Gordon-type arguments in the spectral theory of one-dimensional quasicrystals

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Abstract. We review the recent developments in the spectral theory of discrete one-dimensional Schrödinger operators with potentials generated by substitutions and circle maps. We discuss how occurrences of local repetitive structures allow for estimates of generalized eigenfunctions. Among the recent applications of this general approach are almost sure and uniform results on the absence of eigenvalues as well as continuity of the spectral measures with respect to Hausdorff measures.

Introduction

In 1983, two groups independently proposed and investigated a simple model, a discrete one-dimensional Schrödinger operator with a potential taking only two values, which was shown to exhibit unexpected and spectacular behavior. Kohmoto et al. [**KKT**] and Ostlund et al. [**OPRSS**] employed dynamical systems methods to study the scaling properties that were embodied in their model problem. It was argued that the eigenfunctions display a critical behavior in that they are neither extended nor localized. Moreover, this critical behavior seemed to be a universal feature of the model being quite independent of a modulation of the strength of the potential values. This was in sharp contrast to another popular model, the Harper operator, which exhibits such critical eigenfunctions only for some fixed modulation which in fact represents a sharp transition from extended to localized states.

One year later, in 1984, Shechtman et al. published their discovery of a structure which has long range order without being globally translation invariant [**SBGC**]. This discovery was essentially the birth of a whole new field, the investigation of quasicrystals, structures whose existence has been absolutely unexpected up to then, and it triggered intensive research activities including a reconsideration of the nature and definition of order, compare, for example, [**Ba99a**].

Further structures with this property were soon discovered and on the theoretical side, models were proposed which reflect the observed phenomena. Chief among these were quasiperiodic structues that are constructed by a cut-and-project mechanism, which basically projects an ordered structure from a higher-dimensional space to the physical space. In 1986, Luck and Petritis showed in [**LP86**] that the model proposed by Kohmoto et al. and Ostlund et al. is naturally associated to

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such a cut-and-project structure. The features of this model were then generalized to related models by Kalugin et al. [**KKL86**] and Gumbs and Ali [**GA**]. All these works concluded the critical behavior of eigenfunctions as well as the nowhere dense structure of the set of allowed energies which was in fact claimed to be a set of zero Lebesgue measure (compare in particular [**KO**]). Also in 1986, using symbolic dynamics methods, Casdagli presented a fine anaysis of the Fibonacci dynamical system which further supported this claim [**Ca**].

In a 1987 paper, Sütő pursued a rigorous study of these phenomena for the basic original model, the *Fibonacci Hamiltonian* [**Sü87**]. He was able to confirm parts of the observations, namely delocalized states and thus absence of point spectrum, and, for sufficiently large modulation of the potential, nowhere dense structure of the spectrum. He complemented these results in 1989 by showing that the spectrum is indeed always a set of measure zero [**Sü89**]. Absence of absolutely continuous spectrum then follows immediately. In the same year, Bellissard et al. obtained these results for a more general class by essentially the same strategy [**BIST**]. It was thus rigorously established that those operators exhibit purely singular continuous zero-measure spectrum, reflecting the fact that they model one-dimensional structures being intermediate between periodic (leading to absolutely continuous spectrum) and disordered (leading to pure point spectrum).

From a mathematical point of view, the occurrence of purely singular continuous spectrum, however, was still considered to be a curiosity joining the "constructed" toy model examples by Pearson [**Pe78**]. Only in the mid-90's did Simon and co-workers aim at a conceptual understanding of singular continuous spectral measures. In a series of papers [**Si95a, DMS, JS, DJLS, SS, Si96a, Si96b**], they not only exhibited many new examples of operators with this spectral type, they even showed that the occurrence of purely singular continuous spectrum is generic in an appropriate sense.

Moreover, in the course of this decade, building upon the landmark papers mentioned above, the occurrence of purely singular continuous spectrum has been shown to be a universal feature of operators associated to structures displaying aperiodicity at an intermediate level. To review these developments is the purpose of the present article. In 1994, Sütő contributed a review in similar spirit to a Les Houches winter school [**Sü95**]. We therefore put particular emphasis on ideas and approaches that were introduced since then. Moreover, we shall center these approaches around one core idea which is due to Gordon and dates back to the 70's [**Go76**]. It was recently realized that the philosophy embodied in Gordon's paper may serve as a universal tool in the theory as the necessary input, local repetitive structures, is always present in the proposed models. For further background and an introduction to the relevant basics in operator theory, we refer the reader to Sütő's article and, as further introductory reading, we want to mention [**BG95, D97**].

Our main goal here is to explain what quantities to look at when studying spectral properties of one-dimensional quasicrystal models, to present strategies and methods of their investigation which either have been successfully applied in the past or which seem promising when tackling some of the open problems, and to list some of these problems which appear to be important and interesting.

The organization is as follows. We start by presenting, in Section 1, essential parts of the theory of ergodic families of discrete one-dimensional Schrödinger operators which provides a very useful framework to work within due to strong general results and proof strategies. Section 2 presents two classes of such families which we will focus on, both being natural generalizations of the family of Fibonacci operators. It is shown how these classes fit into the framework, and known results for these families are recalled. The fundamental Gordon idea, upon which the core message of this paper is based, is discussed in Section 3. A certain variant of the Gordon method suggests investigating traces of some unimodular 2×2-matrices associated to an operator. Section 4 explains how useful bounds on these traces can be obtained by studying a dynamical system which is induced by hierarchical structures in the potential value arrangement. A recently introduced general strategy for obtaining results that hold for all members of a family of operators is presented in Section 5. It basically emphasizes a combinatorial point of view as opposed to measure theoretical type of arguments suggested by ergodic theory. Using this method, one may investigate quantities that do not behave nicely under very weak perturbations, such as the point spectrum. Consequently, Section 6 deals with results on the absence of point spectrum. We present three types of results which are of increasing completeness, the most complete, of course, being uniform results. The latter is shown to be accessible by combining the Gordon method, the bounds on the traces, and the combinatorial point of view. Parts of the results obtained in the proofs of absence of eigenvalues can in fact be used to show that the spectrum has Lebesgue measure zero as explained in Section 7. Section 8 is concerned with transport properties of one-dimensional quasicrystals and hence with the unitary groups generated by the operators. We discuss recent results and possible ways to obtain bounds on the time evolution. Finally, we present some open problems in Section 9.

1. Ergodic families of Schrödinger operators

In this section we recall the concept of ergodic families of Schrödinger operators which has proved to provide a convenient framework for the operators that are of interest in this article. For more detailed presentations we refer the reader to the books by Carmona-Lacroix [**CL**] and Cycon et al. [**CFKS**]. In particular, every application of the fundamental Kotani theory requires this framework, and this theory is at the heart of all results concerning absence of absolutely continuous spectrum and the zero-measure property for the operators under consideration. Our presentation does not strive for greatest generality but rather for an appropriate notion of ergodic family which comprises all the examples we want to discuss here.

Definition 1.1. Let Ω be a compact metric space and let $T : \Omega \to \Omega$ be a homeomorphism. The pair (Ω, T) is called a *topological dynamical system*. Given some $\omega \in \Omega$, the set $\{T^n\omega : n \in \mathbb{Z}\}$ is called the *orbit* of ω. Denote by $\mathcal{B}$ the Borel σ-algebra of Ω. A Borel probability measure μ is called *stationary* if $\mu(T(B)) = \mu(B)$ for every $B \in \mathcal{B}$. A Borel set B is called *shift invariant* if $T(B) = B$. A stationary measure is called *ergodic* if any shift invariant set has measure zero or one. The topological dynamical system (Ω, T) is called *minimal* if the orbit of every $\omega \in \Omega$ is dense. It is called *uniquely ergodic* if there exists a unique ergodic measure, and it is called *stricly ergodic* if it is both minimal and uniquely ergodic.

It is in fact well known that if there is a unique stationary measure μ, then μ is necessarily ergodic [**W**]. The two examples below will be of major importance in what follows.

EXAMPLE 1.2. Let $\Omega = \mathbb{T} \simeq [0,1)$ and $T\omega = \omega + \alpha \mod 1$ for some irrational $\alpha \in (0,1)$. It is well known that the Lebesgue measure on $\mathbb{T}$ is the unique stationary measure (i.e., the system is uniquely ergodic) and that every orbit is dense (i.e., the system is minimal). Hence, (Ω, T) is strictly ergodic.

Before turning to the next example let us introduce some notation.

DEFINITION 1.3. Let $\mathcal{A} = \{a_1, \ldots, a_s\}$ be a finite set, called *alphabet*. Endow $\mathcal{A}$ with the discrete topology. The a_i are called *symbols* or *letters* , the elements of $\mathcal{A}^* = \bigcup_{k \ge 1} \mathcal{A}^k$ are called *words* . We denote by $|v|$ the *length* of a word $v \in \mathcal{A}^*$ (i.e., $|v| = l$ if $v \in \mathcal{A}^l$). For $v, w \in \mathcal{A}^*$, $\#_v(w)$ denotes the number of occurrences of v in w (e.g., $\#_{aa}(aabaaa) = 3$). Let $\mathcal{A}^{\mathbb{N}}, \mathcal{A}^{\mathbb{Z}}$ denote the sets of one-sided and two-sided infinite sequences, called *infinite words*, over $\mathcal{A}$, both being equipped with product topology which is easily seen to be a metric topology. Given a finite or infinite word w, a finite word v is called a *subword* or *factor* of w if there are (finite or infinite) words r, s such that $w = rvs$, with the obvious definition of concatenation of words. We define $F_w = \{y : y \text{ is a factor of } w\}$ and $F_w(n) = F_w \cap \mathcal{A}^n$, $n \in \mathbb{N}$. The *complexity function* $p_w : \mathbb{N} \to \mathbb{N}_0$ corresponding to w is given by $p_w(n) = |F_w(n)|$, $|\cdot|$ denoting cardinality.

EXAMPLE 1.4. Let $\mathcal{A}$ be an alphabet. The shift T on $\mathcal{A}^{\mathbb{Z}}$ is defined by $(T\psi)_n = \psi_{n+1}$ for $\psi \in \mathcal{A}^{\mathbb{Z}}$. Let $\Omega \subseteq \mathcal{A}^{\mathbb{Z}}$ be closed and invariant under T, that is, $T\Omega = \Omega$. The topological dynamical system (Ω, T) is called a *subshift*. Let us consider subshifts that are generated as follows. Given some $\psi \in \mathcal{A}^{\mathbb{N}}$, we define Ω_ψ to be the set of two-sided sequences having all their subwords occur in ψ, that is,

$$\Omega_\psi \; = \; \{\omega \in \mathcal{A}^{\mathbb{Z}} : F_\omega \subseteq F_\psi\}\,.$$

It is clear that Ω_ψ is closed and invariant. Moreover, unique ergodicity and strict ergodicity of (Ω_ψ, T) can be characterized in terms of frequencies of subwords as follows (cf. [**Q**]). The subshift is uniquely ergodic if and only if for every $v \in F_\psi$ there is a number $d_\psi(v) \ge 0$, the *frequency of* v *in* ψ, such that for every $k \in \mathbb{N}$, we have

$$\frac{\#_v(\psi_k \ldots \psi_{k+n-1})}{n} \longrightarrow d_\psi(v)$$

as $n \to \infty$, uniformly in k. Minimality of (Ω, T) is equivalent to the fact that every word in F_ψ occurs infinitely often in ψ and the gaps between consecutive occurrences are bounded by a constant depending on the word, that is, the occurrences of every word are relatively dense. Thus, the subshift is strictly ergodic if and only if it is uniquely ergodic with strictly positive frequencies, that is, $d_\psi(v) > 0$ for all $v \in F_\psi$. In the uniquely ergodic case, the unique stationary measure μ obeys

$$\mu(\{\omega \in \Omega : \omega_m \ldots \omega_{m+|v|-1} = v\}) \; = \; d_\psi(v) \tag{1.1}$$

for every $v \in F_\psi$ and every $m \in \mathbb{Z}$. That is, the measure of some cylinder set can be determined by studying the frequency of the defining word.

