $\mathcal{G}$–invariant circle iff it contains one of the points $\mathbf{n}_1, \mathbf{n}_1' \in C \cap \mathcal{L}_1$.*

Proof: Let $F(x, y) = (\overline{x}, \overline{y})$ be a nontwist standard–like map of the cylinder, i.e. $F(x, y) = (x + f_\omega(\overline{y}), y + k \sin x)$. $F^{-1}(0, y^*) = (-f(y^*), y^* + k \sin(f(y^*))) \in C$, and $F(0, y^*) = (f(y^*), y^*)$. $(0, y^*)$ is a periodic point of period three iff $F^2(0, y^*) = F^{-1}(0, y^*)$, i.e. iff $f(y^*) + f(y^* + k \sin(f(y^*)) = -f(y^*) + 2m\pi$, $m \in \mathbb{Z}$. But the last relation means $F^{-1}(0, y^*) \in \mathcal{L}_1$. $\square$

Remark 3.3 *Because of the S–symmetry of the nonmonotone circle and of the $\mathcal{G}$–invariant circle γ_s, the point $\mathbf{n}_0'(\pi, y^*) \in C$ is also a three periodic point, and its orbit lies on γ_s iff $F^{-1}(\pi, y^*) \in \mathcal{L}_1$. Obviously in this case the two S–symmetric three-periodic orbits are orbits of the hyperbolic points (Fig.6a).*

The above discussion points out that R–symmetric nonmomonotone points of a nontwist standard–like map can have periodic orbits on the $\mathcal{G}$–invariant circle.

a) b)

Fig. 6: $k = 0.4$; a) $\omega = 0.660446$: hyperbolic points of period three on the $\mathcal{G}$–invariant curve; the fixed points of the reversors SR, SI have heteroclinic orbits to hyperbolic points. b) $\omega = 0.66066$: the meandering curve γ_s that bifurcated from $\mathcal{G}$–invariant curve displayed in a). The position of the points $\mathbf{p}_i$, $\mathbf{p}_i'$, $i = 0, 1$ on γ_s, is also shown.

In the most cases R–symmetric nonmomonotone points have orbits outside the $\mathcal{G}$–invariant circle. Computer experiments shows that these orbits can densely fill an invariant circle, or can have orbits within a Poincaré–Birkhoff chain. In Fig. 7 the points $\mathbf{n}_0, \mathbf{n}_0'$ have six-periodic elliptic orbits on the two sides of the circle γ_s. We chose

Fig. 7: $k = 0.4$, $\omega = 0.1370511$: R–symmetric nonmonotone points $\mathbf{n}_0, \mathbf{n}_0'$ have six–periodic elliptic orbits. Between their Poincaré–Birkhoff chains lies the $\mathcal{G}$–invariant circle.

this example to show that unlike the elliptic fixed points, elliptic periodic orbits can be nonmonotone. More precisely, if $F = (F_1, F_2)$ is a nontwist area preserving map, and (x_0, y_0) a fixed point, then the differential $DF(x_0, y_0)$ has the matrix:

$$J(x_0, y_0) = \begin{bmatrix} \frac{\partial F_1}{\partial x} & 0 \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{bmatrix} (x_0, y_0) \tag{27}$$

Area preserving property implies $(\frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial y})(x_0, y_0) = 1$, and as the consequence $DF(x_0, y_0)$ cannot have complex eigenvalues.

