Operator interpretation of resonances arising in spectral problems for 2×2 operator matrices

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Abstract. We consider operator matrices $\mathbf{H} = \begin{pmatrix} A_0 & B_{01} \\ B_{10} & A_1 \end{pmatrix}$ with self-adjoint entries A_i, $i = 0, 1$, and bounded $B_{01} = B_{10}^*$, acting in the orthogonal sum $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ of Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}_1$. We are especially interested in the case where the spectrum of, say, A_1 is partly or totally embedded into the continuous spectrum of A_0 and the transfer function $M_1(z) = A_1 - z + V_1(z)$, where $V_1(z) = B_{10}(z - A_0)^{-1}B_{01}$, admits analytic continuation (as an operator-valued function) through the cuts along branches of the continuous spectrum of the entry A_0 into the unphysical sheet(s) of the spectral parameter plane. The values of z in the unphysical sheets where $M_1^{-1}(z)$ and consequently the resolvent $(\mathbf{H} - z)^{-1}$ have poles are usually called resonances. A main goal of the present work is to find non-selfadjoint operators whose spectra include the resonances as well as to study the completeness and basis properties of the resonance eigenvectors of $M_1(z)$ in $\mathcal{H}_1$. To this end we first construct an operator-valued function $V_1(Y)$ on the space of operators in $\mathcal{H}_1$ possessing the property: $V_1(Y)\psi_1 = V_1(z)\psi_1$ for any eigenvector ψ_1 of Y corresponding to an eigenvalue z and then study the equation $H_1 = A_1 + V_1(H_1)$. We prove the solvability of this equation even in the case where the spectra of A_0 and A_1 overlap. Using the fact that the root vectors of the solutions H_1 are at the same time such vectors for $M_1(z)$, we prove completeness and even basis properties for the root vectors (including those for the resonances).

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1. INTRODUCTION

In this paper we deal with 2×2 operator matrices

$$\mathbf{H} = \begin{pmatrix} A_0 & B_{01} \\ B_{10} & A_1 \end{pmatrix}, \qquad (1.1)$$

acting in an orthogonal sum $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ of separable Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}_1$. The entries $A_0 : \mathcal{H}_0 \to \mathcal{H}_0$, and $A_1 : \mathcal{H}_1 \to \mathcal{H}_1$, are assumed to be self-adjoint operators with domains $\mathcal{D}(A_0)$ and $\mathcal{D}(A_1)$, respectively. It is assumed that the couplings $B_{ij} : \mathcal{H}_j \to \mathcal{H}_i$, $i, j = 0, 1$,

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$i \neq j$, are bounded operators (i. e., $B_{ij} \in \mathbf{B}(\mathcal{H}_j, \mathcal{H}_i)$) and $B_{01} = B_{10}^*$. Under these assumptions the matrix $\mathbf{H}$ is a self-adjoint operator in $\mathcal{H}$ with domain $\mathcal{D}(\mathbf{H}) = \mathcal{D}(A_0) \oplus \mathcal{D}(A_1)$.

Operators of the form (1.1) arise in many of physical problems (see e. g., [8,11,15,17–19,23,32–34,37,41]), typically as a result of decomposing the Hilbert space $\mathcal{H}$ of a quantum system in two "channel" subspaces. The first one, say $\mathcal{H}_0$, may be interpreted as a space of "external" (for example hadronic as in [8,15,41]) degrees of freedom. The second one, $\mathcal{H}_1$, is associated with an "internal" (for example, quark [8,15,41]) structure of the system. We mention also that spectral problems for a class of 2×2 operator matrices arise in magnetohydrodynamics [12,21].

In the spectral theory of operators of the form (1.1) an important role is played by the *transfer functions*

$$M_i(z) = A_i - z + V_i(z), \quad i = 0, 1, \tag{1.2}$$

where

$$V_i(z) = -B_{ij} R_j(z) B_{ji}, \quad j \neq i. \tag{1.3}$$

Hereafter the notations $R_j(z)$ are used for the resolvents of the operators A_j, $R_j(z) = (A_j - zI_j)^{-1}$ where I_j stands for the identity operator in $\mathcal{H}_j$. A particular role of the transfer functions $M_i(z)$ can be understood already from the fact that the resolvent $\mathbf{R}(z)$ of the operator $\mathbf{H}$, $\mathbf{R}(z) = (\mathbf{H} - z\mathbf{I})^{-1}$ where $\mathbf{I}$ is the identity operator in $\mathcal{H}$, can be expressed explicitly in terms of $M_0(z)$ or $M_1(z)$:

$$\begin{aligned}
\mathbf{R}(z) &= \begin{pmatrix} R_{00}(z) & R_{01}(z) \\ R_{10}(z) & R_{11}(z) \end{pmatrix} \\
&= \begin{pmatrix} M_0^{-1}(z) & -M_0^{-1}(z) B_{01} R_1(z) \\ -R_1(z) B_{10} M_0^{-1}(z) & R_1(z) + R_1(z) B_{10} M_0^{-1}(z) B_{01} R_1(z) \end{pmatrix} \qquad (1.4) \\
&= \begin{pmatrix} R_0(z) + R_0(z) B_{01} M_1^{-1}(z) B_{10} R_0(z) & -R_0(z) B_{01} M_1^{-1}(z) \\ -M_1^{-1}(z) B_{10} R_0(z) & M_1^{-1}(z) \end{pmatrix}.
\end{aligned}$$

It follows from the representations (1.4) that $\mathbf{R}(z)$ and, hence, its components R_{ij}, $i, j = 0, 1$, may partly inherit the singularities of the channel resolvents $R_0(z)$ and $R_1(z)$. However, all the nontrivial singularities of $\mathbf{R}(z)$, differing from those of $R_0(z)$ and $R_1(z)$, are singularities of the inverse transfer functions $R_{00}(z) = M_0^{-1}(z)$ and/or $R_{11}(z) = M_1^{-1}(z)$. Therefore, in studying the spectral properties of the transfer functions one studies at the same time the spectral properties of the initial operator matrix $\mathbf{H}$.

Often the study of the spectral properties of the transfer functions indeed turns out to be a simpler task than an immediate study of the spectral problem for the total matrix (1.1). In particular, the described reduction of the spectral problem $\mathbf{H}\Psi = z\Psi$ for an initial two-channel Hamiltonian $\mathbf{H}$ of the form (1.1) to the channel spectral problems

$$\Big(A_i + V_i(z)\Big)\psi^{(i)} = z\psi^{(i)} \tag{1.5}$$

is common place in quantum physics where the perturbations $V_i(z)$ are called energy-dependent potentials, energy–dependent interactions *etc.* Regarding this subject see, e. g., the papers [26,28,40] discussing some problems related to the use of the energy–dependent potentials in the physics of few-body systems.

In the case where one of the spaces $\mathcal{H}_0$ and $\mathcal{H}_1$ is finite-dimensional, say, the space $\mathcal{H}_i$, the respective transfer function $M_i(z)$ is also known as the Livšic matrix [22] (see Ref. [14] for applications of the Livšic matrices to perturbation theory and for further references).

In the papers [7,26] the following question was raised: *Is it possible to introduce an operator H_i, $i=0,1$, independent of the spectral parameter z, such that its spectrum coincides with the spectrum of Eq.* (1.5) *while the eigenvector of H_i is at the same time an eigenvector of the transfer function M_i (i. e., $H_i\psi^{(i)} = z\psi^{(i)}$ implies that* (1.5) *holds)?* Obviously, having found such an operator one would reduce the spectral problem for the transfer-function $M_i(z)$ to the standard spectral problem for the operator H_i and, thus, the questions regarding completeness and basis properties for the eigenvectors of M_i could be answered in terms of the operator H_i referring to well known facts from operator theory.

A rigorous answer to the above question was found by one of the authors [28–30,32] in the case where the spectra $\sigma(A_0)$ and $\sigma(A_1)$ of the entries A_0 and A_1 can be interwoven with each other but must be strictly separated,

$$\operatorname{dist}\{\sigma(A_0), \sigma(A_1)\} > 0. \tag{1.6}$$

To this end an operator-valued function $V_i(Y_i)$ on the space of linear operators in $\mathcal{H}_i$ was constructed in [28–30,32] such that

$$V_i(Y_i)\psi^{(i)} = V_i(z)\psi^{(i)} \tag{1.7}$$

for any eigenvector $\psi^{(i)}$ corresponding to an eigenvalue z of the operator Y_i. The desired operator H_i was searched for as a solution of the operator equation

$$H_i = A_i + V_i(H_i), \qquad i = 0,1. \tag{1.8}$$

Notice that an equation of the form (1.8) first appeared explicitly in the paper [7] by M. A. Braun. Obviously, if H_i is a solution of Eq. (1.8) and $H_i\psi^{(i)} = z\psi^{(i)}$ then, due to (1.7), automatically $z\psi^{(i)} = \Big(A_i + V_i(H_i)\Big)\psi^{(i)} = \Big(A_i + V_i(z)\Big)\psi^{(i)}$ and, thus, for these z and $\psi^{(i)}$ the equality (1.5) holds. The solvability of the equation (1.8) was announced in [29] and proved in [28,30] under the assumption

$$\|B_{ij}\|_2 < \frac{1}{2}\operatorname{dist}\{\sigma(A_0), \sigma(A_1)\} \tag{1.9}$$

where $\|B_{ij}\|_2$ stands for the Hilbert-Schmidt norm of the couplings B_{ij}. It was found [28,30] that the problem of constructing the operators H_i is closely related to the problem of searching for the invariant subspaces $\mathcal{G}_i$, $i=0,1$, of the matrix $\mathbf{H}$ which admit the graph representations,

$$\begin{aligned}
\mathcal{G}_0 &= \left\{ u \in \mathcal{H} : \; u = \begin{pmatrix} u_0 \\ Q_{10}u_0 \end{pmatrix}, \; u_0 \in \mathcal{H}_0 \right\}, \\
\mathcal{G}_1 &= \left\{ u \in \mathcal{H} : \; u = \begin{pmatrix} Q_{01}u_1 \\ u_1 \end{pmatrix}, \; u_1 \in \mathcal{H}_1 \right\},
\end{aligned} \tag{1.10}$$

with bounded $Q_{ji} : \mathcal{H}_i \to \mathcal{H}_j$ such that $Q_{ij} = -Q_{ji}^*$ and $\mathcal{H} = \mathcal{G}_0 \oplus \mathcal{G}_1$. The point is that under the assumption (1.9) the solutions H_i, $i = 0, 1$, of Eqs. (1.8) read

$$H_i = A_i + B_{ij} Q_{ji} \tag{1.11}$$

where Q_{ji} are contractions just realizing the above representations (1.10). The operators Q_{ji} satisfy the stationary Riccati equations

$$Q_{ji} A_i - A_j Q_{ji} + Q_{ji} B_{ij} Q_{ji} = B_{ji}, \qquad i, j = 0, 1, \quad j \neq i \tag{1.12}$$

while the similarity transform $\mathbf{H}' = \mathcal{Q}^{-1} \mathbf{H} \mathcal{Q}$ with $\mathcal{Q} = \begin{pmatrix} I_0 & Q_{01} \\ Q_{10} & I_1 \end{pmatrix}$ reduces the operator $\mathbf{H}$ to the block-diagonal form $\mathbf{H}' = \operatorname{diag}\{H_0, H_1\}$. Under the condition (1.9) the spectra $\sigma(H_0)$ and $\sigma(H_1)$ do not intersect each other, i. e.,

$$\sigma(H_0) \cap \sigma(H_1) = \emptyset, \tag{1.13}$$

and

$$\sigma(\mathbf{H}) = \sigma(H_0) \cup \sigma(H_1) \tag{1.14}$$

while the H_i, $i = 0, 1$, represent parts (spectral components) of the operator $\mathbf{H}$ in the corresponding invariant subspaces $\mathcal{G}_i$.

The idea of the block diagonalization of the 2×2 operator matrices in terms of the invariant subspaces (1.10) is a rather old one going back to the paper [33] by S. Okubo (regarding applications of Okubo's approach to particle physics see, e. g., [11,17] and Refs. cited therein). In a mathematically rigorous way this idea was used by V. A. Malyshev and R. A. Minlos in their method [23] for the construction of invariant subspaces for a class of selfadjoint operators in statistical physics. Regarding a proof of solvability of the Riccati equations (1.12), the techniques of Ref. [23] are restricted to the case where the norms of the entries B_{ij} are sufficiently small and the separation condition (1.6) holds, too. Recently, the existence of invariant subspaces of the form (1.10) was proved by V. M. Adamyan and H. Langer [2] for arbitrary bounded entries B_{ij} however assuming, instead of the condition (1.6), the essentially different assumption that the spectrum of one of the entries A_i, $i = 0, 1$ is situated strictly below the spectrum of the other one, say

$$\max \sigma(A_1) < \min \sigma(A_0)\,. \tag{1.15}$$

Soon, the result of [2] was extended by V. M. Adamyan, H. Langer, R. Mennicken and J. Saurer [3] to the case where

$$\max \sigma(A_1) \leq \min \sigma(A_0) \tag{1.16}$$

and where the couplings B_{ij} were allowed to be unbounded operators such that, for $\alpha_0 < \min \sigma(A_0)$, the product $(A_0 - \alpha_0)^{-1/2} B_{01}$ makes sense as a bounded operator. The conditions (1.15), (1.16) were then somewhat weakened by R. Mennicken and A. A. Shkalikov [27] in the case of a bounded entry A_1 and the same type of entries B_{ij} as in [3]. Instead of the explicit conditions (1.15), (1.16) on the spectra of A_i, the paper [27] uses a condition on the spectrum of the transfer function $M_1(z)$ requiring the existence of a regular point $\beta > \min \sigma(A_1)$ for M_1 such that

$$M_1(\beta) \le \alpha_1 < 0. \tag{1.17}$$

This condition still allows one to prove the existence of the invariant subspaces of $\mathbf{H}$ in the form (1.10) [27]. It should be noted that the condition (1.17) may hold even in the case where the spectra $\sigma(A_0)$ and $\sigma(A_1)$ weakly overlap but that the requirement above regarding the unboundedness of the entries B_{ij} is strictly necessary in this case (see [27]). For all the cases considered in [2,3,27] the relations (1.13) and (1.14) also hold true. However, if the coupling B_{ij} is unbounded, the Riccati equation for the operator Q_{ji} determining the representations (1.10) must be written in general in a more complicated form as compared to Eq. (1.12) (see details in [3,27]). One can check nevertheless that the spectral component H_i of the matrix $\mathbf{H}$ constructed in [2,3,27] satisfies the equation (1.8), at least in the case where for $j \ne i$ the entry A_j is bounded.

In the present work we study the equation (1.8) in a case which is totally different from the spectral situations considered in [2,3,23,27–30]; namely, we suppose from the beginning that $\sigma(A_0) \cap \sigma(A_1) \ne \emptyset$. In fact, we are especially interested in the case where the spectrum of, say A_1, is partly or totally embedded into the continuous spectrum of A_0. Some remarks concerning the solvability of the equation (1.8) in this case may be found only in Ref. [32].

We work under the assumption that the coupling operators B_{ij} are such that the transfer function $M_1(z)$ admits analytic continuation, as an operator-valued function, under the cuts along the branches of the absolutely continuous spectrum $\sigma_{ac}(A_0)$ of the entry A_0. Among other things, Sect. 2 includes a detailed description of the conditions on B_{ij} making such a continuation of $M_1(z)$ possible.

The problem considered is closely related to the resonances generated by the matrix $\mathbf{H}$. Regarding a definition of the resonance and a history of the subject see, e. g., the books [4,9,5,39]. A recent survey of the literature on resonances can be found in [31]. Throughout the paper we treat resonances as the discrete spectrum of the transfer function $M_1(z)$ situated in the so-called unphysical sheets of its Riemann surface. One can find some definitions regarding the unphysical sheets and the resonances in Sect. 2.

Sect. 3 starts with adjusting the definition of the function $V_1(Y)$ of Refs. [28–30] to the spectral situation considered here. Since we deal with m ($1 \le m < \infty$) distinct intervals of the absolutely continuous spectrum of the entry A_0, we get as a result 2^m variants of the function $V_1(Y)$ and, consequently, 2^m different variants of the equation (1.8) which read now as Eq. (3.7). This circumstance corresponds to the 2^m possible ways of realizing the analytic continuation of the transfer function $M_1(z)$ under m distinct cuts into the unphysical sheet(s) neighboring the physical one. It should be stressed that in this paper we deal only with the neighboring unphysical sheets. For convenience, Eq. (3.7) is referred to as the basic equation in the following. The solvability of this equation is proved[1] under the assumption (3.11) recalling the condition (1.9) but without already requiring the entries B_{ij} to be of the Hilbert-Schmidt class. The solutions of (3.7) represent non-selfadjoint operators the spectrum of which includes the resonances. In general, these operators are not even dissipative.

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[1]It should be noted that having solved the basic equation (3.7) one can find as well some formal solutions for the Riccati equations (1.12). However, in this case the formal solutions Q_{ij} of (1.12) can not be treated in the conventional operator sense. A generalized interpretation of these solutions as well as the construction of the generalized invariant subspaces (1.10) are beyond the scope of the present work and will be a subject of another paper.

In Sect. 4 we first prove the factorization theorem (Theorem 4.1) for the transfer function $M_1(z)$. It follows from this theorem that there exist certain domains surrounding the set $\sigma(A_1)$ and lying partly in the unphysical sheet(s) where the spectrum of M_1 is represented only by the spectrum of the respective solutions of the basic equation (3.7). Since the root vectors of these solutions are also root vectors for M_1, this fact allows us to talk further, in Sect. 5–7, about completeness and basis properties[2] of the root vectors of the transfer function M_1 corresponding to its spectrum in the above domains, including the resonance spectrum. In Sect. 4 we describe as well some relations between the different solutions of (3.7) and some relations between their spectra which reflect the symmetry of the resonance sets with respect to the real axis.

In Sect. 5 we pay special attention to the real point spectrum of the solutions of the basic equation (3.7) and, thereby, to this part of the spectrum of the transfer function M_1 as well. It is found in this section that the real eigenvalues of all the considered solutions of (3.7) are the same. Furthermore, we establish that the real isolated eigenvalues of these solutions correspond to the same algebraic eigenspaces which consist in this case only of eigenvectors. We prove the basis property of these eigenvectors with respect to their closed linear span.

In contrast to Sect. 2–5 we suppose in Sect. 6 and 7 that the entry A_1 only has discrete spectrum.

Sect. 6 is devoted to a detailed consideration of the case where the space $\mathcal{H}_1$ is finite-dimensional. In particular, we describe in this section the relations between the eigenprojections and eigennilpotents corresponding to the resonances.

The results of Sect. 7 are obtained under the assumption that the operator A_1 has a compact resolvent. Here, to prove the completeness and basis properties for the root vector systems of the solutions of Eq. (3.7), we rely mainly on the respective statements regarding non-selfadjoint operators from the books by I. C. Gohberg and M. G. Krein [13] and by T. Kato [16].

In Sect. 8 we present an illustration of the results obtained for a simple example going back to one of the Friedrichs models [10] while Appendices A and B contain some auxiliary material used throughout the paper.

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[2]Note that these results recall those of the Lax–Phillips scattering theory [20] (see also [1,35,36] and Refs. therein) where resonances appear as the spectrum of a dissipative operator representing the generator of the compressed evolution (semi)group and this implies completeness and basis properties of the resonance states in a translationally invariant subspace.

2. TRANSFER FUNCTION: ANALYTIC CONTINUATION THROUGH THE CONTINUOUS SPECTRUM

The transfer function $M_i(z)$, $i=0,1$, considered on the resolvent set $\rho(A_j)$ of the entry A_j, $j\neq i$, represents a particular case of a holomorphic operator-valued function. In the present work we use the standard definition of holomorphy of an operator-valued function with respect to the operator norm topology. Namely:

Let D be a domain in $\mathbb{C}$ and $\mathbf{B}(\mathcal{G}',\mathcal{G}'')$ the Banach space of bounded operators between the Hilbert spaces $\mathcal{G}'$ and $\mathcal{G}''$. A mapping $T:\ D\to\mathbf{B}(\mathcal{G}',\mathcal{G}'')$ is said to be a holomorphic operator-valued function on D if it is differentiable at every $z\in D$ with respect to the operator norm topology.

Let $T(z)=A+F(z)$ with $A:\mathcal{G}'\to\mathcal{G}''$ a closed (in particular self-adjoint if $\mathcal{G}'=\mathcal{G}''\equiv\mathcal{G}$) operator on a domain $\mathcal{D}(A)$ and $F:\ D\to\mathbf{B}(\mathcal{G}',\mathcal{G}'')$. Such a function T is called holomorphic on D if F is holomorphic on D.

Each transfer function $M_i(z)$, $i=0,1$, is holomorphic at least in the resolvent set $\rho(A_j)$ of the entry A_j, $j\neq i$. Since the inverse transfer functions $M_i^{-1}(z)$ coincide with the respective block components $R_{ii}(z)$ of the resolvent $\mathbf{R}(z)$, they are both holomorphic at least in the set $\rho(\mathbf{H})$.

One can extend to operator-valued functions the usual definitions of the spectrum and its components. We recall these definitions here, restricting ourselves to the cases above of holomorphic functions $T(\cdot)$ on a domain $D\subset\mathbb{C}$ where either the function $T(\cdot)$ is bounded itself, $T:\ D\to\mathbf{B}(\mathcal{G},\mathcal{G})$, or $T(\cdot)$ is a sum, $T(\cdot)=A+F(\cdot)$, of a fixed closed operator A in a Hilbert space $\mathcal{G}$ and a bounded function $F:\ D\to\mathbf{B}(\mathcal{G},\mathcal{G})$. A point $\lambda\in D$ is called a regular point of the function T if $T^{-1}(\lambda)\in\mathbf{B}(\mathcal{G},\mathcal{G})$ exists. The set $\rho(T)$ consisting of all the regular points $\lambda\in D$ is called the resolvent set of the function T in the domain D. The set $\sigma(T)=D\setminus\rho(T)$ is called the spectrum of T in D. If $\operatorname{Ker}T(\lambda)\neq\{0\}$, $\lambda\in D$, then one says the λ is an eigenvalue of T in D, $\lambda\in\sigma_p(T)\cap D$. If $x\in\operatorname{Ker}T(\lambda)$, $x\neq 0$, and, thus, $T(\lambda)x=0$, then such an x is called an eigenvector corresponding to the eigenvalue λ. The continuous spectrum $\sigma_c(T)$ of the function T in D is introduced as the set of all those points $\lambda\in D$ for which the image $\mathcal{R}(T(\lambda))$ does not coincide with its closure, $\mathcal{R}(T(\lambda))\neq\overline{\mathcal{R}(T(\lambda))}$ (cf. Ref. [6], §7.3 of Chapter 3). Obviously, the standard definitions for the spectrum of a closed operator A coincide with the definitions above if one takes $T(z)=A-z$ and $D=\mathbb{C}$.

Let E_j be the spectral measure for the entry A_j, $A_j=\displaystyle\int_{\sigma(A_j)}\lambda\,dE_j(\lambda)$, $j=0,1$, $\sigma(A_j)\subset\mathbb{R}$. Then the functions $V_i(z)$ given in (1.3) can be written

$$V_i(z)=B_{ij}\int\limits_{\sigma(A_j)}dE_j(\mu)\frac{1}{z-\mu}B_{ji}.$$

Thus, it is convenient to introduce the quantities

$$\mathcal{V}_j(B)=\|B_{ij}\|^2_{E_j}$$

where, by definition,

$$\|B_{ij}\|^2_{E_j}=\|B_{ji}\|^2_{E_j}=\sup_{\{\delta_k\}}\sum_k\|B_{ij}E_j(\delta_k)B_{ji}\|,$$

with $\{\delta_k\}$ being a finite or countable complete system of pairwise nonintersecting subsets of the spectrum $\sigma(A_j)$ being measurable with respect to E_j (i.e., δ_k are Borel subsets of $\sigma(A_j)$ such that $\delta_k \cap \delta_l = \emptyset$, if $k \neq l$ and $\bigcup\limits_k \delta_k = \sigma(A_j)$). The number $\mathcal{V}_j(B)$ is called the variation of the operators B_{ij} with respect to the spectral measure E_j. At the same time the quantity $\|B_{ij}\|_{E_j} = \|B_{ji}\|_{E_j}$ will be called the norm of the operators B_{01} and B_{10} with respect to this measure. Some properties of such a norm are described below in Appendix A. It follows from the results of this appendix that $\mathcal{V}_j(B)$ satisfies the estimates

$$\|B_{ij}\|^2 \leq \mathcal{V}_j(B) \leq \|B_{ij}\|_2^2$$

where $\|B_{ij}\|_2 = \|B_{ji}\|_2$ is the Hilbert-Schmidt norm of the couplings B_{ij}. The equality $\mathcal{V}_j(B) = \|B_{ij}\|^2$ is attained in the case where A_j is a multiple of the identity operator. The equality $\mathcal{V}_j(B) = \|B_{ij}\|_2^2$ holds if A_j possesses only a pure discrete spectrum which is at the same time simple.

Note that along with the "total" variation $\mathcal{V}_j(B)$ we shall use the "truncated" variations

$$\mathcal{V}_j(B)\Big|_{\Delta} = \sup_{\{\delta_k\}} \sum_k \|B_{ij} E_j(\delta_k \cap \Delta) B_{ji}\|$$

where Δ is a certain Borel subset of $\sigma(A_j)$. Obviously, for any such $\Delta \subset \sigma(A_j)$ one has $\mathcal{V}_j(B)\Big|_{\Delta} \leq \mathcal{V}_j(B)$.

Further, we shall suppose that at least one of the variations $\mathcal{V}_j(B)$, $j = 0, 1$, is finite; say, for example, the variation $\mathcal{V}_0(B)$:

$$\mathcal{V}_0(B) < \infty. \tag{2.1}$$

We assume that the spectrum of the operator A_1 intersects only the continuous spectrum of the operator A_0 and this intersection is only realized on (every of) the pairwise nonintersecting open intervals (see Fig. 1) $\Delta_k^0 = (\mu_k^{(1)}, \mu_k^{(2)}) \subset \sigma_c(A_0)$, $\mu_k^{(1)} < \mu_k^{(2)}$, $k = 1, 2, \ldots, m$, $m < \infty$, and $-\infty \leq \mu_1^{(1)}$, $\mu_m^{(2)} \leq +\infty$. Therefore, we assume that $\Delta_k^0 \cap \sigma(A_1) \neq \emptyset$ for all $k = 1, 2, \ldots, m$ and $\sigma(A_1) \cap \sigma'(A_0) = \emptyset$ where $\sigma'(A_0) = \sigma(A_0) \setminus \bigcup\limits_{k=1}^{m} \Delta_k^0$ denotes the remaining spectrum of A_0.