DEFINITION 1.5. Given a topological dynamical system (Ω, T), an ergodic measure μ, and a measurable function $g : \Omega \to \mathbb{R}$, one defines for each $\omega \in \Omega$ a two-sided infinite sequence $V_\omega : \mathbb{Z} \to \mathbb{R}$ by $V_\omega(n) = g(T^n \omega)$. This gives rise to a discrete one-dimensional Schrödinger operator H_ω on $\ell^2(\mathbb{Z})$ which acts on some $\phi \in \ell^2(\mathbb{Z})$ by

$$(H_\omega \phi)(n) \; = \; \phi(n+1) + \phi(n-1) + V_\omega(n)\phi(n)\,.$$

The family $(H_\omega)_{\omega \in \Omega}$ is called an *ergodic family of Schrödinger operators.*

The striking fundamental result, which motivates the choice of this framework even for deterministic models such as the ones we consider in this paper, is the following result which essentially says that the spectrum and the spectral type are deterministic up to sets of measure zero.

THEOREM 1.6 (Pastur, Kunz-Souillard). *Let $(H_\omega)_{\omega\in\Omega}$ be an ergodic family of Schrödinger operators. Then there exist sets $\Omega_0 \subseteq \Omega$, $\Sigma, \Sigma_{\rm pp}, \Sigma_{\rm sc}, \Sigma_{\rm ac} \subseteq \mathbb{R}$ such that $\mu(\Omega_0) = 1$ and $\sigma(H_\omega) = \Sigma, \sigma_{\rm pp}(H_\omega) = \Sigma_{\rm pp}, \sigma_{\rm sc}(H_\omega) = \Sigma_{\rm sc}, \sigma_{\rm ac}(H_\omega) = \Sigma_{\rm ac}$ for every $\omega \in \Omega_0$.*

A proof of this result can be found in [**Pa, KS**]. However, for a discussion of this result and most of what follows in this section, the reader could also consult the books [**CL, CFKS**]. In fact, an additional assumption even allows for a strengthening of Theorem 1.6.

DEFINITION 1.7. An ergodic family of Schrödinger operators $(H_\omega)_{\omega\in\Omega}$ is called *minimal* if for each pair $\omega_1, \omega_2 \in \Omega$, the sequence V_{ω_1} is a pointwise limit of translates of V_{ω_2}.

REMARK 1.8. Note that if the family is generated by a dynamical system as defined in Example 1.4 along with $g(\omega) = f(\omega_0)$ where $f : \mathcal{A} \to \mathbb{R}$ is arbitrary, then minimality of the family follows from minimality of the dynamical system (Ω, T).

THEOREM 1.9. *Let $(H_\omega)_{\omega\in\Omega}$ be a minimal ergodic family of Schrödinger operators. Then, we have $\sigma(H_\omega) = \Sigma, \sigma_{\rm ac}(H_\omega) = \Sigma_{\rm ac}$ for every $\omega \in \Omega$.*

The statement about the spectrum is already part of the folklore and is essentially contained in [**RS80**]. The result on the absolutely continuous spectrum was recently obtained by Last and Simon [**LS**]. Thus, given a minimal ergodic family, one can pick any member of the family when studying the spectrum or the absolutely continuous spectrum. This is a clear motivation for embedding even a deterministic model into this framework since it may well be that another member of the family, not the one we started with, is easier to study. Let us remark that minimality does not imply constancy of $\sigma_{\rm pp}(H_\omega)$ or $\sigma_{\rm sc}(H_\omega)$ as there are explicit counterexamples [**JS**].

Let us now turn to a beautiful theory which has been termed *Kotani theory*. The results we shall describe below indeed form the core of much of the theory of ergodic families of Schrödinger operators in one dimension and certainly provide a basis for most results on Fibonacci-type operators which have been obtained so far. Given a family $(H_\omega)_{\omega\in\Omega}$, it is often very useful to consider the associated eigenvalue equation in difference sense, that is,

$$\phi(n+1) + \phi(n-1) + V_\omega(n)\phi(n) \;=\; E\phi(n)\,, \tag{1.2}$$

where $E \in \mathbb{C}$ and ϕ is just required to be a two-sided sequence, $\phi : \mathbb{Z} \to \mathbb{C}$. Some of the most useful tools in one-dimensional Schrödinger operator theory are results that establish a link between the behavior of solutions to (1.2) and spectral properties of the operator since the former are to some extent relatively easy to investigate. In fact, Kotani theory provides a link of this kind in the case of the ergodic framework. Connections in the deterministic case have been found by Gilbert, Pearson, and Khan [**GP, Gi, KP**] (see Jitomirskaya-Last [**JL96, JL99a, JL99b**] for an extension of the results and a simplification of the proof) and by

Last and Simon [**LS**]. Let us recall a standard reformulation of (1.2). Given a two-sided sequence ϕ we define $\Phi : \mathbb{Z} \to \mathbb{C}^2$ by

$$\Phi(n) \;=\; \begin{pmatrix} \phi(n+1) \\ \phi(n) \end{pmatrix}.$$

Defining

$$T_{E,\omega}(n) = \begin{pmatrix} E - V_\omega(n) & -1 \\ 1 & 0 \end{pmatrix},$$

$$M_{E,\omega}(n) = \begin{cases} T_{E,\omega}(n) \times \cdots \times T_{E,\omega}(1), & n \ge 1, \\ I, & n = 0, \\ T_{E,\omega}(n+1)^{-1} \times \cdots \times T_{E,\omega}(0)^{-1}, & n \le -1, \end{cases}$$

one may easily check that

$$\phi \text{ solves (1.2)} \;\Leftrightarrow\; \Phi \text{ solves } \Phi(n) = M_{E,\omega}(n)\Phi(0) \text{ for every } n \in \mathbb{Z}.$$

The matrices $M_{E,\omega}(\cdot)$ are called *transfer matrices* . They have determinant 1 since they are products of the *elementary transfer matrices* $T_{E,\omega}(\cdot)$ which obviously have determinant 1. The linear space of solutions to (1.2) for fixed E is two-dimensional, as can be seen from the above relation. Consider the two solutions $\phi_{1,2}$ induced by the initial conditions $\phi_1(0) = \phi_2(1) = 0$, $\phi_1(1) = \phi_2(0) = 1$. Then we also have

$$M_{E,\omega}(n) \;=\; \begin{pmatrix} \phi_1(n+1) & \phi_2(n+1) \\ \phi_1(n) & \phi_2(n) \end{pmatrix}.$$

Thus, the matrix contains all information about $\phi_{1,2}$ and hence all solutions in a pointwise way. In particular, bounds on $\|M_{E,\omega}(n)\|$, for example, yield bounds on $\|\Phi(n)\|$ for all solutions. It turns out to be useful to distinguish between exponential and sub-exponential growth of $\|M_{E,\omega}(n)\|$. Thus, we let

$$\gamma_{\omega,\pm}(E) \;=\; \lim_{n\to\pm\infty} \frac{1}{n} \ln \|M_{E,\omega}(n)\|,$$

provided the limit exists. Regarding existence of this limit, the following has been obtained in [**FK**].

THEOREM 1.10 (Furstenberg-Kesten). *For every $E \in \mathbb{C}$, there exist $\Omega_E \subseteq \Omega$ and $\gamma(E) \in \mathbb{R}$ such that $\mu(\Omega_E) = 1$ and for every $\omega \in \Omega_E$, $\gamma_{\omega,\pm}(E)$ exist and are equal to $\gamma(E)$, that is, $\gamma_{\omega,+}(E) = \gamma_{\omega,-}(E) = \gamma(E)$.*

The number $\gamma(E)$ is called the *Lyapunov exponent* .

THEOREM 1.11 (Osceledec). *Suppose that for some $E \in \mathbb{C}$, $\gamma(E) > 0$. Then, for every $\omega \in \Omega_E$, there exist solutions ϕ_d^+, ϕ_d^- of $H_\omega\phi = E\phi$ such that $\phi_d^\pm$ decays exponentially at $\pm\infty$, respectively, at the rate $-\gamma(E)$. Moreover, every solution which is linearly independent of ϕ_d^+ (resp., ϕ_d^-) grows exponentially at $+\infty$ (resp., $-\infty$) at the rate $\gamma(E)$.*

See [**LS, O, Ru**]. Thus, in the case of a positive Lyapunov exponent, one has a complete understanding of the asymptotics of the solutions at infinity.

Kotani theory now establishes a link between the Lyapunov exponent and the absolutely continuous spectrum. Define

$$A \;=\; \{E \in \mathbb{R} : \gamma(E) = 0\}.$$

The *essential closure* $\overline{S}^{\text{ess}}$ of a set $S \subseteq \mathbb{R}$ is defined by

$$\overline{S}^{\text{ess}} \;=\; \{E \in \mathbb{R} : |(E-\varepsilon, E+\varepsilon) \cap S| > 0 \;\; \forall \varepsilon > 0\},$$

where $|\cdot|$ denotes Lebesgue measure. In particular, $\overline{S}^{\text{ess}} = \emptyset$ for every set S of zero Lebesgue measure.

THEOREM 1.12 (Ishii-Pastur-Kotani). $\Sigma_{\text{ac}} = \overline{A}^{\text{ess}}$.

For a proof of the inclusion "$\subseteq$" the reader may consult [**I, Pa**] (see also [**Bu**] for an alternative proof using Gilbert-Pearson theory), the opposite inclusion has been treated in [**Ko84**] (see [**Si83**] for an adaptation to the discrete case). The following corollary, obtained in [**Ko89**], to the proof given in [**Ko84, Si83**] is of great interest to us since all the potentials we shall be dealing with take only finitely many values. Moreover, the additional assumption of aperiodicity is rather non-restrictive since there is a well-established theory treating the case of periodic potentials; see, for example, [**RS78**].

THEOREM 1.13 (Kotani). *If the potentials V_ω are aperiodic and take only finitely many values, then $|A| = 0$. In particular, $\Sigma_{\text{ac}} = \emptyset$.*

2. Models generated by circle maps and primitive substitutions

In this section we present the two classes of ergodic families we shall discuss in the sequel. Both classes are natural extensions of different aspects of the Fibonacci model, one generalizing its quasiperiodicity (models generated by circle maps) or, when restricted to a subclass, its word complexity properties (models generated by Sturmian sequences), the other one generalizing its self-similar structure (models generated by primitive substitutions). We show how they fit into the general framework presented in the preceding section and recall known results for these classes. For the sake of brevity we introduce the following notions.

DEFINITION 2.1. A family $(H_\omega)_{\omega\in\Omega}$ is called *EFA* if it is an ergodic family of Schrödinger operators such that the potentials V_ω are aperiodic and take only finitely many values. It is called *MEFA* if it is EFA and minimal.

With this convention at hand we infer from the above results that EFA (MEFA) families exhibit almost surely (uniformly) purely singular spectrum, that is, $\Sigma = \Sigma_{\text{pp}} \cup \Sigma_{\text{sc}}$. More precisely, the set A associated to such a family has zero Lebesgue measure. Furthermore, MEFA families have constant spectrum. In fact, all the classes of families $(H_\omega)_{\omega\in\Omega}$ we shall present and discuss now will be MEFA families, so let us bear in mind that when studying the spectral type, we only need to distinguish between point spectrum and singular continuous spectrum since all these families have empty absolutely continuous spectrum for all H_ω, $\omega \in \Omega$; thus the latter does not present any issue at all.

