RIGOROUS PROBABILISTIC ANALYSIS OF EQUILIBRIUM CRYSTAL SHAPES

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ABSTRACT. The rigorous microscopic theory of equilibrium crystal shapes has made enormous progress during the last decade. We review here the main results which have been obtained, both in two and higher dimensions. In particular, we describe how the phenomenological Wulff and Winterbottom constructions can be derived from the microscopic description provided by the equilibrium statistical mechanics of lattice gases. We focus on the main conceptual issues and describe the central ideas of the existing approaches.

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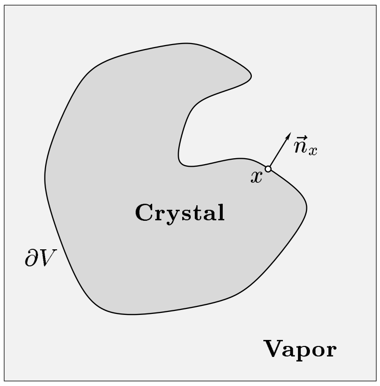
Date: November 15, 1999.



$$\mathcal{W}_\beta(V) \;=\; \int_{\partial V} \tau_\beta(\vec{n}_x)\,\mathrm{d}\mathcal{H}_x^{(d-1)}$$

FIGURE 1. The free energy of the crystal-vapor interface is given by the integral of the anisotropic surface tension τ_β over ∂V. $\mathcal{H}^{(d-1)}$ is the $(d-1)$-dimensional Hausdorff measure.

Part 1. Introduction

1.1. Phenomenological Wulff construction

1.1.1. **Equilibrium crystal shapes.** The phenomenological theory of equilibrated crystals dates back at least to the beginning of the century [Wu]. Suppose that two different thermodynamic phases (say crystal and its vapor) coexist at a certain temperature T. Assuming that the whole system is in equilibrium, in particular that the volume v of the crystalline phase is well defined, what could be said about the region this phase occupies? Of course, the issue cannot be settled in the language of bulk free energies - these do not depend neither on the shape, nor even on the prescribed volume v of the crystal. Instead, possible phase regions are quantified by the value of the free energy of the crystal-vapor interface, or by the total surface tension between the crystal and the vapor[1]. Equilibrium shapes correspond, in this way, to the regions of minimal interfacial energy. This is an isoperimetric-type problem: The surface tension τ_β (where, throughout the article, β denotes the inverse temperature, $\beta = 1/T$) is an anisotropic function of the local direction of the interface. Thus, assuming that the crystal occupies a region $V \subset \mathbb{R}^d$, the corresponding contribution $\mathcal{W}_\beta(V)$ to the free energy is equal to the integral of τ_β over the boundary ∂V of V (Fig. 1).

The Wulff variational problem could then be formulated as follows:

$$(\mathbf{WP})_v \qquad \mathcal{W}_\beta(V) \longrightarrow \min \qquad \text{Given : } \mathrm{vol}(V) = v$$

As in the usual isoperimetric case $(\mathrm{WP})_v$ is scale invariant,

$$\forall s > 0, \qquad \mathcal{W}_\beta\big(\partial(sV)\big) = s^{d-1}\mathcal{W}_\beta\big(\partial V\big).$$

Consequently, any dilatation of an optimal solution is itself optimal, and one really talks here in terms of optimal shapes.

The canonical way to produce an optimal shape is given by the following Wulff construction (Fig. 2): Define

$$\mathcal{K} = \bigcap_{\vec{n}\in\mathbb{S}^{d-1}} \Big\{x \in \mathbb{R}^d \,:\, x \cdot \vec{n} \le \tau_\beta(\vec{n})\Big\} \;\triangleq\; \bigcap_{\vec{n}\in\mathbb{S}^{d-1}} H_\beta(\vec{n})\,. \tag{1.1.1}$$

[1] In this review, our point of view is that of mathematical physics; for an exposition of the problem from the viewpoint of theoretical physics, we refer to [RW] and references therein.

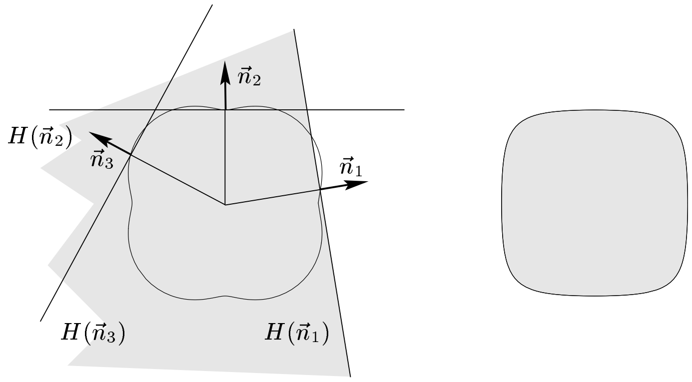


FIGURE 2. Function $\tau_\beta(\vec{n})$ (left) with three half-spaces $H(\vec{n}_1)$, $H(\vec{n}_2)$ and $H(\vec{n}_3)$ (for better visibility, only $H(\vec{n}_1)$ has been shaded). The intersection of **all** such half-spaces gives rise to the corresponding Wulff shape (right).

It would be convenient to normalize $\mathcal{K}$ as

$$\mathcal{K}_1 \triangleq \sqrt[d]{\frac{1}{\mathrm{vol}(\mathcal{K})}}\mathcal{K}.$$

We refer to $\mathcal{K}_1$ as to the normalized, or unit volume, Wulff shape. The variational theory of $(\mathrm{WP})_v$, which we briefly address in the subsequent subsection, states that any solution to $(\mathrm{WP})_v$ can be obtained by a shift of the corresponding dilatation $\mathcal{K}_v \triangleq \sqrt[d]{v}\mathcal{K}_1$ of $\mathcal{K}_1$.

1.1.2. **Variational methods.** The corresponding literature is rather rich and diverse, here we merely attempt to facilitate the orientation of the reader and to introduce some notations which will be useful in the sequel.

Since the half-spaces $H_\beta(\vec{n})$ in (1.1.1) are convex, so is the Wulff shape $\mathcal{K}$. Furthermore, in all the problems we consider here, the surface tension τ_β is bounded above and below,

$$0 < \min_{\vec{n}\in\mathbb{S}^{d-1}} \tau_\beta(\vec{n}) \leqslant \max_{\vec{n}\in\mathbb{S}^{d-1}} \tau_\beta(\vec{n}) < \infty. \tag{1.1.2}$$

Accordingly, equilibrium crystal shapes are bounded and have non-empty interiors, $0 \in \mathrm{int}\big(\mathcal{K}_v\big)$.

The fact that $\mathcal{K}$ is optimal follows from the general Brunn-Minkowski theory: Let τ_β^{**} be the support function of $\mathcal{K}$, $\tau_\beta^{**}(x) = \sup\{y \cdot x \mid y \in \mathcal{K}\}$. Of course, if the homogeneous extension of τ_β

$$\tau_\beta(\vec{x}) \triangleq \|\vec{x}\|_2 \tau_\beta\left(\frac{\vec{x}}{\|\vec{x}\|_2}\right), \tag{1.1.3}$$

is convex, then τ_β and τ_β^{**} coincide. In general τ_β^{**} is the convex lower-semicontinuous regularization of τ_β, in particular $\tau_\beta^{**} \leqslant \tau_\beta$. Nevertheless, for the Wulff shape $\mathcal{K}$,

$$\mathcal{W}_\beta^{**}(\mathcal{K}) \triangleq \int_{\partial\mathcal{K}} \tau_\beta^{**}(\vec{n}_x)\, d\mathcal{H}_x^{(d-1)} = \int_{\partial\mathcal{K}} \tau_\beta(\vec{n}_x)\, d\mathcal{H}_x^{(d-1)}.$$

where, as before, $\vec{n}_x$ is the outward normal to ∂V in x and $\mathcal{H}^{(d-1)}$ is the $(d-1)$ dimensional Hausdorff measure in $\mathbb{R}^d$.

On the other hand, the action of the regularized functional $\mathcal{W}_\beta^{**}$ could be extended to any compact set $V \subset \mathbb{R}^d$ in terms of the mixed volume

$$\mathcal{W}_\beta^{**}(V) \; = \; \liminf_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(\mathrm{vol}(V + \varepsilon \mathcal{K}) - \mathrm{vol}(V) \right),$$

the latter definition coincides with the integral definition of $\mathcal{W}_\beta^{**}$ for regular V. The Brunn-Minkowski inequality [Sch]

$$\mathrm{vol}(A+B) \geqslant \left(\mathrm{vol}(A)^{\frac{1}{d}} + \mathrm{vol}(B)^{\frac{1}{d}} \right)^d ,$$

implies that for any regular V with $\mathrm{vol}(V) = \mathrm{vol}(\mathcal{K})$,

$$\mathcal{W}_\beta(V) \; \geqslant \mathcal{W}_\beta^{**}(V) \; \geqslant d\, \mathrm{vol}(\mathcal{K}) = \mathcal{W}_\beta(\mathcal{K}).$$

Of course, we have been rather sloppy above, and we refer the reader to the works [Ta], [F] and [FM] for the comprehensive discussion and results, including the history of the variational Wulff problem. The language employed in the latter works is that of the geometric measure theory, and we proceed with setting up some of the corresponding notation which will also turn out to be useful for the $\mathbb{L}_1$-approach to the microscopic justification of the Wulff construction, as described in Part 2 of this review. In the latter case, the macroscopic state of the system will be determined by the value of an order parameter which specifies the phase of the system. In the systems that we will consider, the pure phases are characterized by their averaged density, which are encoded by two values $\rho_l(\beta)$ and $\rho_h(\beta)$, for example ρ_h for the crystal and ρ_l for the vapor. (In fact, we shall derive all the results in the symmetrized spin language, in which case the two values will be $\pm m^*(\beta)$, where $m^*(\beta)$ is the spontaneous magnetization (see Section 2) at the inverse sub-critical temperature $\beta > \beta_c$). For a given temperature, it is convenient to replace this order parameter by a parameter with values ± 1. We suppose that the macroscopic region of $\mathbb{R}^d$ where the system is confined is the unit torus $\widehat{\mathbb{T}}^d = (\mathbb{R}/\mathbb{Z})^d$. The macroscopic system is described by a function v taking values ± 1 and the fact that $v_r = 1$ for some r in $\widehat{\mathbb{T}}^d$ means that locally at r the system is in equilibrium in the phase m^*.

For any measurable set V in $\widehat{\mathbb{T}}^d$, the perimeter of V is defined by

$$\mathcal{P}(V) = \sup \left\{ \int_V \mathrm{div}\phi(x)\, dx \quad \Big| \quad \phi \in C^1(\widehat{\mathbb{T}}^d, \mathbb{R}^d), \;\; |\phi| \leqslant 1 \right\} . \tag{1.1.4}$$

A function v with values ± 1 is said to be of bounded variation in $\widehat{\mathbb{T}}^d$ if the perimeter of the set $\{v = 1\}$ is finite. We denote by $\mathrm{BV}(\widehat{\mathbb{T}}^d, \{\pm 1\})$ the set of functions of bounded variation in $\widehat{\mathbb{T}}^d$ with values ± 1 (see [EG] for a review). For any v in $\mathrm{BV}(\widehat{\mathbb{T}}^d, \{\pm 1\})$, there exists a generalized notion of the boundary of $\{v = 1\}$ called reduced boundary and denoted by $\partial^* v$. If $\{v = 1\}$ is a regular set, $\partial^* v$ coincides with the usual boundary ∂v. Furthermore, a blow-up Theorem (see [EG] p. 199) ensures that for all x in $\partial^* v$ an approximate tangent plane can be defined locally. This will imply the existence of a unit vector $\vec{n}_x$ called the measure theoretic unit normal to $\{v = 1\}$ at x. For any x in $\mathbb{R}^d$ and any vector $\vec{n}$, we define the half spaces

$$\begin{aligned} H^+(x, \vec{n}) &= \{ y \in \mathbb{R}^d \mid \quad (y - x) \cdot \vec{n} \geqslant 0 \} , \\ H^-(x, \vec{n}) &= \{ y \in \mathbb{R}^d \mid \quad (y - x) \cdot \vec{n} \leqslant 0 \} . \end{aligned}$$

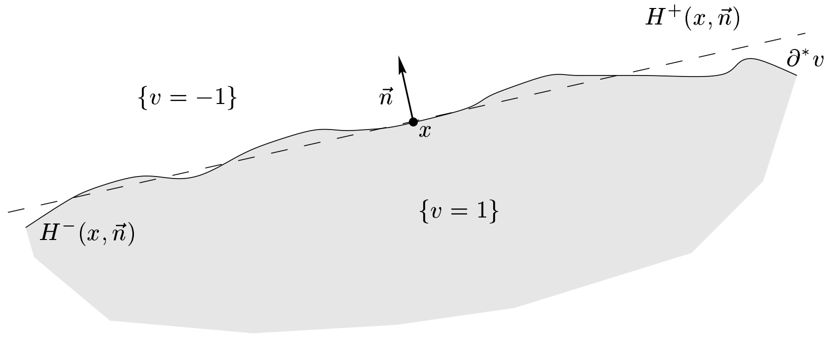


FIGURE 3. Measure theoretic unit normal to $\{v=1\}$ at x

Then for all x in $\partial^* v$, there is a unit vector $\vec{n}_x$ such that

$$\lim_{r\to 0}\frac{1}{r^d}\mathrm{vol}\Big(B(x,r)\bigcap\{v=1\}\bigcap H^+(x,\vec{n})\Big) = 0\,,$$
$$\lim_{r\to 0}\frac{1}{r^d}\mathrm{vol}\Big(B(x,r)\bigcap\{v=-1\}\bigcap H^-(x,\vec{n})\Big) = 0\,,$$

where $B(x,r)$ is the ball of radius r centered in x. The previous property shows that the reduced boundary is not too wild (see Fig. 3). In fact, it is possible to prove that a set of finite perimeter has "measure theoretically a C^1 boundary".

The functional $\mathcal{W}_\beta$ can be extended on $\mathbb{L}_1(\widehat{\mathbb{T}}^d,[-\frac{1}{m^*},\frac{1}{m^*}])$ as follows

$$\mathcal{W}_\beta(v) = \begin{cases} \int_{\partial^* v}\tau(\vec{n_x})\,d\mathcal{H}_x^{(d-1)}, & \text{if } v\in \mathrm{BV}(\widehat{\mathbb{T}}^d,\{\pm 1\})\,,\\ \infty\,, & \text{otherwise.}\end{cases} \tag{1.1.5}$$

Under the assumption that the homogeneous extension (1.1.3) of τ_β is convex, a result by Ambrosio and Braides (see [AmBr], Theorem 2.1) ensures that $\mathcal{W}_\beta$ is lower semi-continuous with respect to $\mathbb{L}_1$ convergence. In certain cases (attractive interactions) the convexity of τ_β can be derived from the properties of the corresponding microscopic system as will be explained later.

To any measurable subset A of $\widehat{\mathbb{T}}^d$, we associate the function $\mathbb{I}_A = 1_{A^c} - 1_A$ and simply write $\mathcal{W}_\beta(A) = \mathcal{W}_\beta(\mathbb{I}_A)$. In this new setting, the isoperimetric problem is to find the minimizers of

$$\min\big\{\mathcal{W}_\beta(v)\;\big|\; v\in \mathrm{BV}(\widehat{\mathbb{T}}^d,\{\pm 1\}),\qquad \Big|\int_{\widehat{\mathbb{T}}^d} m^*\, v_r\, dr\Big| \le m\big\}, \tag{1.1.6}$$

where m belongs to $]\bar{m}(\beta), m^*(\beta)[$. The parameter $\bar{m}$ is chosen such that the minima of the variational problem above are translates of the set $\mathcal{K}_m$ deduced from the Wulff shape $\mathcal{K}$ by dilatation in order to satisfy the volume constraint. This restriction enables us to exclude pathological minimizers which occur from the periodicity. Nevertheless, notice that the precise shape or the uniqueness of the minimizers of the variational problem will be irrelevant for the microscopic derivation of the Wulff construction.

1.1.3. **Stability properties.** In two dimensions Wulff solutions to $(\mathrm{WP})_v$ are stable in the metric of Hausdorff distance: let V be a connected and simply connected subset of $\mathbb{R}^2$ with a rectifiable boundary ∂V. Assume that $\mathrm{Area}(V)\geqslant 1$. Then,

$$\min_x \mathrm{d}_{\mathbb{H}}\left(V, x+\mathcal{K}_1\right) \;\leqslant c_1\sqrt{\mathcal{W}_\beta(V)-\mathcal{W}_\beta(\mathcal{K}_1)}. \tag{1.1.7}$$

This result has been established in [DKS] as a generalization of the classical Bonnesen inequality.

If V consists of several connected and simply connected components, $V = \vee_{i=1}^{n} V_i$, and the total surface tension of V is close to the optimal,

$$\mathcal{W}_\beta(V) \; = \; \sum_{i=1}^{n} \mathcal{W}_\beta(V_i) \; \leqslant \mathcal{W}_\beta(\mathcal{K}_1) + \varepsilon,$$

then, again assuming that $\mathrm{Area}(V) = \sum_{i=1}^{n} \mathrm{Area}(V_i) \geqslant 1$, an easy consequence of (1.1.7) implies (see (2.9.7) and (2.9.8) in [DKS]) that actually all but one components of V are small, and that the only large component, say V_1, is close to a shift of $\mathcal{K}_1$. Namely

$$\sum_{i=2}^{n} \mathrm{Area}(V_i) \; \leqslant c_2 \varepsilon^2 \qquad \text{and} \qquad \sum_{i=2}^{n} \mathcal{W}_\beta(V_i) \; \leqslant c_3 \varepsilon,$$

and V_1 satisfies (1.1.7).

These stability properties are indispensable for a sharp justification of the phenomenological Wulff construction directly from the microscopic assumptions on the local interparticle interactions (see Section 3.5 of Part 3).

As far as we understand, stability properties of higher dimensional isoperimetric problems are much less studied. Already in three dimensions the Hausdorff distance is, of course, not an adequate measure of stability. Trivial rate-free stability properties in $\mathbb{L}_1$ simply follow from the uniqueness of Wulff solutions and the compactness of BV-balls in $\mathbb{L}_1$. On a more qualitative side there are well studied stability properties in the class of convex sets [Sch] and, also, for sets with a smooth boundary [Ha]. We feel, however, that the statistical stability under the microscopic approximations in the problems we consider here might be better than the impartial stability of the corresponding variational problems. A result of this sort is supposed to appear in [BIV].

1.1.4. **Winterbottom problem.** The Wulff variational problem provides a description of an equilibrium crystal shape deep inside a region filled with gas phase. If, however, the spatial extent of the system is finite, it may happen that the boundary of the surrounding vessel exhibits a preference toward the crystal phase. In such a situation, the equilibrium state may not be given by the Wulff shape anymore, but may have the crystal attached to the boundary. We discuss briefly the simplest model of such an interaction between an equilibrium crystal and an attractive substrate. Suppose, for simplicity, that our system is contained in the half-space $H = \{x \in \mathbb{R}^d \, : \, x(d) \geqslant 0\}$; the boundary of this half-space, the hyperplane $\mathfrak{w} = \{x \in \mathbb{R}^d \, : \, x(d) = 0\}$ represents the boundary of the vessel and is called the *wall*. We also suppose to simplify the analysis, and because these assumptions will always be satisfied, that $\tau_\beta(\vec{n}) = \tau_\beta(-\vec{n})$, and that the homogeneous extension of τ_β is convex[2].

To model the degree of attractiveness of the wall, we introduce a new thermodynamical quantity, the *wall free energy* $\tau_{\mathrm{bd}}(\beta,\eta)$, which depends on both the inverse temperature β and the "chemical structure" of the wall η, and modify the free energy functional accordingly,

$$\mathcal{W}_{\beta,\eta}(V) \stackrel{\Delta}{=} \mathcal{W}_\beta(V) + (\tau_{\mathrm{bd}}(\beta,\eta) - \tau_\beta^*)\, \mathcal{H}^{(d-1)}(\partial V \cap \mathfrak{w})\,,$$

[2]In the models we consider in this paper, this is a consequence of FKG inequality.

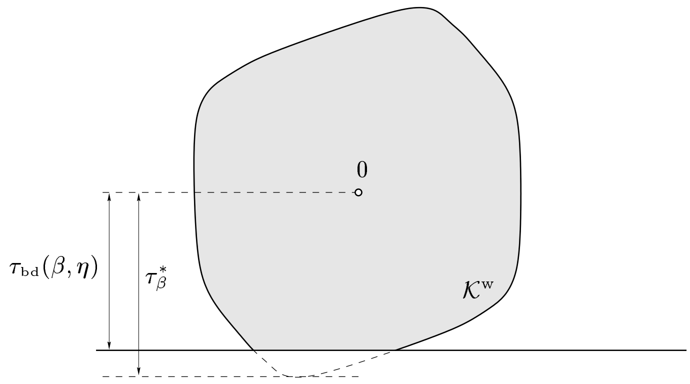


FIGURE 4. The Winterbottom shape is obtained by taking the intersection between the Wulff shape and the half-space $\{x(d) \geqslant -\tau_{\mathrm{bd}}(\beta,\eta)\}$, and rescaling the obtained body.

where $\tau^*_\beta \stackrel{\Delta}{=} \tau_\beta(\vec{e}_d)$, $\vec{e}_d \in \mathbb{R}^d$ with $\vec{e}_d(k) = \delta_{kd}$. The wall free energy replaces therefore the surface tension τ_β along the wall. At equilibrium, a thermodynamical stability argument shows that $\tau_{\mathrm{bd}}(\beta,\eta) \leqslant \tau^*_\beta$ (this can also be proved in some microscopic models, see Part 4), so that this last term is always non-positive. The new variational problem is

$$(\mathbf{WBP})_v \qquad \mathcal{W}_{\beta,\eta}(V) \longrightarrow \min \qquad \text{Given: } V \subset H,\ \mathrm{vol}(V) = v\ .$$

It has first been studied in [Wi] and is called the Winterbottom variational problem. Let us now discuss what its solution looks like. It turns out that there are three cases to consider:

1. $\tau_{\mathrm{bd}}(\beta,\eta) = \tau^*_\beta$

   In this case, $\mathcal{W}_{\beta,\eta}(V) = \mathcal{W}_\beta(V)$ and therefore the solution is the Wulff shape associated to τ_β. The equilibrium crystal is not attached to the wall. This can happen even if *a priori* the chemical structure of the wall is such that it is energetically favorable for the crystal to lay on the wall, see Part 4 for a discussion from a microscopic point of view.

2. $|\tau_{\mathrm{bd}}(\beta,\eta)| < \tau^*_\beta$

   Now the wall is really attractive for the crystal shape. The solution of the variational problem is given by a suitably rescaled version of the following set (see Fig. 4),

   $$\mathcal{K}^{\mathrm{w}} \stackrel{\Delta}{=} \mathcal{K} \cap \{x \in \mathbb{R}^d \ :\ x(d) \geqslant -\tau_{\mathrm{bd}}(\beta,\eta)\}$$

   so that the volume constraint is satisfied (notice that this variational problem is still scale invariant); see [KP] for a simple proof.

3. $\tau_{\mathrm{bd}}(\beta,\eta) = -\tau^*_\beta$

   This is a somewhat pathological case. Indeed, the solution of the variational problem is completely degenerate, the solution being unbounded. A minimizing sequence is, for example,

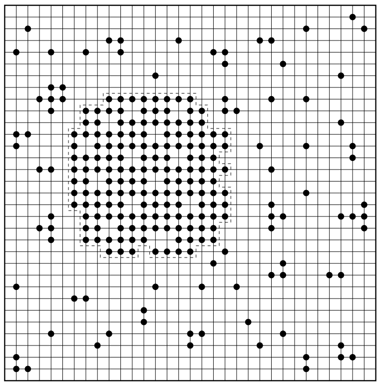
   $$R_n = \{x \in H \ :\ |x(k)| \leqslant n,\ k = 1,\ldots,d-1,\ x(d) \leqslant n^{1-d}\, v\}\ .$$

   As $n \to \infty$, R_n covers the whole wall with a film of vanishingly small width; the limiting value of the surface free energy functional is 0. This describes the regime of so-called *complete wetting* where the wall so strongly prefers the crystal that it wants to prevent any contact with the gas phase.

1.1.5. **Microscopic justification.** Microscopic models we consider here are simple lattice gas type models (in the magnetic interpretation), which are going to be defined precisely in the next section. The prototype situation when the Wulff construction is thought to be recovered as a law of large numbers as the size of the microscopic system tends to infinity could be loosely described as follows: Suppose that the particles of a certain substance live on the vertices of the integer lattice $\mathbb{Z}^d$, so that each vertex of $\mathbb{Z}^d$ could be either occupied by a particle or remain vacant. Thus, various particle configurations n could be labeled by points of $\{0,1\}^{\mathbb{Z}^d}$, where one puts $n_i = 1$ if there is a particle at site $i \in \mathbb{Z}^d$, and $n_i = 0$, otherwise. These random configurations are sampled from a Gibbs distribution $\mathbb{P}$, which takes into account the assumptions on the microscopic interactions between the particles. The strength of the interaction is quantified by the value $\beta = 1/T$ of the inverse temperature; the larger β (respectively the smaller the temperature T) is, the stronger is the interaction. In many instances sufficiently low temperatures give rise to two stable phases - the low density phase (which we call vapor) with an average particle density per site ρ_l and the high density phase (crystal) with a corresponding average density ρ_h, $0 < \rho_l < \rho_h < 1$.

Suppose now that all the particles are confined to a large finite volume vessel $\Lambda_N \subset \mathbb{Z}^d$, where the subindex N indicates the linear size of Λ_N; we put for simplicity $|\Lambda_N| = N^d$. Let us fix $\rho \in (\rho_l, \rho_h)$ and ask what are the typical geometric properties of particle configurations n under the conditional measure $\mathbb{P}\Big(\,\cdot \mid \sum_{i\in\Lambda_N} n_i = \rho N^d\Big)$. In other words, we fix the total number of particles ρN^d in such a way that it falls in-between the two stable values $\rho_l N^d$ and $\rho_h N^d$.

The prototype law of large numbers result we have in mind is schematically:

$$\mathbb{P}\left(\vphantom{\sum}\right.$$

$$\left.\middle|\ \sum_{i\in\Lambda_N} n_i = \rho N^d \right) \longrightarrow 1\,.$$

Thus, with an overwhelming $\mathbb{P}\Big(\,\cdot \mid \sum_{i\in\Lambda_N} n_i = \rho N^d\Big)$-probability particle configurations n on Λ_N, $n \in \{0,1\}^{\Lambda_N}$, obey the following phase segregation pattern: Λ_N splits into two regions, $\Lambda_N = \Lambda_N^h \vee \Lambda_N^l$, where Λ_N^h is occupied by the high density phase, and, respectively, Λ_N^l by the low density one. The relative volume of Λ_N^h can be recovered from the canonical constraint

$$\rho_h \big|\Lambda_N^h\big| \;+\; \rho_l \big|\Lambda_N^l\big| \;=\; \rho N^d$$

and the shape of Λ_N^h is asymptotically Wulff.

There is a long way even towards making the above statement precise - we should define the microscopic models, quantify the notion of phases, in particular of phases over finite volumes, and explain how the surface tension is produced in the large N limit.

1.2. Microscopic Models

1.2.1. **Models with finite-range ferromagnetic 2-body interactions.** We want to introduce mathematically precise realizations of the models discussed in subsection 1.1.5. As described there, our interest lies in models of lattice gases. For simplicity we restrict our attention to a particular subclass of such models, which enjoy several nice properties, the Ising models with finite-range ferromagnetic 2-body interactions.

We consider a family of random variables n_i, $i \in \mathbb{Z}^d$, taking values 0 and 1. Any site i of the lattice $\mathbb{Z}^d$ is either occupied by a particle, in which case $n_i = 1$, or empty, in which case $n_i = 0$. The random variables n_i are called *occupation numbers* and they completely describe a configuration of the lattice gas. We consider a formal Hamiltonian of the form

$$\tfrac{1}{2} \sum_{i,j} K_{ij}\, n_i n_j \,,$$

the 2-body interactions are such that $K_{ij} = K_{\|j-i\|_1}$, $K_{ij} \geqslant 0$ and $K_{ij} = 0$ if $\|i-j\|_1 > r$, where r is the *range* of the interaction. We introduce two parameters, the *chemical potential* μ and the *inverse temperature* β, and set $\Lambda \Subset \mathbb{Z}^d$. The Gibbs measure in Λ with boundary condition $\overline{n} \in \{0,1\}^{\mathbb{Z}^d}$ is the probability measure on $(\{0,1\}^{\mathbb{Z}^d}, \mathcal{A})$, with $\mathcal{A}$ the usual product σ-field, defined by

$$\nu^{\beta}_{\Lambda,\mu,\overline{n}}(n) = \begin{cases} \frac{1}{\mathbf{Z}^{\beta}_{\Lambda,\mu,\overline{n}}} \exp\bigl(\beta\mu \sum_{i\in\Lambda} n_i + \beta \displaystyle\sum_{\{i,j\}\cap\Lambda\neq\varnothing} K_{ij}\, n_i n_j\bigr) & \text{if } n_i = \overline{n}_i, \text{ for all } i \notin \Lambda, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\mathbf{Z}^{\beta}_{\Lambda,\mu,\overline{n}} = \sum_{N\geqslant 0} e^{\beta\mu N} \sum_{\substack{n:\\ \sum_{i\in\Lambda} n_i = N}} e^{\beta \sum_{\{i,j\}\cap\Lambda\neq\varnothing} K_{ij}\, n_i n_j} \,.$$

Two types of boundary conditions are particularly relevant for us, the $\mathbf{1}$ b.c., corresponding to setting $n \equiv 1$, and the $\mathbf{0}$ b.c., $n \equiv 0$. We also need a different kind of boundary conditions: The Gibbs measure in Λ with *free* boundary conditions is the probability measure on $(\{0,1\}^{\Lambda}, \mathcal{F}_{\Lambda})$ defined by

$$\nu^{\beta}_{\Lambda,\mu}(n) = \frac{1}{\mathbf{Z}^{\beta}_{\Lambda,\mu}} \exp\bigl(\beta\mu \sum_{i\in\Lambda} n_i + \beta \sum_{\{i,j\}\subset\Lambda} K_{ij}\, n_i n_j\bigr) \,.$$

These measures describe the lattice gas in the *Grand Canonical Ensemble*, in which the total number of particles, or equivalently the *density* $\rho(n) = \frac{1}{|\Lambda|}\sum_{i\in\Lambda} n_i$, is not fixed. The description of a gas in the *Canonical Ensemble* corresponds to the conditioned measure

$$\nu^{\beta}_{\Lambda,\mu,\overline{n}}(\,\cdot\,|\,\rho(n) = \widetilde{\rho}\,) \,,$$

with $\widetilde{\rho} \in \mathrm{Range}(\rho)$ (this measure is obviously independent of μ). The existence of the Gibbs states $\nu^{\beta}_{\mu,\overline{n}} = \lim_{\Lambda\nearrow\mathbb{Z}^d} \nu^{\beta}_{\Lambda,\mu,\overline{n}}$, for $\overline{n} = \mathbf{0}$, $\mathbf{1}$ or free, can be easily proved using correlations inequalities; moreover, it is unique if $\mu \neq -\frac{1}{2}\sum_j J_{0j}$. Restricting the chemical potential to the particular line $\mu = -\frac{1}{2}\sum_j J_{0j}$, it can be proved that there exists a critical value $\infty > \beta_{\rm c} > 0$ such that

- For all $\beta < \beta_{\rm c}$, there is a unique Gibbs state and $\nu^{\beta}_{\mu,\overline{n}}(\rho) = 1/2$.
- For all $\beta > \beta_{\rm c}$, $\rho_h(\beta) \equiv \nu^{\beta}_{\mu,\mathbf{1}}(\rho) > 1/2 > \nu^{\beta}_{\mu,\mathbf{0}}(\rho) \equiv \rho_l(\beta)$.

It is rather convenient to work with another, equivalent, formulation of these models, in which the symmetries present when $\mu = -\frac{1}{2}\sum_j J_{0j}$ are more transparent; this is the *magnetic interpretation*. To do this, we introduce a new family of random variables σ_i, $i \in \mathbb{Z}^d$, defined by

$$\sigma_i = 2n_i - 1\,.$$

The random variables σ_i therefore take values in $\{-1,1\}$; σ_i is called the *spin* at the site i. Expressed in these variables, the model is defined through the following Gibbs measure in Λ with boundary conditions $\overline{\sigma} \in \{-1,1\}^{\mathbb{Z}^d}$,

$$\mu^{\beta}_{\Lambda,\overline{\sigma},\boldsymbol{h}}(\sigma) = \begin{cases} \frac{1}{\mathbf{Z}^{\beta}_{\Lambda,\overline{\sigma},h}} \exp\bigl(\beta \sum\limits_{i\in\Lambda} h_i\,\sigma_i + \beta \sum\limits_{\{i,j\}\cap\Lambda\neq\varnothing} J_{ij}\,\sigma_i\sigma_j\bigr) & \text{if } \sigma_i = \overline{\sigma}_i, \text{ for all } i \notin \Lambda,\\ 0 & \text{otherwise,}\end{cases}$$

where $h_i \in \mathbb{R}$ are called the *magnetic fields* and the *coupling constants* $J_{ij} = J_{\|i-j\|_1}$ satisfy $J_{ij} \geqslant 0$ and $J_{ij} = 0$ if $\|i-j\|_1 > r$. A configuration σ such that $\sigma_i = \overline{\sigma}_i$, for all $i \notin \Lambda$, is said to be *compatible with b.c. $\overline{\sigma}$ in* Λ; the set of all such configurations is denoted by $\Omega_{\Lambda,\overline{\sigma}}$. We are particularly interested in the $+$ and $-$ b.c. corresponding respectively to $\overline{\sigma} \equiv 1$ and $\overline{\sigma} \equiv -1$. The Gibbs measure in Λ with free b.c. is the probability measure on $(\{-1,1\}^{\Lambda}, \mathcal{F}_{\Lambda})$ defined by

$$\mu^{\beta}_{\Lambda,\boldsymbol{h}}(\sigma) = \frac{1}{\mathbf{Z}^{\beta}_{\Lambda,h}} \exp\bigl(\beta \sum_{i\in\Lambda} h_i\,\sigma_i + \beta \sum_{\{i,j\}\subset\Lambda} J_{ij}\,\sigma_i\sigma_j\bigr)\,.$$

Expected value w.r.t. these measures are denoted with brackets notations, $\langle\,\cdot\,\rangle^{\beta}_{\Lambda,\overline{\sigma},\boldsymbol{h}}$, ...

