Convergence to the maximal invariant measure for a zero-range process with random rates.

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Abstract

We consider a one-dimensional totally asymmetric nearest-neighbor zero-range process with site-dependent jump-rates —an *environment*. For each environment p we prove that the set of all invariant measures is the convex hull of a set of product measures with geometric marginals. As a consequence we show that for environments p satisfying certain asymptotic property, there are no invariant measures concentrating on configurations with critical density bigger than $\rho^*(p)$, a critical value. If $\rho^*(p)$ is finite we say that there is phase-transition on the density. In this case we prove that if the initial configuration has asymptotic density strictly above $\rho^*(p)$, then the process converges to the maximal invariant measure.

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1 Introduction

The interest on the behavior of interacting particle systems in random environment has grown recently: Benjamini, Ferrari and Landim

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(1996), Evans (1996) and Ferrari and Krug (1996) observed the existence of phase transition in these models; Benjamini, Ferrari and Landim (1996), Krug and Seppäläinen (1999) and Koukkous (1999) investigated the hydrodynamic behavior of conservative processes in random environments; Landim (1996) and Bahadoran (1998) considered the same problem for non-homogeneous asymmetric attractive processes; Gielis, Koukkous and Landim (1998) deduced the equilibrium fluctuations of a symmetric zero range process in random environment.

In this article we consider a one-dimensional, totally asymmetric, nearest-neighbor zero-range process in a non-homogeneous environment. The evolution can be informally described as follows. Fix $c \in (0,1)$ and provide each site x of $\mathbb{Z}$ with a rate function $p_x \in [c,1]$. If there is at least one particle at some site x, one of these particles jumps to $x+1$ at rate p_x. A rate configuration $p = (p_x : x \in \mathbb{Z})$ is called an *environment* and a measure m on the set of possible environments a *random environment.*

Benjamini, Ferrari and Landim (1996) and Evans (1996) for an asymmetric exclusion process with rates associated to the particles —which is isomorphic to a zero range process with rates associated to the sites— and Ferrari and Krug (1996) for the model considered here, proved the existence of a phase transition in the density. More precisely, they proved that, under certain conditions on the distribution m, specified in Theorem 2.4, there exists a finite critical value ρ^* such that for m-almost-all p there are no product invariant measures for the process with rates p concentrating on configurations with asymptotic density bigger than ρ^* and that there are product invariant measures concentrating on configurations with asymptotic density smaller than or equal to ρ^*. (The density of a configuration is essentially the average number of particles per site and is defined in (7) below).

Our first result is that the set of extremal invariant measures for the process with fixed environment $p = (p_x : x \in \mathbb{Z})$ is the set $\{\nu_{p,v} : v < p_x, \forall x\}$, where $\nu_{p,v}$ is the product measure on $\mathbb{N}^{\mathbb{Z}}$ with marginals

$$\nu_{p,v}\{\xi : \xi(x) = k\} = \left(\frac{v}{p_x}\right)^k \left(1 - \frac{v}{p_x}\right) . \qquad (1)$$

The above result does not surprise specialists in queuing theory. In fact we are dealing with an infinite series of M/M/1 queues with service rate p_x at queue x. The value v can be interpreted as the arrival rate at "queue" $-\infty$. Since Burke's theorem guarantees that in equilibrium the departure process of a M/M/1 queue is the same as the arrival process (both Poisson of rate v), there is an invariant measure for each arrival rate v strictly smaller than all service rates.

Assume $c = \inf_x p_x$ and that the following limits exist. For $v < c$,

$$R(p,v) \;:=\; \lim_{n\to\infty} \frac{1}{n} \sum_{x=-n+1}^{0} \int \nu_{p,v}(d\xi)\, \xi(x) \;=\; \lim_{n\to\infty} \frac{1}{n} \sum_{x=-n+1}^{0} \frac{v}{p_x - v}. \tag{2}$$

We interpret $R(p,v)$ as the global expected left density per site of the configurations distributed according to $\nu_{p,v}$. A consequence of the existence of the limits is that for all $v < c$, $\nu_{p,v}$ concentrates in configurations with asymptotic left density $R(p,v)$:

$$\nu_{p,v}\Big(\lim_{n\to\infty} \frac{1}{n} \sum_{x=-n+1}^{0} \xi(x) = R(p,v)\Big) \;=\; 1\;. \tag{3}$$

It is easy to prove that $R(p,v)$ is a strictly convex increasing function of v, hence the limit

$$\rho^*(p) := \lim_{v\to c} R(p,v) \tag{4}$$

is well defined (but may be infinite). In the sequel we assume $\rho^*(p) < \infty$. We do not assume the existence of the limit in (2) for $v = c$, nor the $\nu_{p,c}$ almost sure convergence of the density.