Circle map models: A circle map model is parametrized by three parameters, namely, an irrational *rotation number* $\alpha \in (0,1)$, an *interval length* $\beta \in (0,1)$, and a *coupling constant* $\lambda \in \mathbb{R}\backslash\{0\}$. There are two ways to choose Ω, T, g, both of which have been used in the past. Their mutual relation is, for example, given in [**DL99c**]. The first way follows Example 1.2. Thus, let $\Omega = \mathbb{T} \simeq [0,1)$ and $T : \Omega \to \Omega$, $\omega \mapsto \omega + \alpha \mod 1$. As noted above, (Ω, T) is strictly ergodic with the Lebesgue measure on $\mathbb{T}$ as unique ergodic measure μ. Let g be given by $g(\omega) = \lambda \cdot \chi_{[1-\beta,1)}(\omega)$. This yields potentials $V_\omega(n) = \lambda \cdot \chi_{[1-\beta,1)}(\alpha n + \omega \mod 1)$. The other possibility of associating a family of potentials starts with the one-sided sequence $v_{\alpha,\beta,\theta}(n) = \chi_{[1-\beta,1)}(n\alpha + \theta \mod 1)$, $n \in \mathbb{N}$, and follows the lines of Example 1.4. Define $\Omega_{\alpha,\beta} = \Omega_{v_{\alpha,\beta,\theta}}$. The notation is justified since the subshift does not depend on θ. It was shown by Hof in [**H**] that the dynamical system $(\Omega_{\alpha,\beta}, T)$ is strictly ergodic. The function g generating the potentials is in this case given by $g(\omega) = f(\omega_0)$, where $f(0) = 0$, $f(1) = \lambda$. In the case $\alpha = \beta$ we write $v_{\alpha,\theta}$ instead of $v_{\alpha,\alpha,\theta}$ and Ω_α instead of $\Omega_{\alpha,\alpha}$, and we call the dynamical system (Ω_α, T) as well as the resulting potentials *Sturmian*. Sturmian potentials have an interesting combinatorial property. Consider a one-sided sequence s. Recall that the complexity function $p_s(n)$ counts the number of factors of length n in s. One can show (see [**Lo83, Lo99**] for this and much more on combinatorics on words in general and Sturmian sequences in particular, compare also [**LP92**]) that $p_s(n) \le n$ for some n implies that s is ultimately periodic. Thus, any non-ultimately periodic sequence s obeys $p_s(n) \ge n + 1$ for every n. Sequences s having $p_s(n) = n + 1$ for all n are called *Sturmian*. The terminology is now motivated by the fact that every $v_{\alpha,\theta}$ is a Sturmian sequence and every $\{0, 1\}$-valued Sturmian sequence coincides with the restriction of some element of an appropriate Ω_α to $\mathbb{N}$.

Our discussion of this relation involving one-sided sequences may appear somewhat awkward, but the corresponding relation for two-sided sequences is not true, that is, there exist $\{0, 1\}$-valued two-sided sequences s with complexity function obeying $p_s(n) = n + 1$ which, however, do not belong to some Ω_α (consider, e.g., the sequence s defined by $s_n = 0$ for $n < 0$ and $s_n = 1$ for $n \ge 0$), compare [**CH**] for a complete characterization.

Let us note the following concerning families generated by circle maps.

PROPOSITION 2.2. *Both ways of generating an operator family $(H_\omega)_{\omega\in\Omega}$ corresponding to the parameters α, β, λ induce MEFA families.*

For fixed parameter values, the operator family induced by the dynamical system on the torus is contained in the operator family obtained from the subshift. Interestingly, the latter family is strictly larger even though both families are minimal [**DL99c**]! The Sturmian model generated by the golden mean $\alpha = \frac{\sqrt{5}-1}{2}$ is called the *Fibonacci model* .

Models generated by primitive substitutions: Let $\mathcal{A}$ be an alphabet. A *substitution* S is a map $S : \mathcal{A} \to \mathcal{A}^*$. S can be extended morphically to $\mathcal{A}^*$ (resp., $\mathcal{A}^{\mathbb{N}}$) by $S(b_1 \dots b_n) = S(b_1) \dots S(b_n)$ (resp., $S(b_1 b_2 b_3 \dots) = S(b_1)S(b_2)S(b_3) \dots$). S is called *primitive* if there exists $k \in \mathbb{N}$ such that for every $a \in \mathcal{A}$, $S^k(a)$ contains every symbol from $\mathcal{A}$. Prominent examples of primitive substitutions are given by

$a \mapsto ab,\ b \mapsto a$	Fibonacci,
$a \mapsto ab,\ b \mapsto aa$	period doubling,
$a \mapsto ab,\ b \mapsto aaa$	binary non-Pisot,
$a \mapsto ab,\ b \mapsto ba$	Thue-Morse,
$a \mapsto ab,\ b \mapsto ac,\ c \mapsto db,\ d \mapsto dc$	Rudin-Shapiro.

A fixed point $u \in \mathcal{A}^{\mathbb{N}}$ of S is called *substitution sequence*. The existence of such a fixed point is ensured by the following conditions,

- there exists a letter $a \in \mathcal{A}$ such that the first letter of $S(a)$ is a,
- $\lim_{n\to\infty} |S^n(a)| = \infty$,

which are easily seen to hold for a suitable power of S if S is primitive. Without loss of generality (since any power of S is primitive if S is primitive), we assume this power to be equal to one. In this case, $u = \lim_{n\to\infty} S^n(a)$ exists and is a substitution sequence. Define Ω by $\Omega = \Omega_u$. If S is primitive, Ω does not depend on the choice of the substitution sequence. The subshift (Ω, T) is called the *substitution dynamical system associated to* S and it is stricly ergodic [**Q**]. Choose some function $f : \mathcal{A} \to \mathbb{R}$ and define $g(\omega) = f(\omega_0)$.

PROPOSITION 2.3. *Suppose that the substitution S is primitive, the substitution sequence u is not ultimately periodic, and the function f takes at least two values. Then the induced operator family $(H_\omega)_{\omega\in\Omega}$ is MEFA.*

In case of the Fibonacci substitution the subshift is equivalent to the Sturmian subshift corresponding to $\alpha = \frac{\sqrt{5}-1}{2}$ via $a \mapsto 1, b \mapsto 0$. Thus the corresponding families of operators coincide (up to a spectral shift).

These two classes of MEFA families have been studied extensively in the past; see [**BIST, BIT, D98a, DKL, DL99a, DL99b, DP86, IT, IRT, HKS, J91, Ka, Ra, Sü87, Sü89**] and [**Be, BBG91, BBG92, BG93, D98b, D98c, D99b, DP91, H, HKS**] for some important contributions in the case of circle map models and substitution models, respectively. The results comprise in particular singular continuity of spectral measures, zero Lebesgue measure of the spectrum, Gap labelling via K-theory, opening of the gaps at low coupling, continuity of gap boundaries with respect to the rotation number, and uniform existence of the Lyapunov exponent for large subclasses.

3. Pointwise methods and variants of the Gordon criterion

This section is concerned with methods in the spectral theory of some fixed Schrödinger operator with particular emphasis on several variants of an idea originally due to Gordon [**Go76**]. The methods we present can be applied, for example, to a fixed member of some ergodic family $(H_\omega)_{\omega\in\Omega}$.

Consider a bounded function $V : \mathbb{Z} \to \mathbb{R}$ and the associated Schrödinger operator

$$(H\phi)(n) = \phi(n+1) + \phi(n-1) + V(n)\phi(n) \tag{3.1}$$

along with the difference equation

$$\phi(n+1) + \phi(n-1) + V(n)\phi(n) = E\phi(n)\,, \tag{3.2}$$

where $E \in \mathbb{C}$. Similarly to the above, we introduce a reformulation of (3.2) in terms of *transfer matrices* $M_E(n)$,

$$\begin{aligned}
\Phi(n) &= \begin{pmatrix} \phi(n+1) \\ \phi(n) \end{pmatrix}, \\
T_E(n) &= \begin{pmatrix} E - V(n) & -1 \\ 1 & 0 \end{pmatrix}, \\
M_E(n) &= \begin{cases} T_E(n) \times \cdots \times T_E(1), & n \geq 1, \\ I, & n = 0, \\ T_E(n+1)^{-1} \times \cdots \times T_E(0)^{-1}, & n \leq -1. \end{cases}
\end{aligned}$$

Let us discuss "Gordon-type arguments," that is, the exploitation of local repetitions. Recall that transfer matrices have determinant 1, independently of the potential V, the energy E, and the site n. Thus, by the Cayley-Hamilton theorem, the following universal equation holds,

$$M_E(n)^2 - \mathrm{tr}(M_E(n))M_E(n) + I \;=\; 0\,. \tag{3.3}$$

Suppose now that V repeats its values on the interval $\{1, \dots, n\}$ once, that is,

$$V(j) \;=\; V(j+n),\; 1 \le j \le n\,. \tag{3.4}$$

Due to the fact that the definition of transfer matrices is local, we infer that for any energy E, we have

$$M_E(n)^2 \;=\; M_E(2n)\,. \tag{3.5}$$

Plugging this into (3.3), we obtain

$$M_E(2n) - \mathrm{tr}(M_E(n))M_E(n) + I \;=\; 0\,. \tag{3.6}$$

Now consider any initial vector $\Phi(0)$, which we may assume to be normalized, that is, $||\Phi(0)|| = 1$. We apply (3.6) to $\Phi(0)$ and get

$$\Phi(2n) - \mathrm{tr}(M_E(n))\Phi(n) + \Phi(0) \;=\; 0\,. \tag{3.7}$$

Since $\Phi(0)$ has norm 1, either $\Phi(2n)$ or $\mathrm{tr}(M_E(n))\Phi(n)$ has to have norm at least $\frac{1}{2}$. Thus,

$$\max(||\Phi(n)||, ||\Phi(2n)||) \;\ge\; \frac{1}{2}\min\left(1, \frac{1}{|\mathrm{tr}(M_E(n))|}\right). \tag{3.8}$$

If we can find, for some fixed energy E, a sequence $n_k \to \infty$ such that the potential repeats the values on $\{1, \dots, n_k\}$ once and the sequence $\mathrm{tr}(M_E(n_k))$ remains bounded, then the right-hand side of (3.8) is strictly bounded away from zero on this sequence of sites for every initial vector! Thus, in this case no solution tends to zero and, in particular, E is not an eigenvalue of H. Let us summarize this in the following lemma.

Lemma 3.1 (two-block method). *Fix a potential V and an energy E. Suppose there is a sequence $n_k \to \infty$ and some $1 \le C < \infty$ such that we have for every k,*

1. *$V(j) = V(j + n_k)$, $1 \le j \le n_k$,*
2. *$|\mathrm{tr}(M_E(n_k))| \le C$.*

Then, E is not an eigenvalue of H and no solution of (3.2) *tends to zero at $+\infty$. More precisely, for every k, every solution obeys*

$$\max(||\Phi(n_k)||, ||\Phi(2n_k)||) \;\ge\; \frac{1}{2C}\,.$$

Remark 3.2. Of course, there are obvious variations on this idea. First of all, it is not important that the squares are aligned at the origin; any other site will do. Similarly, the squares can also be aligned to their right side and one can work on the left half-line. In this case one gets that, once the modified conditions are satisfied, no solution tends to zero at $-\infty$ with similar uniform lower bound for the norms.

But what if a study of transfer matrix traces is not feasible? The answer is simple, just find another block of repetition! The key ingredient in the above argument is the three-term expression in (3.6) which, given largeness of one term, yields largeness of at least one of the others. Now suppose that we have a repetition of $V(1), \dots, V(n)$ and that the trace of $M_E(n)$ is large. Even if we infer "largeness" of the middle term, this may only be due to the trace but not to the matrix (resp., the vector after application of the equation to an initial vector). This complication does not occur for the other terms. So, in case of a large trace, try to find a repetition of the potential values from 1 to n to the left and, if successful, apply (3.6) to $\Phi(-n)$ but retain normalization at the origin! This yields the equation

$$\Phi(-n) - \mathrm{tr}(M_E(n))\Phi(0) + \Phi(n) \; = \; 0\,. \tag{3.9}$$

Now, the middle term is large (a large factor times a normalized vector) and, again, this says that at least one of the other vectors has to be large. Quantitatively, we have the following lemma.