3.2 Collision of even periodic orbits

The even periodic orbits of a nontwist standard–like map F have a different route to collision from the odd ones. First, by Proposition 2.1 ii) an even periodic orbit intersects both components of a basic symmetry line, and none of the second. Moreover, by Proposition 2.3 two S–symmetric even periodic orbits might undergo a hyperbolic/hyperbolic, elliptic/elliptic collision. The behaviour of the map F^{2q} restricted to a nonmonotone annulus of the cylinder is captured by F^2. Because F is R–reversible, and factorizes as $F = IR$, F^2 is doubly reversible, i.e. it has two reversors as we can see from the decompositions: $F^2 = (IRI)R$, $F^2 = I(RIR)$. Moreover F^2 is S–symmetric, too. Hence collision of even periodic orbits of a nontwist standard like map (or conversely the birth of even periodic orbits) is a bifurcation of the fixed points of a nontwist area preserving map having a reversing symmetry group generated by two involutive reversors R_1, R_2 and an involutive symmetry S. Namely, it is the bifurcation of two fixed points, one R_1–symmetric and another R_2–symmetric. The approximating Hamiltonian for the dynamics of such a map on a nonmonotone annulus, was derived in [Simó, 1998]:

$$H(x, y) = \mu y + \frac{1}{3}y^3 - y \cos 2x \tag{28}$$

Next we point out the symmetry breaking or symmetry increasing bifurcations occuring at the collision of equilibrium points of the Hamiltonian vector field X_H, and the global bifurcations underwent by the $\mathcal{G}$–invariant curve.

The vector field X_H is reversible with respect to the reversors R_1, R_2, defined by $R_1(x, y) = (-x, y)$, $R_2(x, y) = (\pi - x, y)$, and S–symmetric, where $S(x, y) = (-x, -y)$. Because of the symmetry $T = R_2 R_1$, which is a rotation along each circle $\mathbb{S}^1 \times \{y\}$,

$T(x, y) = (x + \pi, y)$, the dynamics of the vector field X_H can be recovered from that corresponding to the Hamiltonian $K = H\varphi$, where $\varphi(x, y) = (x/2, y)$:

$$K(x, y) = \mu y + \frac{1}{3}y^3 - y \cos x \tag{29}$$

The new Hamiltonian vector field X_K defined on the cylinder has the reversing symmetry group $\mathcal{G}$, generated by the reversor $R(x, y) = (-x, y)$, and the symmetry $S(x, y) = (x, -y)$. The circle $\mathbb{S}^1 \times \{0\}$ is $\mathcal{G}$–invariant. For $\mu < -1$, on the symmetry line $x = \pi$ there exists a pair of saddle equilibrium points $(\pi, \pm\sqrt{-1-\mu})$, while for $\mu < 1$ on the symmetry line $x = 0$ are located the centers $(0, \pm\sqrt{1-\mu})$ (Fig 8a). At $\mu = -1$, the two saddles collide, and the resulting point $(\pi, 0)$, which is one of the two fixed points of the reversor SR, has $\mathcal{G}$ as the isotropy group (Fig.8b). For $\mu < -1$ the $\mathcal{G}$–invariant curve is the rotational circle $\mathbb{S}^1 \times \{0\}$. At $\mu = -1$, besides this circle, the energy curve $K(x, y) = 0$ has two additional branches, whose union is $\mathcal{G}$–invariant, namely, the upper ($y > 0$) and the lower ($y < 0$) arc of the closed curve $c: \ -1 + \frac{1}{3}y^2 - \cos x = 0$ (Fig. 8c). Because the sense of flowing on these arcs is from the right to left, while on the the arc $\mathbb{S}^1 \times \{0\} \setminus \{(\pi, 0)\}$ is opposite, each of these arcs form a "vortex" along with the circle $\mathbb{S}^1 \times \{0\}$. Hence we have a pair of S–symmetric vortices. When μ crosses -1 a pair of R–asymmetric saddles are born on $y = 0$: $(\arccos \mu, 0)$, $(2\pi - \arccos \mu, 0)$ (Fig.8d). With these saddles are also associated two S–symmetric vortices (Fig. 8e). As μ increases towards 1, the closed curve $c' : \ \mu + \frac{1}{3}y^2 - \cos x = 0$ shrinks, forcing the two encircled centers to approach each other on the symmetry line $x = 0$, and the two R–asymmetric saddles to get closer on $y = 0$. At $\mu = 1$, these four points collide at $(0, 0)$, which is the second fixed point of the reversor SR, and now it is a point of maximal isotropy group. This collision leads to the annihilation of equilibrium points, because for $\mu > 1$, the vector field has no equilibrium points (Fig. 8f).