Our second and main result states that under the condition $\rho^*(p) < \infty$, initial measures concentrating on configurations with asymptotic left density strictly bigger than $\rho^*(p)$ converge towards the maximal invariant measure $\nu_{p,c}$. We do not know in general if this measure concentrates on configurations with density. But if the limit $R(p,c)$ of (2) exists, equals $\rho^*(p)$ and is finite our result says that the process starts with global density strictly above $\rho^*(p)$ and converges to a measure with density $\rho^*(p)$. This behavior is remarkable as the process is *conservative*, i.e. the total number of particles is conserved, but in the above limit "looses mass". Informally speaking, what happens is that many clients remain trapped in far away slow servers. More precisely, denoting by $S_p(t)$ the semigroup of the process, we first show that for any initial measure ν, all weak limits of the sequence $\{\nu S_p(t),\, t \geq 0\}$ are dominated, in the natural partial order, by $\nu_{p,c}$. We then show that if ν is a measure concentrated on configurations with asymptotic left density strictly greater than $\rho^*(p)$, all weak limits of $\nu S_p(t)$ dominate $\nu_{p,c}$. Surprisingly enough, the proof of the second statement is much more demanding than the proof of the first one.

It follows from the two previous results that the domain of attraction of $\nu_{p,c}$ includes all measures with asymptotic density strictly above $\rho^*(p)$. It remains an open question to describe the domain of attraction of a product invariant measure $\nu_{p,v}$ for $0 < v < c$ or to show the convergence to $\nu_{p,c}$ of initial measures with asymptotic density $\rho^*(p)$.

Our results hold m-a.s. for measures m concentrating on environments satisfying (2).

The paper is organized as follows: in Section 2 we introduce the notation and state the main results. In Section 3 we characterize the set of invariant measures and show that the maximal invariant measure dominates all the weak limits of the process. In Section 4 we obtain the asymptotic velocity of a second class particle for the zero-range process in non homogeneous environment and use this result to prove the main theorem.

2 Notation and Results

Fix $0 < c \leq 1$ and consider a sequence $(p_x)_{x\in\mathbb{Z}}$ taking values in $[c, 1]$ such that $c = \inf_x p_x$. We consider a totally asymmetric zero-range process in the environment p. This is a Markov process that can be informally described as follows. We initially distribute particles on the lattice $\mathbb{Z}$. If there is at least one particle at some site x, then at rate p_x one of them jumps to site $x + 1$. To construct a Markov process η_t on $\mathbf{X} = \mathbb{N}^{\mathbb{Z}}$ corresponding to the above description, let $N_x(t)$ $(x \in \mathbb{Z})$ be a collection of independent Poisson processes such that for all $x \in \mathbb{Z}$, $\mathbf{E}(N_x(t)) = p_x t$. The evolution of η_t is now given by the following rule: if the Poisson process $N_x(.)$ jumps at time t and $\eta_{t-}(x) > 0$ then one particle is moved from x to $x + 1$ at that time. That is, for cylinder functions $f : \mathbf{X} \to \mathbb{R}$,

$$df(\eta_t) := \sum_{x\in\mathbb{Z}} dN_x(t)\, \mathbf{1}\{\eta(x) > 0\} \left[f(\eta^x) - f(\eta)\right] ,$$

where $dN_x(t)$ is one if there is an event of the Poisson process N_x at time t, otherwise it is zero. In the above formula $\eta^x = \eta - \mathfrak{d}_x + \mathfrak{d}_{x+1}$, where $\mathfrak{d}_y$ stands for a configuration with just one particle at y and addition of configurations is performed componentwise. To see that the process is well defined by this rule, just note that in any time interval $[0,t]$ for any x there exists with probability 1 a $y < x$ such that $N_y(t) = 0$. Hence the value of $\eta_t(x)$ depends only on the initial configuration and on a finite number of jumps.

The generator L_p of this process, defined by $L_p f(\eta) = d\mathbf{E}[f(\eta_t) \mid \eta_0 = \eta]/dt\Big|_{t=0}$, acts on cylinder functions f as follows:

$$(L_p f)(\eta) = \sum_{x \in \mathbb{Z}} p_x \, \mathbf{1}\{\eta(x) > 0\} \, [f(\eta^x) - f(\eta)] \ . \tag{5}$$

We denote by $\{S_p(t),\, t \geq 0\}$ the semigroup associated to the generator L_p, i.e. $S_p(t)f(\eta) = \mathbf{E}[f(\eta_t) \mid \eta_0 = \eta]$ and by $\mathcal{I}_p$ the set of invariant measures of η_t (the Markov process with generator L_p). Let v be a real number such that $0 < v < p_x$ for all x. Then a standard calculation shows that the product measure $\nu_{p,v}$ with marginals given by (1) is an invariant measure for the process. Benjamini, Ferrari and Landim (1998) raised the question of whether or not there exist invariant measures which are not convex combinations of the $\nu_{p,v}$'s. In Section 2 we prove the following theorem which, combined with Theorem 12.2 in Dynkin (1978), gives a negative answer to that question. In its statement we denote by $(\mathcal{I}_p)_e$ the set of extremal invariant measures for the process.