LEMMA 3.3 (three-block method). *Fix a potential V. Suppose there is a sequence $n_k \to \infty$ such that we have for every k,*

$$V(j - n_k) \; = \; V(j) = V(j + n_k)\,, \quad 1 \le j \le n_k\,.$$

Then for every energy E, we have that E is not an eigenvalue of H and no solution of (3.2) *tends to zero at both $\pm\infty$. More precisely, for every k, every solution obeys*

$$\max(\|\Phi(-n_k)\|, \|\Phi(n_k)\|, \|\Phi(2n_k)\|) \; \ge \; \frac{1}{2}\,.$$

REMARK 3.4. In general it may well happen that for some energy E, every solution decays at either $+\infty$ or $-\infty$. This happens, for example, for energies outside the spectrum of H. Thus, even if one has cubes rather than only squares, the additional investigation of transfer matrix traces pays off in the form of a stronger conclusion.

Lemma 3.3 is very close to the original Gordon result [**Go76**] (see also [**CFKS**]) which, however, requires another block of repetition. It was stated and proved in this form by Delyon and Petritis [**DP86**]. Lemma 3.1 was proved by Sütő in [**Sü87**].

4. Trace map characterization of the spectrum

The two-block version of the Gordon criterion presented in Lemma 3.1 suggests investigating both repetitive structures in the potential and traces of transfer matrices when trying to obtain bounds on solutions of the eigenvalue equation. In the case of Sturmian models or models generated by primitive substitutions, transfer matrix traces can be investigated by studying a (generalized) dynamical system, the *trace map*, which is induced by hierarchical structures in the potentials. These are by definition present in potentials generated by substitutions and their presence in the Sturmian case can be exhibited using continued fraction expansion theory. The present section is concerned with a discussion of the correspondence between the dynamics of trace maps and the spectra of operators from the two classes. For further information on trace maps, we refer the reader to [**AP, BGJ, BR, KN, PWW, Ro, RB**].

The introduction of a trace map follows a universal program which can be summarized as follows.

1. Exhibit a sequence of generating words obeying recursive relations.
2. Consider transfer matrices as being associated to finite words rather than to infinite sequences from the subshift.
3. Translate the recursive relations to the level of transfer matrices.
4. Pass to the traces of these matrices using suitable identities for unimodular 2×2-matrices.

Among the models we are interested in, trace maps have been found for all Sturmian models and all substitution models. Let us indicate how to establish the above steps in these two cases.

Given some one-sided sequence ψ such that the associated subshift (Ω_ψ, T) is minimal, it is easy to check that we have (this is essentially Gottschalk's theorem, see [**Pe83**])

$$\Omega_\psi \;=\; \{\omega \in \mathcal{A}^{\mathbb{Z}} : F_\omega = F_\psi\}\,. \tag{4.1}$$

In the case of ψ being a substitution sequence associated to some primitive substitution S, ψ has the form $\psi = \lim_{n\to\infty} S^n(a)$ for a suitable $a \in \mathcal{A}$. In particular, we have

$$\Omega_\psi \;=\; \{\omega \in \mathcal{A}^{\mathbb{Z}} : \forall w \in F_\omega\ \exists n \in \mathbb{N} \text{ such that } w \in F_{S^n(a)}\}\,. \tag{4.2}$$

Thus, the words $S^n(a)$ entirely determine the hull. Due to primitivity, any other sequence of the form $S^n(b)$, $b \in \mathcal{A}$, can be used. The set of words $S^n(b)$ where n ranges over $\mathbb{N}_0$ and b ranges over $\mathcal{A}$ therefore serves as a good basis for a study of the local properties of the potential value arrangements. Moreover, among these words the presence of recursive relations is immediate from the substitution rule,

$$S^n(b) \;=\; S^{n-1}(S(b)) \;=\; S^{n-1}(c_1 \ldots c_k) \;=\; S^{n-1}(c_1)\ldots S^{n-1}(c_k)\,. \tag{4.3}$$

Note that the concrete expression, that is, the way to pass from the words $S^{n-1}(c_1)\ldots S^{n-1}(c_k)$ to the word $S^n(b)$, is n-independent.

Example 4.1. Let us consider the Fibonacci substitution S_F which acts as $S_F(a) = ab$, $S_F(b) = a$. The recursive relations are given by

$$S_F^n(a) \;=\; S_F^{n-1}(a)S_F^{n-1}(b)\,, \quad S_F^n(b) \;=\; S_F^{n-1}(a)\,. \tag{4.4}$$

Let us now consider a Sturmian subshift Ω_α. Consider the continued fraction expansion of α (for general information on continued fractions, see, e.g., [**Kh, Pe54**]),

$$\alpha \;=\; \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}} \tag{4.5}$$

with uniquely determined $a_n \in \mathbb{N}$. The associated rational approximants $\frac{p_n}{q_n}$ obey

$$p_0 \;=\; 0\,,\; p_1 \;=\; 1\,,\; p_n \;=\; a_n p_{n-1} + p_{n-2}\,, \tag{4.6}$$

$$q_0 \;=\; 1\,,\; q_1 \;=\; a_1\,,\; q_n \;=\; a_n q_{n-1} + q_{n-2}\,. \tag{4.7}$$

To make things formally similar to the substitution case, define words s_n over the alphabet $\mathcal{A} = \{0, 1\}$ by

$$s_n \;=\; v_{\alpha,0}(1)\ldots v_{\alpha,0}(q_n) \tag{4.8}$$

and the one-sided sequence c_α by

$$c_\alpha \;=\; \lim_{n\to\infty} s_n\,. \tag{4.9}$$

Of course, c_α is nothing else than $v_{\alpha,0}$ restricted to $\mathbb{N}$. Note that the words s_n have length q_n. The equation (4.7) now has the following analog on word level,

$$s_n \;=\; s_{n-1}^{a_n} s_{n-2}\,. \tag{4.10}$$

The equation holds in this form for $n \geq 3$. The correct initial conditions are recovered by

$$s_{-1} = 1\,,\; s_0 = 0\,,\; s_1 = s_0^{a_1-1} s_{-1}\,, \tag{4.11}$$

and with these definitions (4.10) also holds for all $n \geq 2$. This can be proved by using the fact that continued fraction approximants $\frac{p_n}{q_n}$ provide the best possible approximation to α; see [**BIST**] for details. The following useful formula can be deduced from (4.10).

PROPOSITION 4.2. *For each $n \geq 2$,*

$$s_n s_{n+1} = s_{n+1} s_{n-1}^{a_n-1} s_{n-2} s_{n-1}\,. \tag{4.12}$$

PROOF. $s_n s_{n+1} = s_n s_n^{a_n+1} s_{n-1} = s_n^{a_n+1} s_n s_{n-1} = s_n^{a_n+1} s_{n-1}^{a_n} s_{n-2} s_{n-1} = s_{n+1} s_{n-1}^{a_n-1} s_{n-2} s_{n-1}$. □

Again we have found a sequence of words obeying recursive relations tending to a limit word c_α such that the hull can be written in the following form,

$$\Omega_\alpha = \{\omega \in \mathcal{A}^{\mathbb{Z}} : F_\omega = F_{c_\alpha}\}\,. \tag{4.13}$$

The next step is to pass from the recursively generated words to the transfer matrices associated to these words. Recall that given a subshift over some alphabet $\mathcal{A}$, the potentials V_ω were generated by replacing the symbols in the sequences by real numbers according to some function $f : \mathcal{A} \to \mathbb{R}$. Thus, one can study local properties of the potentials by introducing the following energy-indexed representation of words as $\mathrm{SL}(2,\mathbb{R})$ matrices. Fix some real energy E and define the map $M_E : \mathcal{A} \to \mathrm{SL}(2,\mathbb{R})$ by

$$M_E(a) = \begin{pmatrix} E - f(a) & -1 \\ 1 & 0 \end{pmatrix}. \tag{4.14}$$

Extend this mapping to $\mathcal{A}^*$ by

$$M_E(a_1 \ldots a_n) = M_E(a_n) \times \cdots \times M_E(a_1)\,. \tag{4.15}$$

For the above two classes, the recursions (4.3) and (4.10) then extend to the associated matrices in a straightforward way. We obtain, for example, in the Sturmian case

$$M_n = M_{n-2} M_{n-1}^{a_n}\,, \tag{4.16}$$

where we set $M_n = M_E(s_n)$. Note that due to (4.15), the order of the factors has been reversed. Depending on the explicit form of a substitution S, we have a similar analog in this case. In principle, we would like to apply the matrix trace to these equations and to study the dynamics, that is, the limit $n \to \infty$, for the traces. This is motivated by a simple argument which we will discuss in a moment. However, since the trace is not multiplicative, this transition is not as straightforward as the transition from words to matrices. This can be remedied by using appropriate identities to break down powers and by extending the set of underlying variables (from a set of size $|\mathcal{A}|$ to a larger, but still finite, set). First of all, all the powers can be broken down to one just by using the characteristic equation of unimodular matrices M, that is,

$$M^2 = \mathrm{tr}(M) M - I\,. \tag{4.17}$$

Moreover, one can pass to products not having multiple occurrences of factors by using the equation (see [**KN**])

$$\mathrm{tr}(MNMO) \;=\; \mathrm{tr}(MN)\mathrm{tr}(MO) + \mathrm{tr}(NO) - \mathrm{tr}(N)\mathrm{tr}(O)\,. \tag{4.18}$$

This set of remaining possible products, the enlarged alphabet $\mathcal{E}$, has cardinality bounded by [**KN**] (see [**AB, ABG**] for improvements)

$$\sum_{l=1}^{|\mathcal{A}|} \frac{|\mathcal{A}|!}{l(|\mathcal{A}|-l)!}\,. \tag{4.19}$$

Although one may obtain messy expressions, this method generates polynomial expressions for the traces of these products of level n in terms of the traces on level $n-1$. In case of a two-letter alphabet $\mathcal{A}=\{a,b\}$ one may choose $\mathcal{E}=\{a,b,ab\}$.

EXAMPLE 4.3. In the Fibonacci case we infer from (4.4) that $x_n(E) = \mathrm{tr}(M_E(S_F^n(a))$, $y_n(E)=\mathrm{tr}(M_E(S_F^n(b))$, $z_n(E)=\mathrm{tr}(M_E(S_F^n(a)M_E(S_F^n(b))$ obey

$$\begin{aligned} x_n(E) &= z_{n-1}(E), \\ y_n(E) &= x_{n-1}(E), \\ z_n(E) &= x_{n-1}(E)z_{n-1}(E) - y_{n-1}(E). \end{aligned}$$

In this example we do not even need the enlarged alphabet since $z_n(E)=x_{n+1}(E)$. Thus, one has the equivalent recursion

$$x_n(E) \;=\; x_{n-1}(E)x_{n-2}(E) - x_{n-3}(E)$$

involving only the x-variables.

Now why should one expect a connection between the trace map and the spectrum Σ? By the combinatorial subshift definitions (4.1) and (4.13) and by the recursions (4.3) and (4.10), any operator H_ω is a strong limit of operators H_n with periodic potentials where the periods have length $|S^n(a)|$ (resp., q_n) and the potential values are obtained by applying f to the symbols in $S^n(a)$ (resp., s_n). The ω-dependence of these operators is solely reflected in the locations of the periods, the actual periodic blocks are the same. Write $x_n(E)$ for $\mathrm{tr}(M_E(S^n(a)))$ (resp., $\mathrm{tr}(M_n)$). By the general periodic theory [**RS78**] we have

$$\sigma(H_n) \;=\; \{E : |x_n(E)| \le 2\}\,. \tag{4.20}$$

Thus, the strong approximation gives (see [**RS80**])

$$\Sigma \;=\; \sigma(H_\omega) \subseteq \bigcap_{k\in\mathbb{N}} \overline{\bigcup_{n\ge k} \{E : |x_n(E)| \le 2\}}\,. \tag{4.21}$$

If one can prove that $\bigcup_{n\ge k}\{E : |x_n(E)| \le 2\}$ is closed, then one ends up with a nice (and useful!) property of energies in the spectrum, namely, boundedness of traces on a subsequence (even with a uniform bound),

$$\Sigma \;\subseteq\; \bigcap_{k\in\mathbb{N}} \bigcup_{n\ge k} \{E : |x_n(E)| \le 2\} = \{E : |x_n(E)| \le 2 \text{ for infinitely many } n\}\,. \tag{4.22}$$

It turns out that the trace map is a good tool to establish such a property. In many cases it can even be shown that equality holds in (4.22). Namely, building upon Sütő [**Sü87**], the following has been shown for Sturmian models by Bellissard et al. [**BIST**].