We shall suppose that the product

$$K_B(\mu) \stackrel{\text{def}}{=} B_{10} E^0(\mu) B_{01} \tag{2.2}$$

where $E^0(\mu)$ stands for the spectral function of A_0, $E^0(\mu) = E_0\Big((-\infty, \mu)\Big)$, is differentiable in μ for all $\mu \in \Delta_k^0$, $k = 1, 2, \ldots, m$, in the operator norm topology, i.e. the limits

$$\lim_{\lambda \to \mu} \left\| \frac{K_B(\lambda) - K_B(\mu)}{\lambda - \mu} - K_B'(\mu) \right\| = 0, \qquad \lambda, \mu \in \Delta_k^0,$$

exist with $K_B'(\mu) \in \mathbf{B}(\mathcal{H}_1, \mathcal{H}_1)$. Obviously, the derivative $K_B'(\mu)$ is non-negative,

$$K_B'(\mu) \geq 0,$$

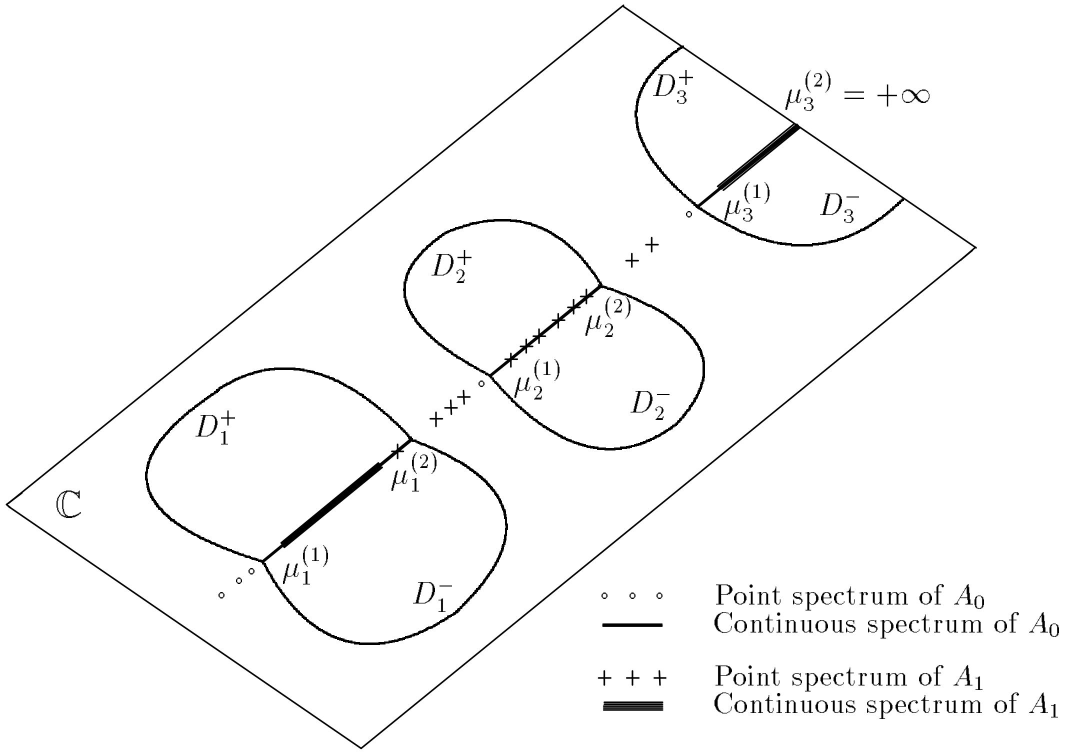

FIG. 1. An example of the spectral situation considered in the paper for the case where $m = 3$ and $\mu_1^{(1)} > -\infty$, $\mu_3^{(2)} = +\infty$.

since $K_B(\mu)$ is a non-decreasing function. Differentiability of $K_B(\mu)$ means that the continuous spectrum of the entry A_0 includes, in each Δ_k^0, $k = 1, 2, \ldots, m$, a branch of the absolutely continuous[3] spectrum $\sigma_{ac}(A_0)$: For any Borel subset $\delta \subset \Delta_k^0$ and for any $u_1 \in \mathcal{H}_1$ the vector $E_0(\delta)B_{01}u_1$ belongs to the invariant subspace $\mathcal{H}_0^{ac} \subset \mathcal{H}_0$ of A_0 corresponding to the absolutely continuous spectrum $\sigma_{ac}(A_0)$ and $E_0(\delta)B_{01}u_1 = E_0^{ac}(\delta)B_{01}u_1$ where E_0^{ac} denotes the part of the spectral measure E_0 corresponding to $\sigma_{ac}(A_0)$. Obviously,

$$\mathcal{V}_0(B)\Big|_{\Delta_k^0} = \int\limits_{\Delta_k^0} d\mu\, \|K_B'(\mu)\|.$$

---

[3]Recall, for convenience of the reader, definition of the absolutely continuous spectrum of a selfadjoint operator A acting in a Hilbert space $\mathcal{G}$. Let $E^A(\mu)$ be the spectral function of A, $E^A(\mu) = E_A\Big((-\infty, \mu)\Big)$, $\mu \in \mathbb{R}$, where E_A stands for the spectral measure of A. Denote by $\mathcal{G}_{ac}$ the invariant subspace of A consisting of all the vectors $f \in \mathcal{G}$ for which the function $\mu \to \langle E^A(\mu) f, f\rangle$ is absolutely continuous in $\mu \in \mathbb{R}$. Then the spectrum of the restriction $A\Big|_{\mathcal{G}_{ac}}$ of A on the subspace $\mathcal{G}_{ac}$ is called absolutely continuous spectrum $\sigma_{ac}(A)$ of the operator A. Also, one says the subspace $\mathcal{G}_{ac}$ corresponds to the spectrum $\sigma_{ac}(A)$. For more details see, e. g., Ref. [16], § X.1.2, and Ref. [38], Section VII.2. It should be noted that in most of physical applications all the continuos spectrum of selfadjoint operators involved is typically absolutely continuous.

Further, we suppose that the function $K'_B(\mu)$ is continuous within the closed intervals $\overline{\Delta^0_k}$ and, moreover, that it admits analytic continuation from each of these intervals to a simply connected domain situated, say, in $\mathbb{C}^-$. For the interval Δ^0_k, let this domain be called D^-_k (see Fig 1). We assume that the boundary of each domain D^-_k includes the entire spectral interval Δ^0_k. Let $K'^{(k)}_B(\mu)$ denote the continuation of K'_B from Δ^0_k into D^-_k. The presence of the index k in this notation is related to the fact that in general the continuation of K'_B to the same domain of $\mathbb{C}$ can be different if one starts from different intervals Δ^0_k. Thus, the notation $K'^{(k)}_B(\mu)$ relates to the distinct branches of the function K'_B. In the case where $D^-_j \cap D^-_k \neq \emptyset$ for $j \neq k$ and $K'^{(j)}_B(\mu) \neq K'^{(k)}_B(\mu)$ for $\mu \in D^-_j \cap D^-_k$ the points $z \in D^-_j$ and $z \in D^-_k$ must be considered as distinct (namely, one must assume that these points belong to different sheets of the Riemann surface of the function K'_B). To avoid unnecessary complications regarding such a treatment of the different branches of K'_B, we shall further assume that *the domains D^-_k for the different k do not intersect each other,* i. e.,

$$D^-_j \cap D^-_k = \emptyset, \quad j \neq k\,. \tag{2.3}$$

It is implied that one can always consider smaller initial holomorphy domains of the function K'_B if it is necessary. The assumption (2.3) allows us to drop the branch identification (the index k) in the notation above for the analytic continuation of K'_B and this is done throughout the paper.

Since $K'_B(\mu)$ represents a self-adjoint operator for $\mu \in \Delta^0_k$ and $\Delta^0_k \subset \mathbb{R}$, the function $K'_B(\mu)$ also automatically admits analytic continuation from Δ^0_k into the domain D^+_k, symmetric to D^-_k with respect to the real axis, $D^+_k = \{z : \overline{z} \in D^-_k\}$. For the continuation into D^+_k we retain the same notation $K'_B(\mu)$. The selfadjoitness of $K'_B(\mu)$ for $\mu \in \Delta^0_k$ implies

$$[K'_B(\mu)]^* = K'_B(\bar{\mu}), \quad \mu \in D^{\pm}_k\,. \tag{2.4}$$

Also, we shall always suppose the $K'_B(\mu)$ satisfies the following Hölder condition at the (finite) end points $\mu^{(1)}_k$, $\mu^{(2)}_k$ of the spectral intervals Δ^0_k,

$$\|K'_B(\mu) - K'_B(\mu^{(i)}_k)\| \le C|\mu - \mu^{(i)}_k|^\gamma, \qquad i = 1,2, \quad \mu \in D^{\pm}_k,$$

with some positive C and γ.

Let $l = (l_1, l_2, \ldots, l_m)$ be a multi-index having the components $l_k = +1$ or $l_k = -1$, $k = 1, 2, \ldots, m$. In what follows we consider the domains $D_l = \bigcup\limits_{k=1}^{m} D^{l_k}_k$ where $D^{l_k}_k$ are the holomorphy domains of K'_B described above. Let $\Gamma^{l_k}_k$ be a rectifiable Jordan curve in $D^{l_k}_k$ resulting from continuous deformation of the interval Δ^0_k, the end points of this interval being fixed (except $\mu^{(1)}_1 = -\infty$ and $\mu^{(2)}_m = +\infty$ which are allowed, if this is possible, to be shifted respectively to $\widetilde{\mu}^{(1)}_1 = -\infty + \mathrm{i}y_{-\infty} \in D^{l_1}_1$ and $\widetilde{\mu}^{(2)}_m = +\infty + \mathrm{i}y_{+\infty} \in D^{l_m}_m$ with some real $y_{-\infty} = l_1|y_{-\infty}|$ and $y_{+\infty} = l_m|y_{+\infty}|$). With the exception of the end points, the closure $\overline{\Gamma}^{l_k}_k$ of the contour $\Gamma^{l_k}_k$ has no other common points with the set $\sigma_c(A_0)$. Note that under the condition (2.3) $\Gamma^{\pm}_j \cap \Gamma^{\pm}_k = \emptyset$ for any $j, k \in \{1, 2, \ldots, m\}$ such that $j \neq k$. By Γ_l, $l = (l_1, l_2, \ldots, l_m)$, we shall denote the union of the contours $\Gamma^{l_k}_k$, $\Gamma_l = \bigcup\limits_{k=1}^{m} \Gamma^{l_k}_k$.

Also, we extend the definition of the variation $\mathcal{V}_0(B)$ to the set $\sigma'(A_0) \cup \Gamma_l$ by introducing the modified variation

$$\mathcal{V}_0(B,\Gamma_l) = \mathcal{V}_0(B)\Big|_{\sigma'(A_0)} + \int\limits_{\Gamma_l} |d\mu|\, \|K'_B(\mu)\| \tag{2.5}$$

with $|d\mu|$ Lebesgue measure on Γ_l. It is clear that if the length ℓ_{Γ_l} of the curve Γ_l is finite (in the case where the set $\overset{m}{\underset{k=1}{\cup}} \Delta_k^0$ is bounded) the value $\mathcal{V}_0(B,\Gamma_l)$ is also finite,

$$\mathcal{V}_0(B,\Gamma_l) \le \mathcal{V}_0(B)\Big|_{\sigma'(A_0)} + \ell_{\Gamma_l} \cdot \max_{\mu\in\Gamma_l} \|K'_B(\mu)\|.$$

We suppose that the operators B_{ij} are such that there exists a contour (contours) Γ_l where the value $\mathcal{V}_0(B,\Gamma_l)$ is finite,

$$\mathcal{V}_0(B,\Gamma_l) < \infty, \tag{2.6}$$

including also the case of the unbounded set $\overset{m}{\underset{k=1}{\cup}} \Delta_k^0$. It is assumed that the inequality (2.6) must hold during the reverse continuous deformation of the contour Γ_l back to the set $\overset{m}{\underset{k=1}{\cup}} \Delta_k^0$. The contours Γ_l satifying the condition (2.6) are said to be K_B-*bounded* contours.

In the following we deal mainly with the analytic continuation of the transfer function $M_1(z)$ and its inverse, $[M_1(z)]^{-1}$, through the spectral intervals Δ_k^0 into the domains D_l. Under the assumed conditions the function $M_1(z)$ admits such a continuation in the conventional sense, i. e., as an operator-valued function. Namely, the following statement holds.

LEMMA 2.1 *The analytic continuation of the transfer function $M_1(z)$, $z \in \mathbb{C}\backslash\sigma(A_0)$, through the spectral intervals Δ_k^0 into the subdomain $D(\Gamma_l) \subset D_l$ bounded by the set $\bigcup\limits_{k=1}^{m} \Delta_k^0$ and a K_B-bounded contour Γ_l is given by*

$$M_1(z,\Gamma_l) = A_1 - z + V_1(z,\Gamma_l) \tag{2.7}$$

where

$$\begin{aligned} V_1(z,\Gamma_l) &= \int\limits_{\sigma'(A_0)\cup\Gamma_l} K_B(d\mu)\, \frac{1}{z-\mu} \qquad (2.8)\\ &\overset{\mathrm{def}}{=} \int\limits_{\sigma'(A_0)} B_{10}E_0(d\mu)B_{01}\, \frac{1}{z-\mu} + \int\limits_{\Gamma_l} d\mu\, K'_B(\mu)\, \frac{1}{z-\mu}. \end{aligned}$$

For $z \in D_k^{l_k} \cap D(\Gamma_l)$ the function $M_1(z,\Gamma_l)$ may be written as

$$M_1(z,\Gamma_l) = M_1(z) + 2\pi\mathrm{i}\, l_k K'_B(z). \tag{2.9}$$

P r o o f . The proof is reduced to the observation that the function $M_1(z,\Gamma_l)$ is holomorphic for $z \in \mathbb{C} \setminus [\sigma'(A_0) \cup \Gamma_l]$ and coincides with $M_1(z)$ for $z \in \mathbb{C} \setminus [\sigma'(A_0) \cup \overline{D(\Gamma_l)}]$. Eq. (2.9) is obtained from (2.8) using the Residue Theorem. □

Remark 2.1 *The definition Eq.* (2.8) *defines the function* $V_1(z,\Gamma_l)$ *and, hence, via Eq.* (2.7) *the function* $M_1(z,\Gamma_l)$ *for* $z\in\mathbb{C}\setminus(\sigma'(A_0)\cup\Gamma_l)$, *the values of* $V_1(z,\Gamma_l)$ *for such* z *being bounded operators in* $\mathcal{H}_1$. *As mentioned above the inverse transfer function* $M_1^{-1}(z)$ *coincides with the block component* $R_{11}(z)$ *of the resolvent* $\mathbf{R}(z)$ *and, thus, it is bounded and holomorphic in* $z\in\mathbb{C}\setminus\sigma(\mathbf{H})$. *Since* $M_1(z,\Gamma_l)=M_1(z)$ *for* $z\in\mathbb{C}\setminus[\sigma'(A_0)\cup\overline{D(\Gamma_l)}]$, *one concludes that* $[M_1(z,\Gamma_l)]^{-1}$ *exists and is bounded and holomorphic in* z *at least for* $z\in\mathbb{C}\setminus[\sigma(\mathbf{H})\cup\overline{D(\Gamma_l)}]$.

It is clear that by varying the contour Γ_l one can represent the continuation of the transfer function $M_1(z)$ in the form (2.8) for any subset of the domain D_l. One notes also that if the subdomains $D(\Gamma_l^{(1)})$ and $D(\Gamma_l^{(2)})$ correspond to two different K_B-bounded contours $\Gamma_l^{(1)}$ and $\Gamma_l^{(2)}$, then automatically $V_1(z,\Gamma_l^{(1)})=V_1(z,\Gamma_l^{(2)})$ and, hence, $M_1(z,\Gamma_l^{(1)})=M_1(z,\Gamma_l^{(2)})$ for $z\in D(\Gamma_l^{(1)})\cap D(\Gamma_l^{(2)})$ because of the uniqueness of the analytic continuation.

The formula (2.9) shows that in general the transfer function M_1 has a multi-sheeted Riemann surface. Properties of this surface such as the number of sheets, the presence of branching points in addition to the points $\mu_k^{(1)},\mu_k^{(2)}$, $k=1,2,\ldots,m$ (if K_B' is considered in a larger domain than $\bigcup\limits_l D_l$) *etc.* are determined by the analytic properties of the function $K_B'(\mu)$ itself. The sheet of the complex plane where the transfer function $M_1(z)$ is considered together with the resolvent $\mathbf{R}(z)$ initially is said to be the *physical sheet.* The remaining sheets of the Riemann surface are said to be *unphysical sheets.*

In the present work we only deal with the unphysical sheets neighbouring the physical one, i. e., with the sheets connected through the intervals Δ_k^0 for some $k\in\{1,2,\ldots,m\}$ immediately to the physical sheet. The index $l=(l_1,l_2,\ldots,l_m)$ can be considered as an identifier of the neighboring sheet. It should be noted however that some of these sheets can turn out to be identical to each other if one is able to consider a wider domain than $\bigcup\limits_l D_l$ but all this depends on a concrete form for the function K_B' and we do not touch on this subject.

Regarding the total resolvent $\mathbf{R}(z)$, it may admit continuation only in a generalized sense. First, one can suppose that there exists a dense subset $\widetilde{\mathcal{H}}_0$ of $\mathcal{H}_0$ such that for any $u_0,v_0\in\widetilde{\mathcal{H}}_0$ the bilinear form $\langle E^0(\mu)u_0,v_0\rangle$ admits analytic continuation in variable μ as a holomorphic function from each interval Δ_k^0 into the respective domain D^{l_k}. So that the previous assumption concerning the holomorphy of the function $K_B'(\mu)$ implies that $B_{01}u_1$ is an element of the subset $\widetilde{\mathcal{H}}_0$ for any $u_1\in\mathcal{H}_1$. Obviously, the analytic continuation of the form $\langle R_0(z)u_0,v_0\rangle$ into D_l reads as (cf. Lemma 2.1)

$$\langle R_0(z)u_0,v_0\rangle|_{D_l}=\int\limits_{\sigma'(A_0)}\frac{d\langle E^0(\mu)u_0,v_0\rangle}{\mu-z}+\int\limits_{\Gamma_l}d\mu\,\frac{K'_{u_0,v_0}(\mu)}{\mu-z}$$

where K'_{u_0,v_0} denotes the derivative of the analytic continuation $K_{u_0,v_0}(\mu)$ of the form $\langle E^0(\mu)u_0,v_0\rangle$. Using the Residue Theorem one can verify

$$\langle R_0(z)u_0,v_0\rangle\Big|_{D_k^{l_k}\cap D_l}=\langle R_0(z)u_0,v_0\rangle+2\pi\mathrm{i}\,l_k\,K'_{u_0,v_0}(z)$$

where $\langle R_0(z)u_0,v_0\rangle$ stands for the conventional bilinear form of the resolvent $R_0(z)$, i. e., this form is taken for $R_0(z)$ in the phyisical sheet.

The analytic continuation of the resolvent $\mathbf{R}(z)$ is understood in terms of the continuation of the bilinear form $\langle \mathbf{R}(z)u, v\rangle$ where $u = (u_0, u_1)$, $v = (v_0, v_1)$ are elements of $\mathcal{H}$ with $u_0, v_0 \in \widetilde{\mathcal{H}}_0$ and $u_1, v_1 \in \mathcal{H}_1$. It follows from the representation (1.4) that such a generalized continuation is indeed possible if the function $K'_{u_0,v_0}(\mu)$ is holomorphic in D_l for any $u_0, v_0 \in \widetilde{\mathcal{H}}_0$. The holomorphy domain of the continuation of $\mathbf{R}(z)$ into D_l thus has to coincide with just such a domain for the continuation of the inverse transfer function $R_{11}(z) = [M_1(z)]^{-1}$.

The spectral problem for the continued transfer function $M_1(z, \Gamma)$, that is, the problem

$$[A_1 + V_1(z, \Gamma)]\, u_1 = z\, u_1\,, \quad u_1 \in \mathcal{H}_1, \tag{2.10}$$

will be referred to in the following as the *initial spectral problem.*

3. THE BASIC EQUATION. SOLUTIONS $H_1^{(l)}$

If an operator-valued function $T : \sigma'(A_0) \cup \Gamma \to \mathbf{B}(\mathcal{H}_1, \mathcal{H}_1)$ satisfies the Lipschitz condition [the inequality (B.4) of Appendix B] on $\sigma'(A_0)$ and is continuous and bounded on a K_B-bounded contour Γ,

$$\|T\|_{\infty,\Gamma} = \sup_{\mu \in \sigma'(A_0) \cup \Gamma} \|T(\mu)\| < \infty, \tag{3.1}$$

then the integral

$$\int\limits_{\sigma'(A_0)\cup\Gamma} K_B(d\mu)\, T(\mu) \stackrel{\text{def}}{=} \int\limits_{\sigma'(A_0)} B_{10}E_0(d\mu)B_{01}T(\mu) + \int\limits_{\Gamma} d\mu\, K'_B(\mu)\, T(\mu), \tag{3.2}$$

exists in the sense of the operator norm topology (see Appendix B) and

$$\Big\| \int\limits_{\sigma'(A_0)\cup\Gamma} K_B(d\mu) T(\mu) \Big\| \le \mathcal{V}_0(B, \Gamma) \cdot \|T\|_{\infty,\Gamma}. \tag{3.3}$$

In particular, if $T(z)$ is the resolvent of an operator Y, $T(z) = (Y - zI_1)^{-1}$, the spectrum of which has no common points with $\sigma'(A_0) \cup \Gamma$, then one can define the operator

$$V_1(Y, \Gamma) = \int\limits_{\sigma'(A_0)\cup\Gamma} K_B(d\mu)(Y - \mu I_1)^{-1}. \tag{3.4}$$

This operator is bounded, $V_1(Y, \Gamma) \in \mathbf{B}(\mathcal{H}_1, \mathcal{H}_1)$, and, because of (3.3), its norm admits the estimate

$$\|V_1(Y, \Gamma)\| \le \mathcal{V}_0(B, \Gamma_l) \cdot \sup_{\mu \in \sigma'(A_0)\cup\Gamma} \|(Y - \mu I_1)^{-1}\|. \tag{3.5}$$

According to the definition (3.4), the operator-valued function $V_1(Y, \Gamma)$ of the operator variable $Y : \mathcal{H}_1 \to \mathcal{H}_1$ possesses the following important property: If $u_1 \in \mathcal{H}_1$ is an eigenvector of Y corresponding to an eigenvalue z, $Yu_1 = zu_1$, then

$$V_1(Y,\Gamma)u_1 = \int\limits_{\sigma'(A_0)\cup\Gamma} K_B(d\mu)\frac{1}{z-\mu}u_1 \equiv V_1(z,\Gamma)u_1. \qquad (3.6)$$

In what follows we consider the equation[4]

$$Y = A_1 + V_1(Y,\Gamma). \qquad (3.7)$$

We deal with this equation, since it possesses the following characteristic property: If an operator H_1 is a solution of (3.7) and u_1 is an eigenvector of H_1 corresponding to an eigenvalue z, $H_1u_1 = zu_1$, then automatically (cf. Sect. 1)

$$zu_1 = A_1u_1 + V_1(H_1,\Gamma)u_1 = A_1u_1 + V_1(z,\Gamma)u_1. \qquad (3.8)$$

This implies that any eigenvalue z of such an operator H_1 is automatically an eigenvalue for the initial spectral problem (2.10) and u_1 a corresponding eigenvector. Thus, having found the solution(s) of the equation (3.7) one obtains an effective means of studying the spectral properties of the transfer function $M_1(z,\Gamma)$ itself. This is why the equation (3.7) and its solutions represent one of the main subjects of the present work.

Often it turns out to be convenient to rewrite Eq. (3.7) in the form

$$X = V_1(A_1 + X,\Gamma) \qquad (3.9)$$

where $X = Y - A_1$. Both equation (3.7) and its variant (3.9) will be referred to in the following as the *basic equations*. Sufficient conditions for solvability of these equations are described in the following statement.

THEOREM 3.1 *Let:*

a) a contour Γ be K_B-bounded;

b) the spectrum of the operator A_1 be strictly separated from the set $\sigma'(A_0)\cup\Gamma$,

$$d_0(\Gamma) = \mathrm{dist}\{\sigma(A_1), \sigma'(A_0)\cup\Gamma\} > 0; \qquad (3.10)$$

c) the inequality

$$\mathcal{V}_0(B,\Gamma) < \frac{1}{4}\, d_0^2(\Gamma) \qquad (3.11)$$

be valid. Then Eq. (3.9) is uniquely solvable in any ball $\mathcal{S}_1(r) \subset \mathbf{B}(\mathcal{H}_1,\mathcal{H}_1)$ including operators $X : \mathcal{H}_1 \to \mathcal{H}_1$ the norms of which are bounded as $\|X\| \le r$ with r such that

$$r_{\min}(\Gamma) \le r < r_{\max}(\Gamma) \qquad (3.12)$$

where

$$r_{\min}(\Gamma) = \frac{d_0(\Gamma)}{2} - \sqrt{\frac{d_0^2(\Gamma)}{4} - \mathcal{V}_0(B,\Gamma)} \qquad (0 < r_{\min}(\Gamma) < d_0(\Gamma)/2) \qquad (3.13)$$

---

[4]In the case where the spectra of A_0 and A_1 have no intersection, the equation (3.7) is reduced to the operator Riccati equation (1.12) (see Sect. 1; for details see Refs. [28–30]).

and

$$r_{\max}(\Gamma) = d_0(\Gamma) - \sqrt{\mathcal{V}_0(B,\Gamma)} \qquad (d_0(\Gamma)/2 < r_{\max}(\Gamma) < d_0(\Gamma)). \tag{3.14}$$

The solution X of Eq. (3.9) *is the same for any r satisfying* (3.12) *and in fact it belongs to the smallest ball $\mathcal{S}_1(r_{\min})$, $\|X\| \le r_{\min}(\Gamma)$.*

P r o o f . The proof will be based on the Banach's Fixed Point Theorem.