Theorem 2.1 *Let p be an arbitrary environment then*

$$(\mathcal{I}_p)_e = \{\nu_{p,v} \ : \ v < p_x, \ \forall x \in \mathbb{Z}\}.$$

In this theorem the range of the parameter v may be either $[0,c)$ or $[0,c]$ - when $p_x = c$ for some x or $p_x > c$ for all x, respectively. In the first case to prove the theorem we only need to follow the proof of Theorem 1.11 in Andjel (1982), but in the second case a complementary argument is needed. In both cases the proof relies on the standard partial order for probability measures on $\mathbf{X}$. To define it, first say that $\eta \leq \xi$ if $\eta(x) \leq \xi(x)$ for all $x \in \mathbb{Z}$. Then say that a real valued function f defined on $\mathbf{X}$ is increasing if $\eta \leq \xi$ implies that $f(\eta) \leq f(\xi)$. Finally if μ and ν are two probability measures on $\mathbf{X}$, say that $\mu \leq \nu$ if $\int f d\mu \leq \int f d\nu$ for all bounded increasing cylinder functions f. The complementary argument alluded above depends on the following proposition:

Proposition 2.2 *Assume that p is an environment such that*

$$p_x > c \text{ for all } x \in \mathbb{Z} \text{ and } \liminf_{x \to -\infty} p_x = c\,, \tag{6}$$

and let ν be an arbitrary probability measure on $\mathbf{X}$. Then the set of measures $\{\nu S_p(t) \ : \ t > 0\}$ is tight and its weak limits as t goes to infinity are bounded above by $\nu_{p,c}$.

An immediate corollary of Proposition 2.2 is that under (6) all invariant measures are dominated by $\nu_{p,c}$.

To state our main result let η be an element of $\mathbf{X}$ and consider

$$\begin{aligned} \underline{D}(\eta) &= \liminf_{n\to\infty} \frac{1}{n} \sum_{x=-n+1}^{0} \eta(x)\ , \\ \overline{D}(\eta) &= \limsup_{n\to\infty} \frac{1}{n} \sum_{x=-n+1}^{0} \eta(x)\ , \end{aligned}$$

the *lower*, respectively *upper asymptotic left density* of η. If both limits are equal to α we say that η has *left density* α and write $D(\eta) = \alpha$.

Assume that p is an environment for which the limits defined in (2) exist. Then, by Kolmogorov's law of large numbers (see *e.g.* Shiryayev (1984), Theorem 2 p. 364) $\nu_{p,v}$ concentrates on configurations with left density $R(p,v)$:

$$\nu_{p,v}\{\eta \in \mathbf{X} : D(\eta) = R(p,v)\} = 1 \tag{7}$$

for all $v < c$.

The values assumed by $R(p,v)$ for $v < c$ are crucial for the characterization of the set of invariant measures for the process with rates p. If $\lim_{v\to c} R(p,v) = \infty$, then the range of allowed densities is $[0,\infty)$ or $[0,\infty]$. The first case occurs when $p_x = c$ for some x. In this case $\nu_{p,v}$ is defined for any $v < c$, but not for $v = c$. Moreover, since $R(p,\cdot)$ is continuous and increases to ∞ as $v \to c$, then for all $\rho \in [0,\infty)$ there exists $v = v(p,\rho)$ such that $\nu_{p,v}\{\eta \in \mathbf{X} : D(\eta) = \rho\} = 1$. The second case occurs when $p_x > c$ for all x. In this case $\nu_{p,c}$ is well defined and concentrates on configurations with infinite asymptotic left density, and for any $\rho \in [0,\infty]$ there exists $v = v(p,\rho)$ such that $\nu_{p,v}\{\eta \in \mathbf{X} : D(\eta) = \rho\} = 1$.

If $\lim_{v\to c} R(p,v) = \rho^*(p) < \infty$ and $p_x > c$ for all x, the measure $\nu_{p,c}$ is well defined and Theorem 2.1 tells us that there are no invariant measures bigger than $\nu_{p,c}$. Our next theorem describes what happens in this case when one starts with a density strictly bigger than $\rho^*(p)$. This is our main result.

Theorem 2.3 *Let p be an environment satisfying (6) such that $\rho^*(p) < \infty$ and η be a configuration such that $\underline{D}(\eta) > \rho^*(p)$. Then*

$$\lim_{t\to\infty} \delta_\eta S_p(t) = \nu_{p,c}\,,$$

where δ_η is the measure giving weight one to the configuration η.

As a corollary to Theorem 2.3 we obtain the asymptotic behavior of the system when the environment is randomly chosen. Let m be the distribution of a stationary ergodic sequence p on $[c,1]$ such that $m(\{p : p_0 = c\}) = 0$, $m(\{p : c < p_0 < c+\varepsilon\}) > 0$ for all $\varepsilon > 0$. The measure $m\nu_{\cdot,v}$ defined by $m\nu_{\cdot,v}f = \int m(dp)\int \nu_{p,v}(d\eta)f(\eta)$ is an ergodic distribution on $\mathbf{X}$ and, by the Ergodic Theorem, for all $v < c$ and for m-almost all p, the asymptotic density exists $\nu_{p,v}$ a.s. and is equal to:

$$R(v) = \int \frac{v}{p_0 - v} m(dp).$$

Let $\rho^* := \lim_{v\to c} R(v)$ and assume $\rho^* < \infty$. In this case for m-almost all environment p any invariant measure for L_p is dominated by $\nu_{p,c}$. The following theorem concerns the behavior of the process when the initial measure concentrates on configurations with density strictly higher than ρ^*.