THEOREM 4.4 (Trace map characterization of spectra of Sturmian models). *Let* $(H_\omega)_{\omega\in\Omega}$ *be a Sturmian family corresponding to* α, λ. *Define* $C_\lambda = 2+\sqrt{8+\lambda^2}$. *Then, we have*

$$\Sigma \;=\; \{E : |x_n(E)| \le 2 \text{ for infinitely many } n\} = \{E : |x_n(E)| \le C_\lambda \;\forall n\}\,.$$

Under certain assumptions, so-called *semi-primitvity* of the trace map and the occurrence of a square in the substitution sequence, which were shown to be satisfied by many prominent examples including Fibonacci, period doubling, binary non-Pisot, and Thue-Morse, Bovier and Ghez obtained a trace map characterization of the spectrum in the primitive substitution case. We refer the reader to their paper [**BG93**] and also its precursors [**Be, BBG91**] for a precise statement of the results. We note, however, that they do not obtain boundedness of orbits for energies from the spectrum in their general setting but rather boundedness on a subsequence. In this sense the trace maps associated to primitive substitution models exhibit only a partial analogy to Theorem 4.4.

5. Partitions and uniform results

In this section we focus on the local structures of sequences in a subshift which is induced by a recursively generated infinite word. As we saw above, this class comprises subshifts generated by substitution sequences and Sturmian words. We exhibit a uniform combinatorial property for the elements in the subshift. We then discuss how to employ this uniform combinatorial property to obtain uniform spectral properties.

To keep the notation simple we shall focus on Sturmian subshifts. However, the reader may easily verify that the ideas we present here apply to substitution models equally well. Fix some irrational $\alpha \in (0,1)$. Recall the description of a Sturmian hull Ω_α in terms of a subword definition (4.13). From this, one sees that the elements of Ω_α have a uniform combinatorial property, namely, they have the same set of factors which is equal to the set of factors of c_α. Now, apart from its subword structure, c_α has an additional structure. Namely, it is built from blocks s_n which obey a recursive relation. From (4.9) and (4.10) one can infer that for each $n \in \mathbb{N}$, c_α can be decomposed into blocks of type s_n and s_{n-1}. This decomposition is even unique. Moreover, in this decomposition, blocks of type s_{n-1} are always isolated and blocks of type s_n have multiplicity a_{n+1} or $a_{n+1}+1$. It is now natural to ask whether this structure is inherited by the elements of Ω_α. After all, as was discussed in the previous section, we already know that these blocks have useful properties in that the transfer matrices associated to them can be analyzed. It turns out that all these properties persist when passing from c_α to $\omega \in \Omega_\alpha$ [**DL99a**].

PROPOSITION 5.1. *Let* $\omega \in \Omega_\alpha$. *Then for every* $n \in \mathbb{N}$, *there exists a unique decomposition of* ω *into blocks of type* s_n *and* s_{n-1}. *In this decomposition, the multiplicity of each occurrence of* s_n *(resp.,* s_{n-1}*) is* a_{n+1} *or* $a_{n+1}+1$ *(resp., 1).*

This decomposition is called *n-partition of* ω. The above result indicates that to a certain extent, the members of Ω_α are equally well accessible to the pointwise methods introduced above. Indeed, in order to establish spectral properties that cannot be studied by pointwise convergence uniformly, one may apply a pointwise criterion to each ω separately. From Proposition 5.1 we learn that local repetitive structures abound in Sturmian potentials. For instance, we know that for every n, each s_{n-1}-block in the n-partition is followed and preceded by at least a_{n+1} copies of s_n, respectively.

Qualitatively, we have the same phenomena in hulls generated by primitive substitution hulls, the specific properties, however, depending on the concrete substitution given.

6. Absence of eigenvalues: Locating squares and cubes

We now turn to results on absence of point spectrum for models generated by circle maps and primitive substitutions. With one exception – the paper by Hof et al. [**HKS**] which employs a criterion in similar spirit using palindromes rather than powers (see [**Ba99b**] for extensions) – virtually all the known results were obtained by using variants of the Gordon method. In this section we show how the methods and tools presented in the three preceding sections can be combined to yield these results.

Let us begin by discriminating between the types of results that have been obtained. Given a family $(H_\omega)_{\omega\in\Omega}$ such that the underlying dynamical system (Ω, T) is a strictly ergodic subshift over some alphabet $\mathcal{A}$ with unique invariant measure μ, define

$$\Omega_c \;=\; \{\omega \in \Omega : \sigma_{\rm pp}(H_\omega) = \emptyset\}\,.$$

We say that absence of eigenvalues for $(H_\omega)_{\omega\in\Omega}$ is *generic* if Ω_c is a dense G_δ (i.e., a countable intersection of open sets which is dense in Ω), *almost sure* if Ω_c has full μ-measure, and *uniform* if $\Omega_c = \Omega$. Let us recall some general arguments that are useful in this context. First of all, to establish a generic result it is sufficient to exclude eigenvalues for just one $\omega \in \Omega$.

PROPOSITION 6.1. *If Ω_c is non-empty, then it is a dense G_δ.*

PROOF. Simon has shown that Ω_c is a G_δ [**Si95a**]. If Ω_c is not empty, then it contains an entire orbit which is dense by minimality. □

Next, by invariance Ω_c has μ-measure 0 or 1. In order to establish an almost sure result, it therefore suffices to bound the measure of Ω_c from below by a positive number. Here is a more elaborate version of this which is useful in connection with local investigations.

PROPOSITION 6.2. *Suppose $G(n)$, $n \in \mathbb{N}$, are Borel sets such that*

1. $\limsup_{n\to\infty} G(n) \subseteq \Omega_c$,
2. $\limsup \mu(G(n)) > 0$.

Then, $\mu(\Omega_c) = 1$.

PROOF. The assertion follows from

$$\mu(\limsup_{n\to\infty} G(n)) \;\geq\; \limsup_{n\to\infty} \mu(G(n))\,,$$

which is readily verified. □

As pointed out earlier, one cannot expect a general way to establish uniform absence of eigenvalues for the models we consider here by inspecting a set of ω's which is strictly smaller than Ω. To a certain extent, this can be understood in view of the discreteness of the potential values and the well-known and heavily studied sensitivity of point spectrum with respect to rank one perturbations [**SW, DMS,**

Go94, Si95b]. Thus one is led to consider each ω individually and to apply pointwise methods.

It seems interesting to note that a study of the eigenvalue problem motivates three different viewpoints, namely, topological arguments for generic results, measure-theoretical arguments for almost sure results, and combinatorial arguments for uniform results.

Let us now combine these general strategies with the Gordon-type criteria from Section 4. We will treat generic, almost sure, and uniform results separately.

6.1. Generic results. We immediately deduce as a first application a criterion for generic absence of eigenvalues as follows.

PROPOSITION 6.3. *Suppose there exists $\omega \in \Omega$ such that $V = V_\omega$ obeys the assumption of either Lemma 3.1 or Lemma 3.3. Then, Ω_c is a dense G_δ.*

A single element with Gordon-type symmetries was found in the Fibonacci case [**Sü87**], in the general Sturmian case [**BIST, D98a**], and for a class of substitution models including period doubling and binary non-Pisot [**D98d**]. We also want to remark that the work by Hof et al. provides a method to prove generic absence of eigenvalues by studying palindromic structures in the potentials [**HKS**]. However, their method seems to be restricted to sets of measure zero [**DZ, D99b**] and is thus not able to establish almost sure or uniform results.

6.2. Almost sure results. Similarly, one gets a criterion for almost sure absence of eigenvalues as follows. We start with the three-block method. Define

$$G(n) \;=\; \{\omega \in \Omega : V_\omega(k-n) = V_\omega(k) = V_\omega(k+n)\,,\ 1 \le k \le n\}\,.$$

Obviously, the $G(n)$ are Borel sets since they are finite unions of cylinder sets. Combining Lemma 3.3 and Proposition 6.2, we obtain the following proposition.

PROPOSITION 6.4. *Suppose*

$$\limsup_{n\to\infty} \mu(G(n)) \;>\; 0\,. \tag{6.1}$$

Then, $\mu(\Omega_c) = 1$.

Moreover, $\mu(G(n))$ can be estimated by inspecting frequencies of cubes of length $3n$ due to equation (1.1). An argument that is often useful is the following.

LEMMA 6.5. *Suppose $\Omega = \Omega_\psi$ and there is a fourth power v^4 occurring in ψ such that $|v| = n$. Then,*

$$\mu(G(n)) \;\ge\; n d_\psi(v^4)\,. \tag{6.2}$$

In particular, the assumption of Proposition 6.4 is satisfied if one can find a constant $B > 0$ and a sequence of words v_k with $|v_k| = n_k \to \infty$ as $k \to \infty$ such that for all k, $v_k^4 \in F_\psi$ and

$$d_\psi(v_k^4) \ge \frac{B}{n_k}. \tag{6.3}$$

This criterion has been applied to circle map models as well as substitution models. In [**DP86**], the following theorem for circle map models has been proved.

THEOREM 6.6 (Delyon-Petritis). *Let $\Omega_{\alpha,\beta}$ be a circle map hull. Suppose that the coefficients a_n in the continued fraction expansion of α obey*

$$\limsup_{n\to\infty} a_n \ \geq \ 5 \,. \tag{6.4}$$

Then for every coupling constant λ, eigenvalues are almost surely absent.

The condition (6.4) has been slightly relaxed by Kaminaga in [**Ka**].

THEOREM 6.7 (Kaminaga). *Let $\Omega_{\alpha,\beta}$ be a circle map hull. Suppose that the coefficients a_n in the continued fraction expansion of α obey*

$$\limsup_{n\to\infty} a_n \ \geq \ 4 \,. \tag{6.5}$$

Then for every coupling constant λ, eigenvalues are almost surely absent.

For substitution models the criterion in [**D98c**] reads as follows.

THEOREM 6.8. *Let u be a fixed point of a primitive substitution S. Suppose u has a fourth power as a factor. Then for the associated operator family $(H_\omega)_{\omega\in\Omega}$, we have $\mu(\Omega_c) = 1$.*

The criterion applies in particular to the binary non-Pisot case. Moreover, the argument in the proof can be modified to include the period doubling case, too (see [**D98b**] for a slightly more direct proof in this case).

Turning now to the two-block method, we can modify the above steps as follows. Define for $n \in \mathbb{N}$ and $C < \infty$,

$$G'(n,C) \ = \ \{\omega \in \Omega : V_\omega(k) = V_\omega(k+n),\ 1 \leq k \leq n,\ |\mathrm{tr}(M_{E,\omega}(n))| \leq C\ \forall E \in \Sigma\} \,.$$

Again the $G'(n,C)$ are finite unions of cylinder sets and hence Borel sets. Lemma 3.1 and Proposition 6.2 now imply the following.

PROPOSITION 6.9. *Suppose there exists $C < \infty$ such that*

$$\limsup_{n\to\infty} \mu(G'(n,C)) \ > \ 0 \,. \tag{6.6}$$

Then, $\mu(\Omega_c) = 1$.

We have the following criterion which is similar to Lemma 6.5.

LEMMA 6.10. *Suppose $\Omega = \Omega_\psi$ and there is a cube v^3 occurring in ψ such that $|v| = n$ and $|\mathrm{tr}(M_E(v))| \leq C$ for every $E \in \Sigma$. Then,*

$$\mu(G'(n,C)) \ \geq \ n d_\psi(v^3) \,. \tag{6.7}$$

In particular, the assumption of Proposition 6.9 is satisfied if one can find constants $B > 0$ and $C < \infty$ and a sequence of words v_k with $|v_k| = n_k \to \infty$ as $k \to \infty$ such that $v_k^3 \in F_\psi$,

$$|\mathrm{tr}(M_E(v_k))| \ \leq \ C \quad \forall E \in \Sigma \,, \tag{6.8}$$

and

$$d_\psi(v_k^3) \ \geq \ \frac{B}{n_k} \,. \tag{6.9}$$

These criteria are particularly useful in the Sturmian case since we have a uniform trace bound for fixed λ; compare Theorem 4.4. Moreover, Proposition 5.1 allows one to estimate frequencies of s_n^3 using (4.12). Putting this together, one obtains the following result.