In the magnetic formulation, the *Canonical Ensemble* corresponds to fixing the value of the *magnetization* (density) $m(\sigma) = \frac{1}{|\Lambda|}\sum_{i\in\Lambda}\sigma_i$,

$$\mu^{\beta}_{\Lambda,\overline{\sigma},\boldsymbol{h}}(\,\cdot\,|\, m(\sigma) = \widetilde{m})\,,$$

where $\widetilde{m} \in \mathrm{Range}(m)$. If $h_i \equiv h$ for all i, then the (infinite-volume) Gibbs states $\mu^{\beta}_{\overline{\sigma},h}$ for $+$, $-$ and free b.c. can be shown to exist; it is always unique when $h \neq 0$. The phase transition statement takes now the following (simpler) form: There exists $\infty > \beta_c > 0$ such that

- For all $\beta < \beta_c$, the Gibbs state is unique and $\langle m\rangle^{\beta}_{\overline{\sigma},0} = 0$.
- For all $\beta > \beta_c$, $m^*(\beta) \equiv \langle m\rangle^{\beta}_{+,0} > 0 > \langle m\rangle^{\beta}_{-,0} = -m^*(\beta)$.

We will use the terminology *Ising models* to refer to the lattice gases in the magnetic formulation. When $h = 0$, we will generally omit it from the notations.

Ferromagnetic models are particularly well-suited for non-perturbative analyses. Indeed, they enjoy several very useful qualitative properties, most of which taking form of correlation inequalities. Of particular importance for us are the following statements ($\sigma_A \triangleq \prod_{i\in A}\sigma_i$):

$$\langle\sigma_A\rangle^{\beta}_{\Lambda,\boldsymbol{h}} \geqslant 0\,,$$
$$\langle\sigma_A\sigma_B\rangle^{\beta}_{\Lambda,\boldsymbol{h}} \geqslant \langle\sigma_A\rangle^{\beta}_{\Lambda,\boldsymbol{h}}\langle\sigma_B\rangle^{\beta}_{\Lambda,\boldsymbol{h}}\,,$$

provided $h_i \geqslant 0$ for all i (1st and 2nd Griffiths', or GKS, inequalities [Gr, KS]); also,

$$\frac{\partial^2}{\partial h_i \partial h_j} \langle \sigma_k \rangle^{\beta}_{\Lambda,\boldsymbol{h}} \leqslant 0\,,$$

for all i, j and k, provided $h_l \geqslant 0$ for all l (GHS inequalities [GHS]); finally

$$\langle fg \rangle^{\beta}_{\Lambda,\boldsymbol{h}} \geqslant \langle f \rangle^{\beta}_{\Lambda,\boldsymbol{h}} \langle g \rangle^{\beta}_{\Lambda,\boldsymbol{h}}\,,$$

for any increasing[3] functions f and g, and any $\boldsymbol{h} \in \mathbb{R}^{\Lambda}$ (FKG inequality [FKG]). Observe that any b.c. can be obtained starting with free b.c. and applying suitable magnetic fields on the spins on the inner boundary of Λ, where the *inner boundary* of a set $A \subset \mathbb{Z}^d$ is defined as

$$\partial_{\mathrm{in}} A \triangleq \{i \in A \,:\, \exists j \notin A,\, i \sim j\}\,,$$

where $i \sim j$ means that $J_{i,j} \neq 0$. Similarly, we define the *(exterior) boundary* of A by

$$\partial A \triangleq \{i \notin A \,:\, \exists j \in A,\, i \sim j\}\,.$$

1.2.2. **2D nearest-neighbors ferromagnetic Ising model.** A particularly simple member of the above-mentioned class of models is the two-dimensional nearest-neighbors Ising model, in which $J_{ij} = 0$ if i and j are not nearest-neighbors, and $J_{ij} = 1$ if they are. This model has still additional remarkable features. First, even though this only plays a very marginal role in this review, it is the only one for which it is possible to compute explicitly various quantities (free energy, surface tension, correlations, ...). Of more importance for our purposes is the property of *self-duality*[4] that it enjoys.

The nearest-neighbors model admit a geometric description in terms of very simple objects, the contours. To define contours in the present context, it is useful to introduce the notion of the dual of the lattice $\mathbb{Z}^2$. The *dual lattice* is the set of dual sites

$$\mathbb{Z}^2_{\star} = \{x \in \mathbb{R}^2 \,:\, x + (\tfrac{1}{2}, \tfrac{1}{2}) \in \mathbb{Z}^2\}\,.$$

To each edge $e = \langle x, y \rangle$, $x, y \in \mathbb{Z}^2$, we associate a dual edge e^* connecting nearest-neighbors dual sites, which is the unique such edge intersecting e (as subset of $\mathbb{R}^2$).

Now, if we consider the Ising model in $\Lambda \Subset \mathbb{Z}^2$ with b.c. $\overline{\sigma}$, a configuration $\sigma \in \Omega_{\Lambda,\overline{\sigma}}$ is entirely determined by giving the following set of dual edges,

$$\{e^* \,:\, e^* \text{ dual to } e = \langle i, j \rangle,\, \{i, j\} \cap \Lambda \neq \varnothing,\, \sigma_i \sigma_j = -1\}\,.$$

The maximal connected components of these dual edges, seen as closed line segments in $\mathbb{R}^2$, are called *contours*. We denote by $\boldsymbol{\gamma}(\sigma)$ the contours of the configuration σ. The *boundary* $\partial\gamma$ of a contour γ is the set of all dual sites belonging to an odd number of the dual edges composing γ. A contour is said to be *closed* if $\partial\gamma = \varnothing$, otherwise it is *open*.

A set $\Lambda \Subset \mathbb{Z}^2$ is *simply connected* if $\bigcup_{i \in \Lambda} \{x \in \mathbb{R}^2 \,:\, \|x - i\|_\infty \leqslant 1/2\}$ is a simply connected subset of $\mathbb{R}^2$.

Given $\Lambda \subset \mathbb{Z}^2$, its *dual* is $\Lambda^* = \{i \in \mathbb{Z}^2_\star \,:\, \exists j \in \Lambda,\, \|j - i\|_\infty = 1/2\}$. A family of contours is said to be Λ^*-*compatible* if they are disjoint (as sets of bonds and sites) and

---

[3] A function $f : \{-1, 1\}^{\mathbb{Z}^d} \to \mathbb{R}$ is *increasing* if $f(\sigma) \geqslant f(\sigma')$ as soon as $\sigma_i \geqslant \sigma'_i$, for all i; it is called *decreasing* if $-f$ is increasing.

[4] The fact that this model is *self*-dual is very convenient, but is not required anywhere. What we need is to be able to control precisely the dual of the model; for example, the Ising model on the hexagonal lattice is not self-dual, but it would be possible to prove the same kind of statements for this model as for the one on the square lattice.

are included in Λ^*. A family of contours $\boldsymbol{\gamma}$ is said to be $(\Lambda,\overline{\sigma})$-*compatible* if there exists a configuration $\sigma \in \Omega_{\Lambda,\overline{\sigma}}$ such that $\boldsymbol{\gamma}(\sigma) = \boldsymbol{\gamma}$. It is easy to show that for simply connected Λ, Λ^*-compatibility of a family of closed contours is equivalent to $(\Lambda,+)$-compatibility.

The measure $\mu^{\beta}_{\Lambda,\overline{\sigma}}$ can be easily written in terms of these objects; for any $\sigma \in \Omega_{\Lambda,\overline{\sigma}}$,

$$\mu^{\beta}_{\Lambda,\overline{\sigma}}(\sigma) = \frac{1}{Z^{\beta}_{\overline{\sigma}}(\Lambda)} \exp\{-2\beta \sum_{\gamma\in\boldsymbol{\gamma}(\sigma)} |\gamma|\}\,, \tag{1.2.1}$$

where $|\gamma|$ is the number of edges in γ and

$$Z^{\beta}_{\overline{\sigma}}(\Lambda) = \sum_{\boldsymbol{\gamma}\ (\Lambda,\overline{\sigma})\text{-comp.}} \exp\{-2\beta \sum_{\gamma\in\boldsymbol{\gamma}} |\gamma|\} \equiv \sum_{\boldsymbol{\gamma}\ (\Lambda,\overline{\sigma})\text{-comp.}} \prod_{\gamma\in\boldsymbol{\gamma}} w(\gamma;\beta)\,. \tag{1.2.2}$$

We now discuss the property of self-duality. Let $\Lambda \Subset \mathbb{Z}^2$ be simply connected. We consider the model at inverse temperature β^* in the box $\Lambda^* \Subset \mathbb{Z}^2_\star$, with free boundary conditions. There exists another graphical representation for this model, the *high-temperature* representation, which results from writing

$$e^{\beta^*\sigma_i\sigma_j} = \cosh\beta^*(1 + \sigma_i\sigma_j \tanh\beta^*)\,,$$

opening all the brackets and expanding. After a simple summation over σ, this yields

$$\begin{aligned} Z^{\beta^*}_{\Lambda^*} = C(\Lambda) \sum_{\boldsymbol{\gamma}\ \Lambda^*\text{-comp.}} (\tanh\beta^*)^{\sum_{\gamma\in\boldsymbol{\gamma}}|\gamma|} &\equiv C(\Lambda) \sum_{\boldsymbol{\gamma}\ \Lambda^*\text{-comp.}} \prod_{\gamma\in\boldsymbol{\gamma}} w^*(\gamma;\beta^*) \\ &\equiv C(\Lambda)\, Z^{\beta^*}(\Lambda^*)\,, \end{aligned} \tag{1.2.3}$$

where $C(\Lambda)$ is some constant which only depends on the set Λ. Setting $\tanh\beta^* = e^{-2\beta}$, we see from (1.2.2) and (1.2.3) that $Z^{\beta}_{+}(\Lambda) = Z^{\beta^*}(\Lambda^*)$, since Λ is simply connected. In the same way, we can expand the 2-point function, for example, and get the following very useful identity

$$\langle\sigma_i\sigma_j\rangle^{\beta^*}_{\Lambda,+} = \sum_{\lambda:i\to j} q^{\beta^*}_{\Lambda^*}(\lambda)\,, \tag{1.2.4}$$

where the sum is over all open contours λ such that $\partial\lambda = \{i,j\}$, and

$$q^{\beta^*}_{\Lambda^*}(\lambda) = w^*(\lambda;\beta^*)\, \frac{Z^{\beta^*}(\Lambda^* \mid \lambda)}{Z^{\beta^*}(\Lambda^*)}\,,$$

$$Z^{\beta^*}(\Lambda^* \mid \lambda) = \sum_{\substack{\boldsymbol{\gamma}\text{ closed}\\ (\boldsymbol{\gamma},\lambda)\ \Lambda^*\text{-comp.}}} \prod_{\gamma\in\boldsymbol{\gamma}} w^*(\gamma;\beta^*)\,.$$

Identity (1.2.4) is the so-called *random-line representation* for the 2-point function of the Ising model, and plays a basic role in the approach to the DKS theory of Part 3 (see [PV2, PV3] for much more details on this topic). What is particularly useful is that the weights $q^{\beta^*}_{\Lambda^*}$, which we have defined for an open contour, can be immediately extended to any family of Λ^*-compatible contours (closed or open). In particular, if $\boldsymbol{\gamma}$ is a family of Λ^*-compatible *closed* contours, then the following identity holds

$$q^{\beta^*}_{\Lambda^*}(\boldsymbol{\gamma}) = \mu^{\beta}_{\Lambda,+}(\boldsymbol{\gamma} \subseteq \boldsymbol{\gamma}(\,\cdot\,))\,.$$

Applications and further results about the random-line representation are given in Section 3.4 and in Part 4. The results stated above also hold when the coupling constants are allowed to vary from edge to edge, provided they remain ferromagnetic; if we denote by $J(e)$ the coupling constant at edge e, then the duality relation takes the form

$$\tanh(\beta^* J^*(e^*)) = e^{-2\beta J(e)} . \tag{1.2.5}$$

1.2.3. **Kac models.** In the original van der Waals Theory, the occurrence of phase transitions is due to long range attractive forces between molecules. In its statistical mechanics formulation, these forces are described by Kac potentials that depend on a positive scaling parameter ε which controls the strength and the range of the potential (see [KUH]). The first probabilistic approach to this model was made in the celebrated paper of Lebowitz and Penrose [LePe].

In dimension d, Ising systems with Kac potentials are defined by Gibbs measures with potentials depending on a scaling parameter $\varepsilon > 0$

$$\forall\, i,j \in \mathbb{Z}^d, \qquad J^{\varepsilon}_{i,j} = \varepsilon^d J(\varepsilon \|i-j\|_2) \ ,$$

and J is a non-negative, smooth function supported by $[0,1]$ and normalized so that

$$\int_{\mathbb{R}^d} dr\; J(\|r\|_2) = 1.$$

The Gibbs measure on the domain Λ is denoted by $\mu^{\beta}_{\varepsilon,\Lambda}$. The constant ε will be so that the system has finite but long range interaction. It is convenient to consider interaction parameters of the form $\varepsilon = 2^{-m}$ (m is typically assumed to be large but fixed).

This model bridges the finite range models and the mean field models. In particular, if the range of the interaction, i.e. ε^{-1}, is scaled proportionally to the number of spins then the statistical properties of the system can be recovered from a mean field functional. In the true thermodynamic limit, when ε is kept fixed while the number of spins goes to infinity, the behavior of the system cannot be described by the mean field continuum limit. Nevertheless, by localizing in finite size regions it is possible to derive some informations from the mean field functional. This strategy was used to recover the phase diagram of the model and to prove that it is arbitrarily close to the one of the mean field model when ε goes to 0. More precisely, let us recall the following result which has been proven by Cassandro, Presutti [CP] and by Bovier, Zaharadnik [BZ] (see also [BP])

Theorem 1.2.1. *For any $\beta > 1$, there is $\varepsilon_0 > 0$ such that for any ε smaller than ε_0 a phase transition occurs and there are at least 2 distinct pure phases μ^+_ε and μ^-_ε.*

If $\beta > 1$, there is a breaking of symmetry and the spontaneous magnetization is denoted by $\mu^+_\varepsilon(\sigma_0) = m^*_\varepsilon$. Define $m^* = \lim_{\varepsilon\to 0} m^*_\varepsilon$. This Theorem was proven via a renormalization procedure which we shall describe in Subsection 2.3.1.

1.2.4. **Surface tension.** We fix $\vec{n}$ a vector in $\mathbb{S}^{d-1}$ and consider an orthonormal basis $(\vec{e}_1,\dots,\vec{e}_{d-1},\vec{n})$. Let $\widehat{\Lambda}(N,M)$ be the parallelepiped of $\mathbb{R}^d$ centered at 0 with side length N for the sides parallel to $(\vec{e}_1,\dots,\vec{e}_{d-1})$ and side length M for the sides parallel to $\vec{n}$. The microscopic counterpart of $\widehat{\Lambda}(N,M)$ is denoted by $\Lambda(N,M)$. The boundary $\partial\Lambda(N,M)$ is split into 2 sets

$$\begin{aligned} \partial^+_{\vec{n}}\Lambda(N,M) &= \{i \in \partial\Lambda(N,M) \mid \vec{i}.\vec{n} \geqslant 0\}, \\ \partial^-_{\vec{n}}\Lambda(N,M) &= \{i \in \partial\Lambda(N,M) \mid \vec{i}.\vec{n} < 0\}. \end{aligned}$$

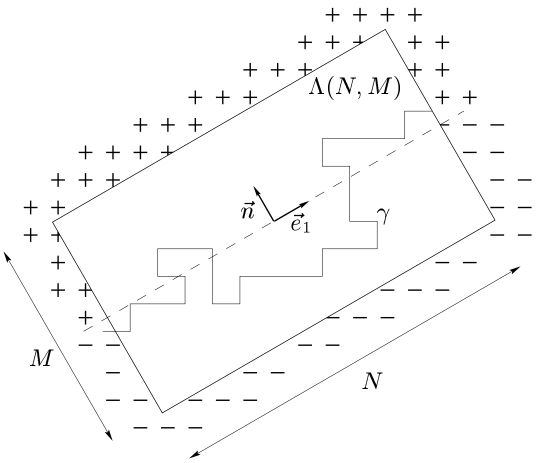


FIGURE 5. Definition of the surface tension.

We fix the boundary conditions outside $\Lambda(N,M)$ to be equal to 1 on $\partial^+_{\vec{n}}\Lambda(N,M)$ and to -1 on $\partial^-_{\vec{n}}\Lambda(N,M)$. The corresponding partition function on $\Lambda(N,M)$ is denoted by $\mathbf{Z}^\beta_{\Lambda(N,M),\vec{n},\pm}$.

Notice that any configuration σ contributing to the partition function $\mathbf{Z}^\beta_{\Lambda(N,M),\vec{n},\pm}$ contains a $\pm$-contour γ which crosses $\Lambda(N,M)$ under the "averaged" direction orthogonal to $\vec{n}$ (Fig. 5). Such a contour is absent in the configurations σ contributing to partition functions $\mathbf{Z}^\beta_{\Lambda(N,M),+}$ with pure boundary conditions on $\partial\Lambda(N,M)$. This contour represents the microscopic $\pm$-interface under the direction $\vec{n}$.

Definition : The surface tension in the direction $\vec{n}\in\mathbb{S}^{d-1}$ is defined[5] by

$$\tau_\beta(\vec{n}) = \lim_{N\to\infty}\lim_{M\to\infty} -\frac{1}{N^{d-1}}\log\frac{\mathbf{Z}^\beta_{\Lambda(N,M),\vec{n},\pm}}{\mathbf{Z}^\beta_{\Lambda(N,M),+}}. \tag{1.2.6}$$

□

The proof the existence of the surface tension can be found in many papers ([Ab2], [Pf2] to mention a few). A general approach has been developed by Messager, Miracle-Sole and Ruiz [MMR]. The core of their proof is the sub-additivity of the sequence of finite-volume approximation to $\tau_\beta(\vec{n})$ which is obtained by means of FKG inequality. The proof is also valid for a wide range of models like Ising models with finite range interactions, Potts and SOS models. Furthermore, they showed that surface tension can be defined with parallelepipeds $\Lambda(N,M_N)$, where M_N is a function of N which diverges as N goes to infinity. More general domains can also be considered provided they contain a parallelepiped of the type $\Lambda(N,M_N)$.

The convexity of the homogeneous extension of τ_β (see (1.1.3)) is a consequence of the pyramidal inequality proven in Theorem 3 of [MMR] : Let $A_0,\dots,A_d$ be $d+1$ points of $\mathbb{R}^d$ and denote by $(\Delta_i)_{i\le d}$ the simplex defined by these points. Let $\vec{n}_i$ be the unit normal

[5]Notice that surface tension is sometimes defined with an extra multiplicative factor $\frac{1}{\beta}$.

to Δ_i and $|\Delta_i|$ its area. Then, the pyramidal inequality says

$$|\Delta_0|\,\tau_\beta(\vec{n}_0) \leqslant \sum_{i=1}^{d} |\Delta_i|\,\tau_\beta(\vec{n}_i).$$

Note also that the homogeneous extension of τ_β is continuous because it is locally bounded and convex. Furthermore, τ_β is uniformly positive on $\mathbb{S}^{d-1}$. This follows from the fact that the surface tension $\tau_\beta(\vec{n}_0)$ in the direction $\vec{n}_0 = (1, 0, \ldots, 0)$ is strictly positive as β is larger than β_c (see Lebowitz and Pfister [LePf]).

1.3. Scope of the theory

The key notion behind the attempts to give a rigorous meaning to the type of the phase segregation phenomena, which have been vaguely discussed in Subsection 1.1.5, is that of **renormalization** or **coarse graining**. The energy (probability) competes with the entropy (number) of microscopic configuration in the corresponding energy shells. Macroscopic quantities like surface tension are produced in the aftermath of the entropy/energy cancelation, which is to say that in order to derive large-N (N-linear size of the system) asymptotics one should renormalize appropriate microscopic objects. The appropriate objects here are, of course, microscopic phase boundaries, which decouple between different "large" microscopic phase regions. These renormalization procedures could follow two different trends, depending on whether the renormalized (mesoscopic) structures keep track of the microscopic or macroscopic state of the system.

1.3.1. **Dobrushin-Kotecký-Shlosman Theory.** The coarse graining of the DKS theory closely follows microscopic phase segregation patterns. Basic tools comprise a fluctuation analysis of the microscopic phase boundaries and sharp uniform local limit estimates over domains encircled by such boundaries. Thus, the notion of finite volume phases is quantified by the rate of the relaxation of the statistics of microscopic observables inside the microscopic phase regions towards the corresponding equilibrium values.

The theory has been developed using the low-temperature cluster expansions in the seminal monograph [DKS]. Our exposition in Part 3 is non-perturbative and follows the works [Pf2], [I1], [I2], [PV2], [ScS3] and [ISc]. By and large the existing results are confined to the simplest two-dimensional models (percolation and nearest neighbor Ising).

1.3.2. $\mathbb{L}_1$**-Theory.** The renormalization approach of the $\mathbb{L}_1$-theory is, in a sense, opposite to that of DKS. In the latter case the principal coarse grained objects (skeletons, see Part 3) are built upon underlying families of large **microscopic** contours. Such information is waved out in the $\mathbb{L}_1$-approach, and the basic renormalization objects here are the local (mesoscopic) order parameters or, in the spin language, locally averaged magnetization on various length scales. The idea is that on sufficiently large scales local averages of the magnetization are, with an overwhelming probability, close to one of the two equilibrium values $\pm m^*$. Thus, under the renormalization, configurations are characterized by their phase labels on different mesoscopic blocks. The objective of the $\mathbb{L}_1$-theory is to describe typical mesoscopic magnetization profiles (or their phase labels) under a relaxed canonical constraint of shell type. Unlike in the DKS case, the mesoscopic phase labels are classified by their proximity to various **macroscopic** states. Combinatorial complexity of this approximation is reduced by an exponential tightness property of the mesoscopic phase labels (for a general claim of this sort see Theorem 2.2.1), which enables to restrict attention only to $\mathbb{L}_1$-compact subsets of feasible macroscopic states, namely to the phase-sets of finite perimeter. The core of the compactness estimates is based on the renormalization decoupling techniques introduced in [Pi1] and on the methods developed to control the phase of small contours by [I2], [PV2], [ScS3] and [ISc]. These techniques are robust enough to be applied on a renormalized scale in any dimensions in a non perturbative setting.

Our exposition in this review is based on the work of [B1] with, though, one exception – we specifically stress that all the relevant estimates of the $\mathbb{L}_1$-theory are obtained on appropriate **finite** scales. The validity of Lemma 2.4.1 up to the slab percolation threshold follows from the results of [CePi].

1.3.3. **Boundary Phenomena.** Parts 2 and 3 provide a derivation of Wulff construction from the basic principles of Equilibrium Statistical Mechanics. Part 4 is concerned with a study of the effect of the boundary conditions on the macroscopic geometry of the phase separation. In particular, it is shown how the interaction with the boundary of the vessel can be analyzed, and used to provide a derivation of Winterbottom construction. The relationship between the macroscopic geometry in this case and the wetting transition is also discussed. The presentation follows [PV2] for the 2D case, and [BIV] for the higher-dimensional ones.

1.3.4. **Bibliographical review.** The rigorous investigation of the macroscopic geometry of phase separation under a canonical constraint certainly started with two seminal papers of Minlos and Sinai in 1967-68 [MS1, MS2]. In these papers, the authors considered nearest-neighbor very low temperature Ising models in arbitrary dimensions $d \geqslant 2$, even though they only wrote down the proof explicitly in the case $d = 2$. Their results could be roughly stated in the following way: At sufficiently low temperatures, typical configurations of the Ising model in the exact canonical ensemble over finite vessels of linear size N, consist of a single large contour whose shape is "nearly a square", whereas the rest of the contours are small, that is at most of the order $\log N$. This is the picture of low temperature excitations of canonical ground states, and it has been treated by the authors as such. In particular, the entropic factor has been frequently suppressed by the microscopic energy cost. However, exact asymptotic results on the level of a microscopic justification of the Wulff construction depend, even at very low but still non-zero temperatures, on a non-trivial entropy/energy competition, and, hence, could not be derived in this way.

Then there followed 15-20 years of a relative stagnation, the only contributions to the area being confined to generalizations of [MS1, MS2] to more complicated models [Ku1]. A popular interest to the problem has been revived towards mid-eighties in the framework of an on-going mingle between probability and statistical mechanics [Sc], [FO], [LeSc], [CCSc].

A breakthrough occurred around 1989, when Dobrushin, Kotecký and Shlosman found a way to derive the Wulff shape in a scaling limit of the low temperature 2D Ising model. They found much more: Essentially the monograph [DKS] sets up a comprehensive mathematical theory of phase segregation. This theory happened to be an intrinsically probabilistic one. The DKS approach is, above all, to quantify the phenomenon of phase separation in terms of probabilistic limit theorems and, accordingly, to study the probabilistic structures related to the canonical states. Thus, in a sharp contrast with most of the preceding works, the ideology of [DKS] has been from the start a very robust one and, actually, pertained to the whole of the phase transition region. It could be implemented, however, only at very low temperatures, since the authors used low temperature cluster expansions as the principal tool for proving the corresponding probabilistic theorems.

The ideas of [DKS] did not wait long to inspire a wave of investigations, even before the draft of the work started to circulate. Two subsequent works of a fundamental importance are [Pf2], where an alternative simplified proof of parts of the DKS results has been given using techniques, which are specific to the 2D Ising model, like self-duality, and [ACC], where the Wulff construction has been derived in the context of the 2D Bernoulli percolation, but in a completely non-perturbative fashion, that is down to the percolation threshold $1/2$. In both instances the exact canonical setting has been substituted by shell-type integral constraints, and, respectively, softer integral type limit results have been used instead of the local estimates of the original DKS theory.

The results and techniques of [ACC] and [Pf2] have been combined with profound renormalization ideas of [Pi1] and lead to an extension of this weak integral approach to the Wulff construction in the whole of the 2D Ising phase coexistence limit [I1], [I2]. Simpler proofs of some of the basic estimates of these two works (e.g estimates in the phases of small contours or skeleton lower bounds) have been found in [ScS1], [ScS2], and the integral version of the two-dimensional DKS theory has been essentially completed in [PV2], the estimates of the latter work being already optimal along the lines of the integral approach. Furthermore, Pfister and Velenik [PV1, PV2] investigated the effect of boundary conditions, and in particular studied the effect of an arbitrary boundary magnetic field, thus providing a derivation of the Winterbottom construction.

In spite of these successes, a non-perturbative treatment of the full DKS theory was still out of reach, because a key ingredient was missing: only rough estimates were available in the phase of small contours. By proving a local limit theorem in the phase of small contours, Ioffe and Schonmann were finally able to provide a non-perturbative version of the strong Wulff theory [ISc]. The techniques of [ISc] are based on improved versions of asymptotic expansions in metastable cutoff phases developed in [ScS3].

In principle, the two-dimensional DKS theory should lead to exact expansions of canonical partition functions up to zero-order terms. This, however, requires a superb control over the statistical behavior of microscopic phase boundaries, which is currently beyond the reach for the Ising model at moderately low temperatures. A certain progress, though, has been reported at very low temperatures [DH], [H] or either in the case of simplified models [HI]. Finally, it should be noted that at moderately low temperatures the success of the DKS theory in two dimensions has been by and large confined to the Ising and percolation models, and that there are serious technical and possibly theoretical challenges to extend it to more general two-dimensional models (see Section 3.6 for more on this).

On the other hand, as it has been communicated to us, an appropriate version of the low temperature DKS theory (as originally developed in [DKS]), should apply to any 2-phase model in the realm of the Pirogov-Sinai theory [Sh].

There is a strong interplay between dynamical properties of the Ising model and its behavior in equilibrium : in absence of phase transition, the correlations at equilibrium are related to the exponential relaxation of the system; instead as a phase transition occurs, the dynamics is driven by the evolution of droplets (nucleation, motion by mean curvature ...). We will not enter into details and simply refer to the seminal paper on metastability by Schonmann and Shlosman [ScS3] and to the lecture notes by Martinelli [Ma] (and references therein) for a survey of the recent works. Let us just mention that, as far as phase coexistence is considered, many dynamical results are only valid in dimension

2 because of the absence of a precise description of the equilibrium properties in higher dimensions.

If the 2D case was subject to rapid progress, the best results for higher dimensions remained for a long time those of Minlos and Sinai.

The turning point of the latest developments should be traced back to the seminal works by Pisztora [Pi1] and by Cassandro and Presutti [CP], where crucial renormalization decoupling estimates have been established in the case of the nearest neighbour Ising and, respectively, Kac interactions.

The basic philosophy of the $\mathbb{L}_1$-approach has been originally developed in the works [ABCP], [BCP], [BBBP], [BBP] in the context of the Ising systems with Kac potentials, and, in a less explicit way, elements and ideas of the theory already appeared in [ACC], [Pi1], [I2] and [PV2].

Using an embedding of the renormalized observables into a continuum setting, Alberti, Bellettini, Cassandro and Presutti [ABCP], [BCP] emphasized the appropriateness of geometric measure theory setting, introduced relevant analytic approximation procedures (see Subsection 2.6.1) and proved large deviation bounds for the appearance of a droplet of the minority phase in a scaling limit when the size of the domain diverges not much faster than the range of the Kac potentials. In this scaling the system can be controlled by a continuum limit via the Γ-convergence of functionals associated to the spins system [ABCP] and by compactness arguments [BCP].

The approach of [ABCP] and [BCP] has been extended by Benois, Bodineau, Butta and Presutti [BBBP], [BBP] to the case when the range of the interaction remains fixed and does not change with the size of the system. The latter works are, already, structured in a way very similar to the one we expose here. Thus the main steps of [BBBP] and [BBP] comprise the coarse-graining of the rescaled magnetization profiles by the $\mathbb{L}_1$-proximity to various continuum sets of finite perimeter, surgery procedures to confine interfaces to tubes around the boundaries of such sets and exponential tightness arguments to reduce the combinatorial complexity of the rescaled problem. The essential model-related input has been provided by the decoupling estimates on the renormalized magnetization [CP], [BZ] and by the result on the instanton structure of Kac interfaces [DOPT1, DOPT2]. The latter structure, however, yields only approximate bounds at each fixed finite interaction range. Consequently, the exact (van der Waals) surface tension could be recovered only when the range of the interaction tends to infinity, that is only in the Lebowitz-Penrose limit. Nevertheless, at long but finite range interactions one could say that the typical mesoscopic configurations concentrate on droplets with $\mathbb{L}_1$-almost spherical shapes.

A complete picture of the higher-dimensional $\mathbb{L}_1$-Wulff construction has been, for the first time, grasped and worked out in a recent remarkable work [Ce1], where the corresponding results have been established in the context of the super-critical 3-dimensional Bernoulli bond percolation. Using novel and unusual renormalization procedures based on the decoupling results of [Pi1], he has essentially rediscovered all the main steps of the $\mathbb{L}_1$-approach as described above. The main turning point of [Ce1] was the introduction of an alternative ingenious definition of the surface tension which happened to be compatible with the setup of $\mathbb{L}_1$-renormalization procedures [6].

The work of [Ce1] triggered a wave of new investigations. In [B1] his ideas on how to define and treat the surface tension have been combined with an appropriate adjustment of

[6] It should be noted, though, that despite relative technical simplicity of this observation, the work [Ce1] most certainly prompted the completion of the $\mathbb{L}_1$-theory by many years.

the renormalization approach of [BBBP] and [BBP], which lead to a relatively short proof of the $\mathbb{L}_1$-Wulff construction for the nearest neighbour Ising model in three and higher dimensions and at sufficiently low temperatures. Most recently, a similar construction has been established up to the FK slab percolation threshold in [CePi]. In the latter article new and important techniques have been developed in order to go around mixed boundary conditions via bulk relaxation properties of the FK-measures.

Although the techniques of the $\mathbb{L}_1$-theory might look "soft" when compared to the local limit setting of the DKS approach, one should bear in mind that there is always a "hard" step needed to initialize the $\mathbb{L}_1$-machinery: The renormalized mesoscopic phase labels have to possess sufficiently good decoupling properties. For the case of Kac models the corresponding estimates have been established in [CP], [BZ], [BMP], and in the case of percolation (including FK for the nearest neighbor Ising model) models in dimension $d \geqslant 3$ in [Pi1], on which both [Ce1],[CePi] and [B1] rely in a fundamental way.

Higher dimensional Winterbottom type shapes have been recovered in the context of effective interface models [BI], [BD], [DGI], [DM] following the original two-dimensional model defined and studied in [CDR].

The results of these works have been also formulated in terms of $\mathbb{L}_1$ concentration properties, but the corresponding approach is quite different from the one we expose here. Thus, the analysis of [BI] heavily relies on specific properties of Gaussian interactions. It should be noted, though, that, unlike in the nearest neighbour higher dimensional Ising case, there is better insight into the fluctuation and relaxation properties of higher dimensional microscopic interfaces [FS], [DGI]. On the other hand, the shapes produced by the effective interface models are much less "physical", in particular the equilibrium shapes are not scale invariant, and the corresponding surface tension is not convex.