Theorem 2.4 *Let m be the distribution of a stationary ergodic sequence $p = (p_x)_{x\in\mathbb{Z}}$ on $(c,1]$ such that $m(\{p : c < p_0 < c+\varepsilon\}) > 0$ for all $\varepsilon > 0$ and for which $\rho^* < \infty$. Let ν be a measure for which ν a.s. $\underline{D}(\eta)$ is strictly bigger than ρ^*. Then, for m-almost all p*

$$\lim_{t\to\infty} \nu S_p(t) = \nu_{p,c}.$$

3 Domination and Invariant measures

In this section we prove Proposition 2.2 and Theorem 2.1.
Proof of Proposition 2.2. Fix an arbitrary site y and let x_n be a decreasing sequence such that $x_1 < y$, $p_{x_n} < p_z$ for $x_n < z \le y$ and

p_{x_n} decreases to c. The existence of such a sequence is guaranteed by (6). Consider a process on $\mathbb{N}^{\{x_n+1,\ldots,y\}}$ with generator given by:

$$
\begin{aligned}
L_{p,n}f(\eta) &= \sum_{z=x_n+1}^{y-1} \mathbf{1}\{\eta(z)>0\}p_z[f(\eta^z)-f(\eta)] \\
&\quad + p_{x_n}[f(\eta+\mathfrak{d}_{x_n+1})-f(\eta)] \qquad\qquad (8) \\
&\quad + \mathbf{1}\{\eta(y)>0\}p_y[f(\eta-\mathfrak{d}_y)-f(\eta)]\ .
\end{aligned}
$$

Let $S_{p,n}$ be the semigroup associated to this process and for an arbitrary probability measure ν let ν_n be its projection on $\mathbb{N}^{\{x_n+1,\ldots,y\}}$. Standard coupling arguments show that

$$(\nu S_p(t))_n \le \nu_n S_{p,n}(t)\ .$$

The coupling of the two processes is done using the same Poisson processes $N_x(t)$ defined in Section 2. The reason why the domination holds is that for the process $S_{p,n}(t)$, each time the Poisson process $N_{x_n}(t)$ jumps, a new particle appears in x_n+1, while the same happens for the process $S_p(t)$ only when there is at least a particle in the site x_n.

The process with generator $L_{p,n}$ is irreducible and has a countable state space, moreover a simple computation shows that the product measure $\mu_{n,p}$ with marginals given by

$$\mu_{n,p}\{\eta:\eta(z)=k\} = \Big(1-\frac{p_{x_n}}{p_z}\Big)\Big(\frac{p_{x_n}}{p_z}\Big)^k,$$

where $x_n < z \le y$, is invariant for the process. Therefore $\nu_n S_{n,p}(t)$ converges to $\mu_{n,p}$ and any weak limit point of $(\nu S_p(t))_n$ is bounded above by $\mu_{n,p}$. Since as n goes to infinity the marginals of $\widetilde{\nu}_{n,p}$ converge to the marginals of $\nu_{c,p}$ the proposition is proved. ■

Proof of Theorem 2.1. Since only the final step of the proof is different from the proof of Theorem 1.11 in Andjel (1982) we refer the reader to that paper. Exactly as there one proves that if ν_p is an extremal invariant measure then for each $v < \inf_x p_x$ either $\nu_p \le \nu_{v,p}$ or $\nu_p \ge \nu_{v,p}$. This implies that either $\nu_p = \nu_{v,p}$ for some v or $\nu_p \ge \nu_{v,p}$ for all v. The latter case cannot occur if there exists x such that $p_x = \inf_y p_y$ because this would imply that $\nu_p\{\eta : \eta(x) > k\} = 1$ for all k. Therefore $\nu_p \ge \nu_{c,p}$ and either $\liminf_{x\to-\infty} p_x = \inf p_y$ or $\liminf_{x\to\infty} p_x = \inf p_y$. In the first of these cases, Proposition 2.2

allows us to conclude immediately. In the second case we argue by contradiction: let $\widetilde{\nu}$ be a probability measure on $\mathbb{N}^{\mathbb{Z}} \times \mathbb{N}^{\mathbb{Z}}$ admitting as first marginal and second marginal ν_p and $\nu_{c,p}$ respectively and such that $\widetilde{\nu}\{(\eta,\xi) : \eta \geq \xi\} = 1$. Consider the standard coupled process with initial measure $\widetilde{\nu}$. Denote by $\overline{S}(t)$ the semigroup associated to this process and assume that for some x, $\widetilde{\nu}\{(\eta,\xi) : \eta(x) > \xi(x)\} > 0$. It then follows that for all $t > 0$

$$\widetilde{\nu}\overline{S}(t)\left\{(\eta,\xi) : \eta(x) > \xi(x) = 0\right\} > 0.$$

Hence

$$\nu_p\left\{\eta : \eta(x) > 0\right\} > \nu_{c,p}\left\{\eta : \eta(x) > 0\right\} = \frac{c}{p_x}.$$

Pick $y > x$ and such that $p_y < p_x \nu_p\{\eta : \eta(x) > 0\}$. Then let $f(\eta) = \sum_{z=x+1}^{y} \eta(z)$. Now a simple calculation shows that $\int L_p f(\eta) d\nu_p > 0$ contradicting the invariance of ν_p. ■

Remark: Proofs of Theorem 2.1 and Proposition 2.2 can easily be extended to a larger class of one-dimensional nearest-neighbors asymmetric zero range processes in non-homogeneous environment. In these systems a particle at site x on configuration η jumps at rate $p_x g(\eta(x))$ to site $x+1$, where $g : \mathbb{N} \to [0,\infty)$ is a non-decreasing bounded function such that $g(0) = 0$.