THEOREM 6.11 (Kaminaga). *Let Ω_α be a Sturmian hull. Then for every λ, $\mu(\Omega_c) = 1$.*

The proof given in [**Ka**] does not use this two-block argument but rather a more elaborate three-block argument. However, using Proposition 5.1 one may also find suitably positioned cubes. We want to point out that the use of the two-block argument yields additional information; compare Remark 3.4.

6.3. Uniform results. This last theorem can even be strengthened. In fact, uniform absence of eigenvalues was recently established for all Sturmian hulls.

THEOREM 6.12. *Let Ω_α be a Sturmian hull. Then for every λ, $\Omega_c = \Omega_\alpha$.*

The proof shows that for all λ, α, Lemma 3.1 is applicable to every V_ω [**DL99a, DKL**]. This is the only uniform singular continuity result in this context that is known so far.

Let us now summarize the known results on absence of eigenvalues with reference to the respective first proof.

Model	generic	almost sure	uniform
Circle maps (every λ, α, $\beta = \alpha$)	[**BIST**]	[**Ka**]	[**DL99a, DKL**]
Circle maps (every λ, β, a.e. α)	[**DP86**]	[**DP86**]	open
Circle maps (every λ, α, β)	[**HKS**]	open	open
Fibonacci substitution	[**Sü87**]	[**Ka**]	[**DL99a**]
Period doubling substitution	[**BBG91**]	[**D98b**]	open
Binary non-Pisot substitution	[**HKS**]	[**D98c**]	open
Thue-Morse substitution	[**DP91**]	open	open
Rudin-Shapiro substitution	open	open	open

7. Zero-measure spectrum

In this section we show how the fact that the spectrum has zero Lebesgue measure can be proved by using the lower bounds that were established when proving absence of eigenvalues. In this sense, the zero-measure property is merely a corollary to a proof of absence of eigenvalues if the latter is based on the two-block method. In fact, this type of argument virtually recovers all the known results on zero-measure spectrum. A more comprehensive discussion of this simple but somewhat surprising fact is given in [**DL99c**].

Recall that A denotes the set of energies where the averaged Lyapunov exponent vanishes and that it has zero Lebesgue measure for the operators under study. The standard way of proving the zero-measure property is to show that the spectrum Σ is contained in this set,

$$\Sigma \subseteq A\,. \tag{7.1}$$

This can be done with the two-block method as follows. Recall that for every E, there exists a full measure set Ω_E such that for $\omega \in \Omega_E$, the pointwise Lyapunov exponent $\gamma_{\omega,+}(E)$ exists and its value is given by $\gamma(E)$. In fact, it has been shown that for $E \in \Sigma$, $\Omega_E = \Omega$ for all substitution models [**H**] and all Sturmian models [**DL99b**]. So to prove (7.1) it is sufficient to show that the two-block method is applicable for some $\omega \in \Omega$ since it yields that no solution is decaying at $+\infty$, whereas, by the Osceledec result Theorem 1.11, a positive Lyapunov exponent would give rise to a solution which is exponentially decaying at $+\infty$! This simple argument can be applied to all Sturmian models [**DL99c**] (thus recovering the main result from [**BIST**]) and it can also be used in a slightly modified form (see, e.g., [**D99a**]) to recover and elucidate results of Bellissard et al. [**BBG91**] and Bovier and Ghez [**BG93**] for substitution models including period doubling, Thue-Morse, and binary non-Pisot.

The main point here is to emphasize that the occurrence of zero-measure and thus Cantor spectrum is natural, given the Kotani result and hierarchical structures in the potential, the latter leading to a trace map characterization of the spectrum. The additional input of the occurrence of squares is unavoidable in the case of Sturmian models and substitution models over a two-letter alphabet, and it appears as a natural further assumption in the result of Bovier and Ghez for substitution models on larger alphabets.

We summarize the known results on zero-measure spectrum together with references to their proofs.

Model	zero-measure spectrum
Circle maps (every $\lambda, \alpha, \beta = \alpha$)	[**BIST**]
Circle maps (every $\lambda, \alpha, \beta \neq \alpha$)	open
Fibonacci substitution	[**Sü89**]
Period doubling substitution	[**BBG91**]
Binary non-Pisot substitution	[**BG93**]
Thue-Morse substitution	[**BBG91**]
Rudin-Shapiro substitution	open

8. Quantum dynamics

How would such spectral properties show up in physical systems (resp., observables)? To give a first hint, we are now concerned with the transport properties of one-dimensional quasicrystal models, that is, with the long time behavior of the unitary groups generated by the operators under study. We have seen that the presence of purely singular continuous spectrum seems to be the rule for circle map and substitution Hamiltonians. Consequently, we discuss recent ideas and results concerning the dynamics of operators with purely singular continuous spectra. An analysis of the quantum dynamics naturally consists of two parts. On the one hand, one tries to identify crucial characteristics of singular continuous spectral measures which enable one to obtain bounds on the dynamics. Certain dimensions, such as Hausdorff dimension and packing dimension, have proved to be useful in this context. On the other hand, one seeks methods to study these dimensions which apply to the models of interest. Investigations in these directions are still in their early stages. In particular the results for concrete operators are very limited as the reader will notice. This, however, should be seen as a challenge.

Let us first recall parts of the general theory. Suppose H is a self-adjoint operator in $\ell^2(\mathbb{Z}^d)$ and $\psi \in \ell^2(\mathbb{Z}^d)$ with $||\psi|| = 1$. The spectral measure μ_ψ of ψ is uniquely defined by

$$\langle \psi, f(H)\psi \rangle = \int_{\mathbb{R}} f(x) d\mu_\psi(x)$$

for any measurable function f. We are interested in the long time behavior of

$$\psi(t) \;=\; e^{-itH}\psi\,.$$

In the singular continuous regime it is convenient to consider time averaged quantities as suggested by the RAGE theorem [**RS79**], a common quantity being the Cesaro mean of the moments of order p of the position operator for ψ, that is,

$$\langle\langle |X|^p\rangle\rangle(T) \;=\; \frac{1}{T}\int_0^T \langle\psi(t), |X|^p\psi(t)\rangle dt\,,$$

where $|X|^p$ is given by

$$|X|^p \;=\; \sum_{n\in\mathbb{Z}^d} |n|^p\langle\delta_n\,,\cdot\rangle\delta_n$$

with the standard orthonormal basis $(\delta_n)_{n\in\mathbb{Z}^d}$ of $\ell^2(\mathbb{Z}^d)$. Several authors have established lower bounds on $\langle\langle |X|^p\rangle\rangle(T)$ in terms of certain continuity properties of μ_ψ. Typical bounds provide a power law behavior where the power depends on the moment, the continuity (measured by some $\alpha\in(0,1)$), and the space dimension in the following way,

$$\langle\langle |X|^p\rangle\rangle(T) \;>\; C_{\psi,p}T^{\frac{p\alpha}{d}}\,. \tag{8.1}$$

The first type of result in this direction is due to Guarneri [**Gu**] and Combes [**Co**]. It requires uniform α-Hölder continuity of μ_ψ, that is, $\mu_\psi(I) < C|I|^\alpha$ for every interval I with $|I|<1$, $|\cdot|$ denoting Lebesgue measure. It was extended by Last in [**La96**] to measures with non-trivial α-continuous component, that is, μ_ψ which are not supported on a set of zero h^α measure, where h^α denotes the α-dimensional Hausdorff measure ; see also [**BCM**] and [**BT**]. A recent result by Guarneri and Schulz-Baldes relaxes the requirement that the bound holds for all times. They are able to prove a similar bound for a sequence of time scales $T_n\to\infty$ in terms of the packing dimension of μ_ψ which sometimes gives a better exponent; see [**GS**] for details. Similar upper bounds purely in terms of Hausdorff dimensional properties of μ_ψ cannot hold true due to an example in [**DJLS**] which shows that even a pure point measure μ_ψ can give rise to a growth rate of $\langle\langle |X|^2\rangle\rangle(T)$ which is arbitrarily close to ballistic (compare, however, [**Si90**]).

In the one-dimensional Schrödinger operator case, Jitomirskaya and Last developed a beautiful way to study such dimensional properies of spectral measures [**JL96, JL99a, JL99b**]. Their method is in fact an extension of the Gilbert-Pearson theory [**GP, Gi, KP**]. It consists of studying the lim inf of

$$\frac{\|\phi_1\|_L^{2-\alpha}}{\|\phi_2\|_L^{\alpha}}$$

as L tends to infinity, where $\phi_{1,2}$ are solutions of (3.2) with "orthogonal" boundary conditions at the origin and

$$\|\phi\|_L \;=\; \left(\sum_{n=1}^{\lfloor L\rfloor}|\phi(n)|^2 + (L-\lfloor L\rfloor)|\phi(\lfloor L\rfloor+1)|^2\right)^{\frac{1}{2}}. \tag{8.2}$$

This approach has been applied to Sturmian models in [**JL96, JL99b, D98a, DKL**]. Those works obtain the bound (8.1) for all elements in the hull in the case where the rotation number has bounded density. It holds for all initial states ψ with a positive α which depends on the rotation number and the coupling contant. The proof essentially consists of three steps. One first linearizes the trace map in order to prove a power-law *upper* bound on (8.2), uniformly for all solutions (see [**IRT, IT**]). Then one proves a similar uniform power-law *lower* bound. Interestingly, in this step a Gordon-type argument is the key ingredient. Finally, one essentially employs the maximum principle to infer the desired property of the whole-line problem from the analysis of the half-line solution behavior.

9. Open problems

In this concluding section we list some open problems. We state explicit questions as well as vague directions that seem interesting and important.

9.1. A constructive proof of Kotani's result. Theorem 1.13, essentially a corollary to Kotani theory, is right at the heart of most of the results presented and discussed in this paper. Specifically, proofs of absence of absolutely continuous spectrum and zero-measure spectrum were possible only after Kotani published this theorem in 1989. Its value to the known results therefore cannot be overrated. However, we feel that an alternative way of understanding these results would be extremely interesting. Kotani's proof of the fact that the set of energies where the Lyapunov exponent vanishes has Lebesgue measure zero is indirect and inconstructive. It does not give any further information as to why the statement of the theorem is true. On the one hand, it would be nice to have an intuitive understanding of the very uniform absence of absolutely continuous spectrum. This phenomenon, of course, relies heavily on the fact that the potentials take only finitely many values. For example, circle map potentials with the discontinuous characteristic function replaced by a smoother function f seem to exhibit a much smoother transition from absolutely continuous spectrum through singular continuous spectrum to pure point spectrum as the coupling constant λ ranges from 0 through finite values to ∞; compare [**J99**] for the completion of the proof of this phenomenon in the case of the almost Mathieu operator (i.e., $f = \cos$) and [**J95, La95**] for the state of the almost Mathieu art as of 1994. In particular, absolutely continuous spectrum is present in this case for non-zero λ. On the other hand, it would be interesting to investigate the Hausdorff dimension of the spectrum. Kotani's proof does not provide any clue how to tackle this problem in the general case. However, in the 1987 Sütő paper [**Sü87**], a constructive proof of Cantor spectrum was given in the Fibonacci case at large coupling ($\lambda > 4$). Extending this approach, Raymond found in [**Ra**] a way to obtain upper bounds on the Hausdorff dimension in this case. His method should extend to some rotation numbers α other than the golden mean, but already the proof in the Fibonacci case requires considerable effort. The study of the dimension is not only interesting from a purely mathematical perspective. Killip et al. have shown that Raymond's study can be used to establish upper bounds on the dynamics in the Fibonacci case [**KKL99**]. We want to stress, however, that these results are limited to large λ, one particular α, and $\beta = \alpha$. This brings us to the next direction of possible future research activity.