Part 2. $\mathbb{L}_1$-Theory

On the macroscopic level the phenomenon of phase segregation is studied in terms of concentration properties of the locally averaged magnetization. Statistical properties of the microscopic phase boundaries are waved out, and the backbone of the $\mathbb{L}_1$-theory are hard model-oriented renormalization estimates, which enable a sharp surface order analysis of the mesoscopic magnetization profiles. Example of such coarse graining procedures in the case of Kac, percolation and Ising models are given in Section 2.3.
The averaging is performed on various mesoscopic scales:

Mesoscopic Notation. All the intermediate scales are of the form $2^k, k \in \mathbb{N}$. For any $M = 2^k$ fixed we split the unit torus $\widehat{\mathbb{T}}^d$ into the disjoint union of the corresponding mesoscopic boxes,

$$\widehat{\mathbb{T}}^d = \bigvee_{x \in \widehat{\mathbb{T}}^d_k} \widehat{\mathbb{B}}_k(x), \tag{2.0.1}$$

where $\widehat{\mathbb{T}}^d_k$ is the scaled embedding of the discrete torus $\mathbb{T}_M = \{1, \dots, M\}^d$ into $\widehat{\mathbb{T}}^d$ as

$$\widehat{\mathbb{T}}^d_k \triangleq \widehat{\mathbb{T}}^d \cap \left(\frac{1}{M} \mathbb{T}_M \right),$$

and, given $x \in \widehat{\mathbb{T}}^d$ the box $\widehat{\mathbb{B}}_k(x) \subset \widehat{\mathbb{T}}^d$ is defined via

$$\widehat{\mathbb{B}}_k(x) \triangleq x + \Big[-\frac{1}{2^{k+1}} , \frac{1}{2^{k+1}} \Big)^d .$$

Let us use $\mathcal{F}_k$ to denote the (finite) algebra of the subsets of $\widehat{\mathbb{T}}^d$ generated by the partition (2.0.1) . Given the size of the system $N = 2^n$, the local magnetization $\mathcal{M}_k$ on the $M = 2^k \leqslant N$ scale is always an $\mathcal{F}_{n-k}$-measurable function. This notation should not be confusing: the subindex k in $\mathcal{M}_k$ measures the "coarseness" of the mesoscopic magnetization profile. Thus, $\mathcal{M}_0$ corresponds to the microscopic configuration, and $\mathcal{M}_n$ identically equals to the averaged total magnetization. In general the local magnetization $\mathcal{M}_k$ is a piecewise constant function on $\widehat{\mathbb{T}}^d$ defined as

$$\forall x \in \widehat{\mathbb{T}}^d_{n-k}, \forall y \in \widehat{\mathbb{B}}_{n-k}(x), \qquad \mathcal{M}_k(\sigma, y) = \frac{1}{M^d} \sum_{j \in \mathbb{B}_M(2^n x)} \sigma_j .$$

Notice that the microscopic counterpart of the box $\widehat{\mathbb{B}}_{n-k}(x)$ is the box $\mathbb{B}_M(2^n x)$ of side length M centered in $2^n x$.

We formulate all the results of Section 2.1 for the nearest neighbor Ising model. Along with the super-critical Bernoulli percolation this is the only instance when a relatively complete $\mathbb{L}_1$-theory has been developed. In both instances, the validity of the $\mathbb{L}_1$-Theory hinges in a crucial way on the validity of Pisztora's coarse graining [Pi1], which is by far the most profound model related fact employed. Nevertheless, the approach itself is rather robust, and in subsequent Subsections we shall try to distinguish between specific model dependent properties and more general results. In particular, compactness properties of local magnetization profiles are discussed in Section 2.2 without any reference to specific models. Instead we briefly indicate how the conditions of the corresponding general exponential tightness Theorem could be verified in several particular cases.

2.1. Results and the strategy of the proof

2.1.1. **Main results.** For simplicity, we restrict to the case of the torus $\mathbb{T}_N$ and denote by μ_N the Gibbs measure with periodic boundary conditions.

Define the total magnetization $\mathbf{M}_{\mathbb{T}_N}$ as

$$\mathbf{M}_{\mathbb{T}_N} \triangleq \frac{1}{N^d} \sum_{i \in \mathbb{T}_N} \sigma_i .$$

Let us define also the set $\mathfrak{B}_p$ as

$\mathfrak{B}_p = \{\beta\ :\ \text{Pisztora's coarse-graining hold for the Ising model at inverse temperature } \beta\}$.

We refer to the original article [Pi1] and [CePi] for the precise relevant definitions (see also remark at the end of the Subsection 2.3.3). It is known that $\mathfrak{B}_p$ contains all except for at most countably many points of the interval $]\tilde{\beta}_c, \infty[$, where $\tilde{\beta}_c$ is the so called slab percolation threshold, which is conjectured to coincide with β_c.

A compact way to state the main result of the $\mathbb{L}_1$-theory is:

Theorem 2.1.1. *For any $\beta \in \mathfrak{B}_p$ and m in $]\bar{m}, m^*[$*

$$\lim_{N \to \infty} \frac{1}{N^{d-1}} \log \mu_N \left(\left| \mathbf{M}_{\mathbb{T}_N} \right| \leqslant m \right) = -\mathcal{W}_\beta(\mathcal{K}_m),$$

where $\bar{m} = \bar{m}(\beta)$ and $\mathcal{K}_m$ were defined in Subsection 1.1.2.

Remark. The above Theorem has been established for $\beta \gg 1$ in [B1]. The only additional ingredient required for an extension of the results of the latter paper to the whole of the temperature range $\tilde{\beta}_c$ was the validity of the Lemma 2.4.1. Such a statement happens to be highly non-trivial, and it has been proven in [CePi] along with an alternative derivation of the claim of Theorem 2.1.1. □

Theorem 2.1.1 looks like a surface order large deviation principle. Such an appellation, however, would not help to explain the structure of the underlying phenomena. In fact Theorem 2.1.1 is essentially equivalent to a seemingly stronger statement on the macroscopic geometry of the phase segregation of local magnetization profiles under the conditional measure $\mu_N \left(\cdot \,\middle|\, \left| \mathbf{M}_{\mathbb{T}_N} \right| \leqslant m \right)$:

For any function v in $\mathbb{L}^1 (\widehat{\mathbb{T}}^d, [-\frac{1}{m^*}, \frac{1}{m^*}])$, the δ-neighborhood of v is denoted by $\mathcal{V}(v, \delta)$

$$\mathcal{V}(v, \delta) \triangleq \left\{ v' \in \mathbb{L}^1 \left(\widehat{\mathbb{T}}^d, [-\frac{1}{m^*}, \frac{1}{m^*}] \right) \ \middle|\ \int_{\widehat{\mathbb{T}}^d} |v'_x - v_x| \, dx \leqslant \delta \right\}.$$

The $\mathbb{L}_1$-Theorem on the phase separation says that for β large enough with $\mu_N \left(. \,\middle|\, \left| \mathbf{M}_{\mathbb{T}_N} \right| \leq m \right)$-probability converging to 1, the function $\mathcal{M}_k$ is close to some translate of the Wulff shape $m^* \mathbb{I}_{\mathcal{K}_m}$.

More precisely, fix a number $\nu < 1/d$.

Theorem 2.1.2. *For any $\beta \in \mathfrak{B}_p$ and m in $]\bar{m}, m^*[$ the following holds:*
For every $\delta > 0$, one can choose a scale $k_0 = k_0(\beta, \delta)$, such that

$$\lim_{N \to \infty} \min_{k_0 \leqslant k \leqslant \nu n} \mu_N \left(\frac{\mathcal{M}_k}{m^*} \in \bigcup_{x \in \widehat{\mathbb{T}}^d} \mathcal{V}(\mathbb{I}_{\mathcal{K}_m + x}, \delta) \,\middle|\, \left| \mathbf{M}_{\mathbb{T}_N} \right| \leqslant m \right) = 1,$$

where $\bar{m}$ and $\mathcal{K}_m$ were defined in Subsection 1.1.2.

The proofs of Theorems 2.1.1 and 2.1.2 are similar and are divided into 2 steps. The first step amounts to prove a compactness Theorem and the second one to derive precise logarithmic asymptotics.

2.1.2. **Exponential tightness.** Recall [EG] that for any a positive, the set

$$K_a \triangleq \left\{v \in \mathrm{BV}(\widehat{\mathbb{T}}^d, \{\pm 1\}) \mid \quad \mathcal{P}(\{v = 1\}) \leqslant a\right\},$$

is compact with respect to convergence in $\mathbb{L}^1(\widehat{\mathbb{T}}^d)$.

Proposition 2.1.1. *Let β be in $\mathfrak{B}_p$. Then there exists a constant $C(\beta) > 0$ such that for all δ positive one can find $k_0(\delta)$*

$$\forall a > 0, \qquad \limsup_{N\to\infty} \frac{1}{N^{d-1}} \max_{k_0(\delta) \leqslant k \leqslant \nu n} \log \mu_N \left(\frac{\mathcal{M}_k}{m^*} \in \mathcal{V}(K_a, \delta)^c\right) \leqslant -C(\beta)\, a,$$

where $\mathcal{V}(K_a, \delta)$ is the δ-neighborhood of K_a in $\mathbb{L}^1(\widehat{\mathbb{T}}^d, [-\frac{1}{m^*}, \frac{1}{m^*}])$.

This proposition tells us that only the configurations close to the compact set K_a have a contribution which is of the surface order. This statement reduces the complexity of the problem : as K_a is compact, it is enough to derive the leading terms in the logarithmic asymptotics for the probability of a finite number of events.

In Section 2.2, we prove that the analog of Proposition 2.1.1 holds for a broad class of models.

2.1.3. **Precise logarithmic asymptotics.** As the minimizers are known, it is sufficient to derive a lower bound for configurations concentrated close to $\mathcal{K}_m$.

Proposition 2.1.2. *Let β be in $\mathfrak{B}_p$ and let m be in $]\bar{m}, m^*[$*

$$\liminf_{N\to\infty} \frac{1}{N^{d-1}} \min_{k_0(\delta) \leqslant k \leqslant \nu n} \log \mu_N \left(\frac{\mathcal{M}_k}{m^*} \in \mathcal{V}(\mathbb{1}_{\mathcal{K}_m}, \delta)\right) \geqslant -\mathcal{W}_\beta(\mathcal{K}_m) - o(\delta)\,,$$

where the function $o(\cdot)$ depends only on β and vanishes as δ goes to 0.

According to proposition 2.1.1, we will prove the upper bound only for a restricted class of events

Proposition 2.1.3. *Let β be in $\mathfrak{B}_p$. Then for all v in $\mathrm{BV}(\widehat{\mathbb{T}}^d, \{\pm 1\})$ such that $\mathcal{W}_\beta(v)$ is finite, one can choose $\delta_0 = \delta_0(v)$, such that uniformly in $\delta < \delta_0$*

$$\limsup_{N\to\infty} \frac{1}{N^{d-1}} \max_{k_0(\delta) \leqslant k \leqslant \nu n} \log \mu_N \left(\frac{\mathcal{M}_k}{m^*} \in \mathcal{V}(v, \delta)\right) \leqslant -\mathcal{W}_\beta(v) + o(\delta)\,.$$

where the function $o(\cdot)$ depends only on β and v and vanishes as δ goes to 0.

The Propositions above ensure that given a precision δ, there is a finite scale $k_0(\delta)$ after which the phases are uniformly segregated with this precision.

2.1.4. **Scheme of the proof.** The scheme of the proof is well known in the soft context of large deviations: one first proves an exponential tightness property and then a weak large deviation principle (Proposition 2.1.2 holds also for any bounded variation function with finite perimeter). To be sure, the proof itself has nothing to do with the theory of large deviations: the central tools here are the renormalization estimates leading to Peierls type bounds and estimate in the phase of small contours, and, of course, the identification methods to produce the macroscopic surface tension in the precise logarithmic asymptotics.

Thus, Proposition 2.1.1 tells us that, under the appropriate renormalization, the occurrence of many small contours or of very large contours is unlikely. It is a straightforward consequence of the general exponential tightness Theorem 2.2.1, which we state in Section 2.2. The statement is reminiscent to the results proven in [BBP], but the proof itself is based on the analysis of the phase of small contours developed in [I2], [ScS1], [PV2].

To prove Propositions 2.1.2 and 2.1.3, we first consider the macroscopic event $\big\{\frac{\mathcal{M}_k}{m^*} \in \mathcal{V}(v,\delta)\big\}$ and by using several localization procedures, we reduce to compute the probability of microscopic events from which, adopting the procedure developed in [Ce1], we can derive the exact surface tension factor. This enables us to avoid the computations related to the microscopic phase boundaries at, however, a principal cost of loosing track of the latter.

Since the most likely configurations in $\big\{\frac{\mathcal{M}_k}{m^*} \in \mathcal{V}(v,\delta)\big\}$ are those for which both phases coexist along the boundary of $\partial^* v$, we would like to prove that a microscopic interface is localized close to the boundary. To derive the lower bound (Proposition 2.1.2), one can enforce such a microscopic interface and then recover the surface tension factor.

This is not the case for the upper bound (Proposition 2.1.3) because the $\mathbb{L}_1$ constraint $\big\{\frac{\mathcal{M}_k}{m^*} \in \mathcal{V}(v,\delta)\big\}$ imposed on the magnetization is not strong enough to localize the interface close to $\partial^* v$: there might be mesoscopic fingers of one phase percolating into the other. To circumvent this problem, we follow an argument developed in [BBBP] and first prove a weak localization on a mesoscopic level. This involves a surgery procedure called the minimal section argument. This procedure ensures that one can chop off the mesoscopic fingers without changing too much the probability of the event and therefore localize the interface on a mesoscopic level. The renormalization is an essential feature of this proof. Once the interface is localized on the mesoscopic level, it remains to identify surface tension.

We now proceed by first defining a coarse graining and deducing the exponential tightness from Theorem 2.2.1. Then we compute the logarithmic asymptotics.

2.2. Coarse graining and mesoscopic phase labels

At every mesoscopic scale $M = 2^k$ the local magnetization $\mathcal{M}_k$ gives a coarse grained representation of the system. Statistical properties of the microscopic configurations are waved out, and instead one keeps track only of the local order parameters over the corresponding mesoscopic blocks. These are quantified by three values ± 1 and 0 according to whether they are sufficiently close to one of the two equilibrium values $\pm m^*$ or not. 0-blocks play the role of the mesoscopic phase boundaries, and the ± 1 blocks of the corresponding mesoscopic phase regions. Thus, the outcome of the renormalization could be schematically represented as the following two-step diagram :

$$\left\{\begin{matrix}\text{Microscopic} \\ \text{configurations}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{Local} \\ \text{magnetization}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{Mesoscopic} \\ \text{phase labels}\end{matrix}\right\}.$$

There are two principal results to be discussed in this Subsection: we show that the $\mathbb{L}_1$-difference between the local magnetization and the corresponding phase labels vanishes on the exponential scale, and we give a general exponential tightness criterion for families of $\{\pm 1, 0\}$-valued phase label functions. In Section 2.3, we will indicate how to construct phase labels in the case of Kac, percolation and nearest neighbor Ising models.

Definition : A $\{\pm 1, 0\}$-valued function u on $\widehat{\mathbb{T}}^d$ is called a mesoscopic phase label, if there exists $k \in \mathbb{N}$, such that u is an $\mathcal{F}_k$-measurable function.

2.2.1. **Tightness theorem for mesoscopic phase labels.** We fix now a sequence of non-negative numbers $\{\rho_k\}$ such that

$$\lim_{k\to\infty} \rho_k \;=\; 0. \tag{2.2.1}$$

The following compactness result holds uniformly in the microscopic scales $N = 2^n$.

Theorem 2.2.1 (Tightness of Mesoscopic Phase Labels). *Let $N = 2^n$ and assume that $\{u_k(\omega, x)\}$ is a sequence of random mesoscopic phase label functions defined on the common probability space $(\Omega_N, \mathcal{A}_N, \mathbb{P}_N)$, such that the realizations of $u_k \in \mathcal{F}_{n-k}$, $k = 1, ..., n$, and for every k the following two conditions hold:*
A. *The distribution of the family of random variables $\{|u_k(\omega, x)|\}_{x\in\widehat{\mathbb{T}}^d_{n-k}}$ is stochastically dominated by the Bernoulli site percolation measure $\mathbb{P}^{\rho_k}_{\text{perc}}$ on $\widehat{\mathbb{T}}^d_{n-k}$. In particular,*

$$\mathbb{P}_N\left(u_k(x_1) = 0, ..., u_k(x_\ell) = 0\right) \;\leqslant\; (\rho_k)^\ell. \tag{2.2.2}$$

B. *If for two different points $x, y \in \widehat{\mathbb{T}}^d_{n-k}$ the corresponding u_k-phase labels have opposite signs, that is if $u_k(x)u_k(y) = -1$, then on any finer scale $k' \leqslant k$ any $*$-connected chain of $\widehat{\mathbb{B}}_{n-k'}$ blocks joining $\widehat{\mathbb{B}}_{n-k}(x)$ to $\widehat{\mathbb{B}}_{n-k}(y)$ contains at least one block with zero k'-label.*

Then for every $a > 0$ and $\delta > 0$ there exists a finite scale $k_0 = k_0(\delta)$, such that

$$\frac{1}{N^{d-1}} \log \mathbb{P}_N\left(u_k \in \mathcal{V}(K_a, 2\delta)^c\right) \;\leqslant\; -\, c_1(d) \min\left\{\delta 2^{n-dk} \;,\; \frac{a}{2^{(d-1)k_0}} \;, \frac{\delta 2^{n-dk_0}}{n^d}\right\} \;, \tag{2.2.3}$$

for all $k \geqslant k_0$.

Remark . The proof of this general theorem is given in Appendix A. Notice that for N sufficiently large we obtain a simpler surface order estimate which, for every $\nu < 1/d$ fixed, holds uniformly in all mesoscopic scales $k_0(\delta) \leqslant k \;\leqslant \nu \log N$,

$$\frac{1}{N^{d-1}} \log \mathbb{P}_N\left(u_k \in \mathcal{V}(K_a, 2\delta)^c\right) \;\leqslant\; -\, c_1(d) \frac{a}{2^{(d-1)k_0}}. \tag{2.2.4}$$

Also an inspection of the proof shows that the tightness of the phase labels on a certain scale k does not depend on the validity of Assumptions A and B on the successive scales $k' > k$. In particular, the estimate (2.2.4) is valid on fixed (large) finite scales $k = k_0$, once the Assumption A is satisfied, and once any $*$-connected sign changing chain of k_0-blocks necessarily contains a 0-block. This simplified version of Theorem 2.2.1 is used in the case of Kac potentials which we discuss in Subsection 2.3.1. □

2.2.2. **Relation to magnetization profiles.** The original Gibbs measure is related to the above abstract setting in the following way: For every $N = 2^n$, one constructs a (possibly enlarged) probability space $(\Omega_N, \mathcal{A}_N, \mathbb{P}_N)$, on which both the spin variables $\sigma \in \{-1,+1\}^{\mathbb{T}_N}$ and various indexed families $\{u_k^\zeta\}$ of mesoscopic phase labels are defined. Such construction should enjoy the following set of properties:

C1. The marginal distribution of σ under $\mathbb{P}_N$ is precisely μ_N.
C2. For every $\zeta > 0$ the family $\{u_k^\zeta\}$ of mesoscopic phase labels satisfies Assumption A of Theorem 2.2.1 with the corresponding sequence $\{\rho_{k,\zeta}\}$ of site percolation probabilities obeying (2.2.1).
C3. For every $k \in \{0, ..., n\}$ and $\zeta > 0$ the local magnetization profile $\mathcal{M}_k$ and the phase label u_k^ζ are related as follows: $\mathbb{P}_N$-a.s.,

$$\Big|\mathcal{M}_k(x) - m^* u_k^\zeta(x)\Big| \leqslant \zeta \qquad \text{whenever } |u_k^\zeta(x)| = 1. \tag{2.2.5}$$

Notice that both functions above are $\mathcal{F}_{n-k}$-measurable, that is (2.2.5) should be verified over the mesoscopic boxes indexed by the points $x \in \widehat{\mathbb{T}}_{n-k}^d$.

Under conditions C1-C3, given any $\delta > 0$ one can choose the accuracy ζ of the coarse graining, a finite scale $k_0 = k_0(\delta, \beta)$ and a sequence of mesoscopic phase labels $\{u_k^\zeta\}$, such that for every $\nu < 1/d$ fixed,

$$\frac{1}{N^{d-1}} \log \mathbb{P}_N \left(\max_{k_0 \leqslant k \leqslant \nu n} \|\mathcal{M}_k - m^* u_k^\zeta\|_1 > \delta \right) \leqslant -c_2\, 2^{(1-d\nu)n}. \tag{2.2.6}$$

Notice that (2.2.6) holds uniformly in the size of the system $N = 2^n$, once Assumptions C1-C3 do so.

Let us check (2.2.6). By the very construction,

$$\|\mathcal{M}_k - m^* u_k^\zeta\|_1 \leqslant \zeta + \frac{2}{|\widehat{\mathbb{T}}_{n-k}^d|} \sum_{x \in \widehat{\mathbb{T}}_{n-k}^d} 1_{u_k^\zeta(x)=0}.$$

Consequently, using the domination by the Bernoulli site percolation (Assumption A),

$$\mathbb{P}_N\Big(\|\mathcal{M}_k - m^* u_k^\zeta\|_1 > \delta\Big) \leqslant \mathbb{P}_N \left(\frac{1}{|\widehat{\mathbb{T}}_{n-k}^d|} \sum_{x \in \widehat{\mathbb{T}}_{n-k}^d} 1_{u_k^\zeta(x)=0} > \frac{\delta - \zeta}{2} \right)$$
$$\leqslant \mathbb{P}_{\text{perc}}^{\rho_{k,\zeta}} \left(\frac{1}{|\widehat{\mathbb{T}}_{n-k}^d|} \sum_{x \in \widehat{\mathbb{T}}_{n-k}^d} 1_{u_k^\zeta(x)=0} > \frac{\delta - \zeta}{2} \right) \leqslant \exp\left\{ -c_1 2^{d(n-k)} \log \frac{\delta - \zeta}{2\rho_{k,\zeta}} \right\}.$$

The latter estimate is of the super-surface order once $\rho_{k,\zeta} \ll (\delta - \zeta)/2$ and $k < n/d$.

2.3. Examples of mesoscopic phase labels

We show that mesoscopic phase labels can be constructed in the case of Kac, percolation and Ising models.

2.3.1. **Kac potentials.** For this model mesoscopic phase labels are defined on the original space of spins $\sigma \in \{-1,+1\}^{\mathbb{T}_N}$: the coarse graining is obtained by averaging locally the magnetization. Recall that we are using dyadic length scales $N = 2^n$.

Phase labels are constructed in three steps. First, for any integer k and $\zeta > 0$, we introduce the block spin variables $\bar{u}_k^\zeta$ which label the boxes $\widehat{\mathbb{B}}_{n-k}$ according to the averaged magnetization over the boxes of the linear size $M = 2^k$. These $\bar{u}_k^\zeta$ are constant on each of the blocks $\widehat{\mathbb{B}}_{n-k}(x)$ with $x \in \widehat{\mathbb{T}}_{n-k}^d$

$$\bar{u}_k^\zeta(\sigma, x) \triangleq \begin{cases} \pm 1 & \text{if} \quad |\frac{1}{M^d}\sum_{i \in \mathbb{B}_M(2^n x)} \sigma_i \mp m^*| < \zeta, \\ 0 & \text{otherwise.} \end{cases}$$

In the Kac case we do not use Theorem 2.2.1 in its full generality, the object of the coarse graining is to choose a finite scale k_0, such that the family of mesoscopic phase labels is exponentially tight in $\mathbb{L}_1$. Recall that the scaling parameter is chosen such that $\varepsilon = 2^{-m}$ with m large but fixed. Eventually finite renormalization scales k_0 are going to satisfy $k_0 = m + a_0$, where a_0 depends on β and ζ, but not on m. The sign of the k_0-label over a box $\widehat{\mathbb{B}}_{n-k_0}(x)$ depends on a more refined information on the fluctuations of the magnetization inside the box : we choose another scale ℓ_0; $\ell_0 = m - b_0$, where, as in the case of a_0, the scale b_0 will eventually depend only on β and ζ, and define the family of modified block spins $\{\tilde{u}_{k_0}^\zeta\}$ on the k_0-scale as

$$\tilde{u}_{k_0}^\zeta(\sigma, x) \triangleq \begin{cases} \pm 1 & \text{if} \qquad \bar{u}_{\ell_0}^\zeta(\sigma, y) = \pm 1, \qquad \forall\, y \in \widehat{\mathbb{T}}_{n-\ell_0}^d \cap \widehat{\mathbb{B}}_{n-k_0}(x) \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we define the mesoscopic phase label functions $\{u_{k_0}^\zeta(\sigma, x)\}$. If $\tilde{u}_{k_0}^\zeta(\sigma, x) = 0$, we set $u_{k_0}^\zeta(\sigma, x) = 0$. If $x, y \in \widehat{\mathbb{T}}_{n-k_0}^d$ are $*$-neighbors, but the corresponding modified blocks spins satisfy $\tilde{u}_{k_0}^\zeta(\sigma, x)\, \tilde{u}_{k_0}^\zeta(\sigma, y) < 0$ then $u_{k_0}^\zeta(\sigma, x) = u_{k_0}^\zeta(\sigma, y) = 0$. Otherwise, we set $u_{k_0}^\zeta(\sigma, x) = \tilde{u}_{k_0}^\zeta(\sigma, x)$.

A consequence of the Peierls estimate proven in [CP] and [BZ] is that assumption A is satisfied, namely

Theorem 2.3.1. *For any $\beta > 1$, there exists $\zeta_0 = \zeta_0(\beta) > 0$, such that the following holds: For any $\zeta < \zeta_0$ one can choose $\varepsilon_0 = \varepsilon_0(\zeta)$, $a_0 = a_0(\zeta)$ and $b_0 = b_0(\zeta)$, such that uniformly in the interaction parameters $\varepsilon = 2^{-m} < \varepsilon_0$,*

$$\mu_{\varepsilon,N}\left(u_{k_0}^\zeta(x_1) = 0, \ldots, u_{k_0}^\zeta(x_r) = 0\right) \leqslant \exp\left(-\frac{c_0}{\varepsilon^d} r\right),$$

where, for every fixed $\varepsilon = 2^{-m} < \varepsilon_0$, the mesoscopic phase labels $u_{k_0}^\zeta$ are constructed on the scales $k_0 = m + a_0(\zeta)$ and $l_0 = m - b_0(\zeta)$.

Remark. A more refined statement implying exponential decay of correlations was proven in [BMP]. Notice that conditions C1-C3 of the previous Section are satisfied by definition of the mesoscopic phase label functions. Notice also that assumption B of Theorem 2.2.1 is automatically satisfied on the k_0-scale. Thus, the family $\{u_{k_0}^\zeta\}$ is exponentially tight in $\mathbb{L}_1$. □

A similar renormalization procedure was carried out by Lebowitz, Mazel and Presutti [LMP] for a system of point particles in $\mathbb{R}^d$ interacting with Kac potentials. In this case the study of phase transition in the continuum is much more involved. Beyond a proof of the liquid-vapor phase transition, their results provide an accurate description of the system in terms of mesoscopic phase labels which represent the liquid and the gaseous phases. Such a coarse graining should be helpful to obtain further results on phase coexistence in the continuum.

2.3.2. **Bernoulli bond percolation.** Bernoulli bond percolation exhibits features similar to the Ising model as phase transition and surface order behavior in a regime of phases coexistence. Nevertheless, as the setting is different from the Ising model, we briefly recall some notation. The set of edges is $\mathbb{E} = \big\{\{x,y\} \mid x \sim y\big\}$, where $x \sim y$ means that the vertices are nearest neighbors. An edge b in $\mathbb{E}$ is open if $\omega_b = 1$ and closed otherwise. To any subset $\Lambda \Subset \mathbb{Z}^d$, we associate $[\Lambda]_e$ the set of edges in Λ. The space of bonds configurations in Λ is $\Omega_\Lambda = \{0,1\}^{[\Lambda]_e}$. For a given p in $[0,1]$, we define the Bernoulli bond percolation measure on Ω_Λ by

$$\Phi^p_\Lambda(\omega) = \prod_{b \in [\Lambda]_e} (1-p)^{1-\omega_b} p^{\omega_b} .$$

For simplicity Φ^p_N denotes the measure on $\Omega_N = \Omega_{\mathbb{T}_N}$.

Let ω be a configuration in Ω, an open path $(x_1, \dots, x_n)$ is a finite sequence of distinct nearest neighbors $x_1, \dots, x_n$ such that on each edge $\omega_{\{x_i, x_{i+1}\}} = 1$. We write $\{A \leftrightarrow B\}$ for the event such that there exists an open path joining a site of A to one of B. The connected components of the set of open edges of ω are called ω-clusters.

A phase transition is characterized by the occurrence of an infinite cluster. Define Θ_p by

$$\Theta_p = \lim_{N \to \infty} \Phi^p_N(\{0 \leftrightarrow \partial\mathbb{T}_N\}) , \tag{2.3.1}$$

then there is a critical value p_c in $]0,1[$ such that for any p below p_c there is no percolation and $\Theta_p = 0$, instead for any p above p_c the occurrence of an infinite cluster starting from 0 has positive probability Θ_p. In the thermodynamic limit, there exists only one limiting Gibbs measure and almost surely a unique infinite cluster with local density Θ_p. In order to mimic the coexistence of 2 phases in the finite domains $\mathbb{T}_N$, we say that one phase is formed by the largest cluster and the other phase by the other clusters.

For this model, Pisztora introduced a renormalization procedure [Pi1], [DePi], [Pi2] which holds as soon as $p > p_c$ and $d \geqslant 3$. The mesoscopic phase labels $\{u^\zeta_k\}$ will be defined for any mesoscopic scale $M = 2^k$, where k is an integer which eventually depends on N. This construction requires 2 steps. The first step is to retain only the main features of the typical configurations on finite size boxes $\mathbb{B}_M$. Then we attribute a sign to the blocks $\widehat{\mathbb{B}}_{n-k}$ according to the phase they represent. Set $M' = 2M$. For any x in $\widehat{\mathbb{T}}^d_{n-k}$, the following events depend only on configurations in the box $\mathbb{B}_{M'}(2^n x)$.

$$U_x = \big\{\omega \in \Omega_N \;\big|\; \text{there is a unique crossing cluster } C^* \text{ in } \mathbb{B}_{M'}(2^n x)\big\} .$$

A crossing cluster is a cluster which intersects all the faces of the box. Let ℓ be an integer smaller than k which will be fixed later

$$\begin{aligned} R_x \;=\; & U_x \bigcap \Big\{\omega \in \Omega_N \;\Big|\; \text{every open path in } \mathbb{B}_{M'}(2^n x) \text{ with diameter larger than } 2^\ell \\ & \qquad \text{is contained in } C^* \Big\}, \end{aligned}$$

where the diameter of a subset A of $\mathbb{Z}^d$ is $\sup_{x,y\in A}\|x-y\|_1$. Finally, we consider an event which imposes that the density of the crossing cluster in $\mathbb{B}_M(2^n x)$ is close to Θ_p with accuracy $\zeta > 0$

$$V_x^\zeta = U_x \bigcap \{\omega \in \Omega_N \mid \quad |C^* \cap \mathbb{B}_M(2^n x)| \in [\Theta_p - \zeta, \Theta_p + \zeta]\, M^d\},$$

where $|\cdot|$ denotes the number of vertices in a set.

Each box $\widehat{\mathbb{B}}_{n-k}(x)$ is labeled by the variable $\tilde{u}_k^\zeta(\omega, x)$

$$\forall x \in \widehat{\mathbb{T}}^d_{n-k}, \qquad \tilde{u}_k^\zeta(\omega,x) \triangleq \begin{cases} 1 & \text{if} \quad \omega \in R_x \cap V_x^\zeta, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\{x_1,\dots,x_r\}$ be vertices in $\widehat{\mathbb{T}}^d_{n-k}$ not $*$-neighbors of x, then [Pi1] implies that for every $p > p_c$, there exists $k_0(p,\zeta)$, and $\ell_0(p)$ such that for all $k \geqslant k_0$ and $k \geqslant \ell \geqslant \ell_0$

$$\Phi_N^p\left(\tilde{u}_k^\zeta(x) = 0 \mid \tilde{u}_k^\zeta(x_1),\dots,\tilde{u}_k^\zeta(x_r)\right) \leqslant \exp(-c_1\, 2^\ell) + \exp(-c_2(\zeta)2^k),$$

From [LSS] (Theorem 1.3), we deduce that for k and ℓ large enough, the random variables $\{\tilde{u}_k^\zeta(x)\}$ are dominated by a Bernoulli site percolation measure $\mathbb{P}^{\rho_k}_{\text{perc}}$

$$\rho_k \leqslant \exp(-c(\zeta)\, 2^\ell). \tag{2.3.2}$$

A straightforward way to recover the previous statement is to partition $\widehat{\mathbb{T}}^d_{n-k}$ into $c(d)$ sub-lattices $\left(\widehat{\mathbb{T}}^d_{n-k-1,i}\right)_{i\le c(d)}$ which are translates of $\widehat{\mathbb{T}}^d_{n-k-1}$. Any collection of vertices $\{x_1,\dots,x_r\}$ in $\widehat{\mathbb{T}}^d_{n-k}$ can be rearrange into $c(d)$ subsets $\{x_1^{(i)},\dots,x_{r_i}^{(i)}\}$ such that each $\{x_1^{(i)},\dots,x_{r_i}^{(i)}\}$ belongs to $\widehat{\mathbb{T}}^d_{n-k-1,i}$. Applying Hölder inequality, we get

$$\Phi_N^p\left(\tilde{u}_k^\zeta(x_1) = 0,\dots,\tilde{u}_k^\zeta(x_r) = 0\right) \leqslant \prod_{i=1}^{c(d)} \Phi_N^p\left(\tilde{u}_k^\zeta(x_1^{(i)}) = 0,\dots,\tilde{u}_k^\zeta(x_{r_i}^{(i)}) = 0\right)^{\frac{1}{c(d)}}.$$

As the vertices in $\widehat{\mathbb{T}}^d_{n-k-1,i}$ are not $*$-neighbors in $\widehat{\mathbb{T}}^d_{n-k}$, the domination by a Bernoulli product measure follows.