4 Convergence

We prove in this section Theorem 2.3. Fix a measure ν on $\mathbf{X}$ concentrated on configurations with lower asymptotic left density strictly greater than $\rho^*(p)$. Theorem 2.3 follows from Proposition 2.2 and from the next lemma, which implies that any invariant measure $\nu_{p,v}$ is dominated by any weak limit of $\nu S_p(t)$. Denote $\{\overline{S}_p(t) : t \geq 0\}$ the semigroup corresponding to the coupling between two versions of the process with (possibly) different initial configurations, by using the same Poisson processes $(N_x(t) : x \in \mathbb{Z})$ in its construction.

Lemma 4.1 *Let p be an environment satisfying (6) and such that $\rho^*(p) < \infty$ and ζ a configuration with lower asymptotic left density $\underline{D}(\zeta) > \rho^*(p)$. Then for any $v < c$,*

$$\lim_{t\to\infty} (\delta_\zeta \times \nu_{p,v})\overline{S}_p(t)\Big\{(\eta,\xi) : \eta(x) < \xi(x)\Big\} \;=\; 0$$

for all x in $\mathbb{Z}$.

The proof of this lemma requires the following result. It states that for each $v < c$ the asymptotic velocity of a second class particle in the zero-range process in the environment p under the invariant measure $\nu_{p,v}$ is strictly positive.

Fix a starting site z and consider a coupled zero range process with initial condition $(\eta, \eta + \mathfrak{d}_z)$ and semigroup $\bar{S}_p(t)$. Under the coupled dynamics the number of sites where the marginals differ does not increase in time. Let X_t^z be the site where the marginals differ at time t. We can think that X_t^z stands for the position of a "second class particle". Indeed, if the second class particle is at x at time t it jumps to $x+1$ at rate $p_x \mathbf{1}\{\eta_t(x) = 0\}$. In other words, the second class particle jumps only if there is no other particle at the site where it is.

For an environment p and a probability measure ν on $\mathbf{X}$, denote by $\mathbf{P}_\nu$ the measure on $D(\mathbb{R}_+, \mathbf{X})$ induced by ν and the Markov process with generator L_p defined in (5). In the next lemma we write $\mathbf{P}_{(\nu,z)}$ for a coupled process whose initial configuration is $(\eta, \eta + \mathfrak{d}_z)$, with η distributed according to ν. Since $R(p,v)$ is convex and strictly increasing

$$\gamma(p,v) := [R'(p,v)]^{-1} \tag{9}$$

exists in a dense subset of $(0,c)$. In the sequel we abuse notation by not writing integer parts where necessary.

Lemma 4.2 *Let p be an environment for which the limits in (2) exist for $v < c$. Pick $v \in (0,c)$ such that $\gamma(p,v)$ exists. Then,*

$$\lim_{t\to\infty} \mathbf{P}_{(\nu_{p,v},-at)}\Big(\Big|\frac{X_t^{-at}}{t} - (\gamma(p,v) - a)\Big| > \varepsilon\Big) \;=\; 1\,, \tag{10}$$

for all $\varepsilon > 0$ if $a > \gamma(p,v)$ and

$$\lim_{t\to\infty} \mathbf{P}_{(\nu_{p,v},-at)}\Big(\frac{X_t^{-at}}{t} > 0\Big) \;=\; 1 \tag{11}$$

if $a < \gamma(p,v)$.

Remark. The more complete result when the starting point a is greater than $\gamma(p,v)$ comes from the fact that in our hypothesis we have only the asymptotic *left* limits (2). If the limits (2) hold for both sides, then (10) is valid for all a.

Proof: Note that it suffices to prove (10), since (11) follows from (10) and the fact that it does not depend on the environment to the right of the origin. For $u < w < c$ let $\bar{\nu}_{p,u,w}$ be the product measure on $\mathbf{X} \times \mathbf{X}$ whose first marginal is equal to $\nu_{p,u}$, whose second marginal is equal to $\nu_{p,w}$ and which is concentrated above the diagonal: $\bar{\nu}_{p,u,w}\{(\eta, \xi) : \eta \leq \xi\} = 1$. Denote by (η_t, ξ_t) the coupled Markov process starting form $\bar{\nu}_{p,u,w}$.