9.2. Spectral dimensions and quantum dynamics. As discussed in an earlier section, there has been considerable progress in the understanding of the dynamics of a Schrödinger time evolution in the presence of peculiar spectral measures, such as purely singular continuous measures with multifractal structures. Such multifractal behavior is expected to be present for Fibonacci-type operators.

Results in this direction for concrete operators, however, are extremely limited, and it will be worthwhile to pursue such investigations. Let us list some questions and problems.

1. Develop methods that allow for the investigation of dimensional properties of spectral measures.
2. Find sufficient criteria for non-trivial dynamical upper bounds.
3. Study dimensions and dynamics for circle map and substitution Hamiltonians.
4. Is it possible to extend the α-continuity result for Sturmian models with bounded density rotation numbers to a larger class or is there in fact a delicate dependence of the transport properties of the operator on the Diophantine properties of the rotation number?
5. A partial answer to the above question could be obtained by an extension of Raymond's bound on the Hausdorff dimension of the spectrum to rotation numbers other than the golden mean since this dimension provides a natural upper bound on the dimension of continuity of the spectral measure. Is it possible, for example, to prove that the Hausdorff dimension of the spectrum is zero for, say, Liouville rotation number and large coupling constant?

9.3. Complexity and spectral theory. Consider two-sided sequences over a two-letter alphabet, that is, elements of $\mathcal{A}^{\mathbb{Z}}$, where $\mathcal{A} = \{x_1, x_2\} \subseteq \mathbb{R}$. Given such a sequence s, we ask what properties of s are crucial in determining the spectral type of the associated Schrödinger operator, $\Delta + s$ in $\ell^2(\mathbb{Z})$, where Δ denotes the discrete Laplacian. It is clear that any spectral type can occur. A possible point of view, namely that combinatorial properties of s might discriminate between the several spectral types, is discussed in this subsection.

Recall the complexity function p_s which measures the subword complexity of some sequence s, that is, $p_s(n)$ equals the number of subwords in s having length n. The condition $p_s(n) \leq n$ for some n is equivalent to p_s being bounded and s being periodic. Moreover, the aperiodic sequences s of minimal complexity (i.e., $p_s(n) = n+1$ for every n) are essentially just the circle map sequences with $\alpha = \beta$ irrational. Restricting our attention to sequences which are *recurrent* in the sense that every subword occurs infinitely often, we can formulate two surprising implications.

1. p_s bounded $\Rightarrow \Delta + s$ has purely absolutely continuous spectrum,
2. $p_s(n) = n + 1 \Rightarrow \Delta + s$ has purely singular continuous spectrum.

As we saw above, sequences generated by primitive substitutions tend to also give rise to operators with purely singular continuous spectrum. This fits very nicely into this picture since their combinatorial complexity is also at the bottom of the hierarchy: It is always bounded by a linear function [**Q**].

On the other complexity extreme, it seems that we encounter a tendency to pure point spectrum. This can be argued as follows. Put some non-trivial probability measure ν on $\mathcal{A}$ (i.e., assign the probability $p \in (0,1)$ to one letter and $1-p$ to the other) and consider the product measure $\bigotimes_{n\in\mathbb{Z}} \nu$ on $\mathcal{A}^{\mathbb{Z}}$. Now it is known that almost every sequence with respect to this measure leads to an operator with pure point spectrum (by results of Carmona et al. on localization for Bernoulli Anderson models [**CKM**]). On the other hand, almost every sequence s has every word from $\mathcal{A}^*$ as a subword and thus has complexity function $p_s(n) = 2^n$. We observe that as complexity is increased, the spectral measures become more singular. This raises several questions.

1. Is there a direct proof of the above observations?
2. Is there a sharp transition from purely singular continuous spectrum to pure point spectrum?
3. Are there examples of potentials in $\mathcal{A}^{\mathbb{Z}}$ where several spectral types coexist?

9.4. The eigenvalue problem. Although there has been considerable effort to study the eigenvalue problem for one-dimensional circle map and substitution models, we feel that there is still room for improving our understanding of this problem. It is extremely puzzling that no counterexample is known to the apparent tendency that the spectrum is purely singular continuous, and yet one is not able to prove absence of eigenvalues for the entire class of potentials. Instead one is currently able to deduce the desired result from certain local symmetries such as powers or palindromes. However, these symmetries are not always present, as the example of the Rudin-Shapiro substitution shows. In fact, this example epitomizes our lack of understanding in that it defies almost all known and well-established approaches. Apart from the absence of absolutely continuous spectrum, essentially nothing is known about this model. Ironically, this again hints at the power of the inconstructive Kotani result. This leads us to the following concrete problems.

1. Find an example with non-empty point spectrum or prove absence of eigenvalues for all models.
2. More modestly, find new ways to prove absence of eigenvalues or to prove presence thereof.
3. Concretely, study the Rudin-Shapiro case.
4. Less concretely, try to prove hierarchical structures in the eigenfunctions using hierarchical structures in the potentials (e.g., for Sturmian models or substitution models) which prevent them from being square-summable. In doing so one would also gain important insight into dimensional issues using, for example, the Jitomirskaya-Last theory. The reader may take a look at the very interesting paper [**J91**] by Jitomirskaya which studies circle map potentials for certain paramter values from this point of view.
5. Finally, since the Thue-Morse substitution is among the most prominent primitive substitutions, it would be nice to prove almost sure absence of eigenvalues also in this case.

9.5. Multi-dimensional models. Although the focus of this paper is on one-dimensional models, we also want to address the problem of extending some of the known results to analogous models in higher dimensions. We hope to have demonstrated that a considerable amount of knowledge and results has been accumulated for one-dimensional Schrödinger operators with circle map or substitution potentials. It is somewhat striking that only very little is known for their multi-dimensional analogs. This is, of course, due to the fact that most results were proved by using the transfer matrix formalism which is a purely one-dimensional concept. However, we believe that some results, such as absence of eigenvalues for certain models, should extend to higher dimensions, and that these extensions should be among the major future objectives in this field.

References

[AP] J.-P. Allouche and J. Peyrière, *Sur une formule de récurrence sur les traces de produits de matrices associés a certaines substitutions*, C. R. Acad. Sci. Paris **302** (1986), 1135–1136.

[AB] Y. Avishai and D. Berend, *Trace maps for arbitrary substitution sequences*, J. Phys. **A 26** (1993), 2437–2443.

[ABG] Y. Avishai, D. Berend and D. Glaubman, *Minimum-dimension trace maps for substitution sequences*, Phys. Rev. Lett. **72** (1994), 1842–1845.

[Ba99a] M. Baake, *A guide to mathematical quasicrystals*, in: "Quasicrystals", Eds. J.-B. Suck, M. Schreiber and P. Häußler, Springer, Berlin (1999), in press; preprint (math-ph/9901014).

[Ba99b] M. Baake, *A note on palindromicity*, to appear in Lett. Math. Phys.; preprint (math-ph/9907011).

[BGJ] M. Baake, U. Grimm and D. Joseph, *Trace maps, invariants, and some of their applications*, Int. J. Mod. Phys. **B 7** (1993), 1527–1550.

[BR] M. Baake and J. A. G. Roberts, *Reversing symmetry group of* $\mathrm{GL}(2,\mathbb{Z})$ *and* $\mathrm{PGL}(2,\mathbb{Z})$ *matrices with connections to cat maps and trace maps*, J. Phys. **A 30** (1997), 1549–1573.

[BCM] J. M. Barbaroux, J. M. Combes and R. Montcho, *Remarks on the relation between quantum dynamics and fractal spectra*, J. Math. Anal. Appl. **213** (1997), 698–722.

[BT] J. M. Barbaroux and S. Tcheremchantsev, *Universal lower bounds for quantum diffusion*, preprint (mp-arc/98-410).

[Be] J. Bellissard, *Spectral properties of Schrödinger's operator with a Thue-Morse potential*, in: "Number Theory and Physics" (Les Houches, 1989), Eds. J. M. Luck, P. Moussa and M. Waldschmidt, Springer, Berlin (1990), pp. 140–150.

[BBG91] J. Bellissard, A. Bovier and J.-M. Ghez, *Spectral properties of a tight binding Hamiltonian with period doubling potential*, Commun. Math. Phys. **135** (1991), 379–399.

[BBG92] J. Bellissard, A. Bovier and J.-M. Ghez, *Gap labelling theorems for one-dimensional discrete Schrödinger operators*, Rev. Math. Phys. **4** (1992), 1–37.

[BIST] J. Bellissard, B. Iochum, E. Scoppola and D. Testard, *Spectral properties of one-dimensional quasicrystals*, Commun. Math. Phys. **125** (1989), 527–543.

[BIT] J. Bellissard, B. Iochum and D. Testard, *Continuity properties of the electronic spectrum of 1D quasicrystals*, Commun. Math. Phys. **141** (1991), 353–380.

[BG93] A. Bovier and J.-M. Ghez, *Spectral properties of one-dimensional Schrödinger operators with potentials generated by substitutions*, Commun. Math. Phys. **158** (1993), 45–66; Erratum: Commun. Math. Phys. **166** (1994), 431–432.

[BG95] A. Bovier and J.-M. Ghez, *Remarks on the spectral properties of tight-binding and Kronig-Penney models with substitution sequences*, J. Phys. **A 28** (1995), 2313–2324.

[Bu] D. Buschmann, *A proof of the Ishii-Pastur theorem by the method of subordinacy*, Univ. Iagel. Acta Math. **34** (1997), 29–34.

[CKM] R. Carmona, A. Klein and F. Martinelli, *Anderson localization for Bernoulli and other singular potentials*, Commun. Math. Phys. **108** (1987), 41–66.

[CL] R. Carmona and J. Lacroix, *Spectral Theory of Random Schrödinger Operators*, Birkhäuser, Boston (1990).

[Ca] M. Casdagli, *Symbolic dynamics for the renormalization group of a quasiperiodic Schrödinger equation*, Commun. Math. Phys. **107** (1986), 295–318.

[Co] J. M. Combes, *Connections between quantum dynamics and spectral properties of time-evolution operators*, in: "Differential Equations with Applications to Mathematical Physics", Eds. W. F. Ames, E. M. Harrel II and J. V. Herod, Academic Press, Boston (1993), pp. 59–68.

[CH] E. M. Coven and G. A. Hedlund, *Sequences with minimal block growth*, Math. Systems Th. **7** (1973), 138–153.

[CFKS] H. L. Cycon, R. Froese, W. Kirsch and B. Simon, *Schrödinger Operators, with Application to Quantum Mechanics and Global Geometry*, Springer, Berlin (1987).

[D97] D. Damanik, *Schrödinger operators with potentials generated by primitive substitutions: An invitation*, Univ. Iagel. Acta Math. **34** (1997), 45–56.

[D98a] D. Damanik, *α-continuity properties of one-dimensional quasicrystals*, Commun. Math. Phys. **192** (1998), 169–182.

[D98b] D. Damanik, *Singular continuous spectrum for the period doubling Hamiltonian on a set of full measure*, Commun. Math. Phys. **196** (1998), 477–483.

[D98c] D. Damanik, *Singular continuous spectrum for a class of substitution Hamiltonians*, Lett. Math. Phys. **46** (1998), 303–311.

[D98d] D. Damanik, *Singulärstetiges Spektrum für Substitutionsoperatoren*, PhD thesis, Univ. Frankfurt, Germany (1998).

[D99a] D. Damanik, *Substitution Hamiltonians with bounded trace map orbits*, preprint (1999).

[D99b] D. Damanik, in preparation.

[DKL] D. Damanik, R. Killip and D. Lenz, *Uniform spectral properties of one-dimensional quasicrystals, III. α-continuity*, preprint (math-ph/9910017).

[DL99a] D. Damanik and D. Lenz, *Uniform spectral properties of one-dimensional quasicrystals, I. Absence of eigenvalues*, Commun. Math. Phys. **207** (1999), 687–696.

[DL99b] D. Damanik and D. Lenz, *Uniform spectral properties of one-dimensional quasicrystals, II. The Lyapunov exponent*, preprint (math-ph/9905008, mp-arc/99-184).