We say that a block $\widehat{\mathbb{B}}_{n-k}(x)$ is regular if $\tilde{u}_k^\zeta(x) = 1$. Finally we define the mesoscopic phase labels u_k^ζ to be equal to 1 on the regular blocks connected to the largest cluster and to -1 on the regular blocks disjoint from the largest cluster. Otherwise, we set $u_k^\zeta(\omega,x) = \tilde{u}_k^\zeta(\omega,x) = 0$. From (2.3.2), the mesoscopic phase labels satisfy assumption A. Notice that if x and y are $*$-neighbors in $\widehat{\mathbb{T}}^d_{n-k}$ the boxes $\mathbb{B}_{M'}(2^n x)$ and $\mathbb{B}_{M'}(2^n y)$ overlap. Choosing the parameter $\ell \leqslant k-3$ we insure that if the boxes $\widehat{\mathbb{B}}_{n-k}(x)$ and $\widehat{\mathbb{B}}_{n-k}(y)$ are both regular, then the crossing clusters in these boxes are connected. This implies that assumption B is satisfied : two blocks with k-labels of different signs cannot be $*$-connected.

The Bernoulli bond percolation model is precisely described by Pisztora's coarse graining, namely on a sufficiently large scale 2^k, the typical configurations have a unique crossing cluster surrounded by small islands of size smaller than 2^ℓ. According to Theorem 2.2.1, the family $\{u_k^\zeta\}$ is exponentially tight in $\mathbb{L}^1$.

2.3.3. **Ising nearest neighbor.** An extension of the preceding renormalization procedure applicable to the Ising model has been also introduced in [Pi1]. Unlike Ising model with Kac potentials, this coarse graining is defined on an enlarged phase space via the FK representation. For a review of FK measures, we refer the reader to [Pi1], [ACCN] and [Gri].

Let us recall the definition of the random cluster measures (or FK measures) which are a generalization of the Bernoulli bond percolation measures with correlated bond distribution. To any subset Λ of $\mathbb{Z}^d$ and π included in $\partial\Lambda$, we associate a set of edges

$$[\Lambda]_e^\pi = \big\{\{x,y\} \mid x \sim y,\ x \in \Lambda,\ y \in \Lambda \cup \pi\big\},$$

and the space of configurations in Λ is $\Omega_\Lambda^\pi = \{0,1\}^{[\Lambda]_e^\pi}$. The first step is to introduce a measure on Ω_Λ^π. A vertex x of Λ is called π-wired if it is connected by an open path to π. We call π-clusters the clusters defined with respect to the boundary condition π : a π-cluster is a connected set of open edges in Ω_Λ^π and we identify to be the same cluster all the clusters which are π-wired, i.e. connected to π. For a given p in $[0,1]$, we define the FK measure on Ω_Λ^π with boundary conditions π by

$$\Phi_\Lambda^{\pi,p}(\omega) = \frac{1}{\mathbf{Z}_\Lambda^{\pi,p}} \left(\prod_{b \in [\Lambda]_e^\pi} (1-p)^{1-\omega_b} p^{\omega_b} \right) 2^{c^\pi(\omega)},$$

where $Z_\Lambda^{\pi,p}$ is a normalization factor and $c^\pi(\omega)$ is the number of clusters which are not π-wired. If $\pi = \partial\Lambda$ then the boundary conditions are said to be wired and the corresponding FK measure on $\Omega_\Lambda^{\mathrm{w}}$ is denoted by $\Phi_\Lambda^{\mathrm{w,p}}$. Finally, the periodic measure on the torus $\mathbb{T}_N$ is denoted by $\Phi_N^{\mathrm{per,p}}$ and the phase space by Ω_N^{per}.

In order to recover the Gibbs measure μ_Λ, we fix the percolation parameter $p_\beta = 1 - \exp(-2\beta)$ and generate the edges configuration ω in Ω_N^{per} according to the measure $\Phi_N^{\mathrm{per,p_\beta}}$. Given ω, we equip randomly each ω-cluster with a color ± 1 with probability $\frac{1}{2}$ independently from the others. This amounts to introducing the measure P_N^ω on $\{-1,1\}^{\mathbb{T}_N}$ such that the spin σ_i has the color of the cluster attached to i. The Gibbs measure μ_N can be viewed as the first marginal of the coupled measure $\mathbb{P}_N(\sigma,\omega) = P_N^\omega(\sigma)\Phi_N^{\mathrm{per,p_\beta}}(\omega)$ on the space $\{-1,1\}^{\mathbb{T}_N} \otimes \Omega_N^{\mathrm{per}}$. In the case of π-wired boundary conditions, the spins attached to the π-wired cluster are equal to 1.

As a consequence of this representation, one has for any increasing sequence of sets Λ_N

$$m^* = \lim_{N\to\infty} \mu_{\Lambda_N}^+(\sigma_0) = \lim_{N\to\infty} \Phi_{\Lambda_N}^{\mathrm{w,p_\beta}}(\{0 \leftrightarrow \partial\Lambda_N\}) = \Theta_{p_\beta}.$$

In the following, we use m^* or Θ_{p_β} depending on the context. Furthermore, we suppose that

$$\lim_{N\to\infty} \Phi_{\Lambda_N}^{\mathrm{f,p_\beta}}(\{0 \leftrightarrow \partial\Lambda_N\}) = \lim_{N\to\infty} \Phi_{\Lambda_N}^{\mathrm{w,p_\beta}}(\{0 \leftrightarrow \partial\Lambda_N\}) = \Theta_{p_\beta}. \tag{2.3.3}$$

This property is satisfied for all β outside a subset of $\mathbb{R}$ which is at most countable (see Lebowitz [L] and Pfister [Pf1]).

On the scale $M = 2^k$, we define, in the same way as for Bernoulli bond percolation, the variables $\tilde{u}_k^\zeta(\omega,x)$ which are piecewise constant on each box $\widehat{\mathbb{B}}_{n-k}(x)$ with x in $\widehat{\mathbb{T}}_{n-k}^d$. The mesoscopic phase labels depend on the averaged magnetization in regular blocks. Define the label of $\widehat{\mathbb{B}}_{n-k}(x)$ by

$$u_k^\zeta(\sigma,\omega,x) \triangleq \begin{cases} \mathrm{sign}(C^*) & \text{if} \quad \tilde{u}_k^\zeta(\omega,x)=1 \ \text{ and } \ |\mathcal{M}_k(\sigma,x)-\mathrm{sign}(C^*)\, m^*|<2\zeta, \\ 0 & \text{otherwise,} \end{cases}$$

where C^* is the crossing cluster in $\mathbb{B}_M(2^n x)$.

In a regular box $\widehat{\mathbb{B}}_{n-k}(x)$ (i.e. $\tilde{u}_k^\zeta(x)=1$), the averaged magnetization is controlled by the random coloring of the small clusters included in $\mathbb{B}_M(2^n x)$. So that the averaged magnetization in a regular box is independent of the configurations in the neighboring boxes. In the case of Ising model, the additional parameter $\ell=\ell(k)$ is tuned in order to control the fluctuations of the magnetization over the small clusters. As a consequence of this, assumptions A, B and C1-C3 are satisfied for p_β above a certain non-trivial slab percolation threshold $p_{\tilde{\beta}_c}$, which is conjectured to coincide with p_{β_c} (see [Pi1] for details), and Theorem 2.2.1 holds.

Remark . Using the notations of this Subsection, the set $\mathfrak{B}_p$ introduced in Subsection 2.1.1 could be defined as

$$\mathfrak{B}_p=\{\beta\ :\ \beta>\tilde{\beta}_c \text{ and (2.3.3) holds}\}.$$

2.4. Surface tension

We are going to derive Propositions 2.1.2 and 2.1.3 for Ising model with nearest neighbor interaction. As explained before, the philosophy of the proof is to start from the macroscopic level and to localize successively on finer scales with the help of a coarse graining. The approach itself is quite general. Nevertheless the coarse graining is model dependent, therefore we will need first to state an alternative representation of the surface tension in terms of the FK representation in order to use the estimates which will be obtained from Pisztora's coarse graining. The idea of such definitions has been introduced in [Ce1].

2.4.1. **FK representation.** We fix $\vec{n}$ a vector in $\mathbb{S}^{d-1}$ and study $\tau_\beta(\vec{n})$. Following notation of Subsection 1.2.4, we consider, for any ε positive, the parallelepiped $\widehat{\Lambda}(N,\varepsilon N)$ of $\mathbb{R}^d$ oriented according to $\vec{n}$. Namely, the basis of $\widehat{\Lambda}(N,\varepsilon N)$ with side lengths equal to N is orthogonal to $\vec{n}$ and the other sides have lengths equal to εN. For simplicity its microscopic counterpart $\widehat{\Lambda}(N,\varepsilon N)\cap\mathbb{Z}^d$ will be denoted by $\Lambda_N(\varepsilon)$.

By using the correspondence between the Ising model and the FK representation, one can rewrite τ_β in terms of the bond model. Let $\{\partial^+\Lambda_N(\varepsilon)\not\leftrightarrow\partial^-\Lambda_N(\varepsilon)\}$ be the event such that there is no open path inside $\Lambda_N(\varepsilon)$ joining $\partial^+\Lambda_N(\varepsilon)$ to $\partial^-\Lambda_N(\varepsilon)$. Then,

$$\tau_\beta(\vec{n})=\lim_{N\to\infty}-\frac{1}{N^{d-1}}\log\Phi^{\mathrm{w},\mathrm{p}_\beta}_{\Lambda_N(\varepsilon)}\big(\{\partial^+\Lambda_N(\varepsilon)\not\leftrightarrow\partial^-\Lambda_N(\varepsilon)\}\big). \tag{2.4.1}$$

Notice that the event $\{\partial^+\Lambda_N(\varepsilon)\not\leftrightarrow\partial^-\Lambda_N(\varepsilon)\}$ takes only into account the paths inside $\Lambda_N(\varepsilon)$ and not the identification produced by wired boundary conditions. The relation above will be useful only in the proof of Proposition 2.1.2.

We are now going to state an approximate expression of the surface tension which is weakly dependent on the boundary conditions. It will be used in the derivation of

Proposition 2.1.3. Let $\Lambda'_N(\varepsilon)$ be the the parallelepiped

$$\Lambda'_N(\varepsilon) = \Big\{ i \in \Lambda_N(\varepsilon) \quad | \quad \vec{i} \cdot \vec{n} \in [-\frac{\varepsilon}{4}N, \frac{\varepsilon}{4}N] \Big\}, \tag{2.4.2}$$

and denote by $\partial^{\text{top}}\Lambda'_N(\varepsilon)$ (resp $\partial^{\text{bot}}\Lambda'_N(\varepsilon)$) the face of $\partial^+\Lambda'_N(\varepsilon)$ (resp $\partial^-\Lambda'_N(\varepsilon)$) orthogonal to $\vec{n}$. Let $\{\partial^{\text{top}}\Lambda'_N(\varepsilon) \nleftrightarrow \partial^{\text{bot}}\Lambda'_N(\varepsilon)\}$ be the event such that there is no open path inside $\Lambda'_N(\varepsilon)$ connecting $\partial^{\text{top}}\Lambda'_N(\varepsilon)$ to $\partial^{\text{bot}}\Lambda'_N(\varepsilon)$. One has

Lemma 2.4.1. [[B1] $\beta \gg 1$, [CePi] $\beta \in \mathfrak{B}_p$] *For any $\beta \in \mathfrak{B}_p$*

$$\tau_\beta(\vec{n}) = -\frac{1}{N^{d-1}} \log \Phi^{\pi,\mathrm{p}_\beta}_{\Lambda_N(\varepsilon)} \Big(\{\partial^{\text{top}}\Lambda'_N(\varepsilon) \nleftrightarrow \partial^{\text{bot}}\Lambda'_N(\varepsilon)\} \Big) + c_{\varepsilon,N}(\pi), \tag{2.4.3}$$

where the function $c_{\varepsilon,N}$ goes to 0 as N tends to infinity and ε goes to 0, uniformly over the boundary conditions π and $\vec{n} \in \mathbb{S}^{d-1}$.

As it will be explained in Part 4 on the wetting phenomenon, the system is in fact extremely sensitive to boundary conditions. Nevertheless in the above Lemma, the interface is constrained to be in $\Lambda'_N(\varepsilon)$, so that it does not feel the influence of the boundary : the boundary conditions are screened because the system relaxes to equilibrium in the region $\Lambda_N(\varepsilon) \setminus \Lambda'_N(\varepsilon)$.

Let us first examine the influence of the boundary conditions π on the faces of $\Lambda_N(\varepsilon)$ orthogonal to $\vec{n}$. As $\{\partial^{\text{top}}\Lambda'_N(\varepsilon) \nleftrightarrow \partial^{\text{bot}}\Lambda'_N(\varepsilon)\}$ is a decreasing event, FKG inequality imply that it is enough to check that

$$\tau_\beta(\vec{n}) = \lim_{N\to\infty} -\frac{1}{N^{d-1}} \log \Phi^{\mathrm{f,w,p}_\beta}_{\Lambda_N(\varepsilon)} \big(\{\partial^+\Lambda'_N(\varepsilon) \nleftrightarrow \partial^-\Lambda'_N(\varepsilon)\} \big), \tag{2.4.4}$$

where $\Phi^{\mathrm{f,w,p}_\beta}_{\Lambda_N(\varepsilon)}$ is the FK measure with free boundary conditions on the faces orthogonal to $\vec{n}$ and wired on the others. This can be proved by means of a Peierls argument for β large enough [B1] or by an analysis of the relaxation of the clusters density for β in $\mathfrak{B}_p$ [CePi].

As already noticed in [Ce1] in the context of percolation, the influence of the boundary conditions on the sides of $\Lambda_N(\varepsilon)$ parallel to $\vec{n}$ is negligible as ε goes to 0. This explains that the factor $c_{\varepsilon,N}(\cdot)$ vanishes uniformly over the boundary conditions.

2.4.2. **Extended representation.** We would like to stress that the previous treatment of the surface tension is not satisfactory and a more coherent approach would be to consider a more general definition independent of the model in terms only of mesoscopic phase labels. In fact, a definition of surface tension valid in an abstract setting would be difficult to use because the surgical procedure of the minimal section argument requires a precise knowledge of how the microscopic system is related to the mesoscopic phase labels.

2.5. Lower bound : Proposition 2.1.2

The proof is divided into 3 steps. We first start by approximating the surface $\partial^*\mathcal{K}_m$ by a regular surface $\partial\widehat{K}$ and imposing the condition that a mesoscopic interface exists close to $\partial\widehat{K}$. Then, using the definition of surface tension (2.4.1), we derive Proposition 2.1.2.

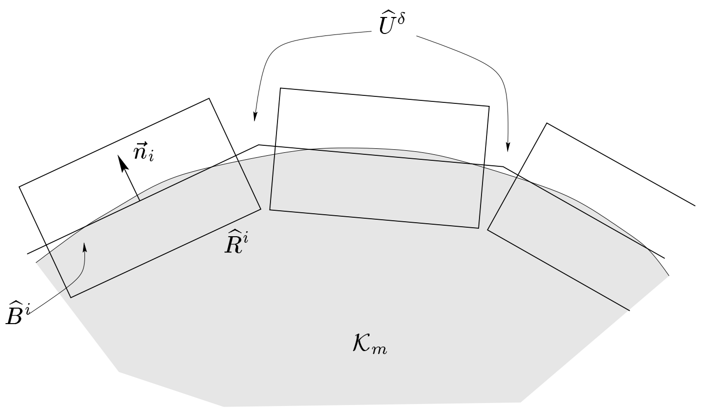


FIGURE 6. Polyhedral approximation.

2.5.1. **Step 1 : Approximation procedure.** A polyhedral set has a boundary included in the union of a finite number of hyper-planes. The surface $\partial^* \mathcal{K}_m$ can be approximated as follows (see Fig. 6)

Theorem 2.5.1. *For any δ positive, there exists a polyhedral set $\widehat{K}$ such that*

$$\mathbb{I}_{\widehat{K}} \in \mathcal{V}(\mathbb{I}_{\mathcal{K}_m}, \delta) \qquad \text{and} \qquad \big|\mathcal{W}_\beta(\widehat{K}) - \mathcal{W}_\beta(\mathcal{K}_m)\big| \leqslant \delta.$$

For any h small enough there are ℓ disjoint parallelepipeds $\widehat{R}^1, \dots, \widehat{R}^\ell$ with basis $\widehat{B}^1, \dots, \widehat{B}^\ell$ included in $\partial\widehat{K}$ of side length h and height δh. Furthermore, the sets $\widehat{B}^1, \dots, \widehat{B}^\ell$ cover $\partial\widehat{K}$ up to a set of measure less than δ denoted by $\widehat{U}^\delta = \partial\widehat{K} \setminus \bigcup_{i=1}^\ell \widehat{B}^i$ and they satisfy

$$\Big|\sum_{i=1}^{\ell} \int_{\widehat{B}^i} \tau_\beta(\vec{n}_i)\, d\mathcal{H}_x^{(d-1)} - \mathcal{W}_\beta(\mathcal{K}_m)\Big| \leqslant \delta,$$

where the normal to $\widehat{B}^i$ is denoted by $\vec{n}_i$.

The proof is a direct application of Reshtnyak's Theorem and can be found in the paper of Alberti, Bellettini [AlBe].

Using Theorem 2.5.1, we can reduce the proof of Proposition 2.1.2 to the computation of the probability of $\{\frac{\mathcal{M}_k}{m^*} \in \mathcal{V}(\mathbb{I}_{\widehat{K}}, \delta)\}$. According to (2.2.6) the estimates can be restated in terms of the mesoscopic phase labels. For any $\delta > 0$, there exists $\zeta = \zeta(\delta)$ and $k_0 = k_0(\delta)$ such that Proposition 2.1.2 will be implied by

$$\liminf_{N\to\infty} \frac{1}{N^{d-1}} \min_{k_0(\delta) \leqslant k \leqslant \nu n} \log \mathbb{P}_N\left(u_k^\zeta \in \mathcal{V}(\mathbb{I}_{\widehat{K}}, \delta)\right) \geqslant -\mathcal{W}_\beta(\widehat{K}) - o(\delta). \tag{2.5.1}$$

2.5.2. **Step 2 : Localization of the interface.** The images of $\widehat{K}$, $\widehat{R}^i$ and $\widehat{U}^\delta$ in $\mathbb{T}_N$ will be denoted by K_N, R_N^i and U_N^δ. In order to enforce a mesoscopic interface which crosses each R_N^i, we define the event

$$\mathcal{A} = \bigcap_{i=1}^{\ell} \{\partial^+ R_N^i \not\leftrightarrow \partial^- R_N^i\}\ .$$

We consider also $\mathcal{B}$ the set of configurations such that the bonds at distance less than 10 of U_N^δ are closed. Notice that these events depend only on bonds variables. One has

$$\mathbb{P}_N\left(u_k^\zeta \in \mathcal{V}(\mathbb{I}_{\widehat{K}}, \delta)\right) \geqslant \mathbb{P}_N\left(\left\{u_k^\zeta \in \mathcal{V}(\mathbb{I}_{\widehat{K}}, \delta)\right\} \cap \mathcal{A} \cap \mathcal{B}\right). \tag{2.5.2}$$

The interface imposed by the event $\mathcal{A}\cap\mathcal{B}$ decouples K_N from its complement, therefore the system is in equilibrium in K_N and K_N^c : a proof similar to the one of Theorem 2.2.1 implies that one can choose $\zeta'=\zeta'(\delta)$ and $k_0'=k_0'(\delta)$ such that

$$\lim_{N\to\infty}\ \max_{k_0'(\delta)\leqslant k\leqslant \nu n}\ \mathbb{P}_N\left(\left(\int_\Lambda |u_k^{\zeta'}(x)-1|\,dx\geqslant\frac{\delta}{2}\ \text{or}\ \int_\Lambda |u_k^{\zeta'}(x)+1|\,dx\geqslant\frac{\delta}{2}\ \Big|\ \mathcal{A}\cap\mathcal{B}\right)=0\,,$$

where Λ stands for $\widehat{K}$ or $\widehat{K}^c$. So that (2.5.2) can be rewritten for N large enough as

$$\min_{k_0(\delta)\leqslant k\leqslant \nu n}\ \mathbb{P}_N\left(u_k^\zeta\in\mathcal{V}(\mathbb{I}_{\widehat{K}},\delta)\right)\ \geqslant\frac{1}{8}\,\Phi_N^{\mathrm{per},\mathrm{p}_\beta}\left(\mathcal{A}\cap\mathcal{B}\right). \tag{2.5.3}$$

2.5.3. **Step 3 : Surface tension.** Combining the definition of surface tension (2.4.1), inequality (2.5.3) and Theorem 2.5.1, we get

$$\liminf_{N\to\infty}\frac{1}{N^{d-1}}\ \min_{k_0(\delta)\leqslant k\leqslant \nu n}\ \log\mathbb{P}_N\left(u_k^\zeta\in\mathcal{V}(\mathbb{I}_{\widehat{K}},\delta)\right)\ \geqslant\ -\sum_{i=1}^{\ell}\int_{\widehat{B}^i}\tau_\beta(\vec{n}_i)\,d\mathcal{H}_x^{d-1}-o(\delta).$$

We have also used the fact that the event $\mathcal{B}$ is supported by at most $c(d,\delta)N^{d-1}$ edges where $c(d,\delta)$ vanishes as δ goes to 0. Therefore the probability of $\mathcal{B}$ is negligible with respect to a surface order.

2.6. Upper bound : Proposition 2.1.3

The proof is divided into 3 steps. First we decompose $\partial^* v$ in order to reduce the proof to local computations in small regions. Then in each region we localize the interface on the mesoscopic level via the minimal section argument. Finally the last step is devoted to the computation of the surface tension factor.

2.6.1. **Step 1 : Approximation procedure.** We approximate $\partial^* v$ with a finite number of parallelepipeds (see Fig. 7).

Theorem 2.6.1. *For any δ positive, there exists h positive such that there are ℓ disjoint parallelepipeds $\widehat{R}^1,\dots,\widehat{R}^\ell$ included in $\widehat{\mathbb{T}}^d$ with basis $\widehat{B}^1,\dots,\widehat{B}^\ell$ of size h and height δh. The basis $\widehat{B}^i$ divides $\widehat{R}^i$ in 2 parallelepipeds $\widehat{R}^{i,+}$ and $\widehat{R}^{i,-}$ and we denote by $\vec{n}_i$ the normal to $\widehat{B}^i$. Furthermore, the parallelepipeds satisfy the following properties*

$$\int_{\widehat{R}^i}|\mathcal{X}_{\widehat{R}^i}(x)-v(x)|\,dx\leqslant\delta\,\mathrm{vol}(\widehat{R}^i)\quad\text{and}\quad\Big|\sum_{i=1}^{\ell}\int_{\widehat{B}^i}\tau_\beta(\vec{n}_i)\,d\mathcal{H}_x^{(d-1)}-\mathcal{W}_\beta(v)\Big|\leqslant\delta,$$

where $\mathcal{X}_{\widehat{R}^i}=1_{\widehat{R}^{i,+}}-1_{\widehat{R}^{i,-}}$ and the volume of $\widehat{R}^i$ is $\mathrm{vol}(\widehat{R}^i)=\delta h^d$.

This Theorem is a rather standard assertion of the geometric measure Theory. A variation of it has been formulated and applied in the context of the $\mathbb{L}_1$-theory of phase segregation in [ABCP] along with a sketch of the proof, which, however, contained a gap (see [B1] for a detailed proof along the lines of [ABCP]). A very clean alternative derivation of a similar result has been given by Cerf [Ce1] using the Vitali covering Theorem.

Theorem 2.6.1 enables us to decompose the boundary into regular sets (see Fig. 7) so that it will be enough to consider events of the type

$$\left\{\frac{\mathcal{M}_k}{m^*}\in\bigcap_{i=1}^{\ell}\mathcal{V}(\widehat{R}^i,\delta\mathrm{vol}(\widehat{R}^i))\right\}\,,$$

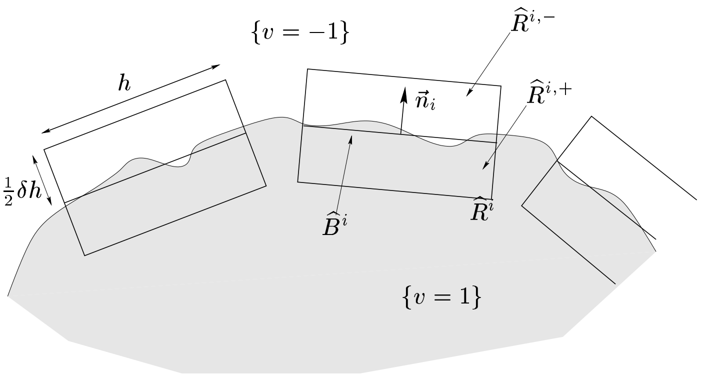


FIGURE 7. Approximation by parallelepipeds.

where $\mathcal{V}(\widehat{R}^i,\varepsilon)$ is the ε-neighborhood of $\mathcal{X}_{\widehat{R}^i}$

$$\mathcal{V}(\widehat{R}^i,\varepsilon) = \left\{ v' \in \mathbb{L}^1\left(\widehat{\mathbb{T}}^d\right) \mid \quad \int_{\widehat{R}^i} |v'(x) - \mathcal{X}_{\widehat{R}^i}(x)|\, dx \leqslant \varepsilon \right\}.$$

Using (2.2.6), we see that to derive Proposition 2.1.3, it is equivalent to prove the following statement for any δ positive and $k_0 = k_0(\delta)$, $\zeta = \zeta(\delta)$

$$\limsup_{N\to\infty} \frac{1}{N^{d-1}} \max_{k_0(\delta)\,\leqslant\, k\,\leqslant\, \nu n} \log \mathbb{P}_N\big(u_k^\zeta \in \bigcap_{i=1}^{\ell} \mathcal{V}(\widehat{R}^i, \delta \mathrm{vol}(\widehat{R}^i))\big) \leqslant -\mathcal{W}_\beta(v) + C(\beta,v)\delta.$$

2.6.2. **Step 2 : Minimal section argument.** The microscopic domain associated to $\widehat{R}^i$ is $R_N^i = N\widehat{R}^i \cap \mathbb{T}_N$. We also set $R_N^{i,+} = N\widehat{R}^{i,+} \cap \mathbb{T}_N$ and $R_N^{i,-} = R_N^i \setminus R_N^{i,+}$. At the scale $M = 2^k$, we associate to any configuration (σ,ω) the set of *bad* boxes which are the boxes $\mathbb{B}_M$ intersecting R_N^i labeled by 0 and the ones intersecting $R_N^{i,+}$ (resp $R_N^{i,-}$) labeled by -1 (resp 1). For any integer j, we set $\widehat{B}^{i,j} = \widehat{B}^i + j\, c(d) 2^{n-k}\, \vec{n}_i$ and define

$$B_N^{i,j} = \big\{ j' \in R_N^i \mid \exists x \in \widehat{B}^{i,j}, \qquad \|j' - Nx\|_1 \leqslant 10 \big\}.$$

Let $\mathcal{B}_i^j$ be the smallest connected set of boxes $\mathbb{B}_M$ intersecting $B_N^{i,j}$. By construction the $\mathcal{B}_i^j$ are disjoint surfaces of boxes. For j positive, let $n_i^+(j)$ be the number of *bad* boxes in $\mathcal{B}_i^j$ and define

$$n_i^+ = \min\big\{ n_i^+(j) : \qquad 0 < j < \frac{\delta h}{2c(d)} 2^{n-k} \big\}.$$

Call j^+ the smallest location where the minimum is achieved and define the minimal section in $R_N^{i,+}$as $\mathcal{B}_i^{j^+}$. For j negative, we denote by $\mathcal{B}_i^{j^-}$ the minimal section in $R_N^{i,-}$ and n_i^- the number of *bad* boxes in $\mathcal{B}_i^{j^-}$ (see Fig. 8).

For any configuration (σ,ω) such that $u_k^\zeta(\sigma,\omega)$ belongs to $\bigcap_{i=1}^{\ell} \mathcal{V}(\widehat{R}^i, \delta \mathrm{vol}(\widehat{R}^i))$, one can bound the number of *bad* boxes in the minimal sections by

$$\sum_{i=1}^{\ell} n_i^+ + n_i^- \leqslant \delta C_1(v) 2^{(d-1)(n-k)}\,. \tag{2.6.1}$$

Such an estimate implies that a mesoscopic interface is mainly located between the 2 minimal sections and that only some mesoscopic fingers attached to the interface may

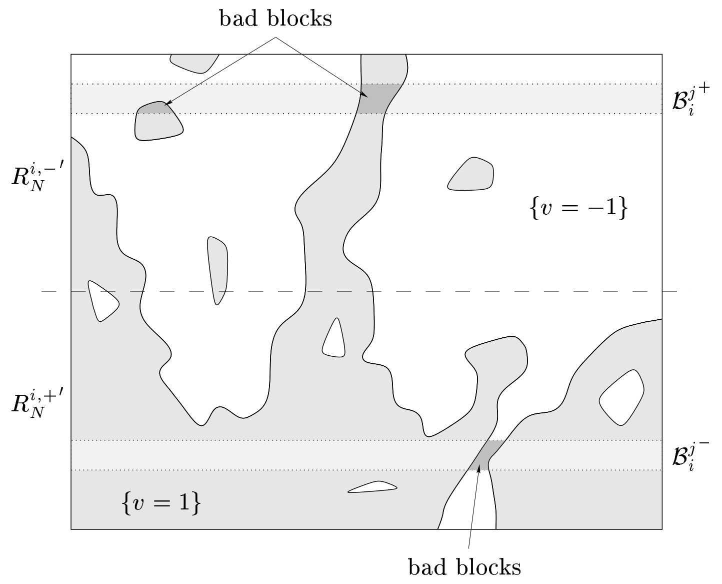


FIGURE 8. Minimal sections.

percolate. As these fingers will cross the minimal sections through *bad* boxes, the strategy is therefore to modify the configuration ω on the *bad* boxes so that no fingers can percolate in the new configuration. More precisely, we introduce the set

$$\mathcal{A} = \Big\{\omega \in \Omega_N^{\mathrm{per}} \;\Big|\; \exists \sigma \text{ such that } u_k^\zeta(\sigma,\omega) \in \bigcap_{i=1}^{\ell} \mathcal{V}(\widehat{R}^i, \delta \mathrm{vol}(\widehat{R}^i))\Big\},$$

and for any ω in $\mathcal{A}$ define $\bar{\omega}$ the configuration with closed edges on the boundary of the *bad* blocks in the minimal sections and equal to ω otherwise. Inequality (2.6.1) implies that ω and $\bar{\omega}$ differ only on at most $\delta C_2(v) N^{d-1}$ edges, so that we can control precisely the cost of the surgical procedure which consists in isolating the *bad* blocks in the minimal sections by closing the edges around them.

$$\begin{aligned} \mathbb{P}_N\left(u_k^\zeta(\sigma,\omega) \in \bigcap_{i=1}^{\ell} \mathcal{V}(\widehat{R}^i, \delta \mathrm{vol}(\widehat{R}^i))\right) \quad &\leqslant \quad \Phi_N^{\mathrm{per},\mathrm{p}_\beta}(\mathcal{A}) \\ &\leqslant \quad \exp\left(\delta\, C_3(v,\beta) N^{d-1}\right) \Phi_N^{\mathrm{per},\mathrm{p}_\beta}(\bar{\mathcal{A}}), \end{aligned} \tag{2.6.2}$$

where $\bar{\mathcal{A}} = \{\bar{\omega} \mid \omega \in \mathcal{A}\}$.

2.6.3. **Step 3 : Surface tension estimates.** Let $\widehat{R}^{i\,\prime}$ be the parallelepiped included in $\widehat{R}^i$ with basis $\widehat{B}^i$ and height $\frac{\delta}{2}h$. Its microscopic counterpart is $R_N^{i\,\prime}$. We are going to check now that $\bar{\mathcal{A}}$ is included in $\bigcap_{i=1}^{\ell}\{\partial^{\mathrm{top}} R_N^{i\,\prime} \not\leftrightarrow \partial^{\mathrm{bot}} R_N^{i\,\prime}\}$. This amounts to say that not only the minimal section argument enables us to find a mesoscopic interface in R_N^i but that in fact this interface exists on the microscopic level. To see this, choose any configuration ω in $\mathcal{A}$ which contains an open path $\mathbf{C}$ joining $\partial^{\mathrm{top}} R_N^{i\,\prime}$ to $\partial^{\mathrm{bot}} R_N^{i\,\prime}$ and suppose that $\mathbf{C}$ crosses the minimal sections without intersecting a *bad* box. Then $\mathbf{C}$ intersects 2 regular boxes $\mathbb{B}_M(2^n x^+)$ and $\mathbb{B}_M(2^n x^-)$ in $\mathcal{B}_i^{j^+}$ and $\mathcal{B}_i^{j^-}$. According to the definition of the coarse graining, this would imply that the crossing clusters of $\mathbb{B}_M(2^n x^+)$ and $\mathbb{B}_M(2^n x^-)$ are connected to $\mathbf{C}$, so that $\tilde{u}_k^\zeta(x^+) = \tilde{u}_k^\zeta(x^-)$. Therefore one of these boxes has to be a *bad* box.

From (2.6.2), we get

$$\mathbb{P}_N\left(u_k^\zeta \in \bigcap_{i=1}^{\ell} \mathcal{V}(\widehat{R}^i, \delta \mathrm{vol}(\widehat{R}^i))\right) \leqslant \exp\left(\delta\, C_3(v,\beta) N^{d-1}\right) \Phi_N^{\mathrm{per},\mathrm{p}_\beta}\Big(\bigcap_{i=1}^{\ell}\{\partial^{\mathrm{top}} {R_N^i}' \not\leftrightarrow \partial^{\mathrm{bot}} {R_N^i}'\}\Big).$$

Conditioning outside each domain R_N^i and using (2.4.3), we derive

$$\limsup_{N\to\infty} \frac{1}{N^{d-1}} \max_{k_0(\delta)\,\leqslant\, k\,\leqslant\,\nu n} \log \mathbb{P}_N\left(u_k^\zeta \in \bigcap_{i=1}^{\ell} \mathcal{V}(\widehat{R}^i, \delta \mathrm{vol}(\widehat{R}^i))\right) \leqslant - \sum_{i=1}^{\ell} \int_{\widehat{B}_i} \tau_\beta(\vec{n}_i)\, d\mathcal{H}_x + C_4(\beta, v)\delta.$$

This concludes the Proposition.