Denote by ζ_t the difference $\xi_t - \eta_t$ and observe that the ζ-particles evolve as second class particles in the sense that a ζ-particle jumps from x to $x + 1$ at rate

$$p_x \left[\mathbf{1}\{\eta(x) + \zeta(x) \geq 1\} - \mathbf{1}\{\eta(x) \geq 1\}\right];$$

that is, when there are no η particles present. In this case we say that the η particles have *priority* over the ζ particles. We label the ζ-particles at time 0 in the following way. Without losing much we can assume that there is a ζ particle at site (integer part of) $-at$. The measure conditioned on this event is absolutely continuous with respect to $\bar{\nu}_{p,u,w}$, an this will be enough for our purposes, as we shall only use laws of large numbers. Call particle 0 this particle, and complete the labeling in such a way that a particle with label j is at the same site or at the left of a particle with label k if $j < k$. Denote by Y_t^j the position at time t of the particle labeled j. By construction, we have $\cdots \leq Y_0^{-1} < Y_0^0 = -at \leq Y_0^1 \leq \cdots$. We let the second class particles evolve in a way to preserve this order. To keep track of the densities involved in the definition we call $Y_t^{u,w} = Y_t^0$.

Consider now a single second class particle for the η process initially at the position of $-at$. This is obtained by considering the coupled initial condition $(\eta, \eta + \mathfrak{d}_{-at})$. Denote the position of the single second class particle at time t by X_t^u (for $u = v$, this has the same law as the particle denoted by X_t^{-at} in the statement of the proposition). Since $Y_0^{u,w} = X_0^u$, in the coupled evolution obtained by using the same Poisson processes $(N_x(t))$ we have $Y_t^{u,w} \leq X_t^u$ for all t. Indeed, in this coupling Y_t^k for $k > 0$ have priority over $Y_t^{u,w}$ while those particles have no priority over X_t^u. Similarly, consider a second class particle for the ξ process and denote it X_t^w. Since Y_t^k for $k < 0$ have priority over X_t^w but not over $Y_t^{u,w}$, $X_t^w \leq Y_t^{u,w}$. Hence, for $0 \leq u < w \leq c$,

$$X_t^w \leq Y_t^{u,w} \leq X_t^u\,, \tag{12}$$

$\mathbf{P}_{\bar{\nu}_{p,u,w}}$ almost surely.

Denote by J_t^1, J_t^{1+2} and J_t^2 the the total number of η, ξ and ζ particles that jumped from $-at$ to $-at+1$ before time t. In particular, $J_t^2 = J_t^{1+2} - J_t^1$. By Burke's theorem, the number of η-particles (resp. ξ-particles) that jump from $-at$ to $-at+1$ is a Poisson process of parameter u (resp. w). Hence the number of ζ-particles that jump from $-at$ to $-at+1$ in the interval $[0,t]$ is the difference of two Poisson process and satisfies the law of large numbers:

$$\lim_{t\to\infty} \frac{J_t^2}{t} = \lim_{t\to\infty} \frac{J_t^{1+2} - J_t^1}{t} = w - u$$

in $\mathbf{P}_{\bar{\nu}_{p,u,w}}$ probability. On the other hand, for every $t \geq 0$,

$$J_t^2 := \sum_{x=-at}^{Y_t^{u,w}} \zeta_t(x) - A_t = \sum_{x=-at}^{Y_t^{u,w}} \xi_t(x) - \sum_{x=-at}^{Y_t^{u,w}} \eta_t(x) - A_t \; .$$

where $|A_t| \leq \zeta_t(Y_t^{u,w})$. Note that $\zeta_t(Y_t^{u,w})$ is stochastically bounded above by a geometric random variable of parameter w/c. Therefore $|A_t/t|$ converges to 0 in in $\mathbf{P}_{\bar{\nu}_{p,u,w}}$ probability as t goes to infinity. As in Ferrari (1992), it follows from the previous equation and the law of large numbers for η_t and ξ_t —that are distributed according to product (invariant) measures with densities $R(p,u)$ and $R(p,w)$ respectively— that for u and w strictly smaller than c,

$$\lim_{t\uparrow\infty} \frac{Y_t^{u,w}}{t} = \frac{w-u}{R(p,w) - R(p,u)} \tag{13}$$

in $\mathbf{P}_{\bar{\nu}_{p,u,w}}$ probability. Hence, from (12) we have

$$\lim_{t\uparrow\infty} \frac{X_t^w}{t} \leq \frac{w-u}{R(w) - R(u)} \leq \lim_{t\uparrow\infty} \frac{X_t^u}{t} \tag{14}$$

in $\mathbf{P}_{\bar{\nu}_{p,u,w}}$ probability. Fixing $w = v$ and taking the limit $u \to v$ and then fixing $u = v$ and taking the limit $u \to v$ and taking account of the differentiability of R in v, we get (10) and (11). ■

We are now in a position to prove Lemma 4.1.

Proof of lemma 4.1 We shall prove that for all $v < c$, any weak limit $\tilde{\nu}$ of $\delta_\eta S_p(t)$ dominates stochastically $\nu_{p,v}$: $\tilde{\nu} \geq \nu_{p,v}$. The result follows from this because $\{\nu_{p,v} : 0 \leq v < c\}$ is an increasing sequence of measures converging to $\nu_{p,c}$.