[DL99c] D. Damanik and D. Lenz, *Half-line eigenfunction estimates and singular continuous spectrum of zero Lebesgue measure*, preprint (math.SP/9905099).

[DZ] D. Damanik and D. Zare, *Palindrome complexity bounds for primitive substitution sequences*, to appear in Discr. Math.; preprint (1999).

[DJLS] R. del Rio, S. Jitomirskaya, Y. Last and B. Simon, *Operators with singular continuous spectrum, IV. Hausdorff dimensions, rank one perturbations, and localization*, J. d'Analyse Math. **69** (1996), 153–200.

[DMS] R. del Rio, N. Makarov and B. Simon, *Operators with singular continuous spectrum, II. Rank one operators*, Commun. Math. Phys. **165** (1994), 59–67.

[DP86] F. Delyon and D. Petritis, *Absence of localization in a class of Schrödinger operators with quasiperiodic potential*, Commun. Math. Phys. **103** (1986), 441–444.

[DP91] F. Delyon and J. Peyrière, *Recurrence of the eigenstates of a Schrödinger operator with automatic potential*, J. Stat. Phys. **64** (1991), 363–368.

[FK] H. Furstenberg and H. Kesten, *Products of random matrices*, Ann. Math. Stat. **31** (1960), 457–469.

[Gi] D. J. Gilbert, *On subordinacy and analysis of the spectrum of Schrödinger operators with two singular endpoints*, Proc. Roy. Soc. Edinburgh **112A** (1989), 213–229.

[GP] D. J. Gilbert and D. B. Pearson, *On subordinacy and analysis of the spectrum of one-dimensional Schrödinger operators*, J. Math. Anal. Appl. **128** (1987), 30–56.

[Go76] A. Gordon, *On the point spectrum of the one-dimensional Schrödinger operator*, Usp. Math. Nauk. **31** (1976), 257–258.

[Go94] A. Gordon, *Pure point spectrum under 1-parameter perturbations and instability of Anderson localization*, Commun. Math. Phys. **164** (1994), 489–505.

[Gu] I. Guarneri, *Spectral properties of quantum diffusion on discrete lattices*, Europhys. Lett. **10** (1989), 95–100.

[GS] I. Guarneri and H. Schulz-Baldes, *Lower bounds on wave packet propagation by packing dimensions of spectral measures*, Math. Phys. Electron. J. **5** (1999), paper 1, 16 pp.

[GA] G. Gumbs and M. K. Ali, *Dynamical maps, Cantor spectra, and localization for Fibonacci and related quasiperiodic lattices*, Phys. Rev. Lett. **60** (1988), 1081–1084.

[H] A. Hof, *Some remarks on discrete aperiodic Schrödinger operators*, J. Stat. Phys. **72** (1993), 1353–1374.

[HKS] A. Hof, O. Knill and B. Simon, *Singular continuous spectrum for palindromic Schrödinger operators*, Commun. Math. Phys. **174** (1995), 149–159.

[IRT] B. Iochum, L. Raymond and D. Testard, *Resistance of one-dimensional quasicrystals*, Physica **A 187** (1992), 353–368.

[IT] B. Iochum and D. Testard, *Power law growth for the resistance in the Fibonacci model*, J. Stat. Phys. **65** (1991), 715–723.

[I] K. Ishii, *Localization of eigenstates and transport phenomena in the one dimensional disordered system*, Supp. Progr. Theor. Phys. **53** (1973), 77–138.

[J91] S. Jitomirskaya, *Singular spectral properties of a one-dimensional discrete Schrödinger operator with quasiperiodic potential*, Adv. Sov. Math. **3** (1991), 215–254.

[J95] S. Jitomirskaya, *Almost everything about the almost Mathieu operator. II.*, in: "XI Intern. Congress of Mathematical Physics" (Paris, 1994), Ed. D. Iagolnitzer, Intern. Press, Cambridge, MA (1995), pp. 373–382.

[J99] S. Jitomirskaya, *Metal-insulator transition for the almost Mathieu operator*, to appear in Ann. of Math.; preprint (mp-arc/99-369).

[JL96] S. Jitomirskaya and Y. Last, *Dimensional Hausdorff properties of singular continuous spectra*, Phys. Rev. Lett. **76** (1996), 1765–1769.

[JL99a] S. Jitomirskaya and Y. Last, *Power law subordinacy and singular spectra, I. Half-line operators*, to appear in Acta Math.; preprint (mp-arc/98-723).

[JL99b] S. Jitomirskaya and Y. Last, *Power law subordinacy and singular spectra, II. Line operators*, preprint (mp-arc/99-364).

[JS] S. Jitomirskaya and B. Simon, *Operators with singular continuous spectrum: III. Almost periodic Schrödinger operators*, Commun. Math. Phys. **165** (1994), 201–205.

[KKL86] P. A. Kalugin, A. Yu. Kitaev and L. S. Levitov, *Electron spectrum of a one-dimensional quasicrystal*, Sov. Phys. JETP **64** (1986), 410–415.

[Ka] M. Kaminaga, *Absence of point spectrum for a class of discrete Schrödinger operators with quasiperiodic potential*, Forum Math. **8** (1996), 63–69.

[KP] S. Khan and D. B. Pearson, *Subordinacy and spectral theory for infinite matrices*, Helv. Phys. Acta **65** (1992), 505–527.

[Kh] A. Khintchine, *Continued Fractions*, Dover, Mineola (1997).

[KKL99] R. Killip, A. Kiselev and Y. Last, in preparation.

[KKT] M. Kohmoto, L. P. Kadanoff and C. Tang, *Localization problem in one dimension: Mapping and escape*, Phys. Rev. Lett. **50** (1983), 1870–1872.

[KO] M. Kohmoto and Y. Oono, *Cantor spectrum for an almost periodic Schrödinger equation and a dynamical map*, Phys. Lett. **A 102** (1984), 145–148.

[KN] M. Kolár and F. Nori, *Trace maps of general substitutional sequences*, Phys. Rev. **B 42** (1990), 1062–1065.

[Ko84] S. Kotani, *Ljapunov indices determine absolutely continuous spectra of stationary random one-dimensional Schrödinger operators*, in: "Stochastic Analysis" (Katata/Kyoto, 1982), Ed. K. Itô, North Holland, Amsterdam (1984), pp. 225–247.

[Ko89] S. Kotani, *Jacobi matrices with random potentials taking finitely many values*, Rev. Math. Phys. **1** (1989), 129–133.

[KS] H. Kunz and B. Souillard, *Sur le spectre des opérateurs aux différences finies aléatoires*, Commun. Math. Phys. **78** (1980), 201–246.

[La95] Y. Last, *Almost everything about the almost Mathieu operator. I.*, in: "XI Intern. Congress of Mathematical Physics" (Paris, 1994), Ed. D. Iagolnitzer, Intern. Press, Cambridge, MA (1995), pp. 366–372.

[La96] Y. Last, *Quantum dynamics and decompositions of singular continuous spectra*, J. Funct. Anal. **142** (1996), 406–445.

[LS] Y. Last and B. Simon, *Eigenfunctions, transfer matrices, and absolutely continuous spectrum of one-dimensional Schrödinger operators*, Invent. Math. **135** (1999), 329–367.

[Lo83] M. Lothaire, *Combinatorics on words*, 2nd ed., Cambridge University Press, Cambridge (1999).

[Lo99] M. Lothaire, *Algebraic combinatorics on words*, in preparation.

[LP86] J. M. Luck and D. Petritis, *Phonon spectra in one-dimensional quasicrystals*, J. Stat. Phys. **42** (1986), 289–310.

[LP92] W. F. Lunnon and P. A. B. Pleasants, *Characterization of two-distance sequences*, J. Austral. Math. Soc. **A 53** (1992), 198–218.

[O] V. Osceledec, *A multiplicative ergodic theorem. Lyapunov characteristic numbers for dynamical systems*, Trans. Moscow Math. Soc. **19** (1968), 197–231.

[OPRSS] S. Ostlund, R. Pandit, D. Rand, H. J. Schellnhuber and E. D. Siggia, *One-dimensional Schrödinger equation with an almost periodic potential*, Phys. Rev. Lett. **50** (1983), 1873–1877.

[Pa] L. Pastur, *Spectral properties of disordered systems in the one-body approximation*, Commun. Math. Phys. **75** (1980), 179–196.

[Pe78] D. B. Pearson, *Singular continuous measures in scattering theory*, Commun. Math. Phys. **60** (1978), 13–36.

[Pe54] O. Perron, *Die Lehre von den Kettenbrüchen*, Teubner, Stuttgart (1954).

[Pe83] K. Petersen, *Ergodic theory*, Cambridge University Press, Cambridge (1983).

[PWW] J. Peyrière, Z.-Y. Wen and Z.-X. Wen, *Polynomes associés aux endomorphismes de groupes libres*, Enseign. Math. **39** (1993), 153–175.

[Q] M. Queffélec, *Substitution Dynamical Systems – Spectral Analysis*, Lecture Notes in Mathematics, Vol. 1284, Springer, Berlin (1987).

[Ra] L. Raymond, *A constructive gap labelling for the discrete Schrödinger operator on a quasiperiodic chain*, preprint (1997).
[RS80] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, Vol. I: Functional Analysis*, 2nd ed., Academic Press, New York (1980).
[RS79] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, Vol. III: Scattering Theory*, Academic Press, New York (1979).
[RS78] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, Vol. IV: Analysis of Operators*, Academic Press, New York (1978).
[Ro] J. A. G. Roberts, *Escaping orbits in trace maps*, Physica **A 228** (1996), 295–325.
[RB] J. A. G. Roberts and M. Baake, *Trace maps as 3D reversible dynamical systems with an invariant*, J. Stat. Phys. **74** (1994), 829–888.
[Ru] D. Ruelle, *Ergodic theory of differentiable dynamical systems*, Publ. IHES **50** (1979), 27–58.
[SBGC] D. Shechtman, I. Blech, D. Gratias and J. V. Cahn, *Metallic phase with long range orientational order and no translational symmetry*, Phys. Rev. Lett. **53** (1984), 1951–1953.
[Si83] B. Simon, *Kotani theory for one dimensional stochastic Jacobi matrices*, Commun. Math. Phys. **89** (1983), 227–234.
[Si90] B. Simon, *Absence of ballistic motion*, Commun. Math. Phys. **134** (1990), 209–212.
[Si95a] B. Simon, *Operators with singular continuous spectrum: I. General operators*, Ann. of Math. **141** (1995), 131–145.
[Si95b] B. Simon, *Spectral analysis of rank one perturbations and applications*, CRM Proc. Lecture Notes **8** (1995), 109–149.
[Si96a] B. Simon, *Operators with singular continuous spectrum: VI. Graph Laplacians and Laplace-Beltrami operators*, Proc. Amer. Math. Soc. **124** (1996), 1177–1182.
[Si96b] B. Simon, *Operators with singular continuous spectrum: VII. Examples with borderline time decay*, Commun. Math. Phys. **176** (1996), 713–722.
[SS] B. Simon and G. Stolz, *Operators with singular continuous spectrum: V. Sparse potentials*, Proc. Amer. Math. Soc. **124** (1996), 2073–2080.
[SW] B. Simon and T. Wolff, *Singular continuous spectrum under rank one perturbations and localization for random Hamiltonians*, Commun. Pure Appl. Math. **39** (1986), 75–90.
[Sü87] A. Sütő, *The spectrum of a quasiperiodic Schrödinger operator*, Commun. Math. Phys. **111** (1987), 409–415.
[Sü89] A. Sütő, *Singular continuous spectrum on a Cantor set of zero Lebesgue measure for the Fibonacci Hamiltonian*, J. Stat. Phys. **56** (1989), 525–531.
[Sü95] A. Sütő, *Schrödinger difference equation with deterministic ergodic potentials*, in: "Beyond Quasicrystals" (Les Houches, 1994), Eds. F. Axel and D. Gratias, Springer, Berlin (1995), pp. 481–549.
[W] P. Walters, *An Introduction to Ergodic Theory*, Springer, New York (1982).

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