2.7. Open problems

We would like mention some open questions related to the $\mathbb{L}_1$-theory

1. Extention of the $\mathbb{L}_1$-theory to general finite range models and to the context of Pirogov-Sinai Theory.
2. Proof of the Wulff construction for continuum models in an $\mathbb{L}_1$-setting.
3. Upgrade of the concentration properties to the Hausdorff distance, based on more delicate versions of the minimal section argument; some results of this sort should appear in [BIV].
4. A more challenging problem would be to provide an accurate description of phase segregation à la DKS. In particular one should understand how to control phase boundaries and prove local limit results with boundary conditions which are only statistically pure.

Part 3. Dobrushin-Kotecký-Shlosman (DKS) theory in 2D

In this part we review and explain the results on phase separation in the two-dimensional nearest neighbor Ising model as enforced by the canonical constraint on the magnetization [DKS], [ISc]. The theory is built upon sharp local estimates over finite volume vessels Λ_N and on the probabilistic analysis of the random microscopic phase separation line. We focus here on the "free" spatial geometry of the phase segregation, that is disregarding the boundary effects. These effects could enter the picture in two different ways: in terms of the boundary conditions on $\partial\Lambda_N$ and in terms of the geometry of Λ_N. In the former case the minority phase could be absorbed by part of the boundary $\partial\Lambda_N$. This and related phenomena are discussed in Part 4. In the second case the finite vessel Λ_N might not be able to accommodate the corresponding optimal crystal shape. Such a geometric constraint is, from the point of view of the microscopic theory, merely a technical nuisance, though, on the macroscopic level, it might lead to formidable variational problems. We go around this domain geometry issue by choosing Λ_N to be of the Wulff shape itself

$$\Lambda_N \;=\; N\mathcal{K}_1 \cap \mathbb{Z}^2,$$

where $\mathcal{K}_1$ is the unit area Wulff shape. Thus, Λ_N accommodates any optimal shape of area smaller than N^2.

The corresponding finite volume canonical Gibbs measure is then defined by

$$\mu^{\beta}_{\Lambda_N,-}\left(\,\cdot\,\big|\, M_N(\sigma) = -N^2 m^* + a_N\right), \tag{3.0.3}$$

where $M_N \triangleq \sum_{i\in\Lambda_N}\sigma_i$ is the total spin, $m^* = m^*(\beta)$ is the spontaneous magnetization, and a_N points inside the phase transition region, $a_N \in (0, 2N^2 m^*)$. In the sequel we shall use the shortcut $\mu^{\beta}_{N,-}$ for the finite volume measure $\mu^{\beta}_{\Lambda_N,-}$.

Notation. The values of positive constants $c_1, c_2, \ldots$ are updated with each subsection.

3.1. Main Result

DKS theory gives a comprehensive solution to the following problem of phase separation:

Problem 1. For $\beta > \beta_c$ and $a_N \in (0, 2N^2 m^*)$ characterize typical spin configurations σ under the canonical measure (3.0.3) .

An ostensibly simpler problem is

Problem 2. For $\beta > \beta_c$ and $a_N \in (0, 2N^2 m^*)$ find sharp local asymptotics of

$$\mu^{\beta}_{N,-}\left(M_N \;=\; -m^* N^2 + a_N\right).$$

In fact both problems are equivalent. In particular, the phenomenon behind the shift of the magnetization is inside the phase transition region not a bulk one (and hence is not in the realm of the usual theory of large deviations), and the crucial role is played by the spatial geometry of symmetry breaking.

3.1.1. **Heuristics.** Under the finite volume pure state $\mu^{\beta}_{N,-}$ the typical maximal size of $\pm$ contours is of order $\log N$. One could then visualize a typical microscopic configuration σ on Λ_N in terms of an archipelago of small (that is of the maximal size $\sim \log N$) "+" islands which could contain still smaller "−" lakes etc. This archipelago spreads out uniformly over Λ_N, and the density of the plus "soil", which spells out in terms of the magnetization $M_N(\sigma)$ as $(|\Lambda_N| + M_N(\sigma))/2|\Lambda_N|$, is close to its equilibrium value

$$\frac{|\Lambda_N| + \langle M_N \rangle^{\beta}_{N,-}}{2|\Lambda_N|} \sim \frac{1-m^*}{2}.$$

Thus, one could think of two different competing patterns behind the a_N-shifts, $a_N \geqslant 0$, of the magnetization M_N from its equilibrium value $\langle M_N \rangle^{\beta}_{N,-} \sim -m^*|\Lambda_N|$:
1) The density of the archipelago increases in a spatially homogeneous fashion without, however, altering the typical sizes of the islands.
2) Spatial symmetry is broken, and an abnormally huge island of the "+" phase of excess area $\sim a_N/2m^*$ appears.
Heuristically, the first scenario corresponds to Gaussian fluctuations, and its price, in terms of probability, should be of order

$$\exp\left(-c_1(\beta)a_N^2/N^2\right).$$

Phase segregation manifests itself in the second scenario, and the probabilistic price for creating such a huge island is proportional to the length of its boundary

$$\exp\left(-c_2(\beta)\sqrt{a_N}\right).$$

A comparison between the two expressions above suggests that the first scenario should be preferred whenever $a_N \ll N^{4/3}$, whereas large shifts $a_N \gg N^{4/3}$ should result in the phase segregation picture described in the second scenario. This indeed happens to be the case, and we refer to [DS] and [ISc] for a complete rigorous treatment[7].

For the sake of the exposition, we shall stick here to the possibly most interesting case of $a_N \sim N^2$, which corresponds also to the macroscopic type of scaling discussed in Part 2. The DKS theory gives then the following sharp characterization of the phase segregation in the canonical ensemble: under $\mu^{\beta}_{N,-}\left(\,\cdot\,\middle| M_N = -m^*N^2 + a_N\right)$ a typical spin configuration σ contains exactly one abnormally large contour γ which decouples between the "+" phase (inside γ) and the "−" phase (outside γ). In particular, the average magnetization inside (respectively outside) γ is close to m^* (respectively $-m^*$), and the area encircled by γ can be thus recovered from the canonical constraint,

$$m^*\left|\mathrm{int}\,(\gamma)\right| - m^*\left(N^2 - \left|\mathrm{int}\,(\gamma)\right|\right) \approx -m^*N^2 + a_N \implies \left|\mathrm{int}\,(\gamma)\right| \approx \frac{a_N}{2m^*}.$$

Under the scaling of Λ_N by $1/N$, that is into the normalized continuous shape $\mathcal{K} \subset \mathbb{R}^2$, the microscopic phase boundary γ sharply concentrates around a shift of the Wulff shape of the corresponding scaled area $a_N/2m^*N^2$ (Fig. 9).

3.1.2. **DKS theorem.** More precisely, for any $r \in \mathbb{R}_+$ let $\mathcal{K}_r$ to denote the Wulff shape of the area r. Also given a number $s \in \mathbb{R}_+$, let us say that a microscopic contour γ is s-large, if $\mathrm{diam}_\infty(\gamma) > s$.

Theorem 3.1.1 ([DKS][8],[ISc]). *Let the inverse temperature $\beta > \beta_c$ be fixed, and let the sequence $\{a_N\}$, $-m * N^2 + a_N \in \mathrm{Range}(M_N)$, be such that the limit*

$$a = \lim_{N\to\infty} \frac{a_N}{N^2} \in (0, 2m^*(\beta))$$

[7] The critical case of $a_N \sim N^{4/3}$ is still an open problem.

[8] In the original monograph [DKS] the corresponding results has been derived in the context of the Ising model with periodic boundary condition.

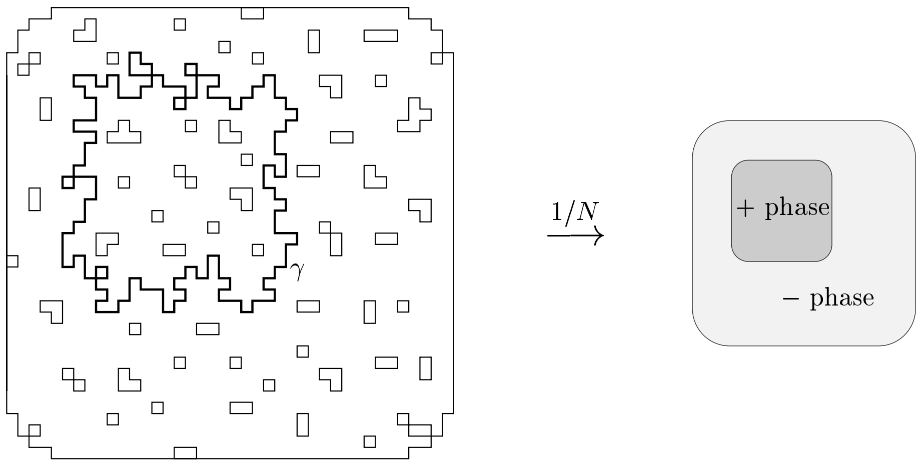


FIGURE 9. DKS picture under the $1/N$ scaling: On the left the microscopic Λ_N box with the unique $K \log N$-large contour γ. On the right the continuous box $\mathcal{K}_1$ with the scaled image of γ.

exists. Then,

$$\log \mu^{\beta}_{N,-}\big(M_N \;=\; -m^* N^2 \;+ a_N \big) \;=\; -\mathcal{W}_\beta\left(\partial \mathcal{K}_{\frac{a_N}{2m^*}}\right)\big(1 \;+\; O\big(N^{-1/2}\log N\big)\big).$$

Moreover, if $K = K(\beta)$ is large enough, with $\mu^{\beta}_{N,-}\big(\,\cdot\,|M_N = -N^2 m^* + a_N\big)$-probability converging to 1 as $N \to \infty$:

1. *There is exactly one $K(\beta)\log N$-large contour γ.*
2. *This γ satisfies*

$$\min_x \frac{1}{N} d_{\mathbb{H}}\big(\,\gamma, x + \partial\mathcal{K}_{\frac{a_N}{2m^*}}\big) \;\leqslant\; c_1(\beta) N^{-1/4}\sqrt{\log N} \tag{3.1.1}$$

and

$$\min_x \frac{1}{N^2}\mathrm{Area}\Big(\;\mathrm{int}\,(\gamma)\,\Delta\left(x + \mathcal{K}_{\frac{a_N}{2m^*}}\right)\;\Big) \;\leqslant\; c_2(\beta) N^{-3/4}\sqrt{\log N}. \tag{3.1.2}$$

3.1.3. **DKS theory.** The DKS theory views the production of the event $\{M_N - m^* N^2 + a_N\}$ in terms of a two-step procedure: On the first stage a length scale $s = s(N)$ is chosen, and **all** the microscopic s-large contours $(\gamma_1, ..., \gamma_n)$ are fixed. If the total area inside these $s(N)$-large contours is smaller than $a_N/2m^*$, then the total magnetization M_N still has to be steered towards the imposed value $M_N = -m^* N^2 + a_N$, but already under the constraint that all the $\pm$ contours different from $(\gamma_1, ..., \gamma_n)$ are $s(N)$-small. The probability $\mu^{\beta}_{N,-}\big(M_N = -m^* N^2 + a_N\big)$ reflects the price of the optimal strategy along these lines.

We record the two steps of the DKS theory as follows:
1) Study the statistics of $s(N)$-large contours under $\mu^{\beta}_{N,-}$.
2) Give local limit estimates on the magnetization in the $s(N)$-restricted phases.
The introduction of $s(N)$-cutoffs leads to the separation of the length scales which has a double impact on the problem: it sets up the stage for the renormalization analysis of microscopic phase boundaries, and it improves the control over the bulk magnetization inside the corresponding microscopic phase regions. Let us try to explain this in more details: As far as the statistics of the $s(N)$-large contours is considered, we are interested

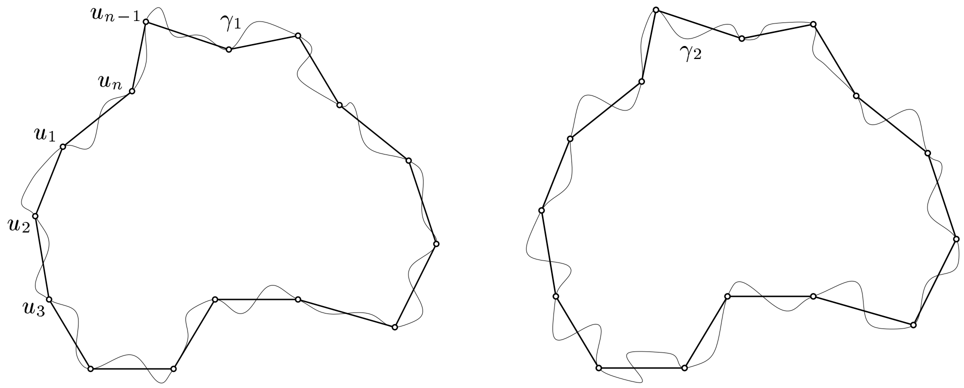


FIGURE 10. Two microscopic contours γ_1 and γ_2 are compatible with the same skeleton $S = (u_1, ...u_n)$.

in giving sharp estimates on the $\mu^{\beta}_{N,-}$-probability of the events of the type

$$\{ \ s(N)\text{-large contours of } \sigma \text{ encircle a certain prescribed area } \} .$$

The point is that the contribution of any particular microscopic contour to the probability of such an event is negligible. In other words, one also has to take into account the entropy (number) of all the contributing contours. The required entropy cancelation (and hence the production of the relevant limiting thermodynamic quantity - surface tension) is achieved by means of a certain coarse graining procedure, the so called skeleton calculus, which we describe in Section 3.4. Roughly, instead of studying the probabilities of individual microscopic contours one considers the packets of all contours passing through the vertices of a given "$s(N)$-skeleton" $S = (u_1, u_2, ..., u_n)$ and staying within a distance of the order $s(N)$ from the closed polygonal line $\mathrm{Pol}(S)$ (Fig. 10). The distance between successive vertices of S complies with the length scale $s(N)$, $\|u_{i+1} - u_i\|_\infty \sim s(N)$. Surface tension is produced on the level of skeletons. In fact, the probability of observing a $\pm$ contour compatible with a given skeleton S admits an asymptotic with $s(N) \nearrow \infty$ description

$$\mu^{\beta}_{N,-}(S) \ \asymp \ \exp\{-\mathcal{W}_\beta(\mathrm{Pol}(S))\} . \tag{3.1.3}$$

We quote the precise result in Section 3.4, which we devote to a general exposition of the skeleton calculus.

Since the vertices of S are $s(N)$-apart, and the surface tension τ_β is strictly positive for all $\beta > \beta_c$, the energy $\mathcal{W}_\beta(\mathrm{Pol}(S))$ controls the number $\#(S)$ of vertices of S as

$$\#(S) \ \leqslant c_3(\beta)\frac{\mathcal{W}_\beta(\mathrm{Pol}(S))}{s(N)}. \tag{3.1.4}$$

When combined with (3.1.3) this leads to the reduction of the combinatorial complexity of the problem: the number of different skeletons of a fixed energy $\widehat{\mathcal{W}}_N$ does not compete with the approximate probability $\exp\{-\widehat{\mathcal{W}}_N\}$ to observe any such skeletons. Thus, the study of $\{M_N = -m^*N^2 + a_N\}$ reduces, in terms of skeletons, to the maximal term estimation. It should be stressed, however, that unlike the coarse graining procedures of the $\mathbb{L}_1$ theory, the mesoscopic objects (skeletons) of the DKS theory closely follow the microscopic structure of phase boundaries.

The local limit estimates in the $s(N)$-restricted phases are, therefore, required uniformly over finite lattice domains whose boundaries are carved with $s(N)$-large contours compatible with not too costly skeletons. This imposes a natural restriction on the length of these boundaries, and we shall describe the appropriate family of domains in Section 3.2 along with the exposition of the corresponding uniform local limit results. Intuitively, long contours are responsible for long range dependencies between spins, and, therefore, the $s(N)$-cutoff constraint improves the mixing properties of the system and helps to extend the validity of classical (Gaussian) behavior of moderate deviations. In Section 3.3 we quote the corresponding relaxation and decay properties which lie in the heart of the local limit estimates. In Section 3.5 we give an outline of the proof of the DKS theorem.

Finally, the (long) list of open problems is briefly addressed in Section 3.6.

3.2. Estimates in the phases of small contours

As it has been mentioned, the estimates in the phase of small contours should be derived uniformly over a family of lattice domains whose boundaries are composed of not too costly $s(N)$-large contours.

Definition Basic family $\mathcal{D}_N$ of subsets $A \subseteq \Lambda_N$: We fix two numbers a (small) and R (big).

$$A \in \mathcal{D}_N \iff aN^2 \leqslant |A| \quad \text{and} \quad |\partial A| \leqslant RN\log N.$$

□

We fix a basic scale $s(N) = K\log N$ of large contours, where $K = K(\beta)$ is a sufficiently large number, so that $K\log N$-contours are highly improbable under the pure state $\mu^{\beta}_{N,-}$. Of course, exactly the same number K appears in the statement of Theorem 3.1.1. The upper bound on ∂A in the definition of the family $\mathcal{D}_N$ states that the configurations with total length of $K\log N$ large contour exceeding $RN\log N$ are ruled out. This conclusion is explained in more detail in Section 3.4 (see the remark following Lemma 3.4.1).

3.2.1. **Structure of local limit estimates.**

Let us turn now to the structure of local limit estimates in the $s(N)$-restricted phases. First of all, given any $A \subset \mathbb{Z}^2$, the s-restricted phase on A is defined via

$$\mu^{\beta,s}_{A,-}(\,\cdot\,) \triangleq \mu^{\beta}_{A,-}\left(\,\cdot\,\Big|\text{All} \pm \text{contours are } s\text{-small}\right).$$

We would like to study the probabilities of deviations $a_N \geqslant 0$ of the total magnetization M_A from the corresponding averaged value $\langle M_A\rangle^{\beta,s}_{A,-}$. Let us define the set of feasible values of such deviations as

$$\mathbf{M}^{+}_{A} = \left\{a_N \geqslant 0 :\ \langle M_A\rangle^{\beta,s}_{A,-} + a_N \in \text{Range}(M_A)\right\}.$$

Roughly, the cutoff s extends the validity of Gaussian moderate deviations for the following reason: The price of shifting the magnetization by a_N on the expense of $s(N)$-small contours is of the order $(a_N/s^2)s \sim a_N/s(N)$. This should be tested against the Gaussian moderate deviation exponent of the order a_N^2/N^2. Thus the Gaussian behavior should prevail once $a_N \ll N^2/s(N)$. Of course, the latter constraint on a_N becomes less stringent as $s(N)$ decreases. On the rigorous mathematical part the classical approach to estimating

$$\mu^{\beta,s}_{A,-}\left(M_A = \langle M_A\rangle^{\beta,s}_{A,-} + a_N\right),$$

amounts to first finding the value of magnetic field

$$g = g(A, s(N), a_N),$$

such that the expected magnetization under the g-tilted state is precisely what we want,

$$\langle M_A\rangle^{\beta,s}_{A,-,g} \;=\; \langle M_A\rangle^{\beta,s}_{A,-} \;+\; a_N, \tag{3.2.1}$$

and, then, to rewrite the $\mu^{\beta,s}_{A,-}$-probability in terms of the $\mu^{\beta,s}_{A,-,g}$ one:

$$\begin{aligned}
&\mu^{s}_{A,-}\Big(M_A = \langle M_A\rangle^{\beta,s}_{A,-} + a_N\Big)\\
&\quad= \exp\big\{ -(\langle M_A\rangle^{s}_{A,-} + a_N)g \;+\; \log\big\langle \mathrm{e}^{gM_A}\big\rangle^{\beta,s}_{A,-}\big\}\; \mu^{\beta,s}_{A,-,g}\big(\; M_A \;=\; \big\langle M_A\big\rangle^{\beta,s}_{A,-,g}\;\big)\\
&\quad= \exp\left\{ -\int_0^g\int_r^g \langle M_A; M_A\rangle^{\beta,s}_{A,-,h}\mathrm{d}h\mathrm{d}r\right\}\; \mu^{\beta,s}_{A,-,g}\Big(\; M_A \;=\; \big\langle M_A\big\rangle^{\beta,s}_{A,-,g}\Big).
\end{aligned} \tag{3.2.2}$$

One then tries to derive sufficiently precise estimates on the semi-invariants of $\mu^{\beta,s}_{A,-,h}$ and to prove a local CLT under $\mu^{\beta,s}_{A,-,g}$. Thus, it is extremely important to understand how the magnetization $\langle M_A\rangle^{\beta,s}_{A,-,g}$ and other semi-invariants of $\mu^{\beta,s}_{A,-,g}$ change with the magnetic field g in the phase of $s(N)$-small contours.

Breaking of the classical limit behavior in the $s(N)$-restricted phase manifests itself by the jump of the magnetization which is related to the appearance of abnormally large $\pm$-contours. Without cutoffs this jump occurs for $g \sim 1/N$, and imposing the $s(N)$ constraint would delay such a jump [ScS3]. It is easy to imagine what should be the critical order of the magnetic field g, at which those large contours should start to be favored in the s-restricted phase: for a $\pm$ contour of the linear size $s(N)$ one wins $\sim s^2g$ on the level of magnetization and loses $\sim s$ on the level of surface energy. These two terms start to be comparable when $sg \sim 1$. Therefore no particular deviation from the classical behavior should be expected as far as $gs(N) \;\ll\; 1$. We refer to [ISc], where all these heuristic considerations have been made precise.

3.2.2. **Basic local estimate on the $K\log N$ scale.** Actually [ISc] it is enough to consider only the basic $K\log N$-scale:

Lemma 3.2.1 ([ISc]). *Assume that a sequence of numbers $\{b_N\}$ satisfies*

$$\lim_{N\to\infty}\frac{b_N\log N}{N^2} \;=\; 0.$$

Then, on the basic scale $s(N) = K\log N$, the estimate

$$\begin{aligned}
&\mu^{\beta,s}_{A,-}\Big(M_A = \langle M_A\rangle^{\beta,s}_{A,-} + a_N\Big)\\
&\quad= \frac{1}{\sqrt{2\pi\chi_\beta|A|}}\exp\big\{ -\frac{a_N^2}{2\chi_\beta|A|} + O\big(\frac{a_N^2}{N^3}(\log N\vee\frac{a_N}{N})\big)\big\}\big(1 \;+\; o(1)\big),
\end{aligned} \tag{3.2.3}$$

holds uniformly in domains $A\in\mathcal{D}_N$ and in $a_N\in\mathbf{M}^+_A\cap[0,b_N]$, where χ_β is the susceptibility under the pure state μ^β_-.

3.2.3. **Super-surface estimates in the restricted phases.** Moderate deviations on the intermediate scales $s(N) \gg \log N$ are, for the purposes of the theory, controlled by the following super-surface order estimate in the phase of small contours (c.f. Lemma 2.5.1 in [ISc])

Lemma 3.2.2. *Let the large contour parameter $s(N) \gg \log N$ be fixed. There exists a constant $c_1 = c_1(\beta) > 0$, such that for all $N > 0$, $A \in \mathcal{D}_N$ and all $a_N \in \mathbf{M}_A^+$,*

$$\mu_{A,-}^{\beta,s}\left(M_A = \langle M_A \rangle_{A,-}^{\beta,s} + a_N \right) \leqslant \exp\left(-c_1 \frac{a_N^2}{N^2} \wedge \frac{a_N}{s(N)} \right). \tag{3.2.4}$$

The idea of the proof is simple: either an area of order $a_N/2m^*$ is exhausted by the $K \log N$ large contours, which, in the $\mu_{N,-}^{\beta s}$-restricted phase, should have a surface tension price with the exponent of the order $a_N/s(N)$, or $K \log N$ large contours cover an area much less than $a_N/2m^*$, which means that the remaining deficit of the magnetization should be compensated in the basic $K \log N$ restricted phase, where we can use Lemma 3.2.1.

3.3. Bulk Relaxation in Pure Phases

The term relaxation is used here in the equilibrium setting in order to describe the approximation of local finite volume statistics by the infinite volume ones. We successively describe the relaxation properties of pure "$-$" states with non-positive and small positive magnetic fields and in the restricted phases of small contours.

3.3.1. **Non-positive magnetic fields $h \leq 0$.** The crucial property of low temperature pure phases could be stated as follows: Let us say that the sites i and j are $*$-neighbors if $\|i-j\|_1 = 1$. Given a spin configuration σ on $\{-1,+1\}^{\mathbb{Z}^2}$, let us say that the sites i and j are $+*$-connected, if there exists a $*$-connected chain of sites $i_1, ..., i_n$, $i_1 = i$ and $i_n = j$, such that $\sigma(i_k) = 1$ for every $k = 1, ..., n$.

Theorem 3.3.1 ([CCSc]). *For every $\beta > \beta_c$ there exists $c_1 = c_1(\beta) > 0$, such that uniformly in subsets $A \subseteq \mathbb{Z}^2$, $i, j \in A$ and in magnetic fields $h \leq 0$,*

$$\mu_{A,-,h}^{\beta}\left(i \overset{+*}{\longleftrightarrow} j \right) \leq \mathrm{e}^{-c_1(\beta)\|i-j\|_\infty}. \tag{3.3.1}$$

Remark. Of course, since $\left\{ i \overset{+*}{\longleftrightarrow} j \right\}$ is a non-decreasing event, the uniformity follows from the FKG ordering, once (3.3.1) is verified for the infinite volume zero-field measure μ_-^β.

Corollary 3.3.1 (Relaxation of local observables). *Fix $k \in \mathbb{Z}$. Uniformly in $A \subseteq \mathbb{Z}^2$, magnetic fields $h \leq 0$ and local observables f with $|\mathrm{supp}(f)| = k$,*

$$\left| \langle f \rangle_{A,-,h}^\beta - \langle f \rangle_{-,h}^\beta \right| \leq c_2(k) \mathrm{e}^{-c_3(\beta)\mathrm{dist}_\infty\left(\mathrm{supp}(f), \partial A\right)} \tag{3.3.2}$$

Furthermore,

Corollary 3.3.2 (Relaxation and decay of semi-invariants). *Fix $n \in \mathbb{Z}$. Uniformly in $A \subseteq \mathbb{Z}^2$, magnetic fields $h \leq 0$ and sites $i_1, ..., i_n \in A$,*

$$\left| \langle \sigma(i_1); ...; \sigma(i_n) \rangle_{A,-,h}^\beta - \langle \sigma(i_1); ...; \sigma(i_n) \rangle_{-,h}^\beta \right| \leq c_4(n) \mathrm{e}^{-c_5(\beta)\mathrm{dist}_\infty\left(\{i_1,...,i_n\}, \partial A\right)} \tag{3.3.3}$$

and

$$\left| \langle \sigma(i_1); ...; \sigma(i_n) \rangle_{A,-,h}^\beta \right| \leq c_6(n) \exp\left\{ -c_7(\beta) \frac{\mathrm{diam}_\infty(i_1, ..., i_n)}{n} \right\}. \tag{3.3.4}$$

Finally,

Corollary 3.3.3 (Asymptotic expansions). *Fix $n \in \mathbb{Z}$. Uniformly in $A \subseteq \mathbb{Z}^2$ and in $i \in A$,*

$$\left| \langle \sigma(i) \rangle^{\beta}_{A,-,h} \;-\; \big(-m^*(\beta) + \sum_{k=1}^{n} \mathfrak{s}_k \frac{h^k}{k!} \big) \right| \;\leq\; c_8(n)|h|^{n+1} + c_9(n) \mathrm{e}^{-c_{10}(\beta)\mathrm{dist}_\infty \left(i, \partial A \right)}, \tag{3.3.5}$$

where $\mathfrak{s}_k$ is the k-th semi-invariant of the zero-field infinite volume measure μ^{β}_{-},

$$\mathfrak{s}_k \;\triangleq\; \sum_{i_1,\dots,i_k \in \mathbb{Z}^2} \langle \sigma(0); \sigma(i_1); \dots; \sigma(i_n) \rangle^{\beta}_{-}.$$

Remark It is possible (and straightforward) to formulate (3.3.3), (3.3.4) and (3.3.5) in the general case of n local observables $f_1, \dots, f_n$. □

3.3.2. **Positive magnetic fields $h > 0$.** Modifying "$-$" states by negative magnetic fields $h < 0$ amounts to moving away from the phase transition region. Relaxation properties of $\mu^{\beta}_{A,-,h}$ with $h > 0$ are radically different - uniformity is lost, and the size of the domain A starts to play a crucial role. Indeed, the unique infinite volume measure $\mu^{\beta}_{-,h} = \mu^{\beta}_{h}$ stochastically dominates μ^{β}_{+} whatever small $h > 0$ is. Thus, for large domains A, the configuration in the bulk is flipped under $\mu^{\beta}_{A,-,h}$ into the "+" dominated state. It is easy to understand on the heuristic grounds what should be the order of the critical size of A for such a "flip" to occur: given $h > 0$, the surface energy of a $\pm$-contour γ is of the order $|\gamma|$ and it competes with the bulk gain inside the contour which, in its turn, is proportional to $h\mathrm{Area}(\gamma)$. The latter factor wins (loses), once the linear size of γ is much larger (respectively much smaller) than $1/h$. Thus the sign of the dominant spin under $\mu^{\beta}_{A,-,h}$ should depend on whether A can accommodate large enough contours, or, in other words, on how the linear size of A relates to $1/h$.

The important and remarkable fact is that exponential relaxation properties of finite volume "$-$" states are uniformly preserved for domains of the sub-critical size.

Theorem 3.3.2 ([ScS3], [ISc]). *There exists a constant $a = a(\beta) > 0$ such that for any $h > 0$ fixed,*

$$\mu^{\beta}_{A,-,h} \left(\, i \xleftrightarrow{+*} j \, \right) \;\leq\; \mathrm{e}^{-c_1(\beta)\|i-j\|_\infty}. \tag{3.3.6}$$

uniformly in domains $A \subset \mathbb{Z}$ such that any connected component of A has diameter bounded above by a/h. As a consequence exponential decay of semi-invariants (3.3.4) *and the asymptotic expansion estimate* (3.3.5) *hold uniformly in such domains as well.*

3.3.3. **Phases of small contours.** Theorem 3.3.2 explains how the cutoff parameter $s(N)$ upgrades the regular behavior of "$-$"-states with positive magnetic fields h: By the definition of the restricted phase $\mu^{\beta,s}_{A,-}$ the diameter of any relevant microscopic domain is at most of the order $s(N)$.

Theorem 3.3.3 ([ScS3], [ISc]). *There exists a constant $a = a(\beta) > 0$ such that for any $h > 0$ and s satisfying $hs \leqslant a(\beta)$,*

$$\mu^{\beta,s}_{A,-,h} \left(\, i \xleftrightarrow{+*} j \, \right) \;\leq\; \mathrm{e}^{-c_1(\beta)\|i-j\|_\infty}\,, \tag{3.3.7}$$

uniformly in domains $A \subseteq \mathbb{Z}$ *.*
Furthermore, the expectations in restricted phase are controlled as follows: for every $k \in \mathbb{Z}$,

$$\left| \langle f\rangle^{\beta,s}_{A,-,h} \; - \; \langle f\rangle^{\beta}_{A\cap\Lambda_s(f),-,h} \right| \; \leq \; c_2(k)\mathrm{e}^{-c_3(\beta)s}, \tag{3.3.8}$$

uniformly in $A \subseteq \mathbb{Z}^2$ *and in local functions* f, $\left|(\mathrm{supp(f)})\right| = k$, *where we have used the following notation:* $\Lambda_s(f) \triangleq \{i : \mathrm{d}_\infty(i, \mathrm{supp}(f)) \leqslant s\}$. *Finally, the decay of the semi-invariants is controlled in the restricted phases as*

$$\left| \langle \sigma(i_1); ...; \sigma(i_n)\rangle^{\beta,s}_{A,-,h} \right| \; \leq \; c_4(n)\exp\left\{ -c_5(\beta)\frac{\mathrm{diam}_\infty(i_1,...,i_n)}{n} \wedge s \right\}. \tag{3.3.9}$$

3.4. Calculus of Skeletons

The renormalization analysis of large $\pm$ contours is performed on various cutoff scales s, the appropriate choice of s typically depending on the linear size N of the system $s = s(N)$. We shall state coarse graining estimates uniformly in finite domains $A \subset \mathbb{Z}^2$ and in the cutoff scales s.

3.4.1. **Definition.** A $\pm$ contour γ is said to be s-large if $\mathrm{diam}_\infty(\gamma) \geqslant s$. Given a cutoff scale $s \in \mathbb{N}$ and an s-large $\pm$ contour γ we say that $S = (u_1, ..., u_n)$ is an s-skeleton of γ, $\gamma \sim S$ if

1. All vertices of S lie on γ.
2. $s(N)/2 \leqslant \|u_i - u_{i+1}\|_\infty \leqslant 2s$, $\forall\, i = 1, ..., n$, where we have identified $u_{n+1} \equiv u_1$.
3. The Hausdorff distance $d_{\mathbb{H}}$ between γ and the polygonal line $\mathrm{Pol}(S)$ through the vertices of S satisfies
$$d_{\mathbb{H}}\big(\gamma, \mathrm{Pol}(S)\big) \; \leqslant \; s(N).$$

Similarly, given the collection $(\gamma_1, ..., \gamma_n)$ of all s-large contours of a configuration $\sigma \in \Omega_{A,-}$, let us say that a collection $\mathfrak{S} = (S_1, ..., S_n)$ of s-large skeletons is compatible with σ, $\sigma \sim \mathfrak{S}$, if $\gamma_i \sim S_i$ for all $i = 1, ..., n$.

Of course, a configuration $\sigma \in \Omega_{A,-}$ has, in general, many different compatible collections of s-skeletons. Nonetheless, for each particular $\mathfrak{S}$ the probability

$$\mu^{\beta}_{A,-}(\mathfrak{S}) \; \triangleq \; \mu^{\beta}_{A,-}(\sigma : \; \sigma \sim \mathfrak{S}) \tag{3.4.1}$$

is well defined.