a) b) c)
d) e) f)

Fig. 8: The route to collision of equilibrium points of the system defined by the Hamiltonian Eq. (29). In a) and b) $\mathbb{S}^1 \equiv [0, 2\pi)$, while in the next figures $\mathbb{S}^1 \equiv [-\pi, \pi)$. a) Two equilibrium points of the same stability type on each symmetry line; b) Collision of the saddles on $y = 0$; c) The $\mathcal{G}$–invariant curve at the threshold of collision of saddles; d) R–asymmetric saddles, and associated vortices; e) $\mathcal{G}$–invariant curve for $\mu \in (-1, 1)$; f) The phase space after annihilation of equilibria.

Note that besides the collision of equilibrium points at $\mu = -1$, respectively at $\mu = 1$, these values of the parameter μ are also thresholds of global bifurcation of the $\mathcal{G}$–invariant energy curve of the Hamiltonian Eq. (29).

In Figs. 9a, 9b we display the $\mathcal{G}$–invariant curve of the model map corresponding to $k = 0.4$ and ω at the point of colision of period six hyperbolic points , respectively, after the birth of R–asymmetric hyperbolic points.

a) b)

Fig. 9: $k = 0.4$. The six periodic chains shown in Fig. 7, entered the nonmonotone annulus and merged; a) At $\omega = 0.153285015$, the parabolic orbit has $\mathcal{G}$ as isotropy group, and the arcs of invariant manifolds form pairs of vortices encircling elliptic points; b) For $\omega = 0.1533$ it is shown the special topological type of the $\mathcal{G}$–invariant curve.

We can conclude that the global bifurcations of the $\mathcal{G}$–invariat curve of a nontwist standard like map are recorded by the fixed points of the orientation preserving involutive reversors SR, and SI. At the threshold of bifurcation to a meander, these special points have heteroclinic orbits to odd periodic hyperbolic points. Thus this threshold is also one of bifurcation of the limit sets of all the points $\mathbf{p}_i, \mathbf{p}'_i$, $i = 0, 1$. In the nongeneric case, at the thresholds of global bifurcations of the $\mathcal{G}$–invariat curve, one of the pair of points $(\mathbf{p}_0, \mathbf{p}'_0)$, $(\mathbf{p}_1, \mathbf{p}'_1)$, and only one, lies on an even periodic orbit of maximal isotropy group. Hyperbolic points collide first, and then elliptic ones. Before collision, each orbit of hyperbolic points has as the isotropy group, the group generated by R. At the collision threshold the isotropy group is $\mathcal{G}$ and the new born R–asymmetric hyperbolic orbits lie on the $\mathcal{G}$–invariant curve but they have trivial isotropy group. At the threshold of collision of elliptic orbits and hyperbolic points coming from transverse direction we have again a full isotropy group for the parabolic orbit.