Let $F(X) = V_1(A_1+X,\Gamma)$ with $X\in\mathcal{S}_1(r)$. To begin with we search for a condition under which the function F maps the ball $\mathcal{S}_1(r)$ into itself. Since, in view of (3.12) and (3.14) the condition $0 < r < d_0$, $d_0 = d_0(\Gamma)$ automatically holds, the spectrum of the operator $A_1 + X$ does not intersect the set $\sigma'(A_0) \cup \Gamma$ because of condition (3.10). This means that for all $\mu \in \sigma'(A_0) \cup \Gamma$ the resolvent $(A_1 + X - \mu I_1)^{-1}$ exists as a bounded operator in $\mathcal{H}_1$. It follows from the estimate (3.5) that

$$\|F(X)\| \le \mathcal{V}_0(B,\Gamma) \sup_{\mu\in\sigma'(A_0)\cup\Gamma} \|(A_1 + X - \mu I_1)^{-1}\|.$$

Using the identity

$$(A_1 + X - \mu I_1)^{-1} = \left(I_1 + (A_1 - \mu I_1)^{-1}X\right)^{-1}(A_1 - \mu I_1)^{-1}, \tag{3.15}$$

one obtains the estimate

$$\begin{aligned}\|(A_1 + X - \mu I_1)^{-1}\| &\le \frac{1}{1 - \|(A_1 - \mu I_1)^{-1}\| \cdot \|X\|} \cdot \|(A_1 - \mu I_1)^{-1}\| \\ &\le \frac{1}{1 - \dfrac{r}{d_0}} \cdot \frac{1}{d_0} = \frac{1}{d_0 - r}.\end{aligned}$$

It follows from this estimate that the ball $\mathcal{S}_1(r)$ is necessarily mapped by the function F into itself if the radius r and the value $\mathcal{V}_0(B,\Gamma)$ are such that

$$\mathcal{V}_0(B,\Gamma) \cdot \frac{1}{d_0 - r} \le r. \tag{3.16}$$

Now, we clarify the conditions for F to be a contraction. To this end we estimate the difference

$$F(X) - F(Y) = \int\limits_{\sigma'(A_0)\cup\Gamma} K_B(d\mu)\, T(\mu)$$

where

$$\begin{aligned}T(\mu) &= (A_1 + X - \mu I_1)^{-1} - (A_1 + Y - \mu I_1)^{-1} \\ &= (A_1 + X - \mu I_1)^{-1}(Y - X)(A_1 + Y - \mu I_1)^{-1}.\end{aligned}$$

Using again the inequality (3.5) we find

$$\|F(X) - F(Y)\| \le$$

$$\le \mathcal{V}_0(B,\Gamma) \cdot \sup_{\mu\in\sigma'(A_0)\cup\Gamma} \|(A_1 + X - \mu I_1)^{-1}\| \cdot \sup_{\mu\in\sigma'(A_0)\cup\Gamma} \|(A_1 + Y - \mu I_1)^{-1}\| \cdot \|Y - X\|$$

$$\leq \mathcal{V}_0(B,\Gamma)\cdot\frac{1}{(d_0-r)^2}\|Y-X\|.$$

Clearly, F is a contraction if

$$\frac{\mathcal{V}_0(B,\Gamma)}{(d_0-r)^2}<1\,. \tag{3.17}$$

Under the condition (c) the inequalities (3.16) and (3.17) considered together are just equivalent to the condition (3.12). Thus if the condition (c) is valid, then F is indeed a contraction of the ball $\mathcal{S}_1(r)$ into itself for any radius r satisfying (3.12). This implies that Eq. (3.9) has a solution in any such ball and this solution is unique. Consequently, the solution is the same for all the radii satisfying (3.12). Moreover, it belongs to the ball $\mathcal{S}_1(r_{\min})$ with the radius $r_{\min}$ given by (3.13).

The proof is complete. □

REMARK 3.1 *It should be noted that the distance* $d_0(\Gamma)=\operatorname{dist}\{\sigma(A_1),\sigma'(A_0)\cup\Gamma\}$ *always satisfies, of course, the inequality* $d_0(\Gamma)\leq\operatorname{dist}\{\sigma(A_1),\mathcal{E}_c(A_0)\}$ *where* $\mathcal{E}_c(A_0)$ *denotes the set of the differing from* $\pm\infty$ *end points* $\mu_k^{(1)}$, $\mu_k^{(2)}$ *of the intervals* Δ_k^0, $k=1,2,\ldots,m$. *Thus, the condition* (3.10) *assumes that the distance between any two of the parts* $\sigma(A_1)\cap\Delta_k^0$ *of the spectrum* $\sigma(A_1)$ *lying inside the different intervals* Δ_k^0, $k=1,2,\ldots,m$ *is greater or equal than* $2d_0(\Gamma)$. *The same is true for the rest part* $\sigma(A_1)\setminus\sigma(A_0)$ *of the spectrum of* A_1 *if it is nonempty:* $\operatorname{dist}\{\sigma(A_1)\setminus\sigma(A_0),\sigma(A_1)\cap\Delta_k^0\}\geq 2d_0(\Gamma)$, $k=1,2,\ldots,m$.

THEOREM 3.2 *Let the conditions of* Theorem 3.1 *be valid for a* K_B*-bounded contour* $\Gamma\subset D_l$ *and let* X *be the solution of Eq.* (3.9) *referred to there. Then the analogous solution* $\widetilde{X}$ *for any other* K_B*-bounded contour* $\widetilde{\Gamma}\subset D_l$ *satisfying the estimate* $\mathcal{V}_0(B,\widetilde{\Gamma})<\widetilde{d}_0^2/4$ *with* $0<\widetilde{d}_0=\operatorname{dist}\{\sigma(A_1),\sigma'(A_0)\cup\widetilde{\Gamma}\}\leq d_0(\Gamma)$ *coincides with* X.

P r o o f. The solution X satisfies the inequality $\|X\|\leq r_{\min}(\Gamma)$ with $r_{\min}(\Gamma)$ given by (3.13). This means $\|X\|<d_0(\Gamma)/2$. Similarly, $\|\widetilde{X}\|<\widetilde{d}_0/2$. The resolvent $(A_1-\widetilde{X}-\mu)^{-1}$ is, therefore, a holomorphic operator-valued function with its values belonging to $\mathbf{B}(\mathcal{H}_1,\mathcal{H}_1)$ for any $\mu\in\mathbb{C}$ such that $\operatorname{dist}\{\mu,\sigma(A_1)\}>\widetilde{d}_0/2$. Recall that we consider only the contours which result from a continuous deformation of respective spectral intervals Δ_k^0, $k=1,2,\ldots,m$. The paths $\Gamma,\widetilde{\Gamma}\subset D_l$ are supposed to belong to this class of contours. So that the contour $\widetilde{\Gamma}$ may be continuously transformed to the path Γ in such a way that for any intermediate paths Γ' one still has $\operatorname{dist}\{\sigma(A_1),\sigma'(A_0)\cup\Gamma'\}\geq\widetilde{d}_0$, since $d_0(\Gamma)\geq\widetilde{d}_0$. In view of holomorphy of the resolvent $(A_1-\widetilde{X}-\mu)^{-1}$ for μ such that $\operatorname{dist}\{\mu,\sigma(A_1)\}>\widetilde{d}_0/2$ one finds immediately that the r. h. side of the equality (3.9) for $\widetilde{X}$, $\widetilde{X}=V_1(A_1+\widetilde{X},\Gamma')$, remains fixed when one transforms Γ' from $\widetilde{\Gamma}$ to Γ, keeping $\operatorname{dist}\{\sigma(A_1),\sigma'(A_0)\cup\Gamma'\}\geq\widetilde{d}_0$ (or even $\operatorname{dist}\{\sigma(A_1),\sigma'(A_0)\cup\Gamma'\}>\widetilde{d}_0/2$). This means that $\widetilde{X}$ satisfies exactly the same equation $\widetilde{X}=V_1(A_1+\widetilde{X},\Gamma)$ as X. According to Theorem 3.1, the solution of Eq. (3.9) is unique and the same in any ball $\mathcal{S}_1(r)\subset\mathbf{B}(\mathcal{H}_1,\mathcal{H}_1)$ with r satisfying (3.12). In particular, the value $r=d_0(\Gamma)/2$ can also be substituted into (3.12). Meanwhile, $\widetilde{X}\in\mathcal{S}_1(\widetilde{r}_{\min})$ with $\widetilde{r}_{\min}=\widetilde{d}_0/2-\sqrt{\widetilde{d}_0^2/4-\mathcal{V}_0(B,\widetilde{\Gamma})}$. Obviously, $\widetilde{r}_{\min}<\widetilde{d}_0/2\leq d_0(\Gamma)/2$, since $\widetilde{d}_0\leq d_0(\Gamma)$ according to our assumption. Hence, $\widetilde{X}$ must coincide with X. This completes the proof. □

COROLLARY 3.1 Theorem 3.2 *shows that, for a fixed multi-index l, the solution of Eq.* (3.9) *referred to in* Theorem 3.1 *is unique and the same for all the K_B-bounded contours $\Gamma_l \subset D_l$ satisfying the inequality* (3.11)*. Moreover this solution satisfies the estimate*

$$\|X\| \leq r_0(B) \tag{3.18}$$

with

$$r_0(B) = \inf_{\Gamma_l:\, \omega(B,\Gamma_l)>0} r_{\min}(\Gamma_l) \tag{3.19}$$

where $r_{\min}(\Gamma_l)$ is given by (3.13) *while*

$$\omega(B, \Gamma_l) = d_0^2(\Gamma_l) - 4\mathcal{V}_0(B, \Gamma_l). \tag{3.20}$$

The value of $r_0(B)$ does not depend on the index l.

This corollary is an immediate consequence of the statement of Theorem 3.2. The only thing we want to show is the independence of the radius $r_0(B)$ on l. To prove this, let us consider a K_B-bounded contour $\Gamma_l \subset D_l$, $\Gamma_l = \bigcup\limits_{k=1}^{m} \Gamma_k^{l_k}$. Denote by $\Gamma_{l'}$ a contour which is obtained from Γ_l by replacing a part of the curves $\Gamma_k^{l_k}$ with the conjugate ones, $\Gamma_k^{(-l_k)} = \{\mu : \overline{\mu} \in \Gamma_k^{l_k}\}$. Obviously, such a replacement generates, in additional to Γ_l, $2^m - 1$ different contours $\Gamma_{l'}$ for $l' = (l'_1, l'_2, \ldots, l'_m)$ with $l'_k = \pm l_k$, $k = 1, 2, \ldots, m$. For any such contour the value of $\mathcal{V}_0(B, \Gamma_{l'})$ is the same, namely

$$\mathcal{V}_0(B, \Gamma_{l'}) = \mathcal{V}_0(B, \Gamma_l). \tag{3.21}$$

Indeed,

$$\int\limits_{\Gamma_k^{(-l_k)}} |d\mu|\, \|K'_B(\mu)\| = \int\limits_{\Gamma_k^{l_k}} |d\overline{\mu}|\, \|K'_B(\overline{\mu})\|, \quad k = 1, 2, \ldots, m.$$

But, according to (2.4),

$$\int\limits_{\Gamma_k^{l_k}} |d\overline{\mu}|\, \|K'_B(\overline{\mu})\| = \int\limits_{\Gamma_k^{l_k}} |d\mu|\, \|[K'_B(\mu)]^*\| = \int\limits_{\Gamma_k^{l_k}} |d\mu|\, \|K'_B(\mu)\|.$$

So that nothing happens to the value of $\int\limits_{\Gamma_l} |d\mu|\, \|K'_B(\mu)\|$ when one replaces Γ_l by $\Gamma_{l'}$. But this just means that Eq. (3.21) holds true and, hence, the infimum (3.19) acquires the same value for any l. $\square$

So, for a given holomorphy domain D_l the solutions X and H_1, $H_1 = A_1 + X$, do not depend on the K_B-bounded contours $\Gamma_l \subset D_l$ satisfying the condition (3.11). But when the index l changes, X and H_1 can also change. For this reason we shall supply them in the following, when it is necessary, with the index l writing, respectively, $X^{(l)}$ and $H_1^{(l)}$, $H_1^{(l)} = A_1 + X^{(l)}$. In fact, it follows from Eq. (3.21) that if the conditions of Theorem 3.1 are valid for a contour Γ_l, then they are valid for the remaining 2^{m-1} contours $\Gamma_{l'}$ described above, too. Therefore, Theorem 3.1 guarantees us, in general, the existence of the 2^m solutions $X^{(l)}$ to the basic equation (3.7) and, hence, the 2^m respective solutions $H_1^{(l)}$ to the basic equation (3.9). In the following we shall deal only with these solutions[5] of (3.7) and (3.9).

LEMMA 3.1 *The above solution $X^{(l)}$ of the basic equation* (3.9)*, independent, for a given D_l, of the contour $\Gamma_l \subset D_l$ satisfying the condition* (3.11)*, is also a solution of this equation for any other K_B-bounded contour $\Gamma_l \subset D_l$ satisfying only the condition*

$$\operatorname{dist}\{\sigma(A_1), \sigma'(A_0) \cup \Gamma_l\} > r_0(B). \tag{3.22}$$

P r o o f of the lemma is reduced to an appropriate continuous deformation of the contours, starting from a contour satisfying (3.11) and finishing with a desired contour satisfying only the condition (3.22). □

Concluding the section we would like to make the following

REMARK 3.2 *If $\Gamma_l \subset D_l$ is a K_B-bounded contour satisfying the condition* (3.11)*, then the resolvent set of the transfer function $M_1(z, \Gamma_l)$ in the domain $D(\Gamma_l)$ bounded by the curve Γ_l and the set $\bigcup_{k=1}^{m} \Delta_k^0$ is not empty. Moreover, the curve*

$$\operatorname{dist}\{z, \sigma(A_1)\} = \frac{d_0(\Gamma_l)}{2} \tag{3.23}$$

including its part belonging to $D(\Gamma_l)$ is entirely embedded into the resolvent set of $M_1(z, \Gamma_l)$. In fact, there is a vicinity of the curve (3.23) *in $D(\Gamma_l)$ which is entirely included in this set.*

It follows from the statement of Remark 3.1 that the curve (3.23) consists of m distinct components surrounding respective parts $\sigma(A_1) \cap \Delta_k^0$ of the spectrum of A_1 lying inside the intervals Δ_k^0, $k = 1, 2, \ldots, m$ and, if $\sigma(A_1) \setminus \sigma(A_0) \neq \emptyset$, another component surrounding the rest of the set $\sigma(A_1)$ lying outside $\sigma(A_0)$. Every such a component is symmetric with respect to the real axis.

Obviously, the component of the curve (3.23) surrounding the set $\sigma(A_1) \setminus \sigma(A_0)$ belongs to the subdomain of $\mathbb{C}$ where $M_1(z, \Gamma_l)$ coincides with the initial transfer function $M_1(z)$. Thus, at least complex points of this component belong to the resolvent set of $M_1(\cdot, \Gamma_l)$. The component of (3.23) surrounding the set $\sigma(A_1) \cap \Delta_k^0$ for some $k = 1, 2, \ldots, m$ is entirely included in the domain $D(\Gamma_k^{l_k}) \cup D(\Gamma_k^{(-l_k)}) \cup \Delta_k^0$ where $D(\Gamma_k^{l_k})$ denotes domain bounded by the contour $\Gamma_k^{l_k}$ and interval Δ_k^0 while $D(\Gamma_k^{(-l_k)})$ stands for domain symmetric to $\Gamma_k^{l_k}$ with respect to the real axis. Since the function $M_1(z, \Gamma_l)$ coincides in $D(\Gamma_k^{(-l_k)})$ with $M_1(z)$ (see Remark 2.1), any point of (3.23) lying in $D(\Gamma_k^{(-l_k)})$ automatically belongs to the resolvent set of $M_1(\cdot, \Gamma_l)$.

Further, we show that any $z \in D(\Gamma_l)$ lying on the curve (3.23) satisfies the inequality

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[5]Surely, Eqs. (3.7) and (3.9) are non-linear equations and, outside the balls $\|X\| < r_{\max}(\Gamma)$, they may, in principle, have other solutions, different from the $X^{(l)}$ or $H_1^{(l)}$ the existence of which is guaranteed by Theorem 3.1.

$$\operatorname{dist}\{z, \sigma'(A_0) \cup \Gamma_l\} \geq \frac{d_0(\Gamma_l)}{2}. \tag{3.24}$$

We prove (3.24) by contradiction. Let us suppose that there is a point $\widetilde{z} \in D(\Gamma_l)$ satisfying (3.23) and such that $\operatorname{dist}\{\widetilde{z}, \sigma'(A_0) \cup \Gamma_l\} < d_0(\Gamma_l)/2$. Note that $\operatorname{dist}\{\widetilde{z}, \sigma'(A_0) \cup \Gamma_l\} = \operatorname{dist}\{\widetilde{z}, \sigma'(A_0) \cup \overline{\Gamma}_l\}$ where as usually overlining in the notation $\overline{\Gamma}_l$ means closure of Γ_l (in the case, making the closure means addition to Γ_l of respective finite end points). Since both sets $\sigma'(A_0) \cup \overline{\Gamma}_l$ and $\sigma(A_1)$ are closed, this implies there exists points $z_0 \in \sigma'(A_0) \cup \overline{\Gamma}_l$ and $z_1 \in \sigma(A_1)$ such that $|\widetilde{z} - z_0| = \operatorname{dist}\{\widetilde{z}, \sigma'(A_0) \cup \Gamma_l\} < d_0(\Gamma_l)/2$ and $|\widetilde{z} - z_1| = \operatorname{dist}\{\widetilde{z}, \sigma(A_1)\} = d_0(\Gamma_l)/2$. Then it follows from the Triangle Inequality that $d_0(\Gamma_l) = \operatorname{dist}\{\sigma(A_1), \sigma'(A_0) \cup \Gamma_l\} \leq |\widetilde{z} - z_0| + |\widetilde{z} - z_1| < d_0(\Gamma_l)$ and, hence, one comes to a contradiction. Consequently, any $z \in D(\Gamma_l)$ lying on the curve (3.23) must satisfy (3.24).

Obviously, for z satisfying (3.23) and (3.24) we have

$$\|V_1(z, \Gamma_l)\| \leq \frac{\mathcal{V}_0(B, \Gamma_l)}{\operatorname{dist}\{z, \sigma'(A_0) \cup \Gamma_l\}}$$

and

$$\|(A_1 - z)^{-1} V_1(z, \Gamma_l)\| \leq \frac{\mathcal{V}_0(B, \Gamma_l)}{\operatorname{dist}\{z, \sigma(A_1)\} \cdot \operatorname{dist}\{z, \sigma'(A_0) \cup \Gamma_l\}} \leq \frac{\mathcal{V}_0(B, \Gamma_l)}{d_0^2(\Gamma)/4} < 1.$$

Thus $M_1(z, \Gamma_l)$ is invertible,

$$M_1^{-1}(z, \Gamma_l) = \Big(I_1 + (A_1 - z)^{-1} V_1(z, \Gamma_l)\Big)^{-1} (A_1 - z)^{-1},$$

and

$$\|M_1^{-1}(z, \Gamma_l)\| \leq \frac{1}{1 - \dfrac{\mathcal{V}_0(B, \Gamma_l)}{d_0^2(\Gamma)/4}} \cdot \frac{1}{d_0(\Gamma_l)/2}.$$

In the same way one can show that any real z lying on the curve (3.23) also belongs to the resolvent set of $M_1(\cdot, \Gamma_l)$.

The last statement of the remark is true due to the fact that each regular point of $M_1(\cdot, \Gamma_l)$ is included in the resolvent set together with a certain open neighborhood.

4. A FACTORIZATION THEOREM AND ITS IMMEDIATE CONSEQUENCES

As a next step we prove the *factorization theorem* for the transfer functions $M_1(z, \Gamma_l)$. This statement will play an important role when we study the spectral properties of the operators $H_1^{(l)}$.

THEOREM 4.1 *Let Γ_l be a K_B-bounded contour satisfying the condition* (3.11) *and $H_1^{(l)} = A_1 + X^{(l)}$ with $X^{(l)}$ the above solution of the basic equation* (3.7), *$\|X^{(l)}\| \leq r_0(B)$. Then, for $z \in \mathbb{C} \setminus (\sigma'(A_0) \cup \Gamma_l)$, the transfer function $M_1(z, \Gamma_l)$ admits the factorization*

$$M_1(z, \Gamma_l) = W_1(z, \Gamma_l)\,(H_1^{(l)} - z) \tag{4.1}$$

where $W_1(z, \Gamma_l)$ is a bounded operator in $\mathcal{H}_1$,

$$W_1(z,\Gamma_l) = I_1 - \int\limits_{\sigma'(A_0)\cup\Gamma_l} K_B(d\mu)\frac{1}{\mu - z}(H_1^{(l)} - \mu)^{-1} \tag{4.2}$$

which is boundedly invertible for $\mathrm{dist}\{z,\sigma(A_1)\} \le d_0(\Gamma_l)/2$ *and*

$$\left\|[W_1(z,\Gamma_l)]^{-1}\right\| \le \frac{1}{1 - \dfrac{\mathcal{V}_0(B,\Gamma_l)}{d_0^2(\Gamma_l)/4}} < \infty. \tag{4.3}$$

It should be noted that the above statement recalls the known factorization theorem by A. I. Virozub and V. I. Matsaev [42] being valid for a class of selfadjoint operator-valued functions (see also [25]). The results of the paper [27] are just based essentially on this theorem. However, in the case we deal with in the present paper, the function $M_1(z,\Gamma_l)$ does not satisfy the conditions of [42]. Moreover, it is not even a selfadjoint operator-valued function in the sense of [42].

P r o o f of Theorem 4.1. For $z \in \mathbb{C}\setminus(\sigma'(A_0)\cup\Gamma_l)$, the boundeness of the operator $W_1(z,\Gamma_l)$ given by (4.2) is evident. To prove the factorization (4.1) we note that for any $z \notin \sigma'(A_0)\cup\Gamma_l$

$$W_1(z,\Gamma_l)\,(H_1^{(l)} - z) = H_1^{(l)} - z - \int\limits_{\sigma'(A_0)\cup\Gamma_l} K_B(d\mu)\frac{1}{\mu - z}(H_1^{(l)} - \mu)^{-1}(H_1^{(l)} - z). \tag{4.4}$$

Since $(H_1^{(l)} - \mu)^{-1}(H_1^{(l)} - z) = I_1 + (\mu - z)(H_1^{(l)} - \mu)^{-1}$, one finds

$$\int\limits_{\sigma'(A_0)\cup\Gamma_l} K_B(d\mu)\frac{1}{\mu - z}(H_1^{(l)} - \mu)^{-1}(H_1^{(l)} - z) =$$

$$= \int\limits_{\sigma'(A_0)\cup\Gamma_l} K_B(d\mu)\frac{1}{\mu - z} + \int\limits_{\sigma'(A_0)\cup\Gamma_l} K_B(d\mu)(H_1^{(l)} - \mu)^{-1}.$$

But according to (2.7) and (2.8)

$$\int\limits_{\sigma'(A_0)\cup\Gamma_l} K_B(d\mu)\frac{1}{\mu - z} = A_1 - z - M_1(z,\Gamma_l)$$

while according to (3.7)

$$\int\limits_{\sigma'(A_0)\cup\Gamma_l} K_B(d\mu)(H_1^{(l)} - \mu)^{-1} = H_1^{(l)} - A_1.$$

Making use of these expressions one immediately obtains Eq. (4.1).

Further, we prove that the factor $W_1(z,\Gamma_l)$ is a boundedly invertible operator if the condition $\mathrm{dist}\{z,\sigma(A_1)\} \le d_0(\Gamma_l)/2$ is valid. Under this condition one finds $|\mu - z| \ge \mathrm{dist}\{z,\sigma'(A_0)\cup\Gamma_l\} \ge d_0(\Gamma_l)/2$, since $\mathrm{dist}\{\sigma(A_1),\sigma'(A_0)\cup\Gamma_l\} = d_0(\Gamma_l)$. On the other hand, $H_1^{(l)} = A_1 + X^{(l)}$ with $\|X^{(l)}\| < d_0(\Gamma_l)/2$ and, for $\mu \in \sigma'(A_0)\cup\Gamma_l$,

$$\|(H_1^{(l)} - \mu)^{-1}\| < \frac{1}{d_0(\Gamma_l)/2}. \tag{4.5}$$

So that

$$\Big\| \int\limits_{\sigma'(A_0)\cup\Gamma_l} K_B(d\mu)\frac{1}{\mu - z}(H_1^{(l)} - \mu)^{-1}\Big\| < \frac{\mathcal{V}_0(B,\Gamma_l)}{(d_0(\Gamma_l)/2)^2} < 1. \tag{4.6}$$

This means that the estimate (4.3) is true. This completes the proof. □

COROLLARY 4.1 *The statement of* Theorem 4.1 *regarding the bounded invertibility of the operator* $W_1(z,\Gamma_l)$ *remains valid for* z *in the domain*

$$D_\varepsilon(\Gamma_l) = \left\{ z : \; z \in D(\Gamma_l), \mathrm{dist}\{z, \sigma(A_1)\} < \frac{d_{\max} - \varepsilon}{2} \right\} \tag{4.7}$$

where

$$d_{\max} = \sup_{\Gamma:\,\omega(B,\Gamma)>0} d_0(\Gamma) \tag{4.8}$$

with $\omega(B,\Gamma)$ *given by Eq.* (3.20) *while the value of* $\varepsilon > 0$ *can be taken arbitrarily close to zero.*

Indeed, we note, first, that the value of $d_{\max}$ does not depend on the index l of the contour in (4.8), for the same reasons that $r_0(B)$, given by (3.19), does not depend on l. Second, according to the definition of $d_{\max}$, for any $\varepsilon > 0$ there exists a K_B-bounded contour $\Gamma_{l,\varepsilon} \subset D_l$ satisfying (3.11) such that $d_0(\Gamma_{l,\varepsilon}) > d_{\max} - \varepsilon$. Therefore, we can apply Theorem 4.1 to $M_1(z,\Gamma_{l,\varepsilon})$. Meanwhile, $M_1(z,\Gamma_l) = M_1(z,\Gamma_{l,\varepsilon})$ for $z \in D(\Gamma_l) \cap D(\Gamma_{l,\varepsilon})$ (see Sect. 2) and $W_1(z,\Gamma_l) = W_1(z,\Gamma_{l,\varepsilon})$ for such z, too. One checks the latter equality simply by deforming the contour Γ_l to $\Gamma_{l,\varepsilon}$ in the explicit formula (4.2). But this just implies that $W_1(z,\Gamma_l)$ is boundedly invertible for any $z \in D_\varepsilon(\Gamma_l)$, since $W_1(z,\Gamma_{l,\varepsilon})$ posesses this property.
□

It is easy to write some simple but useful relations between a part of the operators $H_1^{(l)}$. Namely, we derive such relations between $H_1^{(l)}$ and $H_1^{(-l)}$, $(-l) = (-l_1, -l_2, \ldots, -l_m)$ where l_k, $k = 1, 2, \ldots, m$, stand for the components of the multi-index $l = (l_1, l_2, \ldots, l_m)$. According to our convention, $\Gamma_{(-l)}$, $\Gamma_{(-l)} \subset D_{(-l)}$, is a contour which is obtained from the contour Γ_l by replacing all the components $\Gamma_k^{l_k}$ with the conjugate ones $\Gamma_k^{(-l_k)}$.