3.4.2. **Energy estimate.** As the renormalization scale s grows, the probabilities (3.4.1) start to admit a sharp characterization in terms of the energies $\mathcal{W}_\beta(\mathfrak{S})$,

$$\mathcal{W}_\beta(\mathfrak{S}) \; \triangleq \; \sum_1^n \mathcal{W}_\beta(\mathrm{Pol}(S_i)),$$

for a collection $\mathfrak{S} = (S_1, ..., S_n)$. Below we a give precise version of this crucial statement in terms of the upper and lower bounds on the corresponding probabilities. The first important renormalization energy estimates could be [Pf2] formulated as follows

Lemma 3.4.1 ([Pf2]). *On every skeleton scale* s *and independently of* $A \subset \mathbb{Z}^2$,

$$\mu^{\beta}_{A,-}(\; \mathfrak{S} \;) \; \leqslant \; \exp\{ \; -\mathcal{W}_\beta(\mathfrak{S}) \; \}. \tag{3.4.2}$$

Furthermore, uniformly in $A \subset \mathbb{Z}$, $r > 0$ and cutoff parameters s,

$$\mu^{\beta}_{A,-}\left(\mathcal{W}_{\beta}(\mathfrak{S}) \geqslant r\right) \leqslant \exp\left\{-r\left(1 - \frac{c_1 \log |A|}{s}\right)\right\}. \tag{3.4.3}$$

Energy estimate (3.4.2) provides an upper bound on the probability of observing $\pm$ contours in the vicinity of a skeleton. Before going to a complementary lower bound let us dwell on the sample path structure of the contours which is hidden behind these renormalization estimates.

3.4.3. **Calculus of skeletons.** By definition a contour is a self-avoiding closed path of nearest neighbor bonds of $\mathbb{Z}^2$. For every set $A \subseteq \mathbb{Z}^2$ the Ising measure $\mu^{\beta}_{A,-}$ induces a weight function $q^{\beta^*}_{A^*}$ on the space of such self-avoiding polygons (see Subsection 1.2.2),

$$q^{\beta^*}_{A^*}(\gamma) = \mu^{\beta}_{A,-}\left(\sigma \in \Omega : \ \gamma \text{ is a } \pm \text{ contour of } \sigma\right).$$

In terms of these weights the probability of observing a certain skeleton $S = \{u_1, ..., u_n\}$ could be written as

$$\mu^{\beta}_{A,-}(S) = \sum_{\gamma \sim S} q^{\beta^*}_{A^*}(\gamma).$$

Each microscopic contour γ compatible with S, $\gamma \sim S$, splits into the union of disjoint open self-avoiding lattice paths $\gamma_k : u_k \to u_{k+1}$, $k = 1, ..., n$. The analysis of limit properties of $\mu^{\beta}_{A,-}$ comprises two main steps which could be loosely described as follows: 1) As the renormalization scale s grows, the statistical behavior of different pieces γ_k decouple under $q^{\beta^*}_{A^*}$, that is

$$\sum_{\gamma \sim S} q^{\beta^*}_{A^*}(\gamma) \approx \prod_{k=1}^{n}\left(\sum_{\gamma_k : u_k \to u_{k+1}} q^{\beta^*}_{A^*}(\gamma_k)\right). \tag{3.4.4}$$

2) The $k{-}th$ term ($k = 1, ..., n$) in the above product corresponds to a $\pm$ interface stretched in the direction of the vector $u_{k+1} - u_k \in \mathbb{R}^2$, in other words

$$q^{\beta^*}_{A^*}(\gamma_k) \approx \mathrm{e}^{-\tau_{\beta}(u_{k+1} - u_k)}. \tag{3.4.5}$$

Thus, the skeleton calculus resembles a refined version of the sample path large deviation principle for genuinely two-dimensional random curves. At very low temperatures, a very precise local analysis of the phase separation line has been developed in [DKS],[DS] using the method of cluster expansions. Our approach here pertains to the whole of the phase transition region $\beta > \beta_c$, but is strongly linked to the very specific self-duality properties of the two-dimensional nearest neighbor Ising model. We refer to Subsection 1.2.2 and, eventually, to [PV2, PV3] for comprehensive description and study of the relevant properties of the duality transformation. The output of these techniques could be recorded in the following form

Lemma 3.4.2 (Probabilistic Structure of the Phase Separation Line [PV2]). *Given any $A \subset \mathbb{Z}^2$ and any two compatible self-avoiding paths λ_1 and λ_2,*

$$q^{\beta^*}_{A^*}(\lambda_1 \cup \lambda_2) \geqslant q^{\beta^*}_{A^*}(\lambda_1)\, q^{\beta^*}_{A^*}(\lambda_2). \tag{3.4.6}$$

Furthermore,

$$e^{-c_1(\beta)|\lambda_2|} \leqslant \frac{q_{A^*}^{\beta^*}(\lambda_1 \cup \lambda_2)}{q_{A^*}^{\beta^*}(\lambda_1)} \leqslant e^{-c_2(\beta)|\lambda_2|} \tag{3.4.7}$$

On the other hand, given any $A \subseteq \mathbb{Z}^2$ and any three points $u, v, w \in A^*$, the $q_{A^*}^{\beta^*}$ weight of the paths going from u to v through w is bounded above as [PV2]

$$\sum_{\substack{\lambda: u \to v \\ w \in \lambda}} q_{A^*}^{\beta^*}(\lambda) \leqslant \left(\sum_{\lambda_1: u \to w} q_{A^*}^{\beta^*}(\lambda_1) \right) \left(\sum_{\lambda_2: w \to v} q_{A^*}^{\beta^*}(\lambda_2) \right). \tag{3.4.8}$$

Finally, the weights $q_{A^*}^{\beta^*}$ are non-increasing in A, and are related to the dual connectivities as

$$\sum_{\lambda:\ u \to v} q_{A^*}^{\beta^*}(\lambda) = \langle \sigma(u)\sigma(v) \rangle_{A^*,f}^{\beta^*}. \tag{3.4.9}$$

Relation (3.4.9) is the link to the surface tension: first of all the impact of a particular set A exponentially diminishes with the distance to ∂A [I1],

$$\langle \sigma(u)\sigma(v) \rangle_f^{\beta^*} - \exp\{-c_2(\beta)\mathrm{d}(\{u,v\}, \partial A)\} \leqslant \langle \sigma(u)\sigma(v) \rangle_{A^*,f}^{\beta^*} \leqslant \langle \sigma(u)\sigma(v) \rangle_f^{\beta^*}. \tag{3.4.10}$$

uniformly in $A^* \subseteq \mathbb{Z}^2$ and any $u, v \in A^*$. Moreover the following Ornstein-Zernike type correction formula [Al] holds uniformly in $u, v \in \mathbb{Z}^2$:

$$\exp\{-\tau_\beta(u-v) - c_3(\beta)\log\|u-v\|_\infty\} \leqslant \langle \sigma(u)\sigma(v) \rangle_f^{\beta^*} \leqslant \exp\{-\tau_\beta(u-v)\}, \tag{3.4.11}$$

3.4.4. **Skeleton lower bound.** The energy estimate (3.4.2) is an immediate consequence of the (iterated) sub-multiplicative property (3.4.8), the representation formula (3.4.9) and the right-most inequalities in (3.4.10) and (3.4.11). In order to prove a lower bound one essentially needs to reverse the inequality in (3.4.8). An indirect way to do so is to use the FK representation (see [ScS2] and [ISc]). We shall briefly present here a more direct approach which has been developed in [I1] and [PV2]. Qualitatively it gives the same order of corrections as the FK one, but has a clear advantage of being explicitly related to the statistics of the microscopic phase boundaries at different length scales. The basic idea is that the phase separation line has rather strong mixing properties, in particular paths λ_1 and λ_2 on the right hand side of (3.4.8) should interfere, in the case of (u, v, w) being in a general position, only in a vicinity of w. Thus, at a price of lower order corrections (as we shall see these corrections are logarithmic with the skeleton scale s) the inequality (3.4.8) could be reversed using the super-multiplicativity property (3.4.6). The notion of "general position" simply means that u, w and v do not form too small an angle and live on the same length scale, and it is quantified by the following

Definition. Given a skeleton scale $s \in \mathbb{N}$ and a number $\varepsilon > 0$, let us say that that a triple (u, w, v) of $\mathbb{Z}^2$-lattice points is (s, ε)-compatible, if

$$\frac{s}{2} \leqslant \min\{\|w-u\|_\infty, \|v-w\|_\infty\} \leqslant \max\{\|w-u\|_\infty, \|v-w\|_\infty\} \leqslant 2s,$$

whereas $\cos(w-u, v-w) \geqslant -1+\varepsilon$. □

We shall state the lower bound in terms of the limiting weights $q^{\beta^*}(\cdot) \triangleq \lim_{A^* \nearrow \mathbb{Z}^2_\star} q_{A^*}^{\beta^*}$ (which exist by Lemma 3.4.2).

Lemma 3.4.3. *Fix $\varepsilon > 0$. Then there exists a scale $s = s(\varepsilon)$, such that*

$$\sum_{\substack{\lambda:u\to v\\ w\in\lambda}} q^{\beta^*}(\lambda) \geqslant \exp\left\{-\left(\tau_\beta(w-u)+\tau_\beta(v-w)\right)-c_1(\beta)\log s\right\}, \tag{3.4.12}$$

uniformly in all skeleton scales $s \geqslant s(\varepsilon)$ and in all (s,ε)-compatible triples (u,w,v).

We sketch the proof of this lemma in Appendix B. Iterating (3.4.12) we arrive to the following lower bound on the probability of observing a certain regular skeleton:
Definition. A skeleton $S = (u_1, ..., u_n)$ is said to be (s,ε)-regular, if any triple (u_{i-1}, u_i, u_{i+1}) of successive points of S is (s,ε)-compatible, and the distance between any two non-neighboring intervals $[u_i, u_{i+1}]$ and $[u_j, u_{j+1}]$ exceeds εs. □

Lemma 3.4.4. *For every $\varepsilon > 0$, there exists a number $c_2 = c_2(\varepsilon) < \infty$, such that uniformly in the skeleton scales s and in all (s,ε)-regular skeletons S,*

$$\begin{aligned}
\mu^\beta_{N,-}&\left(\exists\ a\ \pm\ \textit{contour}\ \gamma:\ d_{\mathbb{H}}(\gamma, \mathrm{Pol}(S)) \le K(\beta)\sqrt{s}\log s\right)\\
&\ge \exp\left\{-W_\beta\left(\mathrm{Pol}(S)\right) - c_2(\varepsilon)\#(S)\log s\right\}\\
&\ge \exp\left\{-W_\beta\left(\mathrm{Pol}(S)\right)\left(1 - c_3(\varepsilon,\beta)\frac{\log s}{s}\right)\right\},
\end{aligned} \tag{3.4.13}$$

where $\#(S)$ denotes the number of vertices in S, and the last inequality follows from (3.1.4).

In fact we need lower bounds only for a very specific set of s-skeletons, namely on those approximating the Wulff shape $\mathcal{K}_{a_N/2m^*}$. These skeletons always satisfy the conditions of the above theorem. An academic attempt to prove a lower bound for all possible shapes will lead to annoying, though solvable, technicalities, but will fail to contribute much to the microscopic theory of phase separation, as we see it.

3.5. Structure of The Proof

In order to give a probabilistic characterization of the microscopic canonical state $\mu^\beta_{N,-}\left(\,\cdot\,\middle|M_N = -m^*N^2 + a_N\right)$ one first derives a sharpest possible lower bound on the probability $\mu^\beta_{N,-}\left(M_N = -m^*N^2 + a_N\right)$, and then rules out those geometric events (in terms of skeletons, but with an eventual translation to the language of microscopic spin configurations), which happen to qualify as improbable when compared with this lower bound.

3.5.1. **Lower bound.** The best lower bound comes as an outcome of the optimal combination of the basic local limit Lemma 3.2.1 and the skeleton lower bound (3.4.13). We choose a skeleton approximation of the corresponding Wulff shape $\mathcal{K}_{a_N/2m^*}$, and using local limit estimates steer the magnetization towards the desirable value $-m^*N^2 + a_N$. Optimality reflects the choice of the best possible skeleton scale: Notice that the estimate (3.4.13) becomes sharper with the growth of the cutoff parameter $s(N)$. On the other hand, the area of the microscopic phase region is controlled, with respect to the area inside $\mathrm{Pol}(S) \sim a_N/2m^*$, up to a $N\sqrt{s(N)}\log s(N)$ correction (see Appendix B or [ISc]), which, of course, makes the local limit step more expensive for large values of $s(N)$. It happens that the bounds are balanced on the skeleton scale $s(N) \sim \sqrt[4]{a_N}$.

Theorem 3.5.1 ([ISc]). *Uniformly in $a_N \in \mathbf{M}_N^+$, that is for all $a_N \geqslant 0$, such that $-m^*N^2 + a_N \in \mathrm{Range}(M_N)$,*

$$\mu_{N,-}^{\beta}\left(M_N = -m^*N^2 + a_N\right) \;\geqslant\; \exp\left\{-\sqrt{\frac{a_N}{2m^*}}\mathcal{W}_\beta\left(\partial\mathcal{K}_1\right) - c_1(\beta)\sqrt[4]{a_N}\log N\right\}. \quad (3.5.1)$$

3.5.2. **Upper bounds.** First of all, one derives an upper bound on the shift of the magnetization. On any skeleton scale,

$$\mu_{N,-}^{\beta}\left(M_N = -m^*N^2 + a_N\right) \;\leqslant\; \sum_{\mathfrak{S}} \mu_{N,-}^{\beta}\left(M_N = -m^*N^2 + a_N\;;\;\mathfrak{S}\right). \quad (3.5.2)$$

Due to the intrinsic entropy cancelation under the skeleton coarse graining, and in view of the lower bound (3.5.1) and the energy estimate (3.4.2) one could, for example, shoot for the maximal term in the above sum. If the phase volume (see [DKS] for the precise definition) of $\mathfrak{S}$ is much less than $a_N/2m^*$, then the deficit of the magnetization should be compensated in the phase of $s(N)$-small contours, which, by Lemma 3.2.2 exerts a super-surface price in the exponent. On the other hand, if the phase volume of $\mathfrak{S}$ is close to $a_N/2m^*$, then by the isoperimetric inequality and by the energy estimate (3.4.2), the best possible price one should be prepared to pay is already close to $\exp\left\{-\mathcal{W}_\beta\left(\mathcal{K}_{a_N/2m^*}\right)\right\}$. Again the resulting estimate is subject to an optimization via a careful choice of the skeleton scale $s(N)$.

Theorem 3.5.2 ([ISc]). *Uniformly in $a_N \sim N^2$,*

$$\mu_{N,-}^{\beta}\left(M_N = -m^*N^2 + a_N\right) \;\leqslant\; \exp\left\{-\sqrt{\frac{a_N}{2m^*}}\mathcal{W}_\beta\left(\partial\mathcal{K}_1\right) + c_1(\beta)\sqrt[4]{a_N}\log N\right\}. \quad (3.5.3)$$

A more delicate study [DKS],[ISc] of the typical sample properties of the microscopic configuration σ under $\mu_{N,-}^{\beta}\left(\,\cdot\,\middle|M_N = -m^*N^2 + a_N\right)$ is again based on the analysis of (3.5.2). At this point the stability Bonnesen-type estimates (see Subsection 1.3 of the Introduction) for the Wulff variational problem become important - they enable to quantify the conclusion that only those collections $\mathfrak{S}$, which are close to the shifts of the Wulff shape $\mathcal{K}_{a_N/2m^*}$, have a chance to survive a comparison with the lower bound (3.5.1). A step further, involving local limit estimates of Lemma 3.2.1, is to conclude that all these collections actually contain exactly one large skeleton, which corresponds to the unique large contour as asserted by the DKS theorem.

3.6. Open Problems

There are still important open problems even in the nearest neighbor Ising case. Notably, one knows how to control precise fluctuations of the phase separation line only at very low temperatures, that is using the method of cluster expansions [DH]. This is a serious gap in the theory, since large scale statistics of microscopic phase boundaries are ultimately responsible for exact (up to zero order terms) expansions of canonical partition functions [H]. So far qualitative probabilistic results have been obtained either for very low temperature models [H], or in the simplified setting of self-avoiding polygons [I3], [HI] or Bernoulli bond percolation [CI]. Another interesting and apparently important problem is to understand sample path properties of spin configurations in a situation when a canonical constraint is imposed in the restricted phase. Apart from giving rise to a potentially fascinating probabilistic structure, this question is closely related to the issue of the dynamical spinodal decomposition.

There is absolutely no matching probabilistic study of the phase separation in multi-phase two-dimensional models, for example q-states Potts models. Some results in this direction are reported in [V], but this issue is almost entirely open even in the context of the $\mathbb{L}_1$-theory. In particular, the corresponding phenomena is still not worked out on the level of macroscopic variational problems, see, however [ABFH], [MoS] and the references therein.

The key issue, however, which we feel is largely misunderstood is that at moderately low temperatures the DKS theory of two-dimensional phase segregation, say in the general context of finite range ferromagnetic models with pair interactions is far from being complete. What currently exists is an example of how these ideas could be implemented in the nearest neighbor case. At least from the mathematical point of view, the nearest neighbor case is a degenerate one, in a sense that it enables a reduction to pure boundary conditions over decoupled microscopic regions even at temperatures only moderately below critical. This should not be the case for more general range of interactions. In this respect the assertion that low temperature expansions should go through for general interactions much along the same lines as they do for the nearest neighbor model, seems to be rather irrelevant - the real issue is not to kill mixed boundary conditions, but to understand how they should be incorporated into the DKS theory.

Part 4. Boundary effects

In the previous parts, we explained how the thermodynamical variational problem describing the macroscopic geometry of coexisting phases can be derived in various lattice models of statistical physics. To simplify the analysis, we restricted our attention to periodic boundary conditions or to systems contained in a Wulff-shaped box, avoiding thus a discussion of the effect of a confining geometry on the behavior of the system. In this part, we would like to explain what happens when we take such effects into account. Boundary conditions play a particularly important role in the kind of problems presented in this review, since they concern the asymptotic behavior of large but finite systems and therefore the boundary cannot be simply "sent to infinity" as usually done. We will see that taking care of boundary effects not only provides a complete description of the geometry of these constrained systems thus allowing a rigorous description of the interaction between an equilibrium crystal and a substrate, but also allows to study the effect of so-called *boundary phase transitions.*

For simplicity, we only discuss the case of the Ising model with nearest neighbors interaction.

4.1. Wall free energy

The vessel containing the system has not only the property of confining it, but can also act in an asymmetric way on the various phases inside, favoring some of them; indeed this is what happens typically in real systems. In fact, this is precisely the reason one introduces boundary conditions in the first place: To impose the equilibrium phase the system realizes. It appears to be convenient to have a parameter allowing a fine-tuning of the asymmetry, interpolating between pure $+$ or $-$ boundary conditions. Let us now describe how this is done.

Let $\Sigma = \{i \in \mathbb{Z}^d : i(d) = 0\}$ and $\mathbb{L}^d = \{i \in \mathbb{Z}^d : i(d) \geqslant 0\}$. The vessel of our system is the box

$$\mathbb{D}_{N,M} = \{i \in \mathbb{L}^d : -N \leqslant i(n) \leqslant N,\, n = 1,\ldots,d-1,\, 0 \leqslant i(d) \leqslant M\}\,,$$

and the *wall* is $\Sigma_N = \mathbb{D}_{N,M} \cap \Sigma$.

Let $\eta \in \mathbb{R}$; we consider the following Hamiltonian,

$$\mathbf{H}^{\eta}_{\mathbb{D}_{N,M}}(\sigma) = -\sum_{\substack{\langle i,j\rangle \subset \mathbb{L}^d \\ \langle i,j\rangle \cap \mathbb{D}_{N,M} \neq \varnothing}} \sigma_i\sigma_j - \eta \sum_{i\in\Sigma_N} \sigma_i\,.$$

Let $\overline{\sigma} \in \{-1,1\}^{\mathbb{L}^d}$; the Gibbs measure in $\mathbb{D}_{N,M}$ with boundary condition $\overline{\sigma}$ is the following probability measure on $\{-1,1\}^{\mathbb{L}^d}$ [9],

$$\mu^{\beta,\eta}_{\mathbb{D}_{N,M},\overline{\sigma}}(\sigma) = \begin{cases} (\mathbf{Z}^{\beta,\eta}_{\mathbb{D}_{N,M},\overline{\sigma}})^{-1} \exp[-\beta\, \mathbf{H}^{\eta}_{\mathbb{D}_{N,M}}(\sigma)] & \text{if } \sigma_i = \overline{\sigma}_i,\, \forall i \notin \mathbb{D}_{N,M}\,, \\ 0 & \text{otherwise.} \end{cases}$$

We'll usually use the short-hand notations $\mu^{\beta,\eta}_{N,M,\overline{\sigma}}$, $\mathbf{Z}^{\beta,\eta}_{N,M,\overline{\sigma}}$, ... As usual, we write $+$ for $\overline{\sigma} \equiv 1$ and $-$ for $\overline{\sigma} \equiv -1$. We therefore distinguish one of the sides of the box

[9] Note that we could equivalently consider $\mu^{\beta,\eta}_{\mathbb{D}_{N,M},\overline{\sigma}}$ as a probability measure on $\{-1,1\}^{\mathbb{Z}^d}$ by extending the b.c. $\overline{\sigma}$ by $\overline{\sigma}_i = 1$ for all $i \in \mathbb{Z}^d \setminus \mathbb{L}^d$; it is then possible to replace the boundary magnetic field η by a coupling constant: $\eta\sum_{i\in\Sigma_N}\sigma_i = \eta\sum_{\langle i,j\rangle:\, i\in\Sigma_N,\, j\notin\mathbb{L}^d}\sigma_i\sigma_j$. This will be used when dealing with negative boundary field, see Subsection 4.4.1.

$\mathbb{D}_{N,M}$, Σ_N, which we call the "wall". Notice that instead of usual boundary conditions, a *boundary magnetic field* η is acting on Σ_N; since setting $\eta = 1$ produces + b.c. on the wall, while setting $\eta = -1$ results in − b.c., this provides the promised interpolation parameter. Of course, we could also consider more complicated situations, where (possibly inhomogeneous) boundary magnetic fields act on the whole boundary of the box. However, for simplicity, we restrict our attention to this particular case, which will turn out to be general enough that the basic phenomena induced by the use of boundary fields can already be analyzed.

To quantify the preference of the wall toward one of the phases, it is convenient to introduce a new thermodynamic quantity, the *wall free energy*,

$$\tau_{\mathrm{bd}}(\beta,\eta) \triangleq \lim_{\substack{N\to\infty\\ M\to\infty}} \frac{1}{|\Sigma_N|} \log \frac{\mathbf{Z}^{\beta,\eta}_{N,M,+}}{\mathbf{Z}^{\beta,\eta}_{N,M,-}}\,. \tag{4.1.1}$$

The existence of this quantity, and the remarkable fact that the two limits can be taken in any order, has been established in [FP1]; the proof relies on the simple identity

$$\tau_{\mathrm{bd}}(\beta,\eta) = \lim_{\substack{N\to\infty\\ M\to\infty}} \beta \int_{-\eta}^{\eta} \frac{1}{|\Sigma_N|} \sum_{i\in\Sigma_N} \langle\sigma_i\rangle^{\beta,\eta'}_{N,M,+}\, \mathrm{d}\eta'\,. \tag{4.1.2}$$

We'll return to this formula in the next section. The heuristics behind the definition of $\tau_{\mathrm{bd}}(\beta,\eta)$ is that the free energy $F^{\beta,\eta}_{N,M,+(-)} = -\log \mathbf{Z}^{\beta,\eta}_{N,M,+(-)}$ of the + (−) phase can be decomposed in the following way:

$$F^{\beta,\eta}_{N,M,+} = f_{\mathrm{b}}(\beta)\,|\mathbb{D}_{N,M}| + f^{+}_{\mathrm{s}}(\beta)\,|\partial\mathbb{D}_{N,M}\setminus\Sigma_N| + f^{+}_{\mathrm{w}}(\beta,\eta)\,|\Sigma_N| + o(|\partial\mathbb{D}_{N,M}|,|\Sigma_N|)\,,$$
$$F^{\beta,\eta}_{N,M,-} = f_{\mathrm{b}}(\beta)\,|\mathbb{D}_{N,M}| + f^{-}_{\mathrm{s}}(\beta)\,|\partial\mathbb{D}_{N,M}\setminus\Sigma_N| + f^{-}_{\mathrm{w}}(\beta,\eta)\,|\Sigma_N| + o(|\partial\mathbb{D}_{N,M}|,|\Sigma_N|)\,,$$

where

$$f_{\mathrm{b}}(\beta) \triangleq -\lim_{N,M\to\infty} |\mathbb{D}_{N,M}|^{-1}\,\log \mathbf{Z}^{\beta,\eta}_{N,M,\overline{\sigma}}\,,$$
$$f^{+}_{\mathrm{s}}(\beta) \triangleq -\lim_{N,M\to\infty} |\partial\mathbb{D}_{N,M}|^{-1}\left(\log \mathbf{Z}^{\beta,1}_{N,M,+} - f_{\mathrm{b}}(\beta)|\mathbb{D}_{N,M}|\right),$$
$$f^{+}_{\mathrm{w}}(\beta,\eta) \triangleq -\lim_{N,M\to\infty} |\Sigma_N|^{-1}\left(\log \mathbf{Z}^{\beta,\eta}_{\mathbb{D}_{N,M},\overline{\sigma}} - f_{\mathrm{b}}(\beta)|\mathbb{D}_{N,M}| - f^{+}_{\mathrm{s}}(\beta)|\partial\mathbb{D}_{N,M}\setminus\Sigma_N|\right),$$

(and similarly for $f^{-}_{\mathrm{s}}(\beta)$ and $f^{-}_{\mathrm{w}}(\beta,\eta)$). As the notations suggest, $f_{\mathrm{b}}(\beta)$ is independent of η and $\overline{\sigma}$, $f^{+}_{\mathrm{s}}(\beta)$ is independent of η and by symmetry $f^{+}_{\mathrm{s}}(\beta) = f^{-}_{\mathrm{s}}(\beta)$. Therefore, we see that $\tau_{\mathrm{bd}}(\beta,\eta) = \lim_{\substack{N\to\infty\\ M\to\infty}} \frac{1}{|\Sigma_N|}\,(F^{\beta,\eta}_{N,M,-} - F^{\beta,\eta}_{N,M,+}) = f^{-}_{\mathrm{w}}(\beta,\eta) - f^{+}_{\mathrm{w}}(\beta,\eta)$ is nothing else than the leading order term of the difference in free energy between the two phases in the presence of the wall.

The ultimate justification of (4.1.1) however is that this quantity plays exactly the role of its thermodynamical analogue in the variational problem describing the macroscopic geometry of phase coexistence, see Theorems 4.3.2 and 4.3.3 below.

The following Theorem states basic properties of $\tau_{\mathrm{bd}}(\beta,\eta)$; since $\tau_{\mathrm{bd}}(\beta,\eta)$ is obviously odd in η, we just state them for $\eta \geqslant 0$ (also $\tau_{\mathrm{bd}}(\beta,0) = 0$).

Theorem 4.1.1. [FP2] *Let $\tau^*_\beta = \tau_\beta(\vec{e}_d)$ and suppose $\eta \geqslant 0$. Then*

- *$\tau_{\mathrm{bd}}(\beta,\eta)$ is a non-negative, increasing function of β and η, concave in η; moreover, if $\eta > 0$,*
$$\tau_{\mathrm{bd}}(\beta,\eta) > 0 \Leftrightarrow \beta > \beta_{\mathrm{c}}\,.$$

- *For all β and η, $\tau_{\mathrm{bd}}(\beta,\eta) \leqslant \tau^*_\beta$.*
- *For all $\beta > \beta_{\mathrm{c}}$, there exists $1 \geqslant \eta_{\mathrm{w}}(\beta) > 0$ such that*

$$\tau_{\mathrm{bd}}(\beta,\eta) < \tau^*_\beta \Leftrightarrow \eta < \eta_{\mathrm{w}}(\beta)\,.$$

In the case of the 2D Ising model, $\eta_{\mathrm{w}}(\beta)$ can be computed explicitly, see [Ab1, MW] and Fig. 11.

The following terminology is standard[10]: when $\eta \geqslant \eta_{\mathrm{w}}(\beta)$, we say that the system is in the *complete drying* regime; when $|\eta| < \eta_{\mathrm{w}}(\beta)$, it is in the *partial wetting* regime; and when $\eta \leqslant -\eta_{\mathrm{w}}(\beta)$, it is in the *complete wetting* regime. The reason for this terminology should become clear later.

4.2. Surface phase transition

In this section, we will see that the boundary magnetic field can trigger *surface phase transitions*: The behavior of the system in the vicinity of the wall depends dramatically on $|\eta|$ being greater or smaller than $\eta_{\mathrm{w}}(\beta)$. A more detailed discussion of these issues can be found in [PV1].

The state of the system in the middle of a big box $\mathbb{D}_{N,M}$ is entirely determined by the boundary conditions, and is independent of the value of the boundary field, so that the usual (infinite volume) Gibbs state simply doesn't provide any information on the behavior of the system close to the wall. To analyze the behavior of the system "in the vicinity" of the wall, it is therefore useful to introduce the notion of *surface Gibbs states*; these differ from the Gibbs states usually considered in these models by the fact that one does not work with a sequence of boxes converging to $\mathbb{Z}^d$, but instead converging only to the half-space $\mathbb{L}^d$. More precisely, the surface Gibbs states are the weak limits of the measures $\mu^{\beta,\eta}_{N,M,\overline{\sigma}}$ when $N, M \to \infty$ (observe that $\mathbb{D}_{N,M} \nearrow \mathbb{L}^d$). Two of them are of particular importance for our discussion, $\mu^{\beta,\eta}_{\mathbb{L}^d,+}$ and $\mu^{\beta,\eta}_{\mathbb{L}^d,-}$, obtained respectively by taking weak limits of the measures with $+$ and $-$ boundary conditions. It is not difficult to show [FP1] that these two measures exist, are extremal, and are invariant under translations parallel to the wall; moreover, there is uniqueness of the surface Gibbs state if and only if $\mu^{\beta,\eta}_{\mathbb{L}^d,+} = \mu^{\beta,\eta}_{\mathbb{L}^d,-}$.

There is a close relation between $\tau_{\mathrm{bd}}(\beta,\eta)$ and the behavior of the system near the wall; this can be most easily seen from the following identity, consequence of (4.1.2) and symmetry [FP1],

$$\tau_{\mathrm{bd}}(\beta,\eta) = \int_{-\eta}^{\eta} \langle\sigma_0\rangle^{\beta,\eta'}_{\mathbb{L}^d,+}\,\mathrm{d}\eta' = \int_0^{\eta} \big(\langle\sigma_0\rangle^{\beta,\eta'}_{\mathbb{L}^d,+} - \langle\sigma_0\rangle^{\beta,\eta'}_{\mathbb{L}^d,-}\big)\,\mathrm{d}\eta'\,. \tag{4.2.1}$$

Using (4.2.1), it is possible to prove the following Theorem showing that a surface phase transition occurs at $\eta = \eta_{\mathrm{w}}(\beta)$; this is the so-called *wetting transition*.

Theorem 4.2.1. [FP2] *There is a unique surface Gibbs state if and only if $|\eta| \geqslant \eta_{\mathrm{w}}(\beta)$.*

Let us briefly discuss the heuristics behind this result. The $+$ and $-$ boundary conditions fix the phase present in the bulk (i.e. in the middle of a big box $\mathbb{D}_{N,M}$). However, Theorem 4.2.1 shows that when $\eta \geqslant \eta_{\mathrm{w}}(\beta)$, the surface Gibbs state is unique, and therefore the state of the system near the wall is *independent* of the boundary conditions, i.e. of the phase

[10] This terminology only makes sense once we have chosen one of the equilibrium phase as reference; here it is the $-$ phase.

present in the bulk. The mechanism responsible for this is the following. Suppose that $\eta < 0$ and consider $+$-boundary conditions; then it is natural to regard the boundary field as a negative b.c., and therefore to introduce an open contour with boundary $\partial\Sigma_N$ separating the $-$ phase favored by the wall from the $+$ phase present in the bulk (see Section 4.4 for more details). As long as $\eta > -1$, there is a competition between two effects: On the one hand it is energetically favorable for the open contour to follow the wall, on the other hand this would lead to a loss in entropy, since there is less room for fluctuations. When $\eta \leqslant -\eta_{\mathrm{w}}(\beta)$, the entropy wins: The contour is repelled away from the wall, at a distance diverging with the size of the box; this is the phenomenon of *entropic repulsion*. The surface Gibbs state then describes the behavior of the system below this surface, i.e. a mesoscopic film of $-$ phase along the bottom wall. The fact that the contour is sent away from the wall explains why we recover the surface tension, $\tau_{\mathrm{bd}}(\beta,\eta) = \tau^*_\beta$. When $\eta > -\eta_{\mathrm{w}}(\beta)$ energy wins, and this modifies completely the behavior of the microscopic surface: it sticks to the wall, making only small excursions away from it; in this case, the phase in the bulk can reach the wall and the surface Gibbs state depends on the choice of boundary conditions.

Part of these heuristics can be made quite precise in the 2D case. Consider $+$ boundary conditions. When $0 > \eta > -\eta_{\mathrm{w}}(\beta)$, one can prove that the probability that a connected piece I of the wall is not touched by the open contour is bounded above by $K\exp[-(\tau^*_\beta - \tau_{\mathrm{bd}}(\beta,\eta))\,|I|]$, showing that the phase separation line really sticks to the wall [PV2]. The informations available when $\eta \leqslant -\eta_{\mathrm{w}}(\beta)$ are much less precise; the magnetization profile computed in [Ab1] shows that there is a film of width of order $\sqrt{N}$ along the wall. A related, much more precise result, which holds at sufficiently low temperature and for $\eta = -1$ is that the phase separation line, once suitably rescaled, converges weakly to the Brownian excursion [D]; this should be true for any $\eta \leqslant -\eta_{\mathrm{w}}(\beta)$.