The proof is performed via coupling. We start with two different initial configurations η and ξ with marginal distributions ν and

$\nu_{p,v}$, respectively. Hence η has lower asymptotic density bigger than $\rho^*(p)$ and ξ has asymptotic density $R(p,v)$. We use the same Poisson processes for both processes and call (η_t, ξ_t) the coupled process. The configurations η and ξ are in principle not ordered: there are (possibly an infinite number of) sites z such that $(\eta(z) - \xi(z))^+ > 0$ and (possibly an infinite number of) sites y such that $(\eta(y) - \xi(y))^- > 0$. We say that we have $\eta\xi$ discrepancies in the first case and $\xi\eta$ discrepancies in the second. The number of coupled particles at site x at time t is given by

$$\bar{\xi}_t(x) := \min\{\eta_t(x), \xi_t(x)\} \tag{15}$$

The $\bar{\xi}$ particles move as regular (first class) zero range particles. There is at most one type of discrepancies at each site at time zero. Discrepancies of both types move as second class particles with respect to the already coupled particles. When a $\eta\xi$ discrepancy jumps to a site z occupied by at least one $\xi\eta$ discrepancy, the $\eta\xi$ discrepancy and one of the $\xi\eta$ discrepancies at z coalesce into a coupled $\bar{\xi}$ particle in z. The coupled particle behaves from this moment on as a regular (first class) particle. The same is true when the roles of ξ and η are reversed.

The above description of the evolution implies in particular that a tagged discrepancy can not go through a region occupied by the other type of discrepancies.

We will choose a negative site y such that the jump rate from $y-1$ to y is close to c. Then we follow the $\xi\eta$ discrepancies belonging to two disjoint regions of $\mathbb{Z}$ at time 0 and give upper bounds on the probability of finding them at y at time t.

Roughly speaking, a $\xi\eta$ discrepancy at y cannot come from a region "close" to y because we prove that there is a minimum positive velocity for the $\xi\eta$ discrepancies to go. This velocity is given by the velocity of a second class particle under $\nu_{p,v}$. On the other hand, the $\xi\eta$ discrepancy cannot come from a region "far" from the origin because due to the difference of densities, a lot of $\eta\xi$ discrepancies will be between it and y and hence they must pass site $y-1$ before it. But since we have chosen a small rate for this site, a traffic rush will prevent them to pass. With this idea in mind, we have to choose the "close" and "far" regions and the value of the rate at $y-1$.

Fix $v < c$ such that $R(p,\cdot)$ is differenciable in v. Let $\gamma = \gamma(p,v)$, the (strictly positive) asymptotic speed of a second class particle under $\nu_{p,v}$ in the sense of (9). Denote by β the difference between the asymptotic lower density of η and $R(p,v)$:

$$\beta = \beta(p,v) \; = \; \lim_{n\to\infty} \frac{1}{n} \sum_{x=-n}^{0} [\eta(x) - R(p,v)] \, .$$

For reasons that will become clear later (cf. display (25)), we let

$$b \; = \; b(p,v) \; = \; \frac{R'(p,v)(c-v)}{\rho^*(p) - R(p,v)} \; < \; 1 \, ,$$

by the convexity of R; recall that $\rho^*(p) = \lim_{v\to c} R(p,v)$. With this choice,

$$\beta\gamma b - c + v \; = \; \frac{c-v}{\rho^*(p) - R(p,v)} \Big\{ \beta - [\rho^*(p) - R(p,v)] \Big\} \; > \; 0 \, . \quad (16)$$

This allows us to fix $\varepsilon = \varepsilon(v)$ satisfying

$$0 \; < \; \varepsilon(v) \; < \; \beta\gamma b - c + v \, .$$

Finally, choose a negative site $y = y(v)$ such that $p_{y-1} < c + \varepsilon$. We shall prove that

$$\lim_{t\to\infty} (\nu \times \nu_{p,v}) \overline{S}_p(t) \Big\{ (\eta,\xi) : \eta(y) < \xi(y) \Big\} \; = \; 0 \, . \quad (17)$$

We can order the $\xi\eta$ discrepancies and assume without loss of generality that the order is preserved in future times as we did in Lemma 4.2. Of course some of the discrepancies will disappear. Let $Z^k = Z^k_t(\xi,\eta)$ the positions of the ordered $\xi\eta$ discrepancies at time t with the convention that $Z^k_t = \infty$ if the corresponding discrepancy coalesced with a $\eta\xi$ one giving place to a $\bar{\xi}$ coupled particle. Let

$$\begin{aligned} & A_{\gamma,t}(\eta,\xi) \\ & \quad := \; \{\text{a } \xi\eta \text{ discrepancy in the box } [y - (t\gamma\bar{b}), y] \text{ at time } 0 \\ & \qquad\quad \text{has moved to site } y \text{ at time } t\} \\ & \quad := \; \cup_k \Big\{ Z^k_0 \in \big[y - (t\gamma\bar{b}), y\big] \, , \; Z^k_t = y \Big\} \end{aligned} \quad (18)$$

where $\bar{b} := (1+b)/2 \in (b,1)$. Hence

$$\mathbf{P}(A_{\gamma,t}(\eta,\xi)) \; \le \; \mathbf{P}\Big(\min \Big\{ Z^k_t : Z^k_0 \in \big[y - (t\gamma\bar{b}), y\big] \Big\} \le y \Big) \quad (19)$$