LEMMA 4.1 *Let* $\Gamma_l \subset D_l$ *be a* K_B*-bounded contour for which the conditions of* Theorem 3.1 *are valid. Then for any* $z \in \mathbb{C} \setminus \left(\sigma'(A_0) \cup \Gamma_l\right)$ *the following equality holds true:*

$$W_1(z,\Gamma_l)\left(H_1^{(l)} - z\right) = \left(H_1^{(-l)*} - z\right)[W_1(\overline{z},\Gamma_{(-l)})]^*. \tag{4.9}$$

P r o o f. For $M_1(z,\Gamma_l)$ we have the factorization formula (4.1). The same factorization holds as well for $M_1(\overline{z},\Gamma_{(-l)})$,

$$M_1(\overline{z},\Gamma_{(-l)}) = W_1(\overline{z},\Gamma_{(-l)})\left(H_1^{(-l)}-\overline{z}\right). \tag{4.10}$$

It is easy to check that for $\overline{z}\notin\sigma'(A_0)\cup\Gamma_{(-l)}$ and, thus, for $z\notin\sigma'(A_0)\cup\Gamma_l$

$$\left[M_1(\overline{z},\Gamma_{(-l)})\right]^* = M_1(z,\Gamma_l). \tag{4.11}$$

The equality (4.9), thus, follows immediately from Eqs. (4.1), (4.10) and (4.11). The proof of the lemma is complete. □

It is worth noting that

$$\left[W_1(\overline{z},\Gamma_{(-l)})\right]^* = I_1 - \int\limits_{\sigma'(A_0)\cup\Gamma_l}\left(H_1^{(-l)*}-\mu\right)^{-1}\frac{1}{\mu-z}\,K_B(d\mu) \tag{4.12}$$

while the $X^{(-l)*}$ determining $H_1^{(-l)*} = A_1 + X^{(-l)*}$ satisfies the equation

$$X^{(-l)*} = \int\limits_{\sigma'(A_0)\cup\Gamma_l}\left(A_1+X^{(-l)*}-\mu\right)^{-1}K_B(d\mu)\,. \tag{4.13}$$

One supposes here that $\mathrm{dist}\{\sigma(A_1),\sigma'(A_0)\cup\Gamma_{(-l)}\} > r_0(B)$. If, additionally, the condition (3.11) is valid for Γ_l, then $X^{(-l)*}$ is the only solution of this equation. The proof of this statement repeats literally the proof of Theorem 3.1.

THEOREM 4.2 *The spectrum $\sigma(H_1^{(l)})$ of the operator $H_1^{(l)} = A_1 + X^{(l)}$ belongs to the closed $r_0(B)$-vicinity $\mathcal{O}_{r_0}(A_1)$ of the spectrum of A_1, $\mathcal{O}_{r_0}(A_1) = \{z\in\mathbb{C}:\ \mathrm{dist}\{z,\sigma(A_1)\}\le r_0(B)\}$. If a contour $\Gamma_l\subset D_l$ satisfies (3.11), then the complex spectrum of $H_1^{(l)}$ belongs to $D_l\cap\mathcal{O}_{r_0}(A_1)$ while outside D_l the spectrum of $H_1^{(l)}$ is pure real. Moreover, the spectrum $\sigma(H_1^{(l)})$ coincides with a (subset of the) spectrum of the transfer function $M_1(z,\Gamma_l)$. More precisely, the spectrum of $M_1(z,\Gamma_l)$ in $\mathcal{O}_{d_0/2}(A_1) = \{z:\ z\in\mathbb{C},\mathrm{dist}\{z,\sigma(A_1)\}\le d_0(\Gamma_l)/2\}$ is represented only by the spectrum of $H_1^{(l)}$, i. e. $\sigma\Big(M_1(\cdot,\Gamma_l)\Big)\cap\mathcal{O}_{d_0/2}(A_1) = \sigma(H_1^{(l)})$. Also, the following more detailed relations hold:*

$$\sigma_p(H_1^{(l)}) = \sigma_p\Big(M_1(\cdot,\Gamma_l)\Big)\cap\mathcal{O}_{d_0/2}(A_1), \tag{4.14}$$

$$\sigma_c(H_1^{(l)}) = \sigma_c\Big(M_1(\cdot,\Gamma_l)\Big)\cap\mathcal{O}_{d_0/2}(A_1). \tag{4.15}$$

P r o o f. The spectrum of $H_1^{(l)}$ belongs to $\mathcal{O}_{r_0}(A_1)$, since the estimate (3.18) is valid for $X^{(l)}$. The statement regarding the spectrum of $M_1(z,\Gamma_l)$ follows immediately from the factorization formula (4.1)

$$[M_1(z,\Gamma_l)]^{-1} = (H_1^{(l)}-z)^{-1}[W_1(z,\Gamma_l)]^{-1}, \tag{4.16}$$

since $[W_1(z,\Gamma_l)]^{-1}$ exists and is bounded in $\mathcal{O}_{d_0/2}(A_1)$. Since outside $\overline{D(\Gamma_l)}$ the transfer function $M_1(z,\Gamma_l)$ coincides with the physical-sheet transfer function $M_1(z)$ (see Remark 2.1), the spectrum of $M_1(\cdot,\Gamma_l)$ belongs to $\mathbb{R}$ or $\overline{D(\Gamma_l)}$. But, as we have already established, the spectrum $\sigma(H_1^{(l)})$ represents all the spectrum of $M_1(\cdot,\Gamma_l)$ situated in $\mathcal{O}_{d_0/2}(A_1)$. Hence, the points $z\in\sigma(H_1^{(l)})$ also belong to $\mathbb{R}$ or $\overline{D(\Gamma_l)}$. This just means that for complex $z\in\sigma(H_1^{(l)})$ we have $z\in\mathcal{O}_{r_0}(A_1)\cap\overline{D_l}$.

According to (4.16), not only the location of the singularities of $[M_1(z,\Gamma_l)]^{-1}$ and $(H_1^{(l)}-z)^{-1}$ coincides, but the properties of these singularities are also the same and, hence, Eqs. (4.14) and (4.15) are valid.

The proof is complete. □

COROLLARY 4.2 *It follows from* Corollary 4.1 *that, in fact, the complex spectrum of the transfer function $M_1(z,\Gamma_l)$ is only represented by the spectrum of $H_1^{(l)}$ even in a widen domain than in the statement of* Theorem 4.2; *namely, in the domain* $D(\Gamma_l)\cap\mathcal{O}_{d_{\max}/2}(A_1)$.

THEOREM 4.3 *The spectrum of the operator $H_1^{(-l)*}$ coincides with the spectrum of the operator $H_1^{(l)}$. Moreover,*

$$\sigma_p(H_1^{(-l)*})=\sigma_p(H_1^{(l)})=\sigma_p(M_1(\cdot,\Gamma_l))\cap\mathcal{O}_{d_0/2}(A_1), \tag{4.17}$$

$$\sigma_c(H_1^{(-l)*})=\sigma_c(H_1^{(l)})=\sigma_c(M_1(\cdot,\Gamma_l))\cap\mathcal{O}_{d_0/2}(A_1), \tag{4.18}$$

where Γ_l stands for an arbitrary K_B-bounded contour $\Gamma_l\subset D_l$ satisfying the condition (3.11).

P r o o f. This statement is an immediate consequence of Lemma 4.1. For a K_B-bounded contour $\Gamma_l\subset D_l$ satisfying the condition (3.11) both operators $W_1(z,\Gamma_l)$ and $[W_1(\overline{z},\Gamma_{(-l)})]^*$ are boundedly invertible for $z\in\mathcal{O}_{d_0/2}(A_1)$ and, recall, $d_0(\Gamma_l)/2>r_0(B)$. Meanwhile, according to Eq. (4.9)

$$(H_1^{(l)}-z)^{-1}[W_1(z,\Gamma_l)]^{-1}=[W_1(\overline{z},\Gamma_{(-l)})]^{*-1}(H_1^{(-l)*}-z)^{-1}. \tag{4.19}$$

Therefore, the singularities of the resolvents $(H_1^{(l)}-z)^{-1}$ and $(H_1^{(-l)*}-z)^{-1}$ have the same location and properties as those of $[M_1(z,\Gamma_l)]^{-1}$ in $\mathcal{O}_{d_0/2}(A_1)$. This assertion implies the statement of Theorem. □

THEOREM 4.4 *Suppose that two different domains $D_{l'}$ and $D_{l''}$ include the same subdomain $D_k^{l_k}$ for some $k=1,2,\ldots,m$, i. e., $l'_k=l''_k=l_k$. Then the spectra of the operators $H_1^{(l')}$ and $H_1^{(l'')}$ in $D_k^{l_k}$ coincide,*

$$\sigma_s(H_1^{(l')})\cap D_k^{l_k}=\sigma_s(H_1^{(l'')})\cap D_k^{l_k} \tag{4.20}$$

where $s=p$ or $s=c$.

P r o o f. This statement follows again from Eq. (4.16) and from the identity of $M_1(z,\Gamma_{l'})$ and $M_1(z,\Gamma_{l''})$ for $z\in D(\Gamma_{l'})\cap D(\Gamma_{l''})\cap D_k^{l_k}$, $\Gamma_{l'}$ and $\Gamma_{l''}$ being arbitrary K_B-bounded contours satisfying (3.11). □

Corollary 4.3 (Symmetry of the resonance spectrum with respect to the real axis) *The spectra of any two operators* $H^{(l')}$ *and* $H^{(l'')}$ *for* $l'=(l'_1,l'_2,\ldots,l'_m)$ *and* $l''=(l''_1,l''_2,\ldots,l''_m)$ *are related to each other as follows*

$$\begin{aligned}\sigma_s(H_1^{(l'')})\cap D_k^{l''_k} &= \sigma_s(H_1^{(l')})\cap D_k^{l'_k} \quad \text{if} \quad l''_k=l'_k,\\ \sigma_s(H_1^{(l'')})\cap D_k^{l''_k} &= \sigma_s^*(H_1^{(l')})\cap D_k^{l''_k} \quad \text{if} \quad l''_k=-l'_k\end{aligned}$$

where the symbol "$*$" denotes complex conjugation and, as previously, $s=p$ *or* $s=c$.

In the following we shall use the operators

$$\Omega^{(l)}=\int\limits_{\sigma'(A_0)\cup\Gamma_l}(H_1^{(-l)*}-\mu)^{-1}K_B(d\mu)\,(H_1^{(l)}-\mu)^{-1} \tag{4.21}$$

acting in $\mathcal{H}_1$, where Γ_l stands for a K_B-bounded contour satisfying the condition (3.11). The operator $\Omega^{(l)}$ does not depend (for a fixed l) on the choice of such a Γ_l. In the same way as we came to the estimate (4.6) one can obtain the following estimate for $\Omega^{(l)}$ (see Lemma B.4 of Appendix B):

$$\|\Omega^{(l)}\|<\frac{\mathcal{V}_0(B,\Gamma_l)}{(d_0(\Gamma_l)/2)^2}<1. \tag{4.22}$$

Obviously, we have the equality

$$\Omega^{(l)*}=\Omega^{(-l)}. \tag{4.23}$$

It should be noted that in the case where the spectra of the entries A_0 and A_1 do not overlap, the operators (4.21) as well as the operators $H_1^{(l)}$ do not depend on l. In this case one has $H_1=A_1+B_{01}Q_{01}$ with the contraction

$$Q_{01}=\int\limits_{\sigma(A_0)}E_0(d\mu)B_{01}(H_1-\mu)^{-1},$$

$Q_{01}:\mathcal{H}_1\to\mathcal{H}_0$ (see Theorem 5 of Ref. [30]; cf. [2,3,27]), and $\Omega=Q_{01}^*Q_{01}$. Changing the inner product in $\mathcal{H}_1$ to $[\cdot,\cdot]=\langle(I_1+\Omega)\cdot,\cdot\rangle$ turns H_1 into a self-adjoint operator. However, in the case we consider in the present work this is in general not true since $H_1^{(l)}$ can have a complex spectrum.

Theorem 4.5 *The operators* $\Omega^{(l)}$ *possess the following properties*[6]*:*

$$-\frac{1}{2\pi\mathrm{i}}\int\limits_{\gamma}dz\,[M_1(z,\Gamma_l)]^{-1}=(I_1+\Omega^{(l)})^{-1} \tag{4.24}$$

---

[6]For the case where $A_1=\lambda I_1$ with $\lambda\in\mathbb{R}$ and $\dim\mathcal{H}_1<\infty$ one can find formulas similar to those in Eqs. (4.24) and (4.25) in the final part of Sect. 4 of Ref. [14]. See also [25,27,42].

and

$$-\frac{1}{2\pi \mathrm{i}}\int\limits_{\gamma} dz\, z\,[M_1(z,\Gamma_l)]^{-1} = (I_1+\Omega^{(l)})^{-1}H_1^{(-l)*} = H_1^{(l)}(I_1+\Omega^{(l)})^{-1} \tag{4.25}$$

where γ stands for an arbitrary rectifiable closed (including the points at infinity if the entry A_1 is unbounded) contour going in the positive direction around the spectrum of $H_1^{(l)}$ inside the set $\mathcal{O}_{d_0(\Gamma)/2}(A_1)$. The integration over γ is understood in the strong sense.

P r o o f . First, we note that, using the Closed Graph Theorem and the definition (B.7) of the integral (4.21), one can easily check that for any $u_1 \in \mathcal{D}(H_1^{(-l)*}) = \mathcal{D}(A_1)$ the image $\Omega^{(l)}u_1$ belongs to $\mathcal{D}(H_1^{(l)}) = \mathcal{D}(A_1)$. And due to (4.22) the operator $I_1+\Omega^{(l)}$ is a bijection of $\mathcal{D}(A_1)$ on $\mathcal{D}(A_1)$.

Further, we prove the validity of Eq. (4.24). At the beginning we recall that if $z \in \mathcal{O}_{d_0(\Gamma_l)/2}(A_1)$, then the factorization (4.16) and (4.19) holds for $[M_1(z,\Gamma_l)]^{-1}$ with the holomorphic functions $[W_1(z,\Gamma_l)]^{-1}$ and $[W_1(\overline{z},\Gamma_{(-l)})]^{*-1}$ taking their values in $\mathbf{B}(\mathcal{H}_1,\mathcal{H}_1)$. Meanwhile the product $\Omega^{(l)}(H_1^{(l)}-z)^{-1}$ can be written as

$$\Omega^{(l)}(H_1^{(l)}-z)^{-1} = F_1(z)+F_2(z)$$

with

$$F_1(z) = \int\limits_{\sigma'(A_0)\cup\Gamma_l} \frac{(H_1^{(-l)*}-\mu)^{-1}\,K_B(d\mu)\,(H_1^{(l)}-\mu)^{-1}}{\mu-z} \tag{4.26}$$

and

$$F_2(z) = \Big([W_1(\overline{z},\Gamma_{(-l)})]^* - I_1\Big)(H_1^{(l)}-z)^{-1},$$

since

$$(H_1^{(l)}-\mu)^{-1}(H_1^{(l)}-z)^{-1} = [(H_1^{(l)}-\mu)^{-1}-(H_1^{(l)}-z)^{-1}](\mu-z)^{-1}$$

and since $[W_1(\overline{z},\Gamma_{(-l)})]^*$ is given by (4.12). Therefore,

$$\begin{aligned}
&(I_1+\Omega^{(l)})\,[M_1(z,\Gamma_l)]^{-1} = \\
&= [M_1(z,\Gamma_l)]^{-1} + F_1(z)[W_1(z,\Gamma_l)]^{-1} + ([W_1(\overline{z},\Gamma_{(-l)})]^*-I_1)[M_1(z,\Gamma_l)]^{-1} \\
&= F_1(z)\,[W_1(z,\Gamma_l)]^{-1} + (H_1^{(-l)*}-z)^{-1}.
\end{aligned}$$

One should notice that the function $F_1(z)$ is holomorphic in z inside the contour γ, $\gamma \subset \mathcal{O}_{d_0(\Gamma_l)/2}(A_1)$, since the argument μ of the integrand in (4.26) belongs to $\sigma'(A_0)\cup\Gamma_l$ and, thereby, always $|z-\mu| \geq d_0(\Gamma_l)/2 > 0$. Thus the term $F_1(z)[W_1(z,\Gamma_l)]^{-1}$ makes no contribution to the integral

$$-\frac{1}{2\pi \mathrm{i}}\int\limits_{\gamma} dz (I_1+\Omega^{(l)})[M_1(z,\Gamma_l)]^{-1}$$

while the resolvent $(H_1^{(-l)*}-z)^{-1}$ gives the identity I_1. Therefore, we have proved that Eq. (4.24) is indeed valid.

Regarding Eq. (4.25) one finds

$$-\frac{1}{2\pi \mathrm{i}} \int\limits_{\gamma} dz (I_1 + \Omega^{(l)})\, z\, [M_1(z,\Gamma_l)]^{-1} =$$

$$= -\frac{1}{2\pi \mathrm{i}} \int\limits_{\gamma} dz\, z\, F_1(z)\, [W_1(z,\Gamma_l)]^{-1} - \frac{1}{2\pi \mathrm{i}} \int\limits_{\gamma} dz\, z\, (H_1^{(-l)*} - z)^{-1}$$

where only the last integral is non-zero, giving a contribution just equal to $H_1^{(-l)*}$. The second equation of (4.25) can be checked in the same way. The proof is complete. □

REMARK 4.1 *The formula* (4.25) *implies*

$$H_1^{(l)*} = (I_1 + \Omega^{(-l)})\, H_1^{(-l)}\, (I_1 + \Omega^{(-l)})^{-1} .$$

THEOREM 4.6 *Let λ be an isolated eigenvalue of the operator $H_1^{(l)}$ and, consequently, of the operator $H_1^{(-l)*}$ and of the transfer function $M_1(z,\Gamma_l)$ taken for a K_B-bounded contour Γ_l satisfying the condition* (3.11)*. Denote by $\mathsf{P}_\lambda^{(l)}$ and $\mathsf{P}_\lambda^{(-l)*}$ the respective eigenprojections of the operators $H_1^{(l)}$ and $H_1^{(-l)*}$ and by $P_\lambda^{(l)}$, the residue of $M_1(z,\Gamma_l)$ at $z = \lambda$,*

$$\mathsf{P}_\lambda^{(l)} = -\frac{1}{2\pi \mathrm{i}} \int\limits_{\gamma} dz\ (H_1^{(l)} - z)^{-1}, \tag{4.27}$$

$$\mathsf{P}_\lambda^{(-l)*} = -\frac{1}{2\pi \mathrm{i}} \int\limits_{\gamma} dz\ (H_1^{(-l)*} - z)^{-1}$$

and

$$P_\lambda^{(l)} = -\frac{1}{2\pi \mathrm{i}} \int\limits_{\gamma} dz\ [M_1(z,\Gamma_l)]^{-1}$$

where γ stands for an arbitrary rectifiable closed contour situated in a sufficiently close vicinity of the point λ and going in the positive direction around λ so that $\gamma \cap \Gamma_l = \emptyset$ and no points of the spectrum of $M_1(\cdot,\Gamma_l)$, except the eigenvalue λ, lie inside γ. Then the following relations are valid:

$$P_\lambda^{(l)} = (I_1 + \Omega^{(l)})^{-1}\ \mathsf{P}_\lambda^{(-l)*} = \mathsf{P}_\lambda^{(l)}\ (I_1 + \Omega^{(l)})^{-1} . \tag{4.28}$$

P r o o f is carried out in the same way as the proof of the relation (4.24) in Theorem 4.5, only the path of integration is changed. □

5. SOME PROPERTIES OF REAL EIGENVALUES

If λ is a real eigenvalue of $H_1^{(l)}$, then it can not belong to the spectrum $\sigma'(A_0)$ of the entry A_0 lying outside $\bigcup\limits_{k=1}^{m} \Delta_k^0$. Indeed, according to Theorem 4.2, the spectrum of $H_1^{(l)}$

for arbitrary l is situated in the $r_0(B)$-vicinity $\mathcal{O}_{r_0}(A_1)$ of the set $\sigma(A_1)$ and in any case $r_0(B) < \frac{1}{2}\operatorname{dist}\{\sigma'(A_0), \sigma(A_1)\}$ so that automatically

$$\sigma'(A_0) \cap \sigma(H_1^{(l)}) = \emptyset \qquad \text{and in particular} \qquad \sigma'(A_0) \cap \sigma_p(H_1^{(l)}) = \emptyset\,. \tag{5.1}$$

Hence, such a λ belongs either to the resolvent set $\rho(A_0)$ of the entry A_0 or it is embedded into the continuous spectrum of A_0 in $\bigcup\limits_{k=1}^{m} \Delta_k^0$.

LEMMA 5.1 *If a vector $\psi^{(1)} \in \mathcal{D}(A_1)$ is an eigenvector of $H_1^{(l)}$ corresponding to a real eigenvalue $\lambda \in \rho(A_0)$, $H_1^{(l)}\psi^{(1)} = \lambda\psi^{(1)}$, then the vector $\Psi = (\psi^{(0)}, \psi^{(1)}) \in \mathcal{H}$ with*

$$\psi^{(0)} = -R_0(\lambda)B_{01}\psi^{(1)} \tag{5.2}$$

is an eigenvector of $\mathbf{H}$, $\mathbf{H}\Psi = \lambda\Psi$. The converse statement is also true: if λ, $\lambda \in \rho(A_0)$, is a real eigenvalue of $H_1^{(l)}$ and $\mathbf{H}\Psi = \lambda\Psi$ for some $\Psi = (\psi^{(0)}, \psi^{(1)})$ with $\psi^{(0)} \in \mathcal{D}(A_0)$ and $\psi^{(1)} \in \mathcal{D}(A_1)$, then $\psi^{(0)}$ is related to $\psi^{(1)}$ as in (5.2) *(and, therefore, $\psi^{(1)}$ can not be zero) and $H_1^{(l)}\psi^{(1)} = \lambda\psi^{(1)}$.*

P r o o f . Let us consider a K_B-bounded contour $\Gamma_l \subset D_l$ satisfying the estimate (3.11). Since $\lambda \in \rho(A_0) \cap \mathbb{R}$, and therefore $\lambda \notin \overline{D(\Gamma_l)}$, we have

$$M_1(\lambda, \Gamma_l) = M_1(\lambda). \tag{5.3}$$

So that, according to the factorization formula (4.1), the eigenvector $\psi^{(1)}$ of $H_1^{(l)}$ is automatically an eigenvector of the initial transfer function $M_1(\cdot)$,

$$(A_1 - \lambda - B_{10}(A_0 - \lambda)^{-1}B_{01})\psi^{(1)} = 0.$$

Introducing $\psi^{(0)}$ via (5.2) one immediately finds that $\Psi = (\psi^{(0)}, \psi^{(1)})$ turns out to be an eigenvector for $\mathbf{H}$, $\mathbf{H}\Psi = \lambda\Psi$.

For the converse statement one first observes that if $\mathbf{H}\Psi = \lambda\Psi$ with $\Psi = (\psi^{(0)}, \psi^{(1)})$ (and, thus, $A_0\psi^{(0)} + B_{01}\psi^{(1)} = \lambda\psi^{(0)}$), then Eq. (5.2) holds true. But this means that $M_1(\lambda)\psi^{(1)} = 0$ and, hence, $M_1(\lambda, \Gamma_l)\psi^{(1)} = 0$ is also true. Then, due to Eq. (4.1) and invertibility of $W_1(\lambda, \Gamma_l)$ (see Theorem 4.1), $\psi^{(1)}$ is an eigenvector of $H_1^{(l)}$, $H_1^{(l)}\psi^{(1)} = \lambda\psi^{(1)}$. $\square$

If an eigenvalue λ of $H_1^{(l)}$ belongs to $\Delta_k^0 = (\mu_k^{(1)}, \mu_k^{(2)})$ for some $k = 1, 2, \ldots, m$, then

$$|\lambda - \mu_k^{(i)}| \geq \operatorname{dist}\{\mu_k^{(i)}, \sigma(A_1)\} - r_0(B), \quad i = 1, 2,$$

and, therefore, the λ is situated in this case strictly inside the interval Δ_k^0. Recall that according to our assumption the entry A_0 has no point spectrum inside Δ_k^0. Since Δ_k^0 is a part of the continuous spectrum of A_0, the resolvent $R_0(z) = (A_0 - z)^{-1}$ for $z = \lambda \pm i0$ exists being however an unbounded operator. Nevertheless a statement analogous to Lemma 5.1 is valid in this case, too.

LEMMA 5.2 *If a vector* $\psi^{(1)} \in \mathcal{D}(A_1)$ *is an eigenvector of* $H_1^{(l)}$ *corresponding to a real eigenvalue* $\lambda \in \Delta_k^0 = (\mu_k^{(1)}, \mu_k^{(2)})$, $k = 1, 2, \ldots, m$, $H_1^{(l)}\psi^{(1)} = \lambda\psi^{(1)}$, *then either*

a) $E^0(\mu)B_{01}\psi^{(1)} = 0$ *for all* $\mu \le \mu_k^{(2)}$

or

b) $E^0(\mu)B_{01}\psi^{(1)} \ne 0$ *for any* $\mu \in \Delta_k^0$,

c) the function $\|E^0(\mu)B_{01}\psi^{(1)}\|$ *is differentiable in* μ *on* Δ_k^0

and

d) $\left.\dfrac{d}{d\mu}\|E^0(\mu)B_{01}\psi^{(1)}\|\right|_{\mu=\lambda} = 0.$

In both cases the vector $\psi^{(0)}$ *given by* (5.2) *belongs to* $\mathcal{D}(A_0)$ *and* $\Psi = (\psi^{(0)}, \psi^{(1)})$ *is an eigenvector of* $\mathbf{H}$, $\mathbf{H}\Psi = \lambda\Psi$.