In higher dimensions, much less is known. When $\eta > -\eta_{\mathrm{w}}(\beta)$, one can show that the probability that the open contour touches the middle of the wall is bounded away from 0 uniformly in the size of the box [FP2]. When $\eta \leqslant -\eta_{\mathrm{w}}(\beta)$, very little is known,except in the simpler case of SOS models. Also, if it is known in dimension 2 that $\eta_{\mathrm{w}}(\beta) < 1$ (since the exact expression for $\eta_{\mathrm{w}}(\beta)$ has been computed [Ab1]), this is an open problem in higher dimensions.

Theorem 4.2.1 gives a first explanation of the terminology introduced above: when the system is in the complete drying regime, the equilibrium phase along the wall is the $+$ phase, whatever the phase in the bulk is; when there is complete wetting, it is the $-$ phase; only in the regime of partial wetting can both phases be present near the wall. The fact that the phase transition is determined by $\eta_{\mathrm{w}}(\beta)$ (i.e. the characterization of the partial wetting regime by $\tau^*_\beta = |\tau_{\mathrm{bd}}(\beta,\eta)|$) is known as *Cahn's criterion.*

4.3. Derivation of the Winterbottom construction

In this section, we show how Winterbottom construction, describing the equilibrium shape of a crystal in the presence of an attractive substrate, can be recovered from a microscopic theory. To do this, we consider the measure $\mu^{\beta,\eta}_{N,rN,+}$, for some $r \in \mathbb{R}$, conditioned with some canonical constraint (exact or approximate, see below). Of course, the situation here is more complicated than the one described in the introduction, since instead of an infinite wall, the system is contained in a finite vessel. This, of course, makes the problem more difficult: When the solution of the Winterbottom variational problem does not fit inside the box $\widehat{\mathbb{D}}^d_r \triangleq \{x \in \mathbb{R}^d \,:\, |x(n)| \leqslant 1,\, n = 1,\dots,d-1,\, 0 \leqslant x(d) \leqslant r\}$, the solution of

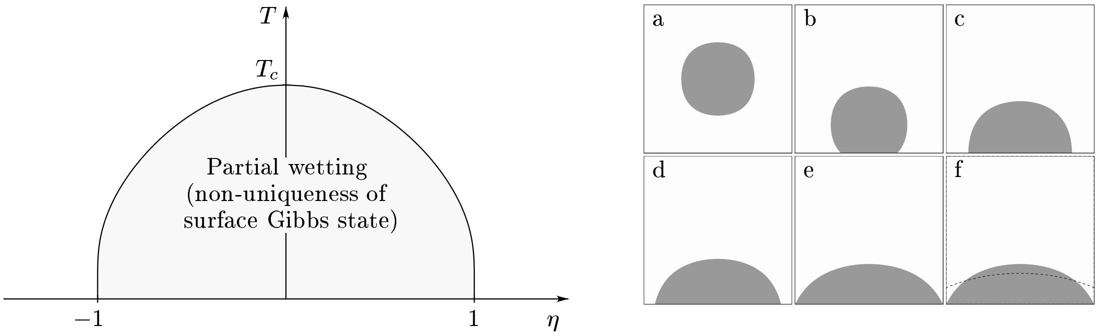


FIGURE 11. The case of the 2D Ising model. Left: The phase diagram; the region of non-uniqueness of the surface Gibbs state is shaded. In the other region, there is a single surface Gibbs state. Right: A sequence of equilibrium shapes.

the constrained problem will differ from Winterbottom shape. In fact, the general solution of the constrained problem is not known. In the way we state them below, the derivation of this variational problem from statistical mechanics still applies in the case when the solution is not known.

Before stating the main Theorems of this Part, we briefly describe how the wetting transition manifests itself in the macroscopic geometry of phase separation. To do this, let $\beta > \beta_c$ be fixed, and choose a value m for the canonical constraint so that the corresponding Wulff shape is small enough to be placed inside the box $\widehat{\mathbb{D}}_r^d$. If $\eta \geqslant \eta_w(\beta)$, then $\tau_{bd}(\beta, \eta) = \tau_\beta^*$, and the typical configurations will consist of a macroscopic droplet of $-$ phase, with Wulff shape, immersed in a background of $+$ phase; in particular, the shape of the droplet is independent of the value of the boundary field (Fig. 11 a). This behavior persists up to the value $\eta = \eta_w(\beta)$. Notice that as soon as $\eta < 1$, it becomes *energetically* more favorable for the droplet to touch the wall. In dimension 2, however, since $\eta_w(\beta) < 1$, the droplet stays away from the wall, because entropy loss is not compensated by energy gain until η reaches the value $\eta_w(\beta)$. It is an interesting open problem to decide whether $\eta_w(\beta) = 1$ for $d > 2$. When $\eta < \eta_w(\beta)$, the typical configurations consist of a macroscopic droplet, with Winterbottom shape, tied to the wall. The shape of the droplet now depends on the value of η, and decreasing the boundary field amounts to letting the droplet spread more and more (Fig. 11 b–e). For some value $\widetilde{\eta}$, the droplet covers for the first time the entire wall (Fig. 11 e). From this point on, the shape of the droplet is left unchanged when η is decreased (Fig. 11 f; the dashed line represent part of a possible "true" equilibrium shape for the unconstrained problem).

From this discussion, we see that the wetting transition at $\eta_w(\beta)$ has a macroscopic manifestation in the canonical ensemble. Because of the confined geometry, however, the second transition, at $\eta = -\eta_w(\beta)$ cannot be seen. To be able to detect it, one has to consider mesoscopic droplets (in the form of large moderate deviations, see the remark after Theorem 4.3.2).

This also explains pretty well the terminology introduced previously: In the complete drying regime, the droplet stays away from the wall, and so the wall is completely dry w.r.t. the $-$ phase; in the partial wetting regime, the droplet touches the wall, and both the $+$ and $-$ phase are in contact with it (provided $\eta < \widetilde{\eta}$). The complete wetting regime cannot be distinguished from the partial wetting regime in this setting, but see the remark after Theorem 4.3.2 for a discussion of this issue.

4.3.1. **2D Ising model.** Let $r \in \mathbb{R}$. The aim of this subsection is to describe the typical configurations under the measure

$$\mu^{\beta,\eta}_{N,rN,+}\big(\,\cdot \mid M_N = m\,|\mathbb{D}_{N,rN}|\big)\,,$$

where $m \in (-m^*, m^*)$ and $M_N = \sum_{i\in\mathbb{D}_{N,rN}} \sigma_i$; we will simplify the notations further by writing simply $\mu^{\beta,\eta}_{N,+}$ (r being kept fixed). As in Part 3, it is possible to obtain precise asymptotics for the large deviations, in the form of the following generalization of the first part of Theorem 3.1.1. Let $\mathcal{W}^\star_{\beta,\eta}(m)$ be the infimum of the functional $\mathcal{W}_{\beta,\eta}$ on subsets of $\widehat{\mathbb{D}}^2_r$ with volume $\frac{m^*-m}{2m^*}|\widehat{\mathbb{D}}^2_r|$.

Theorem 4.3.1. *Let the inverse temperature $\beta > \beta_c$ and the boundary magnetic field $\eta \in \mathbb{R}$ be fixed; let the sequence $\{a_N\}$; $-m * |\mathbb{D}_{N,rN}| + a_N \in \mathrm{Range}(M_N)$, be such that the limit*

$$a \;=\; \lim_{N\to\infty} \frac{a_N}{|\mathbb{D}_{N,rN}|} \;\in\; (0, 2m^*(\beta))$$

exists. Then,

$$\log \mu^{\beta,\eta}_{N,+}\big(M_N = m^*|\mathbb{D}_{N,rN}| - a_N\big) = -\mathcal{W}^\star_{\beta,\eta}\,(1 + O(N^{-1/2}\log N)).$$

A version of this Theorem, in an approximate canonical ensemble (as in (4.3.1)), has been proven in [PV2]; this stronger version can be obtained by combining the techniques of [PV2] and of [ISc], see Section 4.4.

In Theorem 4.3.1, we have made no statement about the asymptotic description of the typical configurations under the conditioned measure. The reason is the following: These strong concentration results require the knowledge of stability properties of the variational problem in the form, for example, of Bonnesen inequality. However, in the present case, one does not always have that much information about the variational problem; in fact, even its solution is not always known. This prevents us from translating the energy estimates on the skeletons (see (4.4.8), (4.4.10) and (4.4.11)) into strong concentration properties of the microscopic contours. Of course, in the situations when such stability properties are known ([KP] contains a simple derivation of such a result for many situations), it is possible to obtain statements of the same kind as those of Part 3.

This illustrates the fact that although the probabilistic theory in the 2D case is complete, in the sense that all the relevant information on the *microscopic scale* is available, the sharpness of the statements one can make on the macroscopic scale still depends on *macroscopic* stability properties, which are logically separated from the probabilistic aspect of the analysis.

However, even without information about the stability properties of the variational problem, it is still possible to derive weak concentration properties, in a $\mathbb{L}_1$ setting close to the one of Part 2. We present such a result in the way it is stated in [PV2]. In this paper, an approximate canonical ensemble was considered, i.e. the measure was $\mu^{\beta,\eta}_{N,+}(\,\cdot \mid \mathcal{A}(m;c))$, where

$$\mathcal{A}(m;c) = \Big\{\sigma \;:\; \Big|\,|\mathbb{D}_{N,rN}|^{-1} M_N(\sigma) - m\Big| \leqslant N^{-c}\Big\}\,, \tag{4.3.1}$$

with $-m^* < m < m^*$, and c is some real number not too large (see Theorem 4.3.2 below). We are going to prove that the phases concentrate near macroscopic droplets which belong to the set $\mathcal{D}(m)$

$$\mathcal{D}(m) = \Big\{V \subset \widehat{\mathbb{D}}^2_r \;:\; |V| = \frac{m^*-m}{2m^*}|\widehat{\mathbb{D}}^2_r|\,,\; \mathcal{W}_{\beta,\eta}(\partial V) = \mathcal{W}^\star_{\beta,\eta}(m)\Big\}\,,$$

Recall that to each $V \in \mathcal{D}(m)$, we associate the function $\mathbb{1}_V = 1_{V^c} - 1_V$.

To state this phase segregation Theorem, we use the mesoscopic notation introduced in Part 2. Recall that $N = 2^n$. For any $a < 1$, we define a magnetization profile $\mathcal{M}_{[an]}(\sigma, x)$ at the $2^{[an]}$-scale which is piecewise constant on boxes $\widehat{\mathbb{B}}_{n-[an]}(x)$ with $x \in \widehat{\mathbb{T}}^d_{n-[an]}$,

$$\mathcal{M}_{[an]}(\sigma, x) = 2^{-d[an]} \sum_{i \in \mathbb{B}_{[an]}(2^n x)} \sigma_i \,. \tag{4.3.2}$$

We get

Theorem 4.3.2. [PV2] *Let $\beta > \beta_c$, $\eta \in \mathbb{R}$, $-m^* < m < m^*$ and $1/4 > c > 0$. Then there exist a function $\delta(N)$ such that $\lim_{N\to\infty} \delta(N) = 0$, a real number $\kappa > 0$ and a coarse-graining parameter $1 > a > 0$ such that for N large enough*

$$\mu^{\beta,\eta}_{N,-}\Big(\frac{\mathcal{M}_{[an]}}{m^*} \in \bigcup_{V \in \mathcal{D}(m)} \mathcal{V}(\mathbb{1}_V, \delta(N)) \,\Big|\, \mathcal{A}(m;c)\Big) \geqslant 1 - \exp\{-O(N^\kappa)\}\,.$$

Remark: In this case, it should also be possible to study the whole range of moderate deviations, combining the techniques of [ISc] and [PV2], although this has not been done explicitly. We briefly describe the results obtained for large deviations sufficiently close to volume order [V].

As long as $\eta > -\eta_w(\beta)$, the results are similar to those obtained in the setting of Part 3: The measure concentrates on configurations containing a single large droplet of $-$ phase, with Wulff or Winterbottom shape depending on η; in particular, the order of the large moderate deviations is still $\exp\{-O(\sqrt{a_N})\}$. There should not be any problem to extend this to the whole large deviations regime ($a_N \gg N^{4/3}$).

More interesting is the case $\eta \leqslant -\eta_w(\beta)$. For those values of the boundary field, the system is in the complete wetting regime ($\tau_{bd}(\beta,\eta) = -\tau^*_\beta$), and the solution of the unconstrained variational problem is degenerate. The solution of the constrained variational problem in $\widehat{\mathbb{D}}^2_r$ is however still well-defined for every N; it is obtained by extracting the cap of a Wulff shape and rescaling it so that the basis of the cap completely covers the wall and the rescaled cap has the required volume. When N goes to infinity, this droplet spreads out to become a thin film in the limit (covering the entire wall, hence the terminology complete wetting), and the corresponding value of the surface free energy functional goes to zero. As a result of this, the scale of the large moderate deviations *is not the same as when* $\eta < \eta_w(\beta)$; indeed the leading term of the asymptotics can again be computed explicitly, and is found to be of order $\exp\{-O((a_N)^2 N^{-3})\}$. In particular, we see that the large moderate deviations cannot extend up to $a_N \sim N^{4/3}$, since $(a_N)^2 N^{-3}$ is of order 1 already when $a_N \sim N^{3/2}$. This should not be surprising since, in the complete wetting regime, the volume under the microscopic contour is expected to have typical fluctuations of order $N^{3/2}$ (this can be shown when $\eta = -1$ and β is very large using the convergence to Brownian excursion stated in [D]). Therefore, typical fluctuations of magnetization in the complete wetting regime are not governed by bulk fluctuations anymore, but by fluctuations of the microscopic phase separation line. To prove that this behavior is valid up to $a_N \sim N^{3/2}$ might be a non-trivial task. □

4.3.2. **Ising model in $D \geqslant 3$.** Let $r \in \mathbb{R}$ and let $\mathcal{D}(m)$ be the set of macroscopic droplets at equilibrium in $\widehat{\mathbb{D}}_r^d$,

$$\mathcal{D}(m) = \left\{ V \subset \widehat{\mathbb{D}}_r^d \,:\, |V| = \frac{m^* - m}{2m^*} |\widehat{\mathbb{D}}_r^d| \,,\, \mathcal{W}_{\beta,\eta}(\partial V) = \mathcal{W}^\star_{\beta,\eta}(m) \right\}.$$

The rest of the notations were introduced in Part 2. The main result is the following

Theorem 4.3.3. [BIV] *For any β in $\mathfrak{B}_p$, any $\eta \in \mathbb{R}$, any m in $(-m^*, m^*)$, the following holds: For any $\delta > 0$, there is $k_0 = k_0(\delta)$ such that for $\nu < \frac{1}{d}$*

$$\lim_{N\to\infty} \min_{k_0 \leqslant k \leqslant \nu n} \mu^{\beta,\eta}_{N,+} \left(\frac{\mathcal{M}_k}{m^*} \in \bigcup_{V \in \mathcal{D}(m)} \mathcal{V}(\mathbb{I}_V, \delta) \;\Big|\; M_N \leqslant m\, |\mathbb{D}_{N,rN}| \right) = 1\,.$$

4.4. The tools

In this Section, we explain how the procedures described in Parts 2 and 3 have to be modified to take into account the effect of the boundary.

4.4.1. **2D Ising model.** We describe the main modifications one needs to apply to the proofs of Part 3 in order to get the results stated in Theorems 4.3.1 and 4.3.2. We split this Subsection into two parts, one dealing with the lower bound on $\mu^{\beta,\eta}_{N,-}(\mathcal{A}(m;c))$ or $\mu^{\beta,\eta}_{N,-}(M_N = -m^*|\mathbb{D}_{N,rN}| + a_N)$, the other one with the upper bound.

<u>The lower bound.</u> The constrained variational problem is more difficult than the usual one. In fact, as noted above, the solution (and *a fortiori* its stability) is not known in general, although it is in many cases. This prevents us from proceeding as in Part 3, where the lower bound follows from summing over large contours fluctuating around the Wulff shape. It would then appear necessary to make the same kind of proof, but for any configurations of droplets surrounding the right volume (all potential solutions to the variational problem). This, however, would be tricky; indeed, since we want our results to hold for large, but *finite* boxes, it is compulsory to obtain estimates *uniform* over the droplet in the chosen set! Fortunately, properties of the surface tension and wall free energy allow us to restrict our analysis to a small class of well-behaved droplets: The solution of the variational problem is necessarily taken on a *single convex* droplet. This is a consequence of the convexity of τ_β (use Jensen inequality) and the fact that $\tau_{\mathrm{bd}}(\beta,\eta) \leqslant \tau^*_\beta$, which imply that replacing a droplet by its convex hull cannot increase the surface free energy; rescaling the resulting droplet decreases the energy even more. It is thus enough to prove the following

Proposition 4.4.1. [PV2] *Let $\beta > \beta_c$ and $\eta \in \mathbb{R}$. There exists $N_0 = N_0(\beta,\eta,m,c,r)$ and a constant C such that, for any simple closed rectifiable curve $\mathcal{C}$ which is the boundary of a convex body of volume $|\widehat{\mathbb{D}}_r^2|(m^*(\beta)+m)/2m^*(\beta)$ contained in $\widehat{\mathbb{D}}_r^2$, and for all $N \geqslant N_0$,*

$$\mu^{\beta,\eta}_{N,-}(\mathcal{A}(m;c)) \geqslant \exp\{-\mathcal{W}_{\beta,\eta}(\mathcal{C})\, N - \beta\, C\, N^{1/2} \log N\}\,.$$

A completely analogous statement holds in the case of the exact canonical ensemble.

The proof of Proposition 4.4.1 is similar to the proof of Theorem 3.5.1. We sketch now the main changes needed to deal with the boundary conditions. The case $\eta \leqslant 0$ requires a slightly more complicated proof than the case $\eta > 0$ so we first consider the latter.

First case: $\eta > 0$

As in the usual case, we want to approximate $\mathcal{C}$ with some polygonal curve with vertices on the dual lattice, and then sum over all contours going through the latter; this would

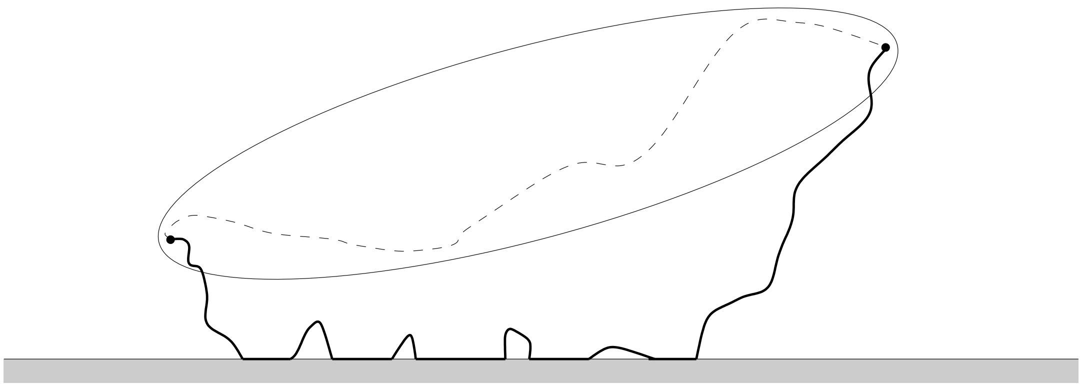


FIGURE 12. When $\tau_{\mathrm{bd}}(\beta,\eta) < \tau_\beta^*$, the open contour connecting two sites close enough to the wall might not stay inside an elliptical set as in the bulk (dashed contour), but instead might get pinned by the wall (full contour). In such a case, the exponential decay-rate is in general not given by τ_β or $\tau_{\mathrm{bd}}(\beta,\eta)$.

allow us to extract, for each piece of the contour, the surface tension of the corresponding part of the polygonal line. Here, however, we want to be able to extract the wall free energy when the curve $\mathcal{C}$ follows the wall. There are some complications related to this: If two vertices are close to the wall, but don't belong to it[11], the sum over the corresponding piece of contour might not yield simply τ_β or $\tau_{\mathrm{bd}}(\beta,\eta)$, but some complicated mixture, since typical such contours might first go down to the wall, then follow it on some length, and only then go up to the other vertex, see Fig. 12; this kind of behavior has been studied in details in [PV3]. It turns out that it is possible to construct a polygonal approximation to the curve $\mathcal{C}$ whose surface tension is not too large in comparison with that of $\mathcal{C}$, while removing these possible pathologies.

The idea is the following. Let $\delta_N = N^{-1/2}\log N$, and set

$$\widehat{\mathbb{D}}_r^2(N) = \{x \in \widehat{\mathbb{D}}_r^2 \,:\, \min_{y\notin\widehat{\mathbb{D}}_r^2} \|y-x\|_1 > \delta_N\}\,.$$

Let V be the convex body with boundary $\mathcal{C}$ and set $\mathcal{C}_N = \partial(V\cap\widehat{\mathbb{D}}_r^2(N))$ We first construct a polygonal approximation for each of the components of $\mathcal{C}_N \cap \widehat{\mathbb{D}}_r^2(N)$ with segments of length δ_N (apart from at most 8 of them which may be shorter). Set $[x,y] = \{z\in\mathcal{C}_N \,:\, z(2)=\delta_N\}$. If $[x,y]\neq\varnothing$, we connect the two corresponding pieces of polygonal lines by a broken line from x to $(x(1),0)$, then to $(y(1),0)$, and finally to y; we divide the segment between $(x(1),0)$ and $(y(1),0)$ into segments of length $\delta_N/2$ (except possibly for the last one which can be shorter). We repeat this construction for the three other sides of the box. The resulting closed polygonal line is denoted by $\widehat{\mathcal{P}}_N$ (see Fig. 13). Notice that by construction there exists an absolute constant C such that

$$\begin{aligned}\mathcal{W}_{\beta,\eta}(\mathcal{C}) &\geqslant \mathcal{W}_{\beta,\eta}(\widehat{\mathcal{P}}_N) - C\beta\delta_N\,,\\ |\mathrm{vol}(\mathcal{C}) - \mathrm{vol}\widehat{\mathcal{P}}_N| &\leqslant C\,|\widehat{\mathbb{D}}_r^2|\,\delta_N\,.\end{aligned}$$

We then rescale the polygonal line $\widehat{\mathcal{P}}_N$ by a factor N and if necessary move slightly the rescaled vertices so that they belong to the dual lattice; the rescaled polygons is denoted by $\mathcal{P}_N$. We then define a class $\mathfrak{G}$ of closed contours going through the vertices of $\mathcal{P}_N$

[11] Consider, for example, a family of curves $\mathcal{C}$ getting closer and closer to the wall; since we need estimates uniform in all such curves, one has to be able to deal with such a situation.

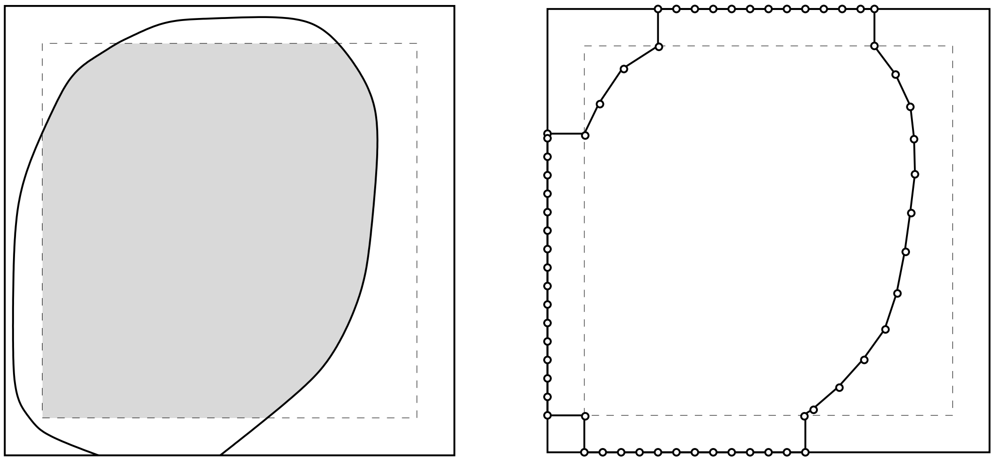


FIGURE 13. Left: The curve $\mathcal{C}$; the shaded area represents the convex body whose boundary is $\mathcal{C}_N$ and the dashed line is the boundary of $\widehat{\mathbb{D}}_r^2(N)$. Right: The polygonal approximation $\widehat{\mathcal{P}}_N$, the dots representing its vertices.

(in the right order), and staying in some small boxes along its edges. For all edges of length smaller than $N\delta_N$, as well as for the (up to 8) pieces we added above to join $\mathcal{C}_N$ to the boundary, we impose that the corresponding piece of the contour is a fixed length-minimizing path between the vertices.

The rest of the argument proceeds in a similar way as in the standard case. The estimates in the phase of small contours carry over without any problems since in that case the effect of the boundary field cannot propagate far away from the wall.

We still have to explain how one can extract the correct surface tension for $\widehat{\mathcal{P}}_N$ from the sum over contours in the class $\mathfrak{G}$ introduced above. To do this, we use several results about the *random-line representation*, proved in [PV2, PV3]. To lighten the notation, we simply write $q_N^{\beta^*,\eta^*}$ instead of $q_{\mathbb{D}^\star_{N,rN}}^{\beta^*,\eta^*}$; β^* and η^* are the dual of β and η, see (1.2.5). The first inequality is just the analogue of (3.4.6) in our case, which turns out to be valid for arbitrary ferromagnetic coupling constants: The weight of any high-temperature contour $\gamma \in \mathfrak{G}$ satisfies ([PV2], Lemma 5.4)

$$q_N^{\beta^*,\eta^*}(\gamma) \geqslant \prod q_N^{\beta^*,\eta^*}(\gamma_k)$$

where γ_k denotes the piece of the contour γ between the kth and $k+1$th vertices of $\mathcal{P}_N$. The next step is to replace $q_N^{\beta^*,\eta^*}(\gamma_k)$ by the corresponding infinite-volume quantity. First, for any γ_k joining vertices not belonging to $\Sigma_N^\star \triangleq \{i \in \mathbb{D}_{N,rN}{}^* \,:\, i(2) = -\frac{1}{2}\}$ (note that γ_k stays necessarily at a distance $O(N\delta_N)$ from $\Sigma_N^\star$)

$$q_N^{\beta^*,\eta^*}(\gamma_k) \geqslant (1 - e^{-O(N\delta_N)})\, q^{\beta^*}(\gamma_k)\,;$$

second, for the pieces γ_k joining two sites of $\Sigma_N^\star$, we use

$$q_N^{\beta^*,\eta^*}(\gamma_k) \geqslant q_{\mathbb{L}_\star^d}^{\beta^*,\eta^*}(\gamma_k)\,,$$

where $\mathbb{L}_\star^d \triangleq \{i \in \mathbb{Z}_\star^d \,:\, i(2) \geqslant -\frac{1}{2}\}$ (both results are proved in [PV2], Lemma 5.3). Finally, the remaining pieces have a length at most $8N\delta_N$, so that their total weight is larger than $e^{-CO(N\delta_N)}$.

The last step is to extract the surface free energy. The basic tool to do this is, as in the proof of Theorem 3.4.4, concentration properties for open contours between 2 fixed dual sites. For the pieces γ_k not touching the boundary, we can use the usual infinite volume results based on (5.2.12), setting $s = N\delta_N$. For the pieces along the boundary, one can use the following statement ([PV3], Lemma 6.10):

$$\sum_{\substack{\lambda:\, i\to j \\ \lambda\subset \mathbf{N}_K(i,j)\cap\mathbb{L}^d_\star}} q^{\beta^*,\eta^*}_{\mathbb{L}^d_\star}(\lambda) \geqslant \langle\sigma_i\sigma_j\rangle^{\beta^*,\eta^*}_{\mathbb{L}^d_\star}\ (1+o(1)), \tag{4.4.1}$$

where $\mathbf{N}_K(i,j)$ is defined in Appendix B (with $s = N\delta_N$). (In fact, (4.4.1) can be strengthened when $\eta < \eta_{\mathrm{w}}(\beta)$: in this case, the set $\mathbf{N}_K(i,j)\cap\mathbb{L}^d_\star$ can be replaced by the set ([PV3], Lemma 6.13)

$$\{k\in\mathbb{L}^d_\star\ :\ (i(1)\wedge j(1)) - K\log\delta_N \leqslant k(1) \leqslant (i(1)\vee j(1)) + K\log\delta_N,\ k(2)\leqslant K\log\delta_N\}\,,$$

which is compatible with our picture of partial wetting.)

The result then follows from lower bounds on the corresponding 2-point functions. The only new inputs are the following lower bounds on the boundary 2-point function,

$$\langle\sigma_i\sigma_j\rangle^{\beta^*,\eta^*}_{\mathbb{L}^{d*}} \geqslant C\,\frac{\exp\{-\tau_{\mathrm{bd}}(\beta,\eta)\|j-i\|\}}{\|j-i\|^{3/2}} \qquad \forall\eta\geqslant\eta_{\mathrm{w}}(\beta)\,, \tag{4.4.2}$$

$$\langle\sigma_i\sigma_j\rangle^{\beta^*,\eta^*}_{\mathbb{L}^{d*}} \geqslant C\,\exp\{-\tau_{\mathrm{bd}}(\beta,\eta)\|j-i\|\} \qquad \forall\eta<\eta_{\mathrm{w}}(\beta)\,, \tag{4.4.3}$$

for any $i,j\in\Sigma^*\triangleq\{k\in\mathbb{L}^d_\star\ :\ k(2)=-\frac{1}{2}\}$. (4.4.3) is proved in [PV2], Prop. 7.1, while (4.4.2) follows from exact computations in the case $\eta^*=1$ [MW], and [PV2], Prop. 7.1,

$$\langle\sigma_i\sigma_j\rangle^{\beta^*,\eta^*}_{\mathbb{L}^{d*}} \geqslant (\tanh\beta^*)^2\,\langle\sigma_i\sigma_j\rangle^{\beta^*,1}_{\mathbb{L}^{d*}}, \qquad \forall\eta\geqslant 0\,.$$

Second case: $\eta = 0$

This is a somewhat marginal case. The apparent difficulty is that in this case $\eta^*=\infty$. However, this does not create any real complications. One just has to modify the construction of the first case as follows: We replace the polygonal line $\widehat{\mathcal{P}}_N$ by the (possibly open) polygonal line $\widehat{\mathcal{P}}_N\setminus\{u\in\mathbb{R}^2\ :\ u(2)=0\}$; we then sum over contours going through the vertices of this polygonal line (contours which are open if the polygonal line is open). This does not give any contribution for the part of $\mathcal{C}$ along the wall, which is what we want since $\tau_{\mathrm{bd}}(\beta,0)=0$.

Third case: $\eta < 0$

This is slightly more tricky. In this situation, one may be even more pessimistic, since the duality is simply not defined when non-ferromagnetic interactions are present! However, this turns out to be a false problem. Indeed, we can use the following obvious identity to recover ferromagnetic interactions (see footnote 9, p. 53),

$$\mu^{\beta,\eta}_{N,+} = \mu^{\beta,|\eta|}_{N,\pm}\,,$$

where $\pm$ correspond to the boundary condition $\overline{\sigma}_i = 1$ if $i(2)\geqslant 0$ and $\overline{\sigma}_i=-1$ otherwise.

We then construct $\widehat{\mathcal{P}}_N$ as in the first step and set $I=\widehat{\mathcal{P}}_N\cap\{x\in\widehat{\mathbb{D}}^2_r\ :\ x(2)=0\}$. If $I=\varnothing$, then we subdivide the set $\{x\in\widehat{\mathbb{D}}^2_r\ :\ x(2)=0\}$ into segments of length $\delta_N/2$ (except possibly for the last one, which might be shorter); this defines a second (open) polygonal line $\widehat{\mathcal{P}}'_N$ (with all its vertices along the wall) (see Fig. 14). We then introduce a class of *pair* of contours (γ,γ'), γ going through the vertices of $\mathcal{P}_N$ and defined as before, and γ' following the wall, going through the vertices of $\mathcal{P}'_N$ and staying inside small boxes

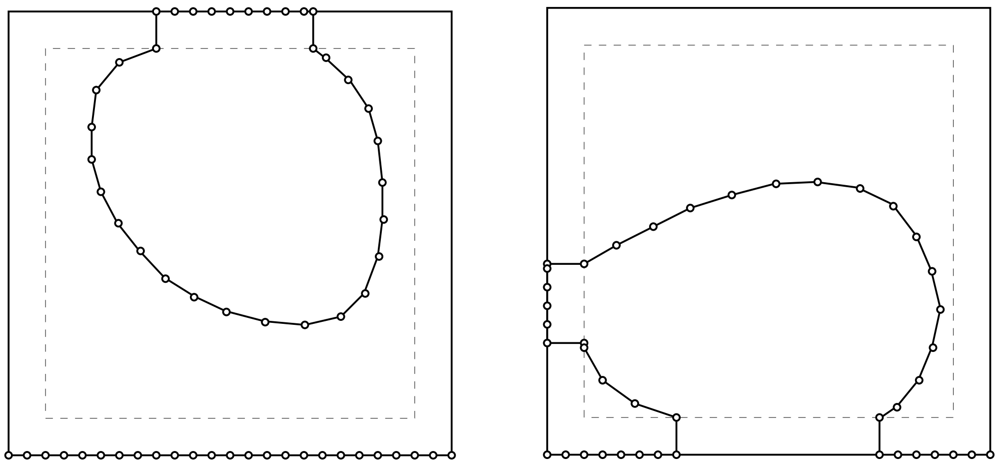


FIGURE 14. The construction for $\eta < 0$. Left: $I = \varnothing$ (two polygonal lines: one open and one closed. Right: $I \neq \varnothing$ (one open polygonal line).

along its edges, similarly as for the other one (γ' is open). By construction γ and γ' are disjoint. Duality then implies the following identity

$$\begin{aligned}\mu_{N,\pm}^{\beta,|\eta|}(\{\gamma,\gamma'\} \subset \boldsymbol{\gamma}(\,\cdot\,)) &= (\mathbf{Z}_{N,\pm}^{\beta,|\eta|})^{-1}\, w(\gamma)w(\gamma') \sum_{\substack{\underline{\zeta}:\\ (\underline{\zeta},\gamma,\gamma')\ \Lambda^*\text{-comp.}}} w(\underline{\zeta}) \\ &= (1 - e^{-O(N)})\, \frac{\mathbf{Z}_{N,+}^{\beta,|\eta|}}{\mathbf{Z}_{N,\pm}^{\beta,|\eta|}}\, q_N^{\beta^*,|\eta|^*}(\gamma,\gamma')\,. \end{aligned} \tag{4.4.4}$$

The factor $(1 - e^{-O(N)})$ comes from the fact that we can apply duality only to simply connected sets, and the exterior of γ is *not* simply connected. We must therefore forbid families $\underline{\zeta}$ for which duality does not hold; since such families must contain at least one contour surrounding γ, we get the above correction.