3.3 Higher order meanders

¿From the analysis of the approximating Hamiltonian Eq. (18) we found out that, for $\mu \in (\mu_c, 0)$, new orbits of the associated vector field X_H are born. As a consequence, in the case of a nontwist standard–like map corresponding to a fixed small value of the parameter k, if p_1, q_1 are relative prime integers, q_1, an odd number, and $\omega_1(p_1/q_1)$ is the point of bifurcation of the $\mathcal{G}$–invariant circle from a graph to a meander, then for $\omega \in (\omega_1(p_1/q_1), \omega_2(p_1/q_1))$, γ_s is a meander. In this case we call it the first order meander associated to the periodic orbits of type (p_1, q_1). In the case of the model map corresponding to $k = 0.3$, $p_1 = 1, q_1 = 1$, we have $\omega_1(1/1) = 0.9065375$ and $\omega_2(1/1) = 1$. As ω increases from ω_1 to ω_2, new orbits are born. By Remark 3.1, there is a "central" source of orbits. Namely, orbits of irrational rotation number, and even periodic orbits appear on the $\mathcal{G}$–invariant circle, while odd periodic orbits of type (p_2, q_2) appear within a narrow strip containing γ_s, but not on γ_s. For p_2/q_2 sufficiently close to the rotation number ρ of the $\mathcal{G}$–invariant circle, i.e. at $\omega_2(p_2/q_2)$ the (p_2, q_2)–type orbits are born as cusps on two S–symmetric invariant circles, and each cusp bifurcates to a pair of points belonging to R–symmetric (p_2, q_2)–type orbits, one elliptic, and another hyperbolic, lying each within a homoclinic chain, not a heteroclinic one. Moreover at the threshold $\omega_2(p_2/q_2)$ the $\mathcal{G}$–invariant curve bifurcates from a first order meander to a second order meander, i.e. γ_s folds further, new q_2 folds being generated. If $\omega_1(p_2/q_2)$

is the point at which the S–symmetric hyperbolic orbits enter the $\mathcal{G}$–invariant curve, then for $\omega \in (\omega_2(p_2/q_2), \omega_1(p_2/q_2))$ the circle γ_s is a second order meander. Moreover, as ω increases in this interval the reverse process occurs, namely orbits are destroyed.

Fig. 10 displays for the model map, the $(p = 2, q = 45)$ orbits as cusps of two S–symmetric invariant circles (Fig. 10a), a part of the second order meander with 45 folds, associated to the periodic orbits of type $(2, 45)$ (Fig. 10b), and the S–symmetric hyperbolic orbits of period 45 on the $\mathcal{G}$ invariant curve (Fig. 10c).

a) b)

c)

Fig. 10: Fig. $k = 0.3$; magnification of the rectangle $[1.5, 2.5] \times [-1.5.0]$ a) $\omega = 0.9299775$: S–symmetric circles with 45 cusps each; this is the threshold of creation of periodic orbits of rotation number $\frac{2}{45}$; b) $\omega = 0.93$: second level meanders on γ_s associated to the periodic orbits of type $(p = 2, q = 45)$ c) $\omega = 0.9300026$: $2/45$–periodic hyperbolic points lie on the $\mathcal{G}$–invariant curve.

Because destruction of periodic orbits of nontwist standard–like maps take place by collision, we **conjecture** that the above described process repeats at every scale. More precisely, if γ_s is a first order meander associated to odd periodic orbits of type (p_1, q_1), then there is a number of subintervals of type $[\omega_2, \omega_1] \subset [\omega_1(p_1/q_1), \omega_2(p_1/q_1)]$ associated to odd periodic orbits that are created for $\omega \in [\omega_1(p_1/q_1), \omega_2(p_1/q_1)]$. For each such subinterval $[\omega_2(p_2/q_2), \omega_1(p_2, q_2)]$ there exists another number of subintervals of type $[\omega_1, \omega_2]$ associated to periodic orbits that are destroyed and so on. In this context, if (p_n/q_n) is a sequence of rational numbers, with (q_n) an increasing sequence of odd numbers, such that:

$$\begin{aligned} &[\omega_1(p_1/q_1), \omega_2(p_1/q_1)] \supset [\omega_2(p_2/q_2), \omega_1(p_2/q_2)] \supset \cdots \\ &\supset [\omega_1(p_{2k-1}/q_{2k-1}), \omega_2(p_{2k-1}/q_{2k-1})] \supset [\omega_2(p_{2k}/q_{2k}), \omega_1(p_{2k}/q_{2k})] \supset \cdots \end{aligned} \tag{30}$$

and $\{\omega_*\}$ is the intersection of the decreasing sequence of compact intervals, we wonder whether for $\omega = \omega_*$ the $\mathcal{G}$–invariant curve is a fractal curve or not.

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