The converse statement is also true. Namely, if a $\Psi = (\psi^{(0)}, \psi^{(1)})$ *with* $\psi^{(0)} \in \mathcal{D}(A_0)$ *and* $\psi^{(1)} \in \mathcal{D}(A_1)$ *is an eigenvector of* $\mathbf{H}$, $\mathbf{H}\Psi = \lambda\Psi$, *corresponding to an eigenvalue* λ *of* $H_1^{(l)}$, $\lambda \in \Delta_k^0$, *then either the condition* (*a*) *is valid or the conditions* (*b* – *d*) *are valid. In both cases the relation* (5.2) *is retained meaning, in particular, that* $\psi^{(1)} \ne 0$ *and* $\psi^{(1)}$ *is an eigenvector of* $H_1^{(l)}$ *corresponding to the eigenvalue* λ.

P r o o f . We first prove the direct statement. To this end we consider the equality

$$\langle H_1^{(l)}\psi^{(1)}, \psi^{(1)}\rangle = \lambda\|\psi^{(1)}\|^2, \qquad \lambda \in \Delta_k^0,$$

which becomes, according to Eqs. (3.7) and (3.8),

$$\langle (A_1 - \lambda)\psi^{(1)}, \psi^{(1)}\rangle + \int\limits_{\sigma'(A_0)\cup\Gamma_l} \frac{\langle K_B(d\mu)\psi^{(1)}, \psi^{(1)}\rangle}{\lambda - \mu} = 0. \tag{5.4}$$

Since the denominator of the integrand is non-zero for $\mu \in \sigma(A_0) \setminus \Delta_k^0$, we can deform the part $\Gamma_l \setminus \Gamma_k^{l_k}$ of the contour Γ_l in (5.4) back into the intervals Δ_i^0, $i = 1, 2, \ldots, m$, $i \ne k$. As a result, Eq. (5.4) acquires the form

$$\begin{aligned} \langle (A_1 - \lambda)\psi^{(1)}, \psi^{(1)}\rangle \;+ & \int\limits_{\sigma(A_0)\setminus\Delta_k^0} \frac{\langle B_{10}E_0(d\mu)B_{01}\psi^{(1)}, \psi^{(1)}\rangle}{\lambda - \mu} \\ & + \int\limits_{\Gamma_k^{l_k}} d\mu\, \frac{\langle K_B'(\mu)\psi^{(1)}, \psi^{(1)}\rangle}{\lambda - \mu} = 0. \end{aligned} \tag{5.5}$$

Obviously, the first and second terms are real, and an imaginary component may appear in the l. h. side of Eq. (5.5) only in the third term. To find this component one can simply transform the integration path to the two intervals $[\mu_k^{(1)}, \lambda - \varepsilon]$ and $[\lambda + \varepsilon, \mu_k^{(2)}]$ and the semicircle $|\mu - \lambda| = \varepsilon$, $l_k \cdot \mathrm{Im}\,\mu \ge 0$, between them. Then taking the limit $\varepsilon \downarrow 0$ one obtains

$$\mathrm{Im} \int\limits_{\Gamma_k^{l_k}} d\mu\, \frac{\langle K_B'(\mu)\psi^{(1)}, \psi^{(1)}\rangle}{\lambda - \mu} = l_k \cdot \pi \mathrm{i}\langle K_B'(\lambda)\psi^{(1)}, \psi^{(1)}\rangle = 0.$$

Therefore, we have

$$\langle K_B'(\lambda)\psi^{(1)}, \psi^{(1)}\rangle = 0.$$

Meanwhile, for any $u_1 \in \mathcal{H}_1$ and $\mu \in \Delta_k^0$

$$\langle K_B'(\mu)u_1, u_1\rangle = \frac{d}{d\mu}\langle B_{10}E^0(\mu)B_{01}u_1, u_1\rangle = \frac{d}{d\mu}\|E^0(\mu)B_{01}u_1\|^2.$$

Thus, the condition $\langle K_B'(\lambda)u_1, u_1\rangle = 0$ implies that either $\|E^0(\lambda)B_{01}u_1\| = 0$ or, if $\|E^0(\lambda)B_{01}u_1\| \neq 0$, then the function $\|E^0(\mu)B_{01}u_1\|$ is differentiable at $\mu = \lambda$ and $\frac{d}{d\mu}\|E^0(\mu)B_{01}u_1\|\Big|_{\mu=\lambda} = 0$.

Since $\|E^0(\mu)B_{01}u_1\|$ is a non-decreasing function of the variable μ, in the first case we conclude that $\|E^0(\mu)B_{01}u_1\| = 0$ for all $\mu \leq \lambda$ and, hence, $\langle K_B(\mu)u_1, u_1\rangle = \|E^0(\mu)B_{01}u_1\|^2 = 0$ for $\mu \leq \lambda$, too. Since $\langle K_B(\mu)u_1, u_1\rangle$ is supposed to be a holomorphic function of $\mu \in D_k^{l_k}$ we find $\langle K_B(\mu)u_1, u_1\rangle \equiv 0$ for $\mu \in D_k^{l_k}$ and, consequently, $\|E^0(\mu)B_{01}u_1\|^2 = \langle K_B(\mu)u_1, u_1\rangle \equiv 0$ for $\lambda < \mu \leq \mu_k^{(2)}$, too. So that we come to the condition (a). Applying this condition to $u_1 = \psi^{(1)}$ we find that in this case the formula (5.2) makes sense and $\Psi = (\psi^{(0)}, \psi^{(1)})$ with $\psi^{(0)} = -R_0(\lambda \pm \mathrm{i}0)B_{01}\psi^{(1)} \in \mathcal{D}(A_0)$ is an eigenvector for $\mathbf{H}$.

In the second case the (non-decreasing) function $\|E^0(\mu)B_{01}\psi^{(1)}\|$ is non-zero (condition (b)), and differentiable at any $\mu \in \Delta_k^0$ (condition (c)), and $\frac{d}{d\mu}\|E^0(\mu)B_{01}\psi^{(1)}\|\Big|_{\mu=\lambda} = 0$ (condition (d)). So that for any finite $\varepsilon, \eta > 0$ such that $[\lambda-\varepsilon, \lambda+\eta] \subset \Delta_k^0$ we have the estimate

$$\Big\|\int\limits_{\lambda-\varepsilon}^{\lambda+\eta} \frac{E_0(d\mu)B_{01}\psi^{(1)}}{\lambda-\mu}\Big\| \leq \int\limits_{\lambda-\varepsilon}^{\lambda+\eta} d\mu\, \frac{\frac{d}{d\mu}\|E_0(d\mu)B_{01}\psi^{(1)}\|}{|\lambda-\mu|} \leq C(\varepsilon,\eta)$$

with some positive $C(\varepsilon,\eta) < \infty$. Consequently, the integral

$$-R_0(\lambda \pm \mathrm{i}0)B_{01}\psi^{(1)} = \int\limits_{\sigma(A_0)} \frac{E_0(d\mu)B_{01}\psi^{(1)}}{\lambda-\mu}$$

exists and determines an element $\psi^{(0)} \in \mathcal{D}(A_0)$ such that again for $\Psi = (\psi^{(0)}, \psi^{(1)})$ one finds $\mathbf{H}\Psi = \lambda\Psi$.

Let us now prove the converse statement. First, we note that if $\mathbf{H}\Psi = \lambda\Psi$ with $\Psi = (\psi^{(0)}, \psi^{(1)})$, $\psi^{(0)} \in \mathcal{D}(A_0)$, $\psi^{(1)} \in \mathcal{D}(A_1)$, then

$$(A_0 - \lambda)\psi^{(0)} = -B_{01}\psi^{(1)}. \tag{5.6}$$

Let $E_{ac}^0(\mu)$ be the spectral function corresponding to the absolutely continuous spectrum of A_0, $E_{ac}^0(\mu) = E_0^{ac}\Big((-\infty, \mu)\Big)$. Applying the projection $E_0^{ac}(\delta)$ with $\delta \subset \Delta_k^0$ to both parts of Eq. (5.6) we obtain

$$\int\limits_{\delta} (\mu-\lambda)dE_{ac}^0(\mu)\psi^{(0)} = -\int\limits_{\delta} dE^0(\mu)B_{01}\psi^{(1)}$$

(recall that we assume $E_0(\delta)B_{01} = E_0^{ac}(\delta)B_{01}$ for any Borel set $\delta \subset \bigcup_{k=1}^m \Delta_k^0$). Thus, for the norm squares, one finds

$$\int\limits_{\delta} (\mu-\lambda)^2 d\langle E_{ac}^0(\mu)\psi^{(0)}, \psi^{(0)}\rangle = \int\limits_{\delta} d\mu \langle K_B'\psi^{(1)}, \psi^{(1)}\rangle,$$

for an arbitrary interval $\delta \subset \Delta_k^0$. Since the function $\langle E_{ac}^0(\mu)\psi^{(0)}, \psi^{(0)}\rangle$ is absolutely continuous and, hence, almost everywhere differentiable, one further finds

$$\langle K_B'(\mu)\psi^{(1)}, \psi^{(1)}\rangle = (\mu-\lambda)^2 \frac{d}{d\mu}\langle E_{ac}^0(\mu)\psi^{(0)}, \psi^{(0)}\rangle$$

for almost all $\mu \in \Delta_k^0$. Meanwhile, the derivative $\dfrac{d}{d\mu}\langle E_{ac}^0(\mu)\psi^{(0)}, \psi^{(0)}\rangle$ is an element of $L_1\Big(\sigma_{ac}(A_0)\Big)$. That is, the function $\langle K_B'(\mu)\psi^{(1)}, \psi^{(1)}\rangle \cdot (\mu-\lambda)^{-2}$ must also be integrable over any interval $\delta \subset \Delta_k^0$. Surely, this is possible only if $\langle K_B'(\lambda)\psi^{(1)}, \psi^{(1)}\rangle = 0$. Now one need only repeat the respective consideration from the proof of the direct part of the lemma and, as a result, come to the conditions (a) or (b – d). With these conditions the formula (5.2) is again correct. The only thing which must be stressed in the case of the condition (a) is the fact that $B_{01}\psi^{(1)} \neq 0$ if $\psi^{(0)} \neq 0$. Indeed, according to Eq. (5.6) the assumption $B_{01}\psi^{(1)} = 0$ implies that $\lambda \in \sigma_p(A_0)$. But this contradicts our initial assumption regarding continuity of the spectrum of A_0 within the intervals Δ_k^0. Thus, $\psi^{(1)}$ can not be zero and $M_1(\lambda \pm \mathrm{i}0)\psi^{(1)} = 0$. This also means that $M_1(\lambda, \Gamma_l)\psi^{(1)} = 0$ for arbitrary K_B-bounded contour $\Gamma_l \subset D_l$ satisfying the condition (3.11). Then applying Theorem 4.1 we conclude that $H_1^{(l)}\psi^{(1)} = \lambda\psi^{(1)}$. The proof is complete. □

REMARK 5.1 *In the above proof we have also found that if an eigenvalue λ of $H_1^{(l)}$ is embedded into an interval Δ_k^0, $k = 1, 2, \ldots, m$, then the function $\langle K_B'(\mu)\psi^{(1)}, \psi^{(1)}\rangle \cdot (\mu-\lambda)^{-2}$ with $\psi^{(1)}$ an eigenvector of $H_1^{(l)}$ corresponding to the λ is integrable over every interval $\delta \subset \Delta_k^0$. In fact, this means that not only $\langle K_B'(\lambda)\psi^{(1)}, \psi^{(1)}\rangle = 0$ but also*

$$\frac{d}{d\mu}\langle K_B'(\mu)\psi^{(1)}, \psi^{(1)}\rangle\Big|_{\mu=\lambda} = 0. \tag{5.7}$$

The latter statement follows from the fact that the function $\langle K_B'(\lambda)\psi^{(1)}, \psi^{(1)}\rangle$ is holomorphic with respect to the variable μ in any vicinity of the point λ included in $\Delta_k^0 \cup D_k^- \cup D_k^+$ and, moreover, this function is non-negative for $\mu \in \Delta_k^0$. □

COROLLARY 5.1 *The statements of* Lemmas 5.1 *and* 5.2 *imply $\sigma_p(H_1^{(l)}) \subset \sigma_p(\mathbf{H})$. Also, it immediately follows from these lemmas that any eigenvector $\psi^{(1)}$ corresponding to an eigenvalue $\lambda \in \sigma_p(H_1^{(l)}) \cap \mathbb{R}$ of the operator $H_1^{(l)} = A + X^{(l)}$ for a certain $l = (l_1, l_2, \ldots, l_m)$ is such an eigenvector, $H_1^{(l')}\psi^{(1)} = \lambda\psi^{(1)}$, for the remaining 2^{m-1} operators $H_1^{(l')} = A_1 + X^{(l')}$ for $l' = (l_1', l_2', \ldots, l_m')$ with arbitrary $l_k' = \pm 1$, $k = 1, 2, \ldots, m$. Thus, the set $\sigma_p(H_1^{(l)}) \cap \mathbb{R}$ is the same for all the 2^m operators $H_1^{(l)}$.*

LEMMA 5.3 *If some λ, $\lambda \in \mathbb{R}$ is an isolated eigenvalue of the operator $H_1^{(l')} = A_1 + X^{(l')}$ for some $l' = (l_1', l_2', \ldots, l_m')$, then this λ is also such an eigenvalue for the remaining 2^{m-1} operators $H_1^{(l)} = A_1 + X^{(l)}$ for $l = (l_1, l_2, \ldots, l_m)$ with arbitrary $l_k = \pm 1$, $k = 1, 2, \ldots, m$. Moreover, the resolvents for all the 2^m operators $H_1^{(l)}$ have a pole of the first order at $z = \lambda$ allowing the decomposition*

$$(H_1^{(l)} - z)^{-1} = \frac{\mathsf{P}_\lambda^{(l)}}{\lambda - z} + \widetilde{R}_\lambda^{(l)}(z) \tag{5.8}$$

with $\widetilde{R}_\lambda^{(l)}(z)$ *holomorphic in a vicinity of* λ. *Also, the factorization* (4.28) *holds where* $P_\lambda^{(l)}$ *does not depend on* l, *since it is the residue of the initial inverse transfer function* $R_{11}(z) = [M_1(z)]^{-1}$ *at* $z = \lambda$,

$$P_\lambda^{(l)} = u - \lim_{z\to\lambda}(\lambda - z)R_{11}(z)\,. \tag{5.9}$$

P r o o f . As the factorization formula (4.1) is valid for $M_1(z,\Gamma_l)$, any isolated real eigenvalue of $H_1^{(l)}$ is at the same time such an eigenvalue of $M_1(\cdot,\Gamma_l)$. Since in $\mathbb{C}\setminus\overline{D(\Gamma_l)}$ the function $M_1(z,\Gamma_l)$ coincides with the initial (i. e., not continued yet through the continous spectrum of the entry A_0) transfer function $M_1(z)$, the point λ must produce for $M_1^{-1}(z)$ the same singularity as for $[M_1(z,\Gamma_l)]^{-1}$. Meanwhile, due to the representation (1.4) for the resolvent $\mathbf{R}(z) = (\mathbf{H} - z)^{-1}$, any singular point of the block component $R_{11}(z) = M_1^{-1}(z)$ of $\mathbf{R}(z)$ produces a singularity of the $\mathbf{R}(z)$. Since $\mathbf{H}$ is a selfadjoint operator, any such a point of $R_{11}(z)$ can only be a pole which is maximum of the first order (even if it is embedded into the continuous spectrum of $\mathbf{H}$). Since all the above is true for arbitrary index $l = (l_1, l_2, \ldots, l_m)$, $l_k = \pm 1$, $k = 1, 2, \ldots, m$, and since the relations (4.28) hold, these considerations lead us immediately to the statements of the lemma. □

Let $\sigma_{pri}(H_1^{(l)})$ be the set of all real isolated eigenvalues of the operator $H_1^{(l)}$. According to Lemma 5.3 (cf. Corollary 5.1) the set $\sigma_{pri}(H_1^{(l)})$ is the same for all $l = (l_1, l_2, \ldots, l_m)$, $l_k = \pm 1$, $k = 1, 2, \ldots, m$. Moreover, this set coincides with the part $\sigma_{pri}(M_1(\cdot,\Gamma_l)$ of the set of the real isolated eigenvalues of the transfer function $M_1(z,\Gamma_l)$ belonging to $\mathcal{O}_{d_0/2}(A_1)$ for any K_B-bounded contour Γ_l satisfying the condition (3.11),

$$\sigma_{pri}(H_1^{(l)}) = \sigma_{pri}(M_1(\cdot,\Gamma_l)) \cap \mathcal{O}_{d_0/2}(A_1).$$

Since in the remainder of the Section we will consider different eigenvalues $\lambda \in \sigma_{pri}(H_1^{(l)})$, we will use a more specific notation, $\psi_{\lambda,j}^{(1)}$, $j = 1, 2, \ldots, m_\lambda$, for the respective eigenvectors of the $H_1^{(l)}$. The notation m_λ, $m_\lambda \le \infty$, stands for the multiplicity of the eigenvalue λ. Recall that every $\psi_{\lambda,j}^{(1)}$ is an eigenvector simultaneously for all the $H_1^{(l)}$ and $M_1(\lambda \pm \mathrm{i}0, \Gamma_l)$, $l = (l_1, l_2, \ldots, l_m)$ with $l_k = \pm 1$, $k = 1, 2, \ldots, m$ (see Lemmas 5.1 and 5.2). Since, according to Lemma 5.3, the resolvent $(H_1^{(l)} - z)^{-1}$ has at $z = \lambda \in \sigma_{pri}(H_1^{(l)})$ a pole of the first order, the multiplicity m_λ is, in the considered case, both the geometric and algebraic multiplicity of λ (in such a case every element of the subspace $\mathsf{P}_\lambda^{(l)}\mathcal{H}_1$ is an eigenvector of $H_1^{(l)}$ since $(H_1^{(l)} - \lambda)\mathsf{P}_\lambda^{(l)} = 0$). Respective eigenvectors of the total matrix $\mathbf{H}$ will be denoted by $\Psi_{\lambda,j}$, $\Psi_{\lambda,j} = (\psi_{\lambda,j}^{(0)}, \psi_{\lambda,j}^{(1)})$. It will be supposed that the $\psi_{\lambda,j}^{(1)}$ are chosen in such a way that the vectors $\Psi_{\lambda,j}$ are orthonormal, $\langle\Psi_{\lambda,j}, \Psi_{\lambda',j'}\rangle = \delta_{\lambda\lambda'}\delta_{jj'}$. Obviously, the statements of Lemmas 5.1 and 5.2 imply that the eigenvectors $\Psi_{\lambda,j}$, $\lambda \in \sigma_{pri}(H_1^{(l)})$, $j = 1, 2, \ldots, m_\lambda$, form an orthonormal basis in the invariant subspace of the operator $\mathbf{H}$ corresponding to the subset $\sigma_{pri}(H_1^{(l)})$ of the point spectrum $\sigma_p(\mathbf{H})$ of $\mathbf{H}$.

Let $\mathcal{H}_1^{(pri)}$, $\mathcal{H}_1^{(pri)} \subset \mathcal{H}_1$, be the closed linear span of the eigenvectors $\psi_{\lambda,j}^{(1)}$ of $H_1^{(l)}$ corresponding to the spectrum $\sigma_{pri}(H_1^{(l)})$,

$$\mathcal{H}_1^{(pri)} = \overline{\bigvee\{\psi_{\lambda,j}^{(1)},\ \lambda \in \sigma_{pri}(H_1^{(l)}),\ j = 1, 2, \ldots, m_\lambda\}}.$$

The following statement holds.

THEOREM 5.1 *The system of vectors*

$$\psi_{\lambda,j}^{(1)}, \quad \lambda \in \sigma_{pri}(H_1^{(l)}), \quad j = 1, 2, \ldots, m_\lambda, \tag{5.10}$$

forms a Riesz basis of the subspace $\mathcal{H}_1^{(pri)}$.

We first prove an auxiliary assertion.

LEMMA 5.4 *For any* $l = (l_1, l_2, \ldots, l_m)$, $l_k = \pm 1$, $k = 1, 2, \ldots, m$, *the operator* $\Omega^{(l)}$ *defined by Eq.* (4.21) *is non-negative on the subspace* $\mathcal{H}_1^{(pri)}$.

P r o o f . It suffices to prove the assertion for a dense subset of $\mathcal{H}_1^{(pri)}$, say, for elements $u_1 \in \mathcal{H}_1^{(pri)}$ of the form

$$u_1 = \sum_{(\lambda,j)\in\mathcal{I}} c_{\lambda,j}\psi_{\lambda,j}^{(1)}, \qquad c_{\lambda,j} \in \mathbb{C},$$

where $\mathcal{I}$ runs through the finite subsets of the set of all possible pairs (λ, j) with $\lambda \in \sigma_{pri}(H_1^{(l)})$, $j = 1, 2, \ldots, m_\lambda$. We have

$$\langle \Omega^{(l)} u_1, u_1 \rangle = \sum_{(\lambda,j)\in\mathcal{I}} \sum_{(\lambda',j')\in\mathcal{I}} c_{\lambda,j}\overline{c}_{\lambda',j'}\Omega_{\lambda,j;\,\lambda',j'}^{(l)}$$

with

$$\begin{aligned}
\Omega_{\lambda,j;\,\lambda',j'}^{(l)} &= \int\limits_{\sigma'(A_0)\cup\Gamma_l} \langle\ K_B(d\mu)\,(H_1^{(l)} - \mu)^{-1}\psi_{\lambda,j}^{(1)}, (H_1^{(-l)} - \overline{\mu})^{-1}\psi_{\lambda',j'}^{(1)}\rangle \\
&= \int\limits_{\sigma'(A_0)\cup\Gamma_l} \frac{\langle\ K_B(d\mu)\,\psi_{\lambda,j}^{(1)}, \psi_{\lambda',j'}^{(1)}\rangle}{(\mu - \lambda)\,(\mu - \lambda')},
\end{aligned} \tag{5.11}$$

since $\psi_{\lambda,j}^{(1)}$ and $\psi_{\lambda',j'}^{(1)}$ are eigenvectors for both $H_1^{(l)}$ and $H_1^{(-l)}$. Due to Lemma 5.2 (see also Remark 5.1 to that Lemma) one can transform the subcontours $\Gamma_k^{l_k} \subset \Gamma_l$, $k = 1, 2, \ldots, m$, back to the respective intervals Δ_k^0, on which $K_B(d\mu) = B_{10}E_0(d\mu)B_{01}$, even in case $\lambda \in \Delta_k^0$ and/or $\lambda' \in \Delta_k^0$. After such a transformation one can use Eq. (5.2) to express $\psi_{\lambda,j}^{(0)}$ in terms of $\psi_{\lambda,j}^{(1)}$ and $\psi_{\lambda',j'}^{(0)}$ in terms of $\psi_{\lambda',j'}^{(1)}$. As a result, one finds

$$\Omega_{\lambda,j;\,\lambda',j'}^{(l)} = \langle \psi_{\lambda,j}^{(0)}, \psi_{\lambda',j'}^{(0)} \rangle \quad \text{(independent of } l\text{)} \tag{5.12}$$

and, hence,

$$\langle \Omega^{(l)} u_1, u_1 \rangle = \|u_0\|^2 \geq 0$$

with $u_0 = \sum_{(\lambda,j)\in\mathcal{I}} c_{\lambda,j}\psi^{(0)}_{\lambda,j}$. Thus, the operator $\Omega^{(l)}$ is non-negative on a dense subset of $\mathcal{H}_1^{(pri)}$ and, consequently, it is non-negative on the whole subspace $\mathcal{H}_1^{(pri)}$, too. The proof of the lemma is complete. $\square$

Thus, one can introduce a new inner product in $\mathcal{H}_1^{(pri)}$,

$$[u_1, v_1]_{\mathcal{H}_1^{(pri)}} = \langle (I_1 + \Omega^{(l)})u_1, v_1\rangle, \qquad u_1, v_1 \in \mathcal{H}_1^{(pri)}, \tag{5.13}$$

topologically equivalent to the initial inner product $\langle\cdot,\cdot\rangle$, since $I_1 + \Omega^{(l)} \geq I_1$ on $\mathcal{H}_1^{(pri)}$ and since, in view of the estimate (4.22), the operator $I_1 + \Omega^{(l)}$ is boundedly invertible. (One can even check that the restriction of $H_1^{(l)}$ on $\mathcal{D}(A_1)\cap\mathcal{H}_1^{(pri)}$ does not depend on l and is an operator in $\mathcal{H}_1^{(pri)}$ which is self-adjoint with respect to the inner product $[\cdot,\cdot]_{\mathcal{H}_1^{(pri)}}$.)

P r o o f of Theorem 5.1. We prove that the vector system (5.10) is an orthonormal system with respect to the inner product $[\cdot,\cdot]_{\mathcal{H}_1^{(pri)}}$. Indeed, according to Eqs. (5.11) and (5.12) we have

$$\langle \Omega^{(l)}\psi^{(1)}_{\lambda,j}, \psi^{(1)}_{\lambda',j'}\rangle = \Omega^{(l)}_{\lambda,j;\,\lambda',j'} = \langle \psi^{(0)}_{\lambda,j}, \psi^{(0)}_{\lambda',j'}\rangle.$$

Thus,

$$[\psi^{(1)}_{\lambda,j}, \psi^{(1)}_{\lambda',j'}] = \langle \psi^{(1)}_{\lambda,j}, \psi^{(1)}_{\lambda',j'}\rangle + \langle \psi^{(0)}_{\lambda,j}, \psi^{(0)}_{\lambda',j'}\rangle = \langle \Psi_{\lambda,j}, \Psi_{\lambda',j'}\rangle = \delta_{\lambda\lambda'}\delta_{jj'}.$$

In addition, the system (5.10) is complete in $\mathcal{H}_1^{(pri)}$ and the inner product $[\cdot,\cdot]_{\mathcal{H}_1^{(pri)}}$ is topologically equivalent to the initial inner product $\langle\cdot,\cdot\rangle$. According to a theorem of N. K. Bari (Theorem VI.2.1 of [13]) this means that the system (5.10) constitutes a basis of $\mathcal{H}_1^{(pri)}$ equivalent to an orthonormal one, i. e., it is a Riesz basis. The proof is complete. $\square$

6. THE OPERATORS $H_1^{(l)}$ IN THE CASE OF A FINITE-DIMENSIONAL SPACE $\mathcal{H}_1$

If $n_{\mathcal{H}_1} = \dim\mathcal{H}_1 < \infty$, then the operators A_1 and $H_1^{(l)}$ are simply $n_{\mathcal{H}_1}\times n_{\mathcal{H}_1}$ scalar matrices. In this case the resolvent of $H_1^{(l)}$ admits the representation (see e. g., [16], pp. 39–44)

$$(H_1^{(l)} - z)^{-1} = \sum_{i=1}^{s}\left(-\frac{\mathsf{P}_i^{(l)}}{z-\lambda_i^{(l)}} - \sum_{1\leq k\leq n_i-1}\frac{[\mathsf{N}_i^{(l)}]^k}{(z-\lambda_i^{(l)})^{k+1}}\right). \tag{6.1}$$

Here $\lambda_i^{(l)}$, $i = 1, 2, \ldots, s$, $s \leq n_{\mathcal{H}_1}$, stand for the different eigenvalues of $H_1^{(l)}$, $\mathsf{P}_i^{(l)}$ for the eigenprojections and $\mathsf{N}_i^{(l)}$, $\mathsf{N}_i^{(l)} = (H_1^{(l)} - \lambda_i^{(l)})\mathsf{P}_i^{(l)} = \mathsf{P}_i^{(l)}(H_1^{(l)} - \lambda_i^{(l)})$ for the eigennilpotents corresponding to the $\lambda_i^{(l)}$. The n_i, $n_i \geq 1$ denote the pole orders of the resolvent $(H_1^{(l)} - z)^{-1}$ and, consequently, of the inverse transfer function $[M_1(z,\Gamma_l)]^{-1}$ at $z = \lambda_i^{(l)}$; if $n_i = 1$, then $\mathsf{N}_i^{(l)} = 0$ and the eigenvalue $\lambda_i^{(l)}$ is said to be *semisimple*.