We can now proceed as in the first case. The only additional work to do is to analyze the ratio of partition functions in (4.4.4), but this is easy, since by duality

$$\frac{\mathbf{Z}_{N,+}^{\beta,|\eta|}}{\mathbf{Z}_{N,\pm}^{\beta,|\eta|}} = \big(\langle\sigma_{t_{\mathrm{l}}}\sigma_{t_{\mathrm{r}}}\rangle_{\mathbb{D}^\star_{N,rN}}^{\beta^*,|\eta|^*}\big)^{-1} \geqslant e^{\tau_{\mathrm{bd}}(\beta,|\eta|)\,(2N+1)}\,, \tag{4.4.5}$$

where $t_{\mathrm{l}} = (-L - \frac12, -\frac12)$ and $t_{\mathrm{r}} = (L + \frac12, -\frac12)$ are the two dual sites at the lower left and lower right corners of $\mathbb{D}^\star_{N,rN}$, and the last inequality follows from the upper bound (see [PV2] for example)

$$\langle\sigma_i\sigma_j\rangle_{\mathbb{D}^\star_{N,rN}}^{\beta^*,|\eta|^*} \leqslant e^{-\tau_{\mathrm{bd}}(\beta,|\eta|)\|j-i\|}\,, \tag{4.4.6}$$

valid for any $i,j \in \Sigma^\star_N$. We then see that the ratio of partition function cancels the contribution from the sum over the open contour γ', up to an error term $\exp\{\mathcal{O}(N\delta_N)\}$.

If $I \neq \varnothing$, the situation is simpler. Let's write $I = [x,y]$; then we define a new polygonal line $\widehat{\mathcal{P}}_N^{\pm}$: $\widehat{\mathcal{P}}_N^{\pm}$ goes from the lower right corner of $\widehat{\mathbb{D}}_r^2$ to a along the wall, then it follows $\widehat{\mathcal{P}}_N \setminus \{x \in \widehat{\mathbb{D}}_r^2 \,:\, x(2) = 0\}$ up to b and finally goes from b to the lower right corner of $\widehat{\mathbb{D}}_r^2$ (see Fig. 14). We subdivide as usual the part of $\widehat{\mathcal{P}}_N^{\pm}$ along the wall into segments of length

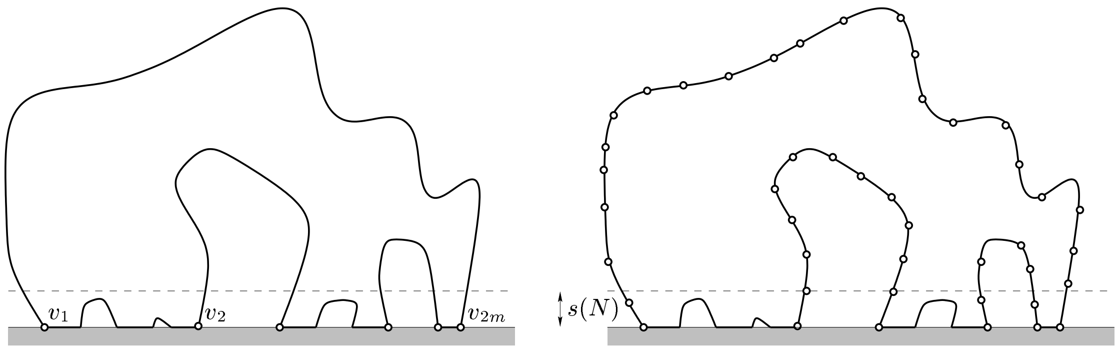


FIGURE 15. Left: A contour touching the wall and the family $(v_1, \dots, v_{2m})$. Right: An s-skeleton for the contour.

$\delta_N/2$ and proceed as in the first case, with $\widehat{\mathcal{P}}_N^{\pm}$ replacing $\widehat{\mathcal{P}}_N$, using (4.4.4). Summing over the open contour going through the vertices of $\mathcal{P}_N^{\pm}$ produces (up to the usual error term) a term $\exp\{-\mathcal{W}_{\beta,|\eta|}(\widehat{\mathcal{P}}_N^{\pm})\,N\}$. Combining this with (4.4.5) and observing that

$$\exp\{2\tau_{\mathrm{bd}}(\beta,|\eta|)\,N\}\ \exp\{-\mathcal{W}_{\beta,|\eta|}(\widehat{\mathcal{P}}_N^{\pm})\,N\} = \exp\{-\mathcal{W}_{\beta,\eta}(\widehat{\mathcal{P}}_N)\,N\}\,,$$

the conclusion follows as in the usual situation.

<u>The upper bound.</u> Let us now turn our attention to the proof of the upper bound. The basic strategy is completely similar to that of the standard case, see Subsection 3.5.2. The only serious modification concerns the energy estimate, which should now associate the functional $\mathcal{W}_{\beta,\eta}$ to the probability of skeletons. Again, the case $\eta \geqslant 0$ is somewhat simpler than the other, so we start with this one.

First case: $\eta \geqslant 0$

The basic problem we encounter when trying to make the energy estimate is the same we met in the proof of the lower bound. Summing over an open contour connecting two dual sites i and j might not yield a decay of order $\exp\{-\tau_\beta(j-i)\}$ or $\exp\{-\tau_{\mathrm{bd}}(\beta,\eta)\|j-i\|\}$ if i and j are close enough to the wall but not on it (see [PV3]). However, the following bound, proven in [PV2], Lemma 5.1, is sufficient to derive the energy estimate,

$$\sum_{\substack{\lambda:\, i\to j\\ \lambda\cap\mathcal{E}(\Sigma_N^\star)=\varnothing}} q_N^{\beta^*,\eta^*}(\lambda) \leqslant \exp\{-\tau_\beta(j-i)\}\,, \tag{4.4.7}$$

for any $\eta \geqslant 0$; $\mathcal{E}(\Sigma_N^\star) = \{e^* \subset \Sigma_N^\star\}$. The definition of skeletons will be done in such a way as to ensure that the additional constraint $\lambda \cap \mathcal{E}(\Sigma_N^\star) = \varnothing$ is automatically satisfied, see below. We also need to extract the wall free energy when summing over contours joining two dual sites belonging to $\Sigma_N^\star$; this however is nothing else as (4.4.6).

Let us now describe the construction of a skeleton $S = (u_1, \dots, u_n)$ of a closed contour γ. Remember that we have to define the skeletons in such a way as to ensure that 1) the piece of the contour between two dual sites not both on the wall must be edge-disjoint from the wall, and 2) the Hausdorff distance between the contour γ and the polygonal line $\mathrm{Pol}(S)$ is smaller than the cutoff parameter $s(N)$.

For contours γ which do not touch the wall, the definition of skeletons is the same as in Part 3. Suppose $\gamma \cap \mathcal{E}(\Sigma_N^\star) \neq \varnothing$. Let us define $(v_1, \dots, v_{2m})$ as the *minimal* family of dual sites satisfying the following properties:

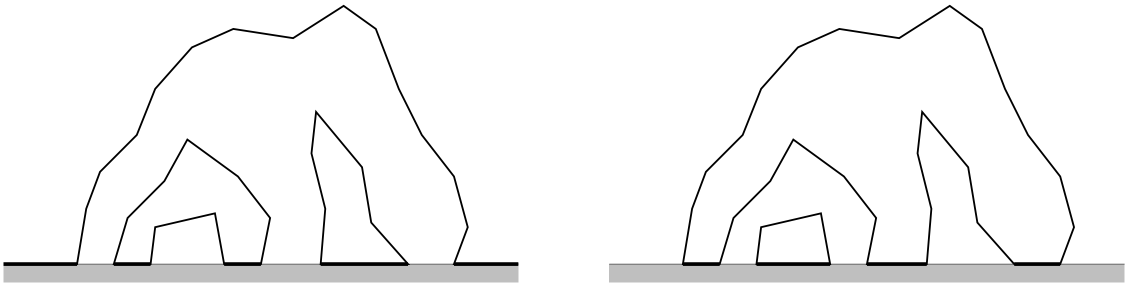


FIGURE 16. Left: The family of polygonal lines associated to $\mathfrak{S}^{\pm}$. Right: The family of *closed* polygonal lines associated to $\mathfrak{S}$.

1. $v_k \in \Sigma_N^\star \cap \gamma$ for $k = 1, \dots, 2m$ and $v_k(1) < v_{k'}(1)$ if $k < k'$;
2. $(v_1, \dots, v_m)$ split γ into pieces $\gamma_1 : v_1 \to v_2, \dots, \gamma_{2m} : v_{2m} \to v_1$, such that
   - $\gamma_{2k} \cap \mathcal{E}(\Sigma_N^\star) = \varnothing$ for all $k = 1, \dots, m$.
   - $d_{\mathbb{H}}(\gamma_{2k}, \{x \in \mathbb{R}^2 \,:\, x(2) = -1/2\}) > s(N)$ for all $k = 1, \dots, m$.
   - $d_{\mathbb{H}}(\gamma_{2k+1}, \{x \in \mathbb{R}^2 \,:\, x(2) = -1/2\}) \leqslant s(N)$ for all $k = 1, \dots, m$.

We then say that $S = (u_1, \dots, u_n)$ is an s-skeleton of γ if

- All vertices of S belong to γ.
- $v_1, \dots, v_{2m}$ are vertices of S.
- The only vertices of S along γ_{2k+1} are v_{2k+1} and v_{2k+2}, for all $k = 1, \dots, m$.
- The distance between any successive pair of vertices u_l, u_{l+1} of S along γ_{2k} satisfies $s(N)/2 \leqslant \|u_l - u_{l+1}\|_\infty \leqslant 2s(N)$, for all $k = 1, \dots, m$.
- $d_{\mathbb{H}}(\gamma, \mathrm{Pol}(S)) \leqslant s(N)$.

This definition has the nice property that either u_l and u_{l+1} both belong to $\Sigma_N^\star$, or the part of γ between these two sites is edge-disjoint from $\Sigma_N^\star$ (see Fig. 15). This allows us to use the estimates (4.4.6) and (4.4.7). This yields the following extension of (3.4.2) [PV2]

$$\mu^{\beta^*,\eta^*}_{\mathbb{D}^\star_{N,rN}}(\mathfrak{S}) \leqslant \exp\{-\mathcal{W}_{\beta,\eta}(\mathfrak{S})\}\,. \tag{4.4.8}$$

The analogue of the energy estimate (3.4.3) then follows easily, since $\tau_{\mathrm{bd}}(\beta,\eta) \geqslant 0$ when $\eta \geqslant 0$ and therefore it is still possible to control the number of vertices of $\mathfrak{S}$ in terms of $\mathcal{W}_{\beta,\eta}(\mathfrak{S})$. This gives

$$\mu^{\beta^*,\eta^*}_{\mathbb{D}^\star_{N,rN}}(\mathfrak{S} \geqslant r) \leqslant \exp\Big\{-r\big(1 - \frac{C\log N}{s(N)}\big)\Big\}\,. \tag{4.4.9}$$

Using this and the estimates in the phase of small contours, which still hold in the presence of a boundary field, the upper bound follows easily.

Second case: $\eta < 0$

As for the lower bound, we have to deal with the fact that, for $\eta < 0$, the duality is not defined. The solution is the same as there: We just change boundary conditions, i.e. we look at the measure $\mu^{\beta,|\eta|}_{N,\pm}$, which was defined when we dealt with the lower bound.

Once we have done this, the main difference is that the family of low-temperature contours of any configurations compatible with these boundary conditions contains exactly one open contour, with endpoints $t_{\mathrm{l}} = (-N - \frac{1}{2}, -\frac{1}{2})$ and $t_{\mathrm{r}} = (N + \frac{1}{2}, \frac{1}{2})$. It is straightforward to generalize the notion of skeleton introduced in the preceding case to the present situation. What we get by this procedure is a family of skeletons $\mathfrak{S}^{\pm} = (S_0, S_1, \dots, S_n)$ containing exactly one skeleton, S_0, with $\mathrm{Pol}(S_0)$ open with endpoints t_{l} and t_{r}.

Since we want to compare the corresponding families of polygonal lines with the solution of the variational problem, i.e. with the boundary of a convex body in $\widehat{\mathbb{D}}_r^2$, it is convenient to introduce another family $\mathfrak{S}$ of skeletons whose associated polygonal lines are *closed*; $\mathfrak{S}$ possesses the same set of vertices (except for t_{l} and t_{r}, but with a different set of edges, which is such that its associated family of polygonal lines satisfies

$$\mathrm{Pol}(\mathfrak{S}) = \mathrm{Pol}(\mathfrak{S}^{\pm}) \triangle \{x \in \mathbb{R}^2 \,:\, -N/2 - \tfrac{1}{2} \leqslant x(1) \leqslant N/2 + \tfrac{1}{2},\, x(2) = -\tfrac{1}{2}\}$$

where $\triangle$ denotes symmetric difference (see Fig. 16).

One then has the following relation

$$\mathcal{W}_{\beta,\eta}(\mathfrak{S}) = \mathcal{W}_{\beta,|\eta|}(\mathfrak{S}^{\pm}) - (2N+1)\; \tau_{\mathrm{bd}}(\beta,|\eta|)\,.$$

In particular, the following version of (4.4.8) holds [PV2]

$$\mu_{N,\pm}^{\beta,|\eta|}(\mathfrak{S}^{\pm}) \leqslant K_1 \, \exp\{-\mathcal{W}_{\beta,\eta}(\mathfrak{S})\} \qquad \eta > -\eta_{\mathrm{w}}(\beta) \tag{4.4.10}$$

$$\mu_{N,\pm}^{\beta,|\eta|}(\mathfrak{S}^{\pm}) \leqslant K_2 N^{3/2} \, \exp\{-\mathcal{W}_{\beta,\eta}(\mathfrak{S})\} \qquad \eta \leqslant -\eta_{\mathrm{w}}(\beta) \tag{4.4.11}$$

The energy estimate (4.4.9) is slightly more delicate now, since the wall free energy is negative. It turns out however that in the partial wetting regime, $\eta > -\eta_{\mathrm{w}}(\beta)$, it is easy to reduce ourselves to a situation similar to the case $\tau_{\mathrm{bd}}(\beta,\eta) \geqslant 0$. The case $\eta \leqslant -\eta_{\mathrm{w}}(\beta)$, i.e. complete wetting, is more subtle, but happens not to give too much problems as long as we consider volume-order large deviations (or, in fact, deviations close enough to volume order).

Let us first consider the case of partial wetting; this regime is characterized by $|\tau_{\mathrm{bd}}(\beta,\eta)| < \tau_\beta^*$. Let us write $\mathcal{W}_{\beta,\eta}(\mathfrak{S}) = T^+ + T^-$, where T^+ (T^-) is the positive (negative) part of the functional. Then, since $T^+ \geqslant (\tau_\beta^*/\tau_{\mathrm{bd}}(\beta,\eta))\, T^-$ and the number of vertices along the wall is at most two-third of the total number $\#(\mathfrak{S})$, we have

$$\#(\mathfrak{S}) \leqslant \frac{K}{s(N)(\tau_\beta^* + \tau_{\mathrm{bd}}(\beta,\eta))}\, \mathcal{W}_{\beta,\eta}(\mathfrak{S})\,,$$

for some absolute constant K. This allows to prove that

$$\mu_{N,-}^{\beta,\eta}(\mathfrak{S} \geqslant r) \leqslant \exp\Big\{-r(1 - \frac{C\log N}{s(N)})\Big\}\,. \tag{4.4.12}$$

When $\eta \leqslant -\eta_{\mathrm{w}}(\beta)$, one cannot establish so good an upper bound. The best we can do is to use the fact that $T^- \geqslant (2N+1)\,\tau_{\mathrm{bd}}(\beta,\eta)$, which turns out to be enough to prove the following, weaker, version of the energy estimate

$$\mu_{N,-}^{\beta,\eta}(\mathfrak{S} \geqslant r) \leqslant \exp\Big\{-r(1 - \frac{C\log N}{s(N)}) + C'\frac{N\log N}{s(N)}\Big\}\,. \tag{4.4.13}$$

The reason why such an estimate is still sufficient to get the desired result is that the relevant values of r are also of order N, so that the first term can always be made to dominate the second one.

Once we have (4.4.12) and (4.4.13), the proof is concluded as usual, after observing that the estimate in the phase of small contours still applies in the presence of the boundary field $|\eta|$.

4.4.2. **Ising model in $D \geqslant 3$.** The proof of Theorem 4.3.3 is based on the $\mathbb{L}_1$-Theory introduced in Part 2. We simply explain how the main ingredients of the proof should be modified and refer to [BIV] for details.

The arguments of geometric measure Theory can be extended easily to this new setting. In particular, it is straightforward to check that the functional $\mathcal{W}_{\beta,\eta}$ is lower semi-continuous and that the approximation Theorems 2.5.1 and 2.6.1 hold.

The main problem is to define proper mesoscopic phase labels for the measures with a boundary magnetic field. If $\eta \geqslant 0$, then the mesoscopic phase labels introduced in Part 2 satisfy the Assumptions A and B, as well as Conditions C1-C3 under the measure $\mu_{N,+}^{\beta,\eta}$. Instead if $\eta < 0$, some problems occur because the FK measure looses its ferromagnetic properties and the random coloring measures are more complicated to deal with. Nevertheless, it is still possible to define mesoscopic phase labels and to derive estimates as in Section 2.2.

Other difficulties have to be overcomed in order to implement the general philosophy of the $\mathbb{L}_1$-Theory. In the case of a negative boundary magnetic field, the interface induced by the field prevents us from applying directly the techniques developed to prove the exponential tightness Theorem 2.1.1. Therefore an alternative approach similar to the one described in Subsection 4.4.1 is required. The analysis of the surface tension needs also some care. We recall that the computation of surface tension is based on a localization procedure along the boundary of functions of bounded variation. For a given test function either locally its boundary is in the bulk and we recover the usual surface tension term or it intersects the wall and arguments similar to those used in the bulk enable us to derive the wall free energy. In this way the complexity of the problem is reduced because the difficult analysis of the fluctuations of the microscopic interface between the wall and the bulk is replaced by soft $\mathbb{L}_1$ estimates.

4.5. Open problems

As in the previous parts, there are still a lot of open problems. Most of those presented before have natural analogues in the present situation. In the following, we restrict ourselves to problems intrinsically related to the topics discussed in this part.

2D nearest-neighbors Ising model. The fact that one is still unable to analyze non-perturbatively the fluctuations of the phase separation line is only strengthened when we would like to study boundary effects. Indeed, a general analysis of typical open paths with endpoints at general positions with respect to the wall has not been done even at low temperature. Problems related to this are the following:

1. Give a non-perturbative proof that the probability measure of a suitably rescaled version of an open contour with endpoints on the wall converges weakly to the measure of Brownian excursion when $\eta \leqslant -\eta_{\text{w}}(\beta)$ (as was sketched in the low-temperature case for $\eta = -1$ in [D]). This would provide a way of analyzing the typical fluctuations of magnetization in the complete wetting regime, and would complete the heuristic picture of the wetting transition in the Grand-Canonical Ensemble.
2. Establish Ornstein-Zernike behavior for the boundary 2-point function without having recourse to explicit computations. Even weaker lower bounds, like those given in [AI], have not been proved in such a constrained geometry.

Another open problem is to investigate the full range of moderate deviations. This may require an understanding of point 1. above.

Higher dimensional nearest-neighbors Ising models. If fluctuations of phase separation lines are not yet understood, the situation is only much worse when considering their higher dimensional counterparts; in fact, even perturbative results are not always available. Here is a far from exhaustive list of related open problems.

1. Give a microscopic description of the behavior of phase boundaries in the partial and complete wetting regimes in the Grand-Canonical Ensemble to put some flesh on the heuristics given above.
2. Decide whether $\eta_{\mathrm{w}}(\beta) = 1$ or not. The corresponding results for the SOS model [Ch] suggest that $\eta_{\mathrm{w}}(\beta) < 1$ in any dimension; numerical investigations confirm this in dimension 3 [BL].

In fact, even much simpler problems related to behavior of higher dimensional interfaces are still open: proof of the existence of a roughening transition in $d = 3$, proof of the unstability of the $(1,1,1)$ interface, ...

In some simpler models of the SOS type some (but not all!) of these problems can be solved, but this does not seem to help in solving the original ones.

The wall. Another type of problems concerns properties of the wall. In particular, it might be interesting to answer the following questions.

1. What happens if the interaction with the wall is more complicated (say, non-nearest neighbor).
2. What happens if the boundary field is not homogeneous (for example, is a "random" configuration of η_1 and η_2 macroscopically equivalent to some well-chosen homogeneous boundary field $\eta = \overline{\eta}$?).

Part 5. Appendix

5.1. Appendix A : Proof of Theorem 2.2.1

Assumption A controls the number of zero u_k-blocks, whereas Assumption B is used to control the geometry of the mesoscopic phase labels. The dependence of k_0 on δ could be described as follows: we choose k_0 so large that

$$\rho_k \leqslant \frac{1}{C(d)}\delta \qquad \text{for every } k \geqslant k_0, \tag{5.1.1}$$

where $C(d)$ is a large enough fixed constant. Three terms on the left hand side on (2.2.3) correspond to three different exponential estimates:

5.1.1. Estimate on the volume of zero u_k-blocks.

The domination by Bernoulli measure (2.2.2) implies that

$$\mathbb{P}_N\left(\#\{x \in \widehat{\mathbb{T}}^d_{n-k} : u_k(x) = 0\} \geqslant \delta\left(\frac{N}{2^k}\right)^d\right) \leqslant c_2 \exp\left\{-\delta\left(\frac{N}{2^k}\right)^d \log\frac{\delta}{\rho_k}\right\}. \tag{5.1.2}$$

Each realization of the phase label function u_k splits $\widehat{\mathbb{T}}^d$ into the disjoint union of three mesoscopic regions:

$$\widehat{\mathbb{T}}^d = \{x : u_k(x) = 1\} \vee \{x : u_k(x) = -1\} \vee \{x : u_k(x) = 0\} \triangleq \mathbf{A}_+ \vee \mathbf{A}_- \vee \mathbf{A}_0.$$

By the choice of the scale k_0 in (5.1.1) the estimate (5.1.2) is non-trivial for every $k \geqslant k_0$, and, in view of the target claim (2.2.3), we can restrict attention only to such realizations of u_k for which

$$|\mathbf{A}_0| = \int_{\widehat{\mathbb{T}}^d} 1_{\{u_k(x)=0\}}\mathrm{d}x < \delta. \tag{5.1.3}$$

This has the following important implication: if $u_k \in \mathcal{V}(K_a, 2\delta)^{\mathrm{c}}$, the area of the boundary of any regular set A such that $\mathbf{A}_+ \subseteq A \subseteq \widehat{\mathbb{T}}^d \setminus \mathbf{A}_-$ is bounded below as

$$|\partial A| \geqslant a. \tag{5.1.4}$$

Using the Assumption B of the Theorem we are going to construct such sets A on the finite k_0 scale; $A \in \mathcal{F}_{n-k_0}$, and in such a fashion that all the boundary k_0-blocks of A will necessarily have zero u_{k_0}-labels. This reduction enables a uniform treatment of all coarser scales $k \geqslant k_0$.

So let $k \geqslant k_0$, and assume that (5.1.3) holds. We denote by A_- (resp. A_+) the set of all boxes $\widehat{\mathbb{B}}_{n-k_0}$ in $\mathbf{A}_-$ (resp $\mathbf{A}_+$). We say that $x \in \widehat{\mathbb{T}}^d_{n-k_0}$ is $-*$ connected to A_-; $x \overset{-*}{\longleftrightarrow} A_-$, if there exists a $*$-connected chain of "$-$" u_{k_0} blocks leading from $\widehat{\mathbb{B}}_{n-k_0}(x)$ (and including it) to A_-. Define now the complement A^{c} as follows:

$$A^{\mathrm{c}} = A_- \bigcup_{x \overset{-*}{\longleftrightarrow} A_-} \widehat{\mathbb{B}}_{n-k_0}(x).$$

By the virtue of the Assumption B, $\mathbf{A}_+ \subseteq A$. Moreover, by construction all the k_0-blocks of A attached to the boundary ∂A^c have zero u_{k_0}-labels. With a slight abuse of notation we proceed to denote this collection of boundary k_0-blocks as ∂A. By (5.1.4) the number of k_0-blocks in ∂A is bounded below by

$$\#_{k_0}(\partial A) \geqslant \frac{c(d)a}{2^{(d-1)k_0}} N^{d-1}. \tag{5.1.5}$$

Since, however, the total number of k_0-blocks in the corresponding decomposition of $\widehat{\mathbb{T}}^d$ equals to $N^d/2^{dk_0}$ the estimate (5.1.5) alone is not sufficient for giving the desirable upper bound on the probability $\mathbb{P}_N\left(u_k \in \mathcal{V}(K_a, 2\delta)^{\mathrm{c}}\right)$. The required entropy cancelation stems from the fact that small connected contours of ∂A cannot surround too much volume.

Let us decompose A to the disjoint union of its maximal connected components:

$$A \;=\; \bigvee_{i=1}^{l} A_i \qquad \text{respectively} \qquad \partial A \;=\; \bigvee_{i=1}^{l} \partial A_i .$$

We shall quantify contours ∂A_i according to the size (or the number of k_0-blocks) in A_i. Namely, the contour ∂A_i is called small, if

$$\#_{k_0}(A_i) \;\leqslant K(d)\log N \qquad \text{or} \qquad |A_i| \;\leqslant K(d)\frac{2^{dk_0}}{N^d}\log N, \tag{5.1.6}$$

where $K(d)$ is a sufficiently large constant. Otherwise, the contour ∂A_i is called large.

We claim that under (5.1.3) the following inclusion is valid:

$$\{u_k \in \mathcal{V}(K_a, 2\delta)^{\mathrm{c}}\} \;\subseteq\; \left\{ \sum_{\partial A_i - \text{small}} |A_i| > \delta \right\} \bigcup \left\{ \sum_{\partial A_i - \text{large}} |\partial A_i| > a \right\}. \tag{5.1.7}$$

Indeed, if the total volume inside small contours is less than δ, then repainting all the small components A_i into "-1" and all the large components A_j into "$+1$" we produce a $\{\pm 1\}$-valued function which is at most at the $\mathbb{L}_1$-distance 2δ from u_k and which, thereby, cannot belong to K_a.

5.1.2. **Peierls estimate on the size of large contours.**

$$\begin{aligned}\mathbb{P}_N\left(\sum_{\partial A_i - \text{large}} |\partial A_i| > a \right) \;&=\; \mathbb{P}_N\left(\sum_{\partial A_i - \text{large}} \#_{k_0}(\partial A_i) > \frac{c(d)a}{2^{(d-1)k_0}} N^{d-1} \right) \\ &\leqslant \exp\left\{ -c_3(d)\frac{a}{2^{(d-1)k_0}} N^{d-1} \right\}.\end{aligned} \tag{5.1.8}$$

This immediately follows from Assumption A, once the constant $K(d)$ in (5.1.6) has been properly chosen.

5.1.3. **Estimate in the phase of small contours.** The volume of small components A_i is related to the total number of k_0-blocks in these components as

$$\sum_{\partial A_i - \text{small}} |A_i| \;=\; \left(\frac{N}{2^{k_0}}\right)^{-d} \sum_{\partial A_i - \text{small}} \#_{k_0}(A_i).$$

On the other hand, for every $l \in [1, ..., n - k_0]$;

$$\begin{aligned}\sum_{\partial A_i - \text{small}} \#_{k_0}(A_i) \;&=\; \sum_{x \in \widehat{\mathbb{T}}^d_{n-k_0}} \sum_{\partial A_i - \text{small}} 1_{\{x \in A_i\}} \\ &=\; \sum_{t \in [0, ..., 2^l)^d} \sum_{x \in \widehat{\mathbb{T}}^d_{n-k_0-l}} \sum_{\partial A_i - \text{small}} 1_{\{\theta_{t\Delta_0} x \in A_i\}},\end{aligned}$$

where $\Delta_0 \triangleq 2^{k_0-n}$ is the step size on the embedded torus $\widehat{\mathbb{T}}^d_{n-k_0}$, and $\theta_\bullet$ is the shift on this torus. Consequently,

$$\mathbb{P}_N\left(\sum_{\partial A_i-\text{small}} |A_i| > \delta\right) \leqslant \max_{t\in[0,\dots,2^l)^d} \mathbb{P}_N\left(\sum_{x\in\widehat{\mathbb{T}}^d_{n-k_0-l}} \sum_{\partial A_i-\text{small}} 1_{\{\theta_{t\Delta_0}x\in A_i\}} > \delta\left(\frac{N}{2^{k_0+l}}\right)^d\right). \tag{5.1.9}$$

If, however, $2^l > K(d)\log N$, then no two distinct points on the torus $\widehat{\mathbb{T}}^d_{n-k_0-l}$ (or any shift of it) can belong to the same small component A_i. This, in view of the domination by the independent Bernoulli site percolation (Assumption A), suggests an application of the B-K inequality. Since, by the choice of the scale k_0 in (5.1.1);

$$\epsilon_{k_0} \triangleq \mathbb{P}^{\rho_{k_0}}_{\text{perc}}\left(\exists \text{ a closed surface of zero } u_{k_0}\text{-blocks around } x\right) < \delta,$$

for every $x \in \widehat{\mathbb{T}}^d_{n-k_0}$, we readily obtain that the right hand side of (5.1.9) is bounded above by

$$c_4(d)\exp\left\{-\delta\left(\frac{N}{2^{k_0+l}}\right)^d \log\left(\frac{\delta}{\epsilon_{k_0}}\right)\right\}.$$

The proof of Theorem 2.2.1 is concluded. ∎

5.2. Appendix B : Proof of the three-point lower bound Lemma 3.4.3

The proof of Lemma 3.4.3 is based on the following positive stiffness property of the surface tension [AA]:

$$\min_{\theta\in[0,2\pi]}\left\{\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\tau_\beta\left(\vec{n}(\theta)\right) + \tau_\beta\left(\vec{n}(\theta)\right)\right\} = \min_{\theta\in[0,2\pi]} R_\beta\left(\vec{n}(\theta)\right) > 0. \tag{5.2.10}$$

where the unit normal $\vec{n}(\theta)$ is defined via $\vec{n}(\theta) = (\cos\theta, \sin\theta)$, and $R_\beta(\vec{n})$ is the radius of curvature of $\partial\mathcal{K}$ at the point supporting the tangent line orthogonal to $\vec{n}$. An integral version of (5.2.10) is the strong triangle inequality [I1], [V]: For any $u, v \in \mathbb{R}^2$:

$$\tau_\beta(u) + \tau_\beta(v) - \tau_\beta(u+v) \geqslant c_1(\beta)\left(\|u\|_2 + \|v\|_2 - \|u+v\|_2\right). \tag{5.2.11}$$

The latter inequality is used to control the fluctuations of the microscopic phase boundaries (in their random line representation of Section 3.4).

Let now an (s,ε)-compatible triple of points (u,w,v) be given. Fix $K = K(\beta)$ large enough and define the "oval" neighborhood $\mathbf{N}_K(u,w)$ of $\{u,v\}$ as:

$$\mathbf{N}_K(u,w) \triangleq \left\{z\in\mathbb{R}^2 : \tau_\beta(z-u) + \tau_\beta(w-z) - \tau_\beta(w-u) \leqslant K\log s\right\}.$$

The oval neighborhood $\mathbf{N}_K(w,v)$ is defined exactly in the same fashion. Relations (3.4.8) and (3.4.11) readily imply that that the main contribution to $\langle\sigma_u\sigma_w\rangle^{\beta^*}_f$ (respectively to $\langle\sigma_w\sigma_v\rangle^{\beta^*}_f$) comes from the paths λ_1 (respectively λ_2) which stay in $\mathbf{N}_K(u,w)$ (respectively $\mathbf{N}_K(w,v)$). More precisely,

$$\sum_{\substack{\lambda_1:u\to w\\ \lambda\in\mathbf{N}_K(u,w)}} q^{\beta^*}(\lambda_1) \geqslant \langle\sigma_u\sigma_w\rangle^{\beta^*}_f\left(1+\mathrm{o}(1)\right), \tag{5.2.12}$$

uniformly in all (s, ε)-compatible triples. Any such path $\lambda_1 = (\lambda_1(0), ..., \lambda_1(n_1))$ could be decomposed as follows: Define

$$n_w \;=\; \max\{k : \lambda_k \in \mathbf{N}_K(u,w) \setminus \mathbf{N}_K(w,v)\},$$

and set $\lambda_1^u = (\lambda_1(0), ..., \lambda_1(n_w))$, $\lambda_1^w = (\lambda_1(n_w+1), ..., \lambda_1(n_1))$; $\lambda_1 = \lambda_1^u \vee \lambda_1^w$. The decomposition $\lambda_2 = \lambda_2^u \vee \lambda_2^w$ is defined in a completely symmetric way. Notice that, by the construction, the paths λ_1^u and λ_2^v are disjoint and compatible, and, by (5.2.11);

$$\max\{\|\lambda_1(n^w) - w\|_2, \|\lambda_2(n^w) - w\|_2\} \;\leqslant c_2(\varepsilon)\log s.$$

The claim of the lemma follows now from (3.4.6) and (3.4.7). $\square$

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