We wish to give and upper bound to the event in the right hand side above. To do so we consider the coupled (η,ξ) process and the ξ

process to which we add a unique second class particle at $y-(t\gamma\bar{b})$, evolving together with jumps occurring at times given by the the same Poisson processes. We denote by X_t^v the position of the second class particle at time t. If the second class particle has reached $y+1$ no latter than time t, then there exists an increasing sequence of random times $0 < T_{y-(t\gamma\bar{b})} < T_{y-(t\gamma\bar{b})+1} < ... < T_y$ such that at each of these times the corresponding site has been emptied of its ξ particles. But this implies that all the $\xi\eta$ discrepancies which at time 0 were in the interval $[y-(t\gamma\bar{b}), y]$ have disappeared or are strictly to the right of y. Therefore:

$$\mathbf{P}\left(A_{\gamma,t}(\eta,\xi)\right) \;\leq\; \mathbf{P}(X_t^{y-t\gamma\bar{b}} \leq y)\,. \tag{20}$$

By (11) this tends to 0 as t tends to infinity because $\gamma\bar{b} < \gamma$. The above argument is independent of the value of p_{y-1}.

It now suffices to check that the probability that a $\xi\eta$ discrepancy, to the left of $y-(t\gamma\bar{b})$ at time 0 reaches y no later than time t, tends to 0 as t tends to infinity. Let

$$\begin{aligned} & B_{\gamma,t}(\eta,\xi) \\ & \quad := \;\; \big\{\text{a } \xi\eta \text{ discrepancy in } (-\infty, y-(t\gamma\bar{b})] \text{ at time } 0 \\ & \qquad\qquad \text{has moved to site } y \text{ at time } t\big\} \\ & \quad := \;\; \cup_k \left\{ Z_0^k \in (-\infty, y-(t\gamma\bar{b})],\; Z_t^k = y \right\} \end{aligned} \tag{21}$$

Call $W_t^k(\eta,\zeta)$ the positions of the $\eta\xi$ discrepancies at time t, being W_0^0 the first $\eta\xi$ discrepancy to the left of the origin. As before set $W_t^k = \infty$ if the kth discrepancy coalesced with a $\xi\eta$ one before t.

Since a $\xi\eta$ discrepancy cannot cross over an $\eta\xi$ discrepancy,

$$\begin{aligned} B_{\gamma,t}(\eta,\xi) \cap \left(\bigcap_{z \leq y-\gamma t\bar{b}} \left\{ \sum_{x=z}^{y-1} (\eta_0(x)-\xi_0(x)) > t\gamma\beta b \right\} \right) & \\ \subset \left\{ J_t^2 - J_t^1 > t\gamma\beta b \right\} & \end{aligned} \tag{22}$$

where J_t^2 and J_t^1 are the number of η, respectively ξ, particles jumping from $y-1$ to y in the interval $[0,t]$. Since

$$\begin{aligned} & \left\{ \bigcap_{z \leq y-\gamma t\bar{b}} \left\{ \sum_{x=z}^{y-1} (\eta_0(x)-\xi_0(x)) > t\gamma\beta b \right\} \right\}^c \\ & \quad = \bigcup_{z \leq y-\gamma t\bar{b}} \left\{ \sum_{x=z}^{y-1} (\eta_0(x)-\xi_0(x)) \leq t\gamma\beta b \right\}, \end{aligned} \tag{23}$$

to bound $\mathbf{P}(B_{\gamma,t}(\eta,\xi))$ it suffices to bound the probabilities of the sets on the right hand side of (22) and (23). For (22) we have

$$\mathbf{P}(J_t^2 - J_t^1 > t\gamma\beta b) \;\leq\; \mathbf{P}(N_t^{c+\varepsilon} \;-\; N_t^v > t\gamma\beta b), \tag{24}$$

where N_t^a is a Poisson process of parameter a. The above inequality holds because the η-particles jump from $y-1$ to y at rate not greater than p_{y-1}, which is by construction less than or equal to $c+\varepsilon$. On the other hand, by Burke's theorem, the number of jumps from $y-1$ to y for the ξ-particles is a Poisson process of rate v. By the law of large numbers for the Poisson processes, we have

$$\lim_{t\to\infty} \frac{1}{t}(N_t^{c+\varepsilon} \;-\; N_t^v) \;\;=\;\; c - v + \varepsilon \;<\; \beta\gamma b\,, \tag{25}$$

because we chose $\varepsilon < \gamma\beta b - c + v$. Hence (24) goes to zero as $t \to \infty$.

On the other hand, the probability of the set in the right hand side of (23) is

$$\mathbf{P}\left(\sup_{z<y-\gamma t\bar{b}} \sum_{x=z}^{y-1} (\eta_0(x) - \xi_0(x)) \leq t\gamma\beta b\right) \tag{26}$$

By the ergodicity of ξ and the fact that η has left density, with probability one:

$$\lim_{t\to\infty} \frac{1}{t} \sum_{x=y-t\gamma\bar{b}}^{y-1} (\eta_0(x) - \xi_0(x)) \;\;=\;\; \gamma\beta\bar{b} \;>\; \gamma\beta b\,, \tag{27}$$

by the way we chose $\bar{b}$. This implies that (26) goes to zero as $t \to \infty$.

This proves (17). Standard arguments (cf. Andjel (1982)) permit to deduce the statement of the lemma from (17). ■

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