Recall that all the eigenvalues $\lambda_i^{(l)}$, $i = 1, 2, \ldots, s$, belong to the set $\mathcal{O}_{r_0(B)}(A_1)$, see Corollary 3.1 to Theorem 3.2.

We assume that the enumeration of the $\lambda_i^{(l)}$ for $H_1^{(l)}$ with different l is co-ordinated in accordance with the statement of Corollary 4.3 to Theorem 4.4: If $l = (l_1, l_2, \ldots, l_m)$ and $l' = (l'_1, l'_2, \ldots, l'_m)$ with $l'_k = l_k$ for a certain $k = 1, 2, \ldots, m$, and $\lambda_i^{(l)} \in D_k^{l_k}$ then $\lambda_i^{(l')} = \lambda_i^{(l)}$. Also, $\lambda_i^{(-l)} = \overline{\lambda}_i^{(l)}$ is accepted. The indication of l in the notation n_i of the pole orders in (6.1) is omitted, since for a given i the pole order does not depend on l according to the factorization formulas (4.16) and (4.19).

The kernel $\mathcal{G}_i^{(l)} = \operatorname{Ker}(H_1^{(l)} - \lambda_i^{(l)})$ is called the geometric eigenspace for the eigenvalue $\lambda_i^{(l)}$, $i = 1, 2, \ldots, s$. For any $u_1 \in \mathcal{G}_i^{(l)}$ one has $H_1^{(l)} u_1 = \lambda_i^{(l)} u_1$. The subspace $\mathcal{M}_i^{(l)} = \mathsf{P}_i^{(l)} \mathcal{H}_1$ is called the algebraic eigenspace for $\lambda_i^{(l)}$; $m_i \equiv \dim \mathcal{M}_i^{(l)} \geq n_i$ and $\mathcal{G}_i^{(l)} \subset \mathcal{M}_i^{(l)}$, so that $g_i \equiv \dim \mathcal{G}_i^{(l)} \leq m_i$. Similarly to the pole order n_i, the *algebraic and geometric multiplicities m_i and g_i of the eigenvalue $\lambda_i^{(l)}$ for a given i do not depend on l.* The system of algebraic eigenspaces $\mathcal{M}_i^{(l)}$ is linearly independent and complete,

$$\mathcal{M}_1^{(l)} \dot{+} \mathcal{M}_2^{(l)} \dot{+} \ldots \dot{+} \mathcal{M}_s^{(l)} = \mathcal{H}_1, \tag{6.2}$$

and, thus,

$$m_1 + m_2 + \ldots + m_s = n_{\mathcal{H}_1}$$

and

$$\mathsf{P}_1^{(l)} + \mathsf{P}_2^{(l)} + \ldots + \mathsf{P}_s^{(l)} = I_1. \tag{6.3}$$

Any vector of $\mathcal{M}_i^{(l)}$ is called a *root vector* of the operator $H_1^{(l)}$ corresponding to the eigenvalue $\lambda_i^{(l)}$.

Recall some properties of the eigenprojections and eigennilpotents:

$$\begin{array}{ll} \mathsf{P}_i^{(l)} \mathsf{P}_j^{(l)} = \delta_{ij} \mathsf{P}_i^{(l)}, & \mathsf{P}_i^{(l)} \mathsf{N}_j^{(l)} = \mathsf{N}_j^{(l)} \mathsf{P}_i^{(l)} = \delta_{ij} \mathsf{N}_j^{(l)}, \\ \mathsf{N}_i^{(l)} \mathsf{N}_j^{(l)} = \delta_{ij} [\mathsf{N}_i^{(l)}]^2, & [\mathsf{N}_i^{(l)}]^{n_i} = 0 \quad \text{but} \quad [\mathsf{N}_i^{(l)}]^{n_i - 1} \neq 0. \end{array} \tag{6.4}$$

The spectral representation for the $H_1^{(l)}$, in terms of $\mathsf{P}_i^{(l)}$ and $\mathsf{N}_i^{(l)}$, is

$$H_1^{(l)} = \sum_{i=1}^{s} (\lambda_i^{(l)} \mathsf{P}_i^{(l)} + \mathsf{N}_i^{(l)}), \tag{6.5}$$

the decomposition being unique.

As we already established in Sect. 5, if $\lambda_i^{(l)} \in \mathbb{R}$, then $n_i = 1$ (see Lemma 5.3) and thus $\mathsf{N}_i^{(l)} = 0$. Therefore, in the case of a real eigenvalue $\lambda_i^{(l)}$, any (root) vector of the subspace $\mathcal{M}_i^{(l)}$ is an eigenvector of the operator $H_1^{(l)}$ corresponding to $\lambda_i^{(l)}$, and $\mathcal{G}_i^{(l)} = \mathcal{M}_i^{(l)}$.

Let $\Gamma_l \subset D_l$ be a K_B-bounded contour satisfying the condition (3.11). According to Theorem 4.2, the spectrum of the transfer function $M_1(\cdot, \Gamma_l)$ is represented in the set $\mathcal{O}_{d_0(\Gamma_l)/2}(A_1)$ [and even in the set $\left(\mathbb{R} \cup D(\Gamma_l)\right) \cap \mathcal{O}_{d_{\max}/2}(A_1)$, according to Corollary 4.2 to the theorem] just by the spectrum of the operator $H_1^{(l)}$. Thus, the transfer function $M_1(\cdot, \Gamma_l)$ has in $\mathcal{O}_{d_0(\Gamma_l)/2}(A_1)$ only discrete spectrum consisting of the eigenvalues $\lambda_i^{(l)}$, $i = 1, 2, \ldots, s$. Due to Eq. (4.16) the inverse function $[M_1(\cdot, \Gamma_l)]^{-1}$ has poles at $z = \lambda_i^{(l)}$ of the same orders n_i as the resolvent $(H_1^{(l)} - z)^{-1}$.

Lemma 6.1 *The eigenprojections* $\mathsf{P}_i^{(l)}$ *and eigennilpotents* $\mathsf{N}_i^{(l)}$, $i = 1, 2, \ldots, s$, *of the operator* $H_1^{(l)}$ *satisfy the equations*

$$M_1(\lambda_i^{(l)}, \Gamma_l)\mathsf{P}_i^{(l)} = \mathsf{N}_i^{(l)} - \sum_{1 \le k \le n_i - 1} \frac{1}{k!} V_1^{(k)}(\lambda_i^{(l)}, \Gamma_l)[\mathsf{N}_i^{(l)}]^k \tag{6.6}$$

where $V_1^{(k)}(\lambda, \Gamma_l)$ *stands for the k-th derivative of the function* $V_1(z, \Gamma_l)$, *defined by Eq.* (2.7), *at* $z = \lambda$,

$$V_1^{(k)}(\lambda, \Gamma_l) = (-1)^k \, k! \int\limits_{\sigma'(A_0) \cup \Gamma_l} K_B(d\mu) \frac{1}{(\lambda - \mu)^{k+1}}, \qquad k = 0, 1, 2, \ldots . \tag{6.7}$$

P r o o f . Write the basic equation (3.7) for $H_1^{(l)}$ as follows

$$A_1 = H_1^{(l)} - \int\limits_{\sigma'(A_0) \cup \Gamma_l} K_B(d\mu) \, (H_1^{(l)} - \mu)^{-1} . \tag{6.8}$$

Multiplying both parts of Eq. (6.8) by $\mathsf{P}_i^{(l)}$ from the right and taking into account equalities

$$H_1^{(l)}\mathsf{P}_i^{(l)} = \lambda_i^{(l)}\mathsf{P}_i^{(l)} + \mathsf{N}_i^{(l)} \tag{6.9}$$

and

$$(H_1^{(l)} - z)^{-1}\mathsf{P}_i^{(l)} = -\frac{\mathsf{P}_i^{(l)}}{z - \lambda_i^{(l)}} - \sum_{1 \le k \le n_i - 1} \frac{[\mathsf{N}_i^{(l)}]^k}{(z - \lambda_i^{(l)})^{k+1}} \tag{6.10}$$

which follow from Eqs. (6.5), (6.1) and (6.4), one finds

$$A_1\mathsf{P}_i^{(l)} = \lambda_i^{(l)}\mathsf{P}_i^{(l)} + \mathsf{N}_i^{(l)} - \int\limits_{\sigma'(A_0) \cup \Gamma_l} K_B(d\mu) \left(-\frac{\mathsf{P}_i^{(l)}}{\mu - \lambda_i^{(l)}} - \sum_{1 \le k \le n_i - 1} \frac{[\mathsf{N}_i^{(l)}]^k}{(\mu - \lambda_i^{(l)})^{k+1}} \right) .$$

But according to Eqs. (2.7) and (6.7) this is just the equation (6.6) which we wanted to prove. □

Remark 6.1 *The eigennilpotents* $\mathsf{N}_i^{(l)}$, $i = 1, 2, \ldots, s$ *also satisfy the equations*

$$\begin{aligned} M_1(\lambda_i^{(l)}, \Gamma_l)[\mathsf{N}_i^{(l)}]^{n_i - p} \; &= \; [\mathsf{N}_i^{(l)}]^{n_i - p + 1} - \sum_{1 \le k \le p - 1} \frac{1}{k!} V_1^{(k)}(\lambda_i^{(l)}, \Gamma_l)[\mathsf{N}_i^{(l)}]^{n_i - p + k}, \\ & p = 1, 2, \ldots, n_i - 1. \end{aligned} \tag{6.11}$$

One obtains the Eqs. (6.11) simply by multiplying both parts of Eqs. (6.6) from the right by $[\mathsf{N}_i^{(l)}]^{n_i - p}$, $p = 1, 2, \ldots, n_i - 1$.

Remark 6.2 *Writing* A_1 *as* $A_1 = H_1^{(l)} - V_1(H_1^{(l)}, \Gamma_l)$ *and then using Eqs.* (6.1), (6.5) *and* (6.7) *one can represent the transfer function* $M_1(z, \Gamma_l)$ *as*

$$M_1(z,\Gamma_l) = \sum_{i=1}^{s} \Big\{ (\lambda_i^{(l)} - z)\mathsf{P}_i^{(l)} + \mathsf{N}_i^{(l)} + V_1(z,\Gamma_l)\mathsf{P}_i^{(l)} \\ - V_1(\lambda_i^{(l)},\Gamma_l)\mathsf{P}_i^{(l)} - \sum_{1\le k\le n_i-1} \frac{1}{k!} V_1^{(k)}(\lambda_i^{(l)},\Gamma_l)[\mathsf{N}_i^{(l)}]^k \Big\}.$$

In fact, the Eqs. (6.6) considered together with the conditions (6.4) determine the eigenprojections $\mathsf{P}_i^{(l)}$ and eigennilpotents $\mathsf{N}_i^{(l)}$, $i = 1, 2, \ldots, s$ uniquely, at least under the additional condition

$$\Big\| \sum_{i=1}^{s} (\lambda_i^{(l)} \mathsf{P}_i^{(l)} + \mathsf{N}_i^{(l)}) - A_1 \Big\| < r_{\max}(\Gamma_l), \tag{6.12}$$

where $r_{\max}(\Gamma_l)$ is given by Eq. (3.14). Namely, the following assertion holds.

Theorem 6.1 *Let* $\Gamma_l \subset D_l$ *be a* K_B*-bounded contour satisfying the condition* (3.11). *Also, let* $\lambda_i^{(l)}$, $i = 1, 2, \ldots, s$ *be the eigenvalues and* n_i, $i = 1, 2, \ldots, s$, *the respective pole orders of the transfer function* $M_1(z, \Gamma_l)$ *in the domain* $\mathcal{O}_{d_0(\Gamma_l)/2}(A_1)$. *Then the system of Eqs.* (6.3), (6.4) *and* (6.6) *for* $i, j = 1, 2, \ldots, s$ *under the condition* (6.12) *determines uniquely the complete system of eigenprojections and eigennilpotents for the operator* $H_1^{(l)}$.

P r o o f . Obviously, one has to prove only the uniqueness of the solution for the system (6.4), (6.6), since the existence of such a solution is already guaranteed by Lemma 6.1.

Let $\widetilde{\mathsf{P}}_i^{(l)}$, $\widetilde{\mathsf{N}}_i^{(l)}$, $i = 1, 2, \ldots, s$ be a solution of the system (6.3), (6.4) and (6.6) under the condition (6.12). Consider the operator

$$\widetilde{H}_1^{(l)} = \sum_{i=1}^{s} (\lambda_i^{(l)} \widetilde{\mathsf{P}}_i^{(l)} + \widetilde{\mathsf{N}}_i^{(l)}). \tag{6.13}$$

Since the $\widetilde{\mathsf{P}}_i^{(l)}$, $\widetilde{\mathsf{N}}_i^{(l)}$ satisfy Eqs. (6.3), (6.4), the equation (6.13) is at the same time the spectral representation for $\widetilde{H}_1^{(l)}$ and, consequently,

$$(\widetilde{H}_1^{(l)} - z)^{-1} = \sum_{i=1}^{s} \left(-\frac{\widetilde{\mathsf{P}}_i^{(l)}}{z - \lambda_i^{(l)}} - \sum_{1\le k\le n_i-1} \frac{[\widetilde{\mathsf{N}}_i^{(l)}]^k}{(z-\lambda_i^{(l)})^k} \right). \tag{6.14}$$

Rewrite Eqs. (6.6) for $\widetilde{\mathsf{P}}_i^{(l)}$ and $\widetilde{\mathsf{N}}_i^{(l)}$ as

$$\lambda_i^{(l)} \widetilde{\mathsf{P}}_i^{(l)} + \widetilde{\mathsf{N}}_i^{(l)} = A_1 \widetilde{\mathsf{P}}_i^{(l)} + V_1(\lambda_i^{(l)},\Gamma_l)\widetilde{\mathsf{P}}_i^{(l)} + \sum_{1\le k\le n_i-1} \frac{1}{k!} V_1^{(k)}(\lambda_i^{(l)},\Gamma_l)[\widetilde{\mathsf{N}}_i^{(l)}]^k \tag{6.15}$$

and represent then the derivatives $V_1^{(k)}(\lambda_i^{(l)}, \Gamma_l)$ by (6.7). Summing over i in (6.15) and taking into account (6.14) one finds

$$\widetilde{H}_1^{(l)} = A_1 + \int\limits_{\sigma'(A_0)\cup\Gamma_l} K_B(d\mu)\,(\widetilde{H}_1^{(l)} - \mu)^{-1}, \tag{6.16}$$

that is, $\widetilde{H}_1^{(l)}$ satisfies the basic equation (3.7) and the difference $\widetilde{X}_1^{(l)} = \widetilde{H}_1^{(l)} - A_1$ the basic equation (3.9). Meanwhile the condition (6.12) means $\|\widetilde{X}_1^{(l)}\| < r_{\max}(\Gamma_l)$. Then it follows from Theorem 3.1 that $\widetilde{X}_1^{(l)} = X_1^{(l)}$ and, hence, $\widetilde{H}_1^{(l)} = H_1^{(l)}$. Since the spectral representation (6.5) for $H_1^{(l)}$ is unique, one must conclude that $\widetilde{\mathsf{P}}_i^{(l)} = \mathsf{P}_i^{(l)}$ and $\widetilde{\mathsf{N}}_i^{(l)} = \mathsf{N}_i^{(l)}$, $i = 1, 2, \ldots, s$ and this completes the proof. □

Thus, the eigenprojections $\mathsf{P}_i^{(l)}$ and eigennilpotents $\mathsf{N}_i^{(l)}$, $i = 1, 2, \ldots, s$ of the operator $H_1^{(l)}$ can be called the eigenprojections and eigennilpotents of the transfer function $M_1(z, \Gamma_l)$ in the domain $\mathcal{O}_{d_0(\Gamma_l)/2}(A_1)$. By Lemma 6.1 and Theorem 6.1 this definition is correct.

Recall that a basis $\{e_j\}_{j=1}^{n_{\mathcal{H}_1}}$ of the space $\mathcal{H}_1$ is said to be *adapted* to the decomposition (6.2) if the first several elements of $\{e_j\}_{j=1}^{n_{\mathcal{H}_1}}$ belong to $\mathcal{M}_1$, the following several elements belong to $\mathcal{M}_2$ and so on. In the case considered here such an adapted basis consists of root vectors corresponding to the eigenvalues $\lambda_i^{(l)}$, $i = 1, 2, \ldots, s$, including the resonances. One can choose, in particular, a basis consisting of the eigenvectors and associated vectors which reduces every eigennilpotent $\mathsf{N}_i^{(l)}$ to Jordan canonical form (see e. g. [16], pp. 22, 43).

To conclude the section we consider the case where all the eigenvalues $\lambda_i^{(l)}$ are semisimple, i. e., $\mathsf{N}_i^{(l)} = 0$, $i = 1, 2, \ldots, s$.

Let $\{\psi_{ij}^{(l)},\ j = 1, 2, \ldots, m_i\}$ be a basis of the eigenspace $\mathcal{G}_i^{(l)}$ ($\mathcal{G}_i^{(l)} = \mathcal{M}_i^{(l)}$ in this case). The union of these bases for $i = 1, 2, \ldots, s$ is a basis of the space $\mathcal{H}_1$. Denote by $\{\varphi_{ij}^{(l)},\ i = 1, 2, \ldots, s;\ j = 1, 2, \ldots, m_i\}$ the biorthogonal basis, $\langle \psi_{ij}^{(l)}, \varphi_{i'j'}^{(l)} \rangle = \delta_{ii'}\delta_{jj'}$. Then the vectors $\varphi_{ij}^{(l)}$, $j = 1, 2, \ldots, m_i$ for a given $i = 1, 2, \ldots, s$ are automatically eigenvectors of the operator $H_1^{(l)*}$ corresponding to the eigenvalue $\overline{\lambda}_i^{(l)}$ while $\mathsf{P}_i^{(l)} = \sum\limits_{j=1}^{m_i} \psi_{ij}^{(l)} \langle \cdot, \varphi_{ij}^{(l)} \rangle$. Eq. (4.25) implies that

$$\psi_{ij}^{(-l)} = (I_1 + \Omega^{(-l)})^{-1} \varphi_{ij}^{(l)}, \quad j = 1, 2, \ldots, m_i$$

are the linearly independent eigenvectors of the operator $H_1^{(-l)}$ corresponding to the eigenvalue $\lambda_i^{(-l)} = \overline{\lambda}_i^{(l)}$ and

$$\langle \psi_{ij}^{(l)}, (I_1 + \Omega^{(-l)}) \psi_{i'j'}^{(-l)} \rangle = \langle (I_1 + \Omega^{(l)}) \psi_{ij}^{(l)}, \psi_{i'j'}^{(-l)} \rangle = \delta_{ii'}\delta_{jj'}\,. \tag{6.17}$$

Therefore, one comes to the following assertion.

Lemma 6.2 *If all the eigenvalues $\lambda_i^{(l)}$, $i = 1, 2, \ldots, s$ of the operator $H_1^{(l)}$ are semisimple, then the spectral projections $\mathsf{P}_i^{(l)}$ can be written as*

$$\mathsf{P}_i^{(l)} = \sum_{j=1}^{m_i} \psi_{ij}^{(l)} \langle \cdot, \psi_{ij}^{(-l)} \rangle (I_1 + \Omega^{(l)}) \tag{6.18}$$

where the eigenvectors $\psi_{ij}^{(l)}$ *and* $\psi_{ij}^{(-l)}$ *of the operators* $H_1^{(l)}$ *and* $H_1^{(-l)}$ $(H_1^{(l)}\psi_{ij}^{(l)} = \lambda_i^{(l)}\psi_{ij}^{(l)}$, $H_1^{(-l)}\psi_{ij}^{(-l)} = \overline{\lambda}_i^{(l)}\psi_{ij}^{(-l)})$ *are normalized according to Eqs.* (6.17). *At the same time*

$$H_1^{(l)} = \sum_{i=1}^{s} \lambda_i^{(l)} \sum_{j=1}^{m_i} \psi_{ij}^{(l)} \langle \cdot, \psi_{ij}^{(-l)} \rangle (I_1 + \Omega^{(l)}).$$

REMARK 6.3 *It follows from the relations* (4.28) *and* (6.18) *that, in the case considered here, the residues* $P_i^{(l)}$ *of the transfer function* $M_1(z,\Gamma)$ *at* $z = \lambda_i^{(l)}$, $i = 1,2,\ldots,s$ *read as follows:*

$$P_i^{(l)} = \sum_{j=1}^{m_i} \psi_{ij}^{(l)} \langle \cdot, \psi_{ij}^{(-l)} \rangle \, .$$

The total sum of these residues represents an invertible operator and

$$\left(\sum_{i=1}^{s} P_i^{(l)} \right)^{-1} = I_1 + \Omega^{(l)} \, .$$

7. COMPLETENESS AND BASIS PROPERTIES OF THE $H_1^{(l)}$ ROOT VECTORS IN THE CASE OF AN INFINITE-DIMENSIONAL SPACE $\mathcal{H}_1$

In the present Section we restrict ourselves to the case where the entry A_1 has pure discrete spectrum only, i.e., the resolvent $R_1(z) = (A_1 - z)^{-1}$ is a compact operator in $\mathcal{H}_1$ for any $z \in \rho(A_1)$.

LEMMA 7.1 *If the entry* A_1 *has compact resolvent, then the operators* $H_1^{(l)}$ *have compact resolvents, too.*

This statement is a simple consequence of Theorem IV.3.17 of [16], since the difference $H_1^{(l)} - A_1 = X^{(l)}$ is a bounded operator (see Theorem 3.1). □

LEMMA 7.2 *If the entry* A_1 *has compact resolvent, then the solutions* $X^{(l)}$ *of the basic equation* (3.9) *are compact operators.*

P r o o f . According to Lemma 7.1, the resolvent $(H_1^{(l)} - \mu)^{-1}$ is a compact operator for any μ belonging to an arbitrary K_B-bounded contour Γ_l satisfying the condition (3.11), since for such a contour $\operatorname{dist}\{\sigma(H_1^{(l)}, \Gamma_l)\} > d_0(\Gamma_l)/2 > 0$. Thus, any finite integral sum for the integral defining $X^{(l)}$, $X^{(l)} = \displaystyle\int_{\sigma'(A_0)\cup\Gamma_l} K_B(d\mu)\,(H_1^{(l)} - \mu)^{-1}$, is a compact operator. But under the K_B-boundedness condition (2.6) the integral sums converge to $X^{(l)}$ with respect to the operator norm topology (see Appendix B). Thus, $X^{(l)}$ must be a compact operator. □

Denote by $\mathcal{H}_{1,\lambda}^{(l)}$ the algebraic eigenspace of $H_1^{(l)}$ corresponding to an eigenvalue λ, $\mathcal{H}_{1,\lambda}^{(l)} = \mathsf{P}_\lambda^{(l)}\mathcal{H}_1$ where the eigenprojection $\mathsf{P}_\lambda^{(l)}$ is given by Eq. (4.27). Let m_λ be the algebraic multiplicity, $m_\lambda = \dim \mathcal{H}_\lambda^{(l)}$, $m_\lambda < \infty$, and $\mathsf{N}_\lambda^{(l)}$ respective eigennilpotent, $\mathsf{N}_\lambda^{(l)} = (H_1^{(l)} - \lambda)\mathsf{P}_\lambda^{(l)}$. The eigenprojections $\mathsf{P}_\lambda^{(l)}$ and eigennilpotents $\mathsf{N}_\lambda^{(l)}$ for different λ again satisfy Eqs. (6.4) as well as Eqs. (6.9) and (6.10) are valid (see [16], § 6.5 of Chapter III). Repeating literally the proof of Lemma 6.1 one can check easily that for the case considered now the statement of this lemma is still valid.

Let $\psi_{\lambda,i}^{(l)}$, $i = 1, 2, \ldots, m_\lambda$, be the root vectors of $H_1^{(l)}$ forming a basis of the algebraic eigenspace $\mathcal{H}_{1,\lambda}^{(l)}$. In the following we will try to give an answer on the question when the union of such bases in λ forms a basis of the total space $\mathcal{H}_1$. But, in any case, we already have an assertion regarding completeness of the system

$$\{\psi_{\lambda,i}^{(l)},\ \lambda \in \sigma(H_1^{(l)}),\ i = 1, 2, \ldots, m_\lambda\}. \tag{7.1}$$

Theorem 7.1 *The closure of the linear span of the system* (7.1) *coincides with* $\mathcal{H}_1$,

$$\overline{\bigvee\{\psi_{\lambda,i}^{(l)},\ \lambda \in \sigma(H_1^{(l)}),\ i = 1, 2, \ldots, m_\lambda\}} = \mathcal{H}_1 .$$

This assertion is a particular case of Theorem V.10.1 from [13]. □

The following statement concerns the case where the basis property of the system (7.1) follows immediately from the general basis property (Theorem 5.1) of the eigenvectors corresponding to the real isolated point spectrum eigenvalues of the operator $H_1^{(l)}$.

Theorem 7.2 *Let the entry A_1 have compact resolvent and be semibounded from below. Suppose the set $\bigcup\limits_{k=1}^{m} \Delta_k^0$ is bounded from above, i. e., $\mu_m^{(2)} < \infty$. Then the operator $H_1^{(l)}$ has only a finite number of complex eigenvalues (resonances). It can be represented as*

$$H_1^{(l)} = H_{1,R}^{(l)} + H_{1,C}^{(l)}$$

with $H_{1,R}^{(l)} = H_1^{(l)}\mathsf{P}_R^{(l)}$ and $H_{1,C}^{(l)} = H_1^{(l)}\mathsf{P}_C^{(l)}$ where $\mathsf{P}_R^{(l)}$ and $\mathsf{P}_C^{(l)}$ stand for the projections on the invariant subspaces $\mathcal{H}_{1,R}^{(l)}$, $\mathcal{H}_{1,R}^{(l)} = \mathsf{P}_R^{(l)}\mathcal{H}_1$, and $\mathcal{H}_{1,C}^{(l)}$, $\mathcal{H}_{1,C}^{(l)} = \mathsf{P}_C^{(l)}\mathcal{H}_1$, corresponding respectively to the real and complex spectrum of $H_1^{(l)}$. The restriction of $H_{1,R}^{(l)}$ to $\mathcal{D}(A_1) \cap \mathcal{H}_{1,R}^{(l)}$ represents an operator which is similar to a selfadjoint one while for the finite-dimensional component $H_{1,C}^{(l)}\Big|_{\mathcal{H}_{1,C}^{(l)}}$ one can find the eigenprojections and eigennilpotents using the statements of Lemma 6.1 *and* Theorem 6.1. *Combining a basis of the subspace $\mathcal{H}_{1,C}^{(l)}$ consisting of the root vectors for $H_{1,C}^{(l)}\Big|_{\mathcal{H}_{1,C}^{(l)}}$ with a Riesz basis of the subspace $\mathcal{H}_{1,R}^{(l)}$ constructed from the eigenvectors of $H_{1,R}^{(l)}\Big|_{\mathcal{H}_{1,R}^{(l)}}$ one gets a Riesz basis of the space $\mathcal{H}_1$.*

P r o o f . According to Theorem 4.2 the complex spectrum of the operator $H_1^{(l)}$ belongs to the set $D_l \cap \mathcal{O}_{r_0(B)}(A_1)$ and even to the domains $D(\Gamma_l)$ restricted by $\bigcup\limits_{k=1}^{m} \Delta_k^0$ and arbitrary K_B-bounded contours $\Gamma_l \subset D_l$ satisfying the condition (3.11). The rest of the spectrum of

$H_1^{(l)}$ is real. Obviously, in the case concerned, the set $D(\Gamma_l) \cap \mathcal{O}_{r_0(B)}(A_1)$ is bounded even if the domain D_l is unbounded. Since the spectrum of $H_1^{(l)}$ is discrete (see Lemma 7.1), only a finite number of the $H_1^{(l)}$ eigenvalues can be situated in $D(\Gamma_l) \cap \mathcal{O}_{r_0(B)}(A_1)$ and these eigenvalues generate the finite-dimensional eigenprojections. Thus, the projection $\mathsf{P}_C^{(l)}$, being a sum of the individual eigenprojections, is finite-dimensional, too. Multiplying both parts of the basic equation for $H_1^{(l)}$, written as Eq. (6.8), by $\mathsf{P}_C^{(l)}$ from the right and separating the eigenprojections and eigennilpotents corresponding to the individual resonances, one further comes to the statements of Lemma 6.1 and Theorem 6.1 restricted to the subspace $\mathcal{H}_{1,C}^{(l)}$. Noting that $\mathcal{H}_1 = \mathcal{H}_{1,R}^{(l)} \dot{+} \mathcal{H}_{1,C}^{(l)}$, and then applying Theorem 5.1 one gets the remaining statements. □

REMARK 7.1 *The statement of* Theorem 7.2 *remains valid if the entry* A_1 *has compact resolvent and is semibounded from above while the set* $\cup_{k=1}^m \Delta_k^0$ *is semibounded from below, i. e.,* $\mu_1^{(1)} > -\infty$.

In what follows we need a few definitions and statements from Chapter VI of the book [13].

Let $\{e_k\}_{k=1}^\infty$ be a basis of a Hilbert space $\mathcal{N}$. If there exists an orthonormal basis $\{e'_k\}_{k=1}^\infty$ of $\mathcal{N}$ such that

$$\sum_{k=1}^{\infty} \|e_k - e'_k\|^2 < \infty,$$

then the basis $\{e_k\}_{k=1}^\infty$ is said to be quadratically close to an orthonormal basis. Also, such a basis is called a Bari basis. Any Bari basis is at the same time a Riesz basis (see Theorem VI.2.3 of [13]).

A sequence $\{\mathcal{N}_k\}_{k=1}^\infty$ of non-zero subspaces $\mathcal{N}_k \subset \mathcal{N}$ is said to be a basis (of subspaces) of the Hilbert space $\mathcal{N}$ if any vector $x \in \mathcal{N}$ can be expanded in a unique way in a series of the form

$$x = \sum_{k=1}^{\infty} x_k$$

where $x_k \in \mathcal{N}_k$, $k = 1, 2, \ldots$.

A sequence $\{\mathcal{N}_k\}_{k=1}^\infty$ of non-zero subspaces $\mathcal{N}_k \subset \mathcal{N}$ is said to be ω-linearly independent if the equality

$$\sum_{k=1}^{\infty} x_k = 0, \quad x_k \in \mathcal{N}_k, \ k = 1, 2, \ldots ,$$

is not possible for

$$0 < \sum_{k=1}^{\infty} \|x_k\|^2 < \infty \, .$$

A sequence $\{\mathcal{N}_k\}_{k=1}^\infty$ of subspaces $\mathcal{N}_k \subset \mathcal{N}$ is said to be quadratically close to an orthogonal basis (of subspaces) of the the space $\mathcal{N}$ if there exists a sequence of pairwise orthogonal subspaces $\mathcal{N}'_k \subset \mathcal{N}$ such that $\bigoplus\limits_{k=1}^{\infty} \mathcal{N}'_k = \mathcal{N}$ and

$$\sum_{k=1}^{\infty} \|\mathsf{P}_{\mathcal{N}_k} - \mathsf{P}_{\mathcal{N}'_k}\|^2 < \infty$$

where $\mathsf{P}_{\mathcal{N}_k}$ and $\mathsf{P}_{\mathcal{N}'_k}$, $k = 1, 2, \ldots$, stand for the orthogonal projections of $\mathcal{N}$ onto $\mathcal{N}_k$ and $\mathcal{N}'_k$, respectively.

The *minimal angle* $\phi(\mathcal{N}', \mathcal{N}'')$, $0 \leq \phi \leq \pi/2$, between two subspaces $\mathcal{N}'$ and $\mathcal{N}''$ is defined as

$$\cos\phi(\mathcal{N}', \mathcal{N}'') = \sup_{\substack{x' \in \mathcal{N}',\ x'' \in \mathcal{N}'' \\ \|x'\| = \|x''\| = 1}} |\langle x', x'' \rangle| \,.$$

THEOREM 7.3 [24] *Let $\{\mathcal{N}_k\}_{k=1}^{\infty}$ be a complete, ω-linearly independent sequence of finite-dimensional subspaces in $\mathcal{N}$ such that*

$$\sum_{\substack{i,j=1 \\ i \neq j}}^{\infty} \cos^2\phi(\mathcal{N}_i, \mathcal{N}_j) < \infty \,. \tag{7.2}$$

Then $\{\mathcal{N}_k\}_{k=1}^{\infty}$ is a basis of the space $\mathcal{N}$, quadratically close to an orthogonal one.

THEOREM 7.4 (Proposition VI.5.6 of [13]) *If the condition* (7.2) *in* Theorem 7.3 *can be replaced with*

$$\sum_{\substack{i,j=1 \\ i \neq j}}^{\infty} \min\{\nu_i, \nu_j\} \cos^2\phi(\mathcal{N}_i, \mathcal{N}_j) < \infty \tag{7.3}$$

where $\nu_i = \dim\mathcal{N}_i$, then the union of orthonormal vector bases of the subspaces $\mathcal{N}_k$, $k = 1, 2, \ldots$, forms a Bari basis of the space $\mathcal{N}$.

The following statement is a particular case of a more general proposition from the final part of § III.7.3 of Ref. [13] regarding criteria for a linear operator to belong to a certain class of compact operators. It also represents Theorem 5 of § 11.5 of Ref. [6].

THEOREM 7.5 *If for a bounded linear operator T acting in a Hilbert space $\mathcal{N}$ the condition*

$$\sum_{k=1}^{\infty} \|Te_k\| < \infty$$

is valid for some orthonormal basis $\{e_k\}_{k=1}^{\infty}$ of the space $\mathcal{N}$ then T is an operator of the trace class.

Let us return to the operators $H_1^{(l)}$, now in the case where the intersection $\left(\bigcup\limits_{k=1}^{m} \Delta_k^0\right) \cap \sigma(A_1)$ includes infinitely many points and, thus, $\dim\mathcal{H}_1 = \infty$.

For the sake of simplicity we assume that the entry A_1 as above, has compact resolvent and is semibounded from below. Then the previous assumption means that at least the interval Δ_m^0 is infinite, $\Delta_m^0 = (\mu_m^{(1)}, +\infty)$. The eigenvalues $\lambda_i^{(A_1)}$, $i = 1, 2, \ldots$, of the operator A_1 will be enumerated in increasing order, $\lambda_1^{(A_1)} < \ldots < \lambda_i^{(A_1)} < \lambda_{i+1}^{(A_1)} < \ldots$ and $\lim\limits_{i\to\infty} \lambda_i^{(A_1)} = +\infty$ exists.

Suppose further that there is a number i_0 such that for any $i \geq i_0$

$$\lambda_i^{(A_1)} - \lambda_{i-1}^{(A_1)} > 2r > 2r_0(B), \tag{7.4}$$

with a fixed value r while $r_0(B)$ is given by (3.19). Let γ_0 be a circle centered at $z = (\lambda_1^{(A_1)} + \lambda_{i_0-1}^{(A_1)})/2$ and having the radius $(\lambda_{i_0-1}^{(A_1)} - \lambda_1^{(A_1)})/2 + r$ while the γ_i for $i \geq i_0$ are the circles with centers $\lambda_i^{(A_1)}$ and the radius r. Obviously, the union $\mathrm{Int}\,\gamma_0 \bigcup\limits_{i\geq i_0} \mathrm{Int}\,\gamma_i$ of the interiors of the circles γ_i, $i = 0, i_0, i_0+1, \ldots$, covers all the spectrum of $H_1^{(l)}$, since $\sigma(H_1^{(l)}) \subset \mathcal{O}_{r_0(B)}(A_1)$. At the same time

$$\overline{\mathrm{Int}\,\gamma_i} \cap \overline{\mathrm{Int}\,\gamma_j} = \emptyset \quad \text{if} \quad i \neq j. \tag{7.5}$$

Thus, one can introduce the projections

$$\mathsf{Q}_i^{(l)} = -\frac{1}{2\pi \mathrm{i}} \int\limits_{\gamma_i} dz\ (H_1^{(l)} - z)^{-1}, \qquad i = 0, i_0, i_0+1, \ldots, \tag{7.6}$$

and, then, the subspaces $\mathcal{N}_i^{(l)} = \mathsf{Q}_i^{(l)}\mathcal{H}_1$ which are invariant under $H_1^{(l)}$. Due to Eqs. (7.5) one has

$$\mathsf{Q}_i^{(l)}\mathsf{Q}_j^{(l)} = \delta_{ij}\mathsf{Q}_i^{(l)}. \tag{7.7}$$

Each projection $\mathsf{Q}_i^{(l)}$ represents a sum of the eigenprojections (4.27) corresponding to the eigenvalues $\lambda^{(l)}$ of $H_1^{(l)}$ belonging to $\mathrm{Int}\,\gamma_i$. Since the algebraic eigenspaces for different eigenvalues are linearly independent, the dimension $\dim \mathcal{N}_i^{(l)}$ coincides with sum of the algebraic multiplicities for the $\lambda^{(l)}$ lying inside γ_i. We introduce also the (orthogonal) projections

$$\mathsf{P}_i^{(A_1)} = -\frac{1}{2\pi \mathrm{i}} \int\limits_{\gamma_i} dz\ (A_1 - z)^{-1}, \qquad i = 0, i_0, i_0+1, \ldots\ . \tag{7.8}$$

Obviously, for $i \geq i_0$ the projections $\mathsf{P}_i^{(A_1)}$ are simply the eigenprojections of the entry A_1 corresponding to the eigenvalues $\lambda_i^{(A_1)}$ while $\mathsf{P}_0^{(A_1)}$ is the sum of the eigenprojections for A_1 corresponding to the eigenvalues $\lambda_1^{(A_1)}, \lambda_2^{(A_1)}, \ldots, \lambda_{i_0-1}^{(A_1)}$. In the following by $\varphi_{ij}^{(A_1)}$, $j = 1, 2, \ldots, n_i^{(A_1)}$, $n_i^{(A_1)} = \dim \mathsf{P}_i^{(A_1)}\mathcal{H}_1 < \infty$ we understand an orthonormal basis of the subspace $\mathsf{P}_i^{(A_1)}\mathcal{H}_1$. For $i \geq i_0$ the vectors $\varphi_{ij}^{(A_1)}$ are automatically eigenvectors of A_1, $A_1\varphi_{ij}^{(A_1)} = \lambda_i^{(A_1)}\varphi_{ij}^{(A_1)}$. The sequence $\{\varphi_{ij}^{(A_1)},\ i = 0, i_0, i_0+1, \ldots,\ j = 1, \ldots, n_i^{(A_1)}\}$ forms an orthonormal basis of $\mathcal{H}_1$.

Lemma 7.3 *Under the condition* (7.4) *the sequence*

$$\{\mathcal{N}_i^{(l)},\ i = 0, i_0, i_0+1, \ldots\} \tag{7.9}$$

of subspaces $\mathcal{N}_i^{(l)} = \mathsf{Q}_i^{(l)}\mathcal{H}_1$ is ω-linearly independent and complete in $\mathcal{H}_1$.

P r o o f . The completeness of the sequence (7.9) follows immediately from Theorem 7.1. Regarding the ω-linear independence of this sequence, it suffices to prove ω-independence for the subsequence $\{\mathcal{N}_i^{(l)}\}_{i=i_0}^{\infty}$. Suppose there is a sequence $\{x_i\}_{i=i_0}^{\infty}$, $x_i \in \mathsf{N}_i^{(l)}$, such that

$$0 < \sum_{i=i_0}^{\infty} \|x_i\|^2 < \infty \tag{7.10}$$

but

$$\lim_{n\to\infty} \sum_{i=i_0}^{n} x_i = 0. \tag{7.11}$$

The condition (7.10) implies that there are nonzero elements among the x_i, say, an element x_k, $k \geq i_0$. Since the projection $\mathsf{Q}_k^{(l)}$ is a continuous operator, the equality

$$\mathsf{Q}_k^{(l)} \lim_{n\to\infty} \sum_{i=i_0}^{n} x_i = \lim_{n\to\infty} \sum_{i=i_0}^{n} \mathsf{Q}_k^{(l)} x_i \tag{7.12}$$

holds. But, due to Eqs. (7.7), $\mathsf{Q}_k^{(l)} x_i = \delta_{ik} x_i$ and the r.h. side of (7.12) gives x_k while the l.h. side gives zero, because of (7.11). Thus, x_k must be zero, too, and one comes to a contradiction which means that the sequence (7.9) is ω-linearly independent. □

LEMMA 7.4 *If, instead of* (7.4), *the condition*

$$\lambda_i^{(A_1)} - \lambda_{i-1}^{(A_1)} > 2r > 4r_0(B) \qquad \forall i \geq i_0 \tag{7.13}$$

holds, then $\dim \mathcal{N}_i^{(l)} = \dim \mathsf{P}_i^{(A_1)} \mathcal{H}_1$, $i = 0, i_0, i_0+1, \ldots$.

P r o o f . The proof is based on ideas from the proof of Theorem V.4.15 in [16]. Our goal is to show that the differences $\mathsf{Q}_i^{(l)} - \mathsf{P}_i^{(A_1)}$, $i = 0, i_0, i_0+1, \ldots$, have norms smaller than unity. Obviously,

$$\begin{aligned} \|\mathsf{Q}_i^{(l)} - \mathsf{P}_i^{(A_1)}\| &= \frac{1}{2\pi} \Big\| \int_{\gamma_i} dz\, (A_1 - z)^{-1} X^{(l)} (H_1^{(l)} - z)^{-1} \Big\| \\ &\leq \frac{1}{2\pi} \int_{\gamma_i} |dz|\, \|(A_1 - z)^{-1}\|\, \|X^{(l)}\|\, \|(H_1^{(l)} - z)^{-1}\|. \end{aligned} \tag{7.14}$$

First, we deal with $i = 0$. One can deform the circle γ_0 in (7.14) into the line $\mathrm{Re}\, z = \lambda_{i_0-1} + r$, since, in the half-plane $\mathrm{Re}\, z \leq \lambda_{i_0-1} + r$, the integrand behaves like $1/z^2$ as $z \to \infty$. On this line, $\|(A_1 - z)^{-1}\| \leq (r^2 + \eta^2)^{-1/2}$ where $\eta = \mathrm{Im}\, z$. At the same time $\|X^{(l)}\| \leq r_0(B)$. Then it follows from the identity (3.15) that $\|(H_1^{(l)} - z)^{-1}\| \leq 1/(\sqrt{r^2+\eta^2} - r_0)$, $r_0 \equiv r_0(B)$, and one obtains the estimate

$$\begin{aligned} \|\mathsf{Q}_0^{(l)} - \mathsf{P}_0^{(A_1)}\| &\leq \frac{r_0}{2\pi} \int_{-\infty}^{\infty} d\eta\, \frac{1}{\sqrt{r^2+\eta^2}(\sqrt{r^2+\eta^2} - r_0)} \\ &= \frac{r_0}{\pi} \int_0^{\infty} d\eta \left(1 + \frac{r_0}{\sqrt{r^2+\eta^2}}\right) \cdot \frac{1}{r^2+\eta^2-r_0^2}. \end{aligned}$$

Estimating the fraction $r_0/\sqrt{r^2+\eta^2}$ by r_0/r and calculating the integral, one obtains

$$\|\mathsf{Q}_0^{(l)} - \mathsf{P}_0^{(A_1)}\| \le \frac{r_0}{2}\left(1+\frac{r_0}{r}\right)\frac{1}{\sqrt{r^2-r_0^2}}.$$

Since $r > 2r_0$, we find finally

$$\|\mathsf{Q}_0^{(l)} - \mathsf{P}_0^{(A_1)}\| \le \frac{\sqrt{3}}{4} < \frac{1}{2}.$$

We can show, further, that

$$\|\mathsf{Q}_i^{(l)} - \mathsf{P}_i^{(A_1)}\| < 1 \tag{7.15}$$

for $i \ge i_0$, too. Indeed, for $z \in \gamma_i$, $i \ge i_0$, we have

$$\|(A_1 - z)^{-1}\| \le \frac{1}{r}, \qquad \|(H_1^{(l)} - z)^{-1}\| \le \frac{1}{r-r_0}.$$

Substitution of these estimates into (7.14) gives

$$\|\mathsf{Q}_i^{(l)} - \mathsf{P}_i^{(A_1)}\| \le \frac{r_0}{r-r_0}$$

and, under the condition (7.13), the inequalities (7.15) hold true.

Thus, we have proved that for any $i = 0, i_0, i_0+1, \ldots$ the estimate (7.15) is valid. But such an estimate implies that the subspaces $\mathcal{N}_i^{(l)}$ and $\mathsf{P}_i^{(A_1)}\mathcal{H}_1$ are isomorphic to each other (see, e. g., [16], § 4.6 of Chapter I) and, consequently, $\dim \mathcal{N}_i^{(l)} = \dim \mathsf{P}_i^{(A_1)}\mathcal{H}_1$. The proof is complete. □

Theorem 7.6 *Assume* $\lambda_{i+1}^{(A_1)} - \lambda_i^{(A_1)} \to \infty$ *as* $i \to \infty$. *Let* i_0 *be a number such that* (7.13) *holds. Then the following limit exists*

$$s-\lim_{n\to\infty} \sum_{i=0,i\ge i_0}^{n} \mathsf{Q}_i^{(l)} = I_1. \tag{7.16}$$

Additionally, assume that

$$\sum_{i=1}^{\infty} (\lambda_{i+1}^{(A_1)} - \lambda_i^{(A_1)})^{-2} < \infty. \tag{7.17}$$

Then (7.16) *is true for any renumbering of* $\mathsf{Q}_i^{(l)}$. *Moreover, there exists a constant* C *such that* $\left\|\sum_{i\in\mathcal{I}} \mathsf{Q}_i^{(l)}\right\| \le C$ *for any finite set* $\mathcal{I}$ *of integers* $i = 0$, $i \ge i_0$.

This theorem represents a slightly extended statement of Theorems V.4.15 and V.4.16 of [16] (which treated only the case where all the eigenvalues $\lambda_i^{(A_1)}$ of the operator A_1 were simple). The proof of Theorem 7.6 is realized in exactly the same way as the proof of the mentioned theorems in [16] and, thus, we omit it.

REMARK 7.2 *Eq.* (7.16) *implies that*

$$s-\lim_{n\to\infty}\sum_{i=0,i\geq i_0}^{n}\sum_{\lambda\in\operatorname{Int}\gamma_i}\mathsf{P}_\lambda^{(l)}=I_1 \tag{7.18}$$

where λ *stand for the eigenvalues of the operator* $H_1^{(l)}$ *and* $\mathsf{P}_\lambda^{(l)}$ *for the respective eigenprojections. If, additionally, the inequality* (7.17) *holds and all the eigenvalues* $\lambda_i^{(A_1)}$ *are simple, then one can renumber the eigenprojections* $\mathsf{P}_\lambda^{(l)}$ *in Eq.* (7.18) *in any way* (*see* Theorem V.4.16 of [16]).

LEMMA 7.5 *As before, assume* $\Delta_m=(\mu_m^{(1)},+\infty)$. *Also, suppose that there is a* K_B*-bounded contour* $\Gamma_l\subset D_l$ *satisfying* (3.11) *and such that a part of its component* $\Gamma_m^{l_m}$ *coincides with the ray* $\widetilde{\Delta}_m^0=[\mu_0,\mathrm{i}b_0+\infty)$ *where* $\mu_0\in D_m^{l_m}$, $\mu_0=a_0+\mathrm{i}b_0$ *with* $a_0,b_0\in\mathbb{R}$. *Additionally, suppose that the remaining part* $\widetilde{\Gamma}_l=\Gamma_l\setminus\widetilde{\Delta}_m^0$ *of the contour* Γ_l *belongs to the half-plane* $\operatorname{Re}\mu<a_0$, *and for* $\mu\in\widetilde{\Delta}_m^0$

$$\|K_B'(\mu)\|\leq\widetilde{C}(1+|\operatorname{Re}\mu|)^{-\theta} \tag{7.19}$$

with $\widetilde{C}>0$ *and* $\theta>1$. *Then the estimate*

$$\|X^{(l)}\mathsf{P}_i^{(A_1)}\|\leq C(1+|\lambda_i^{(A_1)}|)^{-1},\quad i=i_0,i_0+1,\ldots, \tag{7.20}$$

is valid with some $C>0$.

P r o o f . Since the basic equation (3.9) for $X^{(l)}$ can be written as

$$X^{(l)}=\int\limits_{\sigma'(A_0)\cup\Gamma_l}K_B(d\mu)\,[(A_1-\mu)^{-1}-(H_1^{(l)}-\mu)^{-1}X_1^{(l)}(A_1-\mu)^{-1}],$$

one finds

$$X^{(l)}\mathsf{P}_i^{(A_1)}=\int\limits_{\sigma'(A_0)\cup\Gamma_l}\frac{K_B(d\mu)\,T_i(\mu)}{\lambda_i^{(A_1)}-\mu},\qquad i\geq i_0, \tag{7.21}$$

with

$$T_i(\mu)=[I_1-(H_1^{(l)}-\mu)^{-1}X^{(l)}]\mathsf{P}_i^{(A_1)}.$$

Due to the estimates (3.18) and (4.5) the functions $T_i(\mu)$ on $\sigma'(A_0)\cup\Gamma_l$ are bounded, $\|T_i(\mu)\|\leq c$ where the constant c is the same for all $i\geq i_0$ being determined only by $r_0(B)$ and $d_0(\Gamma_l)$. According to one of our assumptions, the condition $\mu\in\widetilde{\Gamma}_l$ implies that $\operatorname{Re}\mu<a_0$ and, thus, $|\lambda_i^{(A_1)}-\mu|^{-1}<|\lambda_i^{(A_1)}-a_0|^{-1}$ for sufficiently large $\lambda_i^{(A_1)}$, $\lambda_i^{(A_1)}>a_0$. Since the condition (2.6) is valid, this assertion immediately implies that the estimate (7.20) holds at least for the contribution to $X^{(l)}\mathsf{P}_i^{(A_1)}$ from the set $\sigma'(A_0)\cup\widetilde{\Gamma}_l$.

As to the contribution from the ray $\widetilde{\Delta}_m^0$, one writes $K_B(d\mu)$ on this ray as $K_B(d\mu)=K_B'(\mu)d\mu$ and applies the inequality (7.19). Here, the elementary estimate

$$\int_0^{+\infty} dx \, \frac{1}{(1+x)^\theta (1+|x-\lambda|)} \le c(\theta) \, \frac{1}{1+|\lambda|}, \qquad \theta > 1, \quad \lambda \in \mathbb{R}, \tag{7.22}$$

is useful with a certain $c(\theta) > 0$ depending only on θ. Using this inequality one easily finds that the estimate (7.20) holds for the contribution to $X^{(l)} \mathsf{P}_i^{(A_1)}$ from the ray $\widetilde{\Delta}_m^0$, too, and this completes the proof. □

THEOREM 7.7 *Let, in addition to the conditions of* Lemma 7.5, *the condition* (7.17) *be valid and the sequence of the differences* $\lambda_{i+1}^{(A_1)} - \lambda_i^{(A_1)}$ *be monotone starting from some* $i = k$, $k \ge 1$, *i. e.,*

$$\lambda_{i+2}^{(A_1)} - \lambda_{i+1}^{(A_1)} \ge \lambda_{i+1}^{(A_1)} - \lambda_i^{(A_1)}, \qquad i \ge k. \tag{7.23}$$

Also, let

$$n_i^{(A_1)} = \dim \mathsf{P}_i^{(A_1)} \mathcal{H}_1 \le n_{\max}^{(A_1)} \tag{7.24}$$

where $n_{\max}^{(A_1)}$ *is a finite number, the same for all* $i = 0, i_0, i_0+1, \ldots$ *. Then* $X^{(l)}$ *is an operator of the trace class.*

To prove this theorem we need the following simple auxiliary statement.

LEMMA 7.6 *Let a sequence* $\{a_n\}_{n=1}^\infty$ *have positive elements,* $a_n > 0$, $n \in \mathbb{N}$ *and starting from some number* N *be monotone, i. e.* $a_{n+1} \ge a_n$ *for* $n \ge N$*. Also, let*

$$\sum_{n=1}^\infty \frac{1}{a_n^2} < \infty \, . \tag{7.25}$$

Then the series $\sum_{n=1}^\infty b_n$ *with* $b_n = (a_1 + a_2 + \ldots + a_n)^{-1}$ *is convergent.*

P r o o f . First, one notes that for $2k \ge 2N$

$$b_{2k} = \frac{1}{a_1 + \ldots + a_k + a_{k+1} + \ldots + a_{2k}} < \frac{1}{a_{k+1} + \cdots + a_{2k}} \le \frac{1}{k \, a_{k+1}}$$

and, similarly,

$$b_{2k+1} < \frac{1}{a_{k+1} + \cdots + a_{2k} + a_{2k+1}} \le \frac{1}{(k+1) \, a_{k+1}} \, .$$

This means that for $m > N$

$$\begin{aligned} \sum_{n=2N}^{2m} b_n &= \sum_{k=N}^{m} b_{2k} + \sum_{k=N}^{m-1} b_{2k+1} \le \sum_{k=N}^{m} \frac{1}{k \, a_{k+1}} + \sum_{k=N}^{m-1} \frac{1}{(k+1) \, a_{k+1}} \\ &\le \left(\sum_{k=N}^{m} \frac{1}{k^2} \right)^{1/2} \left(\sum_{k=N}^{m} \frac{1}{a_{k+1}^2} \right)^{1/2} + \left(\sum_{k=N}^{m-1} \frac{1}{(k+1)^2} \right)^{1/2} \left(\sum_{k=N}^{m-1} \frac{1}{a_{k+1}^2} \right)^{1/2} . \end{aligned} \tag{7.26}$$

Since the condition (7.25) is assumed and the series $\sum_{n=1}^{\infty} n^{-2}$ is convergent, it immediately follows from (7.26) that the series $\sum_{n=1}^{\infty} b_n$ considered is convergent, too, and this completes the proof. $\square$

P r o o f o f Theorem 7.7. Under the conditions (7.17) and (7.23) the series of the inverse eigenvalues of the entry A_1 is convergent

$$\sum_{\substack{i=1 \\ \lambda_i^{(A_1)} \neq 0}}^{\infty} |\lambda_i^{(A_1)}|^{-1} < \infty . \tag{7.27}$$

Indeed if one takes $a_i = \lambda_{i+1}^{(A_1)} - \lambda_i^{(A_1)}$ then the sequence $\{b_i\}_{i=1}^{\infty}$ of Lemma 7.6 with $b_i = (a_1 + \ldots + a_i)^{-1}$ is represented just by $b_i = 1/\lambda_{i+1}^{(A_1)}$ (except the case $\lambda_{i+1}^{(A_1)} = 0$). If all the eigenvalues $\lambda_i^{(A_1)}$ and differences $\lambda_{i+1}^{(A_1)} - \lambda_i^{(A_1)}$ are positive, then to prove (7.27) one can immediately use the statement of Lemma 7.6. In the case of presence of a (finite) number of negative $\lambda_i^{(A_1)}$ and/or of a (finite) number of negative differences $\lambda_{i+1}^{(A_1)} - \lambda_i^{(A_1)}$ one has to omit in the sum in the l.h. side of (7.27) all the negative eigevalues $\lambda_i^{(A_1)}$ and/or all the eigenvalues $\lambda_i^{(A_1)}$ generating negative differences $\lambda_i^{(A_1)} - \lambda_j^{(A_1)}$ with $j < i$. Then, after appropriate shift in numbering of the remaining eigenvalues $\lambda_i^{(A_1)}$, Lemma 7.6 can be applied and thus, the inequality (7.27) will again hold true.

Further, consider the quantity

$$\sum_{i=0, i\geq i_0}^{\infty} \sum_{j=1}^{n_i^{(A_1)}} \|X^{(l)}\varphi_{ij}^{(A_1)}\| = \sum_{i=0, i\geq i_0}^{\infty} \sum_{j=1}^{n_i^{(A_1)}} \|X^{(l)}\mathsf{P}_i^{(A_1)}\varphi_{ij}^{(A_1)}\| \leq \sum_{i=0, i\geq i_0}^{\infty} \|X^{(l)}\mathsf{P}_i^{(A_1)}\| \sum_{j=1}^{n_i^{(A_1)}} \|\varphi_{ij}^{(A_1)}\| .$$

Since the estimate (7.20) as well as the condition (7.24) hold true, one finds

$$\sum_{i=0, i\geq i_0}^{\infty} \sum_{j=1}^{n_i^{(A_1)}} \|X^{(l)}\varphi_{ij}^{(A_1)}\| \leq n_0^{(A_1)}\|X^{(l)}\mathsf{P}_0^{(A_1)}\| + C\, n_{\max}^{(A_1)} \sum_{i=i_0}^{\infty}(1 + |\lambda_i^{(A_1)}|)^{-1}.$$

Due to the inequality (7.27) the above quantity is finite and, thus, according to Theorem 7.5 the operator $X^{(l)}$ is indeed of the trace class. The proof is complete. $\square$

REMARK 7.3 *Under the condition* (7.24)*, the inequality* (7.27) *implies that the resolvent* $(A_1 - z)^{-1}$ *is of the trace class for any* $z \in \rho(A_1)$ *since the sum* $\sum_{j=1}^{n_0^{(A_1)}} \|(A_1 - z)^{-1}\varphi_{0j}^{(A_1)}\|$ *is finite while the series* $\sum_{i=i_0}^{\infty} \sum_{j=1}^{n_i^{(A_1)}} \|(A_1 - z)^{-1}\varphi_{ij}^{(A_1)}\| \leq \sum_{i=i_0}^{\infty} n_i^{(A_1)} |\lambda_i^{(A_1)} - z|^{-1}$ *is convergent* (see Theorem 7.5).

THEOREM 7.8 *Let the condition* (7.17) *and the conditions of* Lemma 7.5 *be valid. Then the inequality* (7.2) *holds for the subspaces* $\mathcal{N}_i^{(l)}$*,* $i = 0, i_0, i_0 + 1, \ldots$ *. This implies that the sequence* (7.9) *forms a basis of the space* $\mathcal{H}_1$*, quadratically close to an orthogonal one.*

If, additionally, the condition (7.24) *holds, then the union of orthonormal vector bases of the subspaces* $\mathcal{N}_i^{(l)}$*,* $i = 0, i_0, i_0 + 1, \ldots$*, forms a Bari basis of* $\mathcal{H}_1$*.*

P r o o f . Let $r_i = \frac{1}{2}\min\{\lambda_{i+1}^{(A_1)} - \lambda_i^{(A_1)}, \lambda_i^{(A_1)} - \lambda_{i-1}^{(A_1)}\}$, $i \geq i_0$. Under the condition (7.17), $r_i \to \infty$ as $i \to \infty$ and, moreover,

$$\sum_{i=i_0}^{\infty} \frac{1}{r_i^2} < \infty\,. \tag{7.28}$$

Denote by $\widetilde{\gamma}_i$, $i \geq i_0$, the circle centered at $z = \lambda_i^{(A_1)}$ and having the radius r_i. Consider the difference $\mathsf{Q}_i^{(l)} - \mathsf{P}_i^{(A_1)}$ replacing γ_i in the definitions (7.6) and (7.8) for $i \geq i_0$ with $\widetilde{\gamma}_i$. Applying the resolvent identities

$$(H_1^{(l)} - z)^{-1} - (A_1 - z)^{-1} = -(H_1^{(l)} - z)^{-1}X^{(l)}(A_1 - z)^{-1} = -(A_1 - z)^{-1}X^{(l)}(H_1^{(l)} - z)^{-1}$$

twice gives

$$\mathsf{Q}_i^{(l)} = \mathsf{P}_i^{(A_1)} + \mathsf{D}_i' + \mathsf{D}_i'' \tag{7.29}$$

where

$$\mathsf{D}_i' = \frac{1}{2\pi \mathrm{i}} \int\limits_{\widetilde{\gamma}_i} dz\, (A_1 - z)^{-1}X^{(l)}(A_1 - z)^{-1}$$

and

$$\mathsf{D}_i'' = -\frac{1}{2\pi \mathrm{i}} \int\limits_{\widetilde{\gamma}_i} dz\, (A_1 - z)^{-1}X^{(l)}(H_1^{(l)} - z)^{-1}X^{(l)}(A_1 - z)^{-1}\,.$$

It is easy to obtain the following estimates (see also the proof of Theorem V.4.16 in [16]):

$$\|\mathsf{D}_i'\| \leq c\,\frac{1}{1 + r_i}, \qquad \|\mathsf{D}_i''\| \leq c\,\frac{1}{(1 + r_i)^2} \tag{7.30}$$

with some $c > 0$, the same for all $i \geq i_0$.

Further, consider the minimal angle $\phi(\mathcal{N}_i^{(l)}, \mathcal{N}_j^{(l)})$ between the subspaces $\mathcal{N}_i^{(l)}$ and $\mathcal{N}_j^{(l)}$, $i, j \geq i_0$, $i \neq j$. To estimate this angle it suffices to evaluate the inner product $\langle \mathsf{Q}_i^{(l)}x, \mathsf{Q}_j^{(l)}y\rangle$ for $x \in \mathcal{N}_i^{(l)}$, $y \in \mathcal{N}_j^{(l)}$, $\|x\| = \|y\| = 1$. Substituting (7.29) one obtains

$$\begin{aligned} |\langle \mathsf{Q}_i^{(l)}x, \mathsf{Q}_j^{(l)}y\rangle| \leq{} & |\langle \mathsf{P}_i^{(A_1)}x, \mathsf{D}_j'y\rangle + \langle \mathsf{D}_i'x, \mathsf{P}_j^{(A_1)}y\rangle| + |\langle \mathsf{P}_i^{(A_1)}x, \mathsf{D}_j''y\rangle| + |\langle \mathsf{D}_i''x, \mathsf{P}_j^{(A_1)}y\rangle| \\ & + |\langle \mathsf{D}_i'x, \mathsf{D}_j'y\rangle| + |\langle \mathsf{D}_i'x, \mathsf{D}_j''y\rangle| + |\langle \mathsf{D}_i''x, \mathsf{D}_j'y\rangle| + |\langle \mathsf{D}_i''x, \mathsf{D}_j''y\rangle|\,. \end{aligned} \tag{7.31}$$

The term $|\langle \mathsf{P}_i^{(A_1)}x, \mathsf{P}_j^{(A_1)}y\rangle|$ is absent in the r. h. side of Eq. (7.31) since $\mathsf{P}_i^{(A_1)}\mathsf{P}_j^{(A_1)} = 0$ for $i \neq j$. Meanwhile, according to (7.30), the last four terms can be estimated together by

$$c\ \frac{1}{1 + r_i} \cdot \frac{1}{1 + r_j} \tag{7.32}$$

with another constant c which does not depend on i, j. The estimation of the terms $|\langle \mathsf{P}_i^{(A_1)}x, \mathsf{D}_j''y\rangle|$ and $|\langle \mathsf{D}_i''x, \mathsf{P}_j^{(A_1)}y\rangle|$ is simple, too. Consider, for example, the term $|\langle \mathsf{D}_i''x, \mathsf{P}_j^{(A_1)}y\rangle| = |\langle \mathsf{P}_j^{(A_1)}\mathsf{D}_i''x, y\rangle|$. For $j \geq i_0$ we find

$$\mathsf{P}_j^{(A_1)}(A_1-z)^{-1}=\mathsf{P}_j^{(A_1)}(\lambda_j^{(A_1)}-z)^{-1} \tag{7.33}$$

and, thus,

$$\langle \mathsf{D}_i''x,\mathsf{P}_j^{(A_1)}y\rangle=-\frac{1}{2\pi\mathrm{i}}\int\limits_{\widetilde{\gamma}_i}dz\;\frac{\langle \mathsf{P}_j^{(A_1)}X^{(l)}(H_1^{(l)}-z)^{-1}X^{(l)}(A_1-z)^{-1}x,y\rangle}{\lambda_j^{(A_1)}-z}.$$

Since $i\neq j$, one observes that $|\lambda_j^{(A_1)}-z|\geq|\lambda_j^{(A_1)}-\lambda_i^{(A_1)}|-r_i$ if $z\in\widetilde{\gamma}_i$. Consequently,

$$|\langle \mathsf{D}_i''x,\mathsf{P}_j^{(A_1)}y\rangle|\leq\frac{\|X^{(l)}\|^2}{(|\lambda_j^{(A_1)}-\lambda_i^{(A_1)}|-r_i)\,(r_i-\|X^{(l)}\|)}.$$

Since

$$|\lambda_j^{(A_1)}-\lambda_i^{(A_1)}|\geq r_i+r_j, \tag{7.34}$$

the term $|\langle \mathsf{D}_i''x,\mathsf{P}_j^{(A_1)}y\rangle|$ can be estimated by (7.32), too, and the same estimate holds for $|\langle \mathsf{P}_i^{(A_1)}x,\mathsf{D}_j''y\rangle|$.

Regarding the first and second terms on the r. h. side of (7.31), they can be greatly simplified by using the identity (7.33) and then the Residue Theorem. As a result one finds

$$\langle \mathsf{P}_i^{(A_1)}x,\mathsf{D}_j'y\rangle+\langle \mathsf{D}_i'x,\mathsf{P}_j^{(A_1)}y\rangle=\frac{\langle (X^{(l)}-X^{(l)*})\mathsf{P}_i^{(A_1)}x,\mathsf{P}_j^{(A_1)}y\rangle}{\lambda_j^{(A_1)}-\lambda_i^{(A_1)}}. \tag{7.35}$$

Applying the inequalities (7.20) and (7.34) one concludes that the term (7.35) can be easily estimated again by (7.32). However, the series (7.28) is convergent. This just implies that

$$\sum_{\substack{i,j=i_0\\ i\neq j}}^{\infty}\cos^2\phi(\mathcal{N}_i^{(l)},\mathcal{N}_j^{(l)})<\infty. \tag{7.36}$$

An almost literal repetition of the previous consideration shows that

$$\sum_{i=i_0}^{\infty}\cos^2\phi(\mathcal{N}_0^{(l)},\mathcal{N}_i^{(l)})<\infty, \tag{7.37}$$

too. The inequalities (7.36) and (7.37) imply that the condition of Theorem 7.3 holds and, thus, the sequence (7.9) indeed forms a basis of the space $\mathcal{H}_1$, quadratically close to an orthogonal one. The second statement of the theorem is now a trivial consequence of Theorem 7.4. The proof is complete. □

8. THE SIMPLEST EXAMPLE

In the present Section we consider the operator matrix (1.1) with the entry A_0 being the multiplication operator,

$$(A_0 u_0)(\mu) = \mu\, u_0(\mu), \tag{8.1}$$

considered in $\mathcal{H}_0 = L_2(0,a)$, $0 < a \le +\infty$. The domain of the operator A_0 is $\mathcal{D}(A_0) = \left\{ u_0 \in L_2(0,a) : \int_0^a d\mu\, \mu^2 |u_0(\mu)|^2 < \infty \right\}$. Surely, if $a < \infty$, then $\mathcal{D}(A_0) = \mathcal{H}_0$. The spectrum of A_0 only consists of absolutely continuous spectrum coinciding with the interval $[0,a]$.

As A_1 we take a diagonal numerical matrix,

$$A_1 = \Lambda = \operatorname{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_k, \ldots\}, \qquad \lambda_k \in \mathbb{R}, \tag{8.2}$$

and $\mathcal{H}_1 = \mathbb{C}^n$, $1 \le n \le \infty$ (by $\mathbb{C}^\infty$ we understand the Hilbert space l_2). The domain of the entry A_1 is given by $\mathcal{D}(A_1) = \left\{ u_1 = (u_1^{(1)}, u_1^{(2)}, \ldots, u_1^{(k)}, \ldots) \in \mathbb{C}^n : \sum_{k=1}^n \lambda_k^2\, u_1^{(k)2} < \infty \right\}$. Of course, if $n < \infty$, then $\mathcal{D}(A_1) = \mathbb{C}^n$.

The coupling operator B_{01} acts on $u_1 \in \mathbb{C}^n$, $u_1 = (u_1^{(1)}, u_1^{(2)}, \ldots, u_1^{(k)}, \ldots)^t$, as

$$(B_{01} u_1)(\mu) \equiv B(\mu) u_1 \equiv \sum_{k=1}^n b_k(\mu) u_1^{(k)} \tag{8.3}$$

where $b_k \in L_2(0,a)$, $k = 1, 2, \ldots$, while $B(\mu)$ stands for the matrix-row of the values $b_k(\mu)$ of the functions b_k for a fixed $\mu \in (0,a)$. Boundedness of the entry B_{01} means

$$\|B_{01}\|^2 = \sup_{\|u_1\|=1} \int_0^a d\mu\, |B(\mu) u_1|^2 < \infty. \tag{8.4}$$

Obviously, the adjoint operator $B_{10} = B_{01}^*$ is given by

$$B_{10} = (\langle \cdot, b_1 \rangle, \langle \cdot, b_2 \rangle, \ldots, \langle \cdot, b_k \rangle, \ldots)^t.$$

Under the condition (8.4) the operator $\mathbf{H}$ is selfadjoint on the domain $\mathcal{D}(\mathbf{H}) = \mathcal{D}(A_0) \oplus \mathcal{D}(A_1)$.

Note that this example is sufficiently universal. In particular the Hamiltonians for the quantum-mechanical two-body systems with internal structure used in Refs. [8,15,18,19,32,34,37,41] can be reduced to just the present example. Note also that for $n = 1$ the operator matrix $\mathbf{H}$ described represents one of the well known Friedrichs models [10].

The definition (8.1) of the entry A_0 represents at the same time its spectral decomposition, that is, the spectral function $E^0(\mu)$ is given by (see, e. g., [6])

$$\Big(E^0(\mu) u_0\Big)(\nu) = \begin{cases} u_0(\nu), & \nu < \mu \\ 0, & \nu \ge \mu \end{cases} \tag{8.5}$$

for $0 < \mu \le a$ and $E^0(\mu) = 0$ for $\mu \le 0$ while $E^0(\mu) = I_0$ for $\mu > a$. Consequently, the product (2.2) for $0 \le \mu \le a$ reads

$$K_B(\mu) = \int\limits_0^{\mu} d\nu\, [B(\nu)]^* B(\nu)$$

and formally

$$K'_B(\mu) = [B(\mu)]^* B(\mu)\,. \tag{8.6}$$

In particular, if $n = 1$, then $K'_B(\mu) = |b_1(\mu)|^2$.

One of our central assumptions of Sect. 2 was the assumption regarding the holomorphy of the function $K'_B(\mu)$ in a vicinity of (a part of) $\sigma_c(A_0)$. Thus, in the case here one has to assume that the function K'_B given by (8.6) takes values in $\mathbf{B}(\mathbb{C}^n, \mathbb{C}^n)$ for any $\mu \in [0, a]$ and admits analytic continuation as $K'_B : D \to \mathbf{B}(\mathbb{C}^n, \mathbb{C}^n)$ on a domain D, $D \supset (0, a)$, symmetric with respect to the real axis, that is $D = D^- \cup D^+ \cup (0, a)$ with $D^\pm \subset \mathbb{C}^\pm$, $D^\pm = \{z : z \in D^\mp\}$, and

$$\mathcal{V}_0(B) = \int\limits_0^a d\mu \left\| [B(\mu)]^* B(\mu) \right\| < \infty\,. \tag{8.7}$$

Obviously, the condition (8.7) implies the inequality (8.4).

The transfer function $M_1(z)$ for the model concerned reads

$$M_1(z) = \Lambda - z + V_1(z) \qquad \text{with} \qquad V_1(z) = \int\limits_0^a d\mu\, \frac{K'_B(\mu)}{z - \mu}.$$

Consider the quantity

$$V_1^{(0)} \stackrel{\text{def}}{=} \sup_{\|u_1\|=1} \left(-\langle V_1(0) u_1, u_1\rangle\right) = \sup_{\|u_1\|=1} \int\limits_0^a d\mu\, \frac{|B(\mu) u_1|^2}{\mu}$$

and, in the case of a finite a, the quantity

$$V_1^{(a)} \stackrel{\text{def}}{=} \sup_{\|u_1\|=1} \langle V_1(a) u_1, u_1\rangle = \sup_{\|u_1\|=1} \int\limits_0^a d\mu\, \frac{|B(\mu) u_1|^2}{a - \mu}.$$

Denote by $\lambda_{\min}$ and $\lambda_{\max}$, respectively, the lower and upper bounds for Λ, $\lambda_{\min} = \inf\limits_{\|u_1\|=1} \langle \Lambda u_1, u_1\rangle$ and $\lambda_{\max} = \sup\limits_{\|u_1\|=1} \langle \Lambda u_1, u_1\rangle$. In the case of a finite n, $\lambda_{\min}$ and $\lambda_{\max}$ coincide, respectively, with the minimal and maximal eigenvalues of Λ.

Considering the quadratic form $\langle M_1(z) u_1, u_1\rangle$ for $z < 0$ and then, if a is finite, for $z > a$ one can easily check that the following assertion holds true.

LEMMA 8.1 *If* $V_1^{(0)}$ *is finite and*

$$V_1^{(0)} < \lambda_{\min}, \tag{8.8}$$

then the operator $\mathbf{H}$ *has no spectrum below* $z = 0$. *If, in the case of a finite* a, $V_1^{(a)}$ *is finite and*

$$V_1^{(a)} < a - \lambda_{\max}, \tag{8.9}$$

then the operator $\mathbf{H}$ *has no spectrum above* $z = a$.

REMARK 8.1 *The inequalities* $V_1^{(0)} < \infty$ *and* $V_1^{(a)} < \infty$ *imply, respectively,* $K_B'(0) = 0$ *and* $K_B'(a) = 0$.

REMARK 8.2 *Suppose* $n = 1$ *and, thus,* $\lambda_{\min} = \lambda_{\max} = \lambda_1$. *Considering the graphs of the functions* $y = z - \lambda$ *and* $y = V_1(z)$ *at* $z < 0$ *one can easily check that if* $\lambda_1 \in (0, a)$, *then for* $V_1^{(0)} > \lambda_1$ *the transfer function* $M_1(z)$ *has a single negative eigenvalue. Respectively, considering the graphs of the same functions at* $z > a$ *one observes that for* $V_1^{(a)} > a - \lambda_1$ *there exists a single eigenvalue situated to the right from* a.

Therefore if the conditions (8.8) and (8.9) are valid the entire spectrum of the operator $\mathbf{H}$ (and, hence, the spectrum of the transfer function $M_1(z)$) must belong to the interval $[0, a]$. Meanwhile, the eigenvalues of Λ which are embedded initially into the continuous spectrum of A_0 can survive in this interval only in exceptional cases. Obviously, if $\lambda \in (0, a)$ and $M_1(\lambda \pm \mathrm{i}0)u_1 = 0$, $u_1 \neq 0$, then the following obvious conditions must hold (also see Lemma 5.2, condition (d), and Eq. (8.5)):

$$\langle K_B'(\lambda)u_1, u_1\rangle = \|B(\lambda)u_1\|^2 = 0, \tag{8.10}$$

$$\langle(\Lambda - \lambda)u_1, u_1\rangle + \mathrm{V.p.}\int_0^a d\mu\, \frac{\langle K_B'(\mu)u_1, u_1\rangle}{\lambda - \mu} = 0\,.$$

These conditions may be hard to satisfy. In particular, for $n = 1$ Eq. (8.10) implies $b_1(\lambda) = 0$. And if one knows that $b_1(\mu) \neq 0$ for any $\mu \in (0, a)$, then no point spectrum of $\mathbf{H}$ can be situated in the interval $(0, a)$. One understands that the embedded eigenvalue does not disappear. It simply shifts into the unphysical sheet(s) and turns into a pair of conjugate resonances which are eigenvalues of the continued transfer function $M_1(z)$.

Suppose that there exist K_B-bounded contours $\Gamma^\pm \subset D^\pm$ (see Sect. 2) such that the condition (3.11) holds. In this case these are contours for which

$$\mathcal{V}_0(B, \Gamma^\pm) = \int_{\Gamma^\pm} |d\mu|\, \|K_B'(\mu)\| < \frac{1}{4}\, d_0^2(\Gamma^\pm) \tag{8.11}$$

where $d_0(\Gamma^\pm) = \mathrm{dist}\Big\{\Gamma^\pm, \{\lambda_k\}_{k=1}^n\Big\}$. Then, according to Theorem 3.1, one can construct two operators $H_1^{(+)}$ and $H_1^{(-)}$ the spectrum of which exhausts the spectrum of the respective (continued) transfer functions $M_1(z, \Gamma^+)$ and $M_1(z, \Gamma^-)$ in the set $\mathcal{O}_{d_{\max}/2}(A_1)$ where $d_{\max}$ is given by (4.8).

Finally, we give an illustration for the assertion of Theorem 3.1 for the simplest case of the model (8.1)–(8.3) with $n = 1$, $a = 2R$, $\lambda_1 = R$ and $b_1(\mu) \equiv \beta$ where R, β are some positive numbers, $R, \beta \in \mathbb{R}^+$. In this case the basic equation (3.7) coincides with the equation $M_1(z, \Gamma^{\pm}) = 0$ and the solutions $H_1^{(\pm)}$ if they exist are operators in $\mathbb{C}$ defined by multiplication by respective resonances. Obviously

$$\mathcal{V}_0(B, \Gamma^{\pm}) = \beta^2 \int\limits_{\Gamma^{\pm}} |d\mu| = \beta^2 \, \ell_{\Gamma^{\pm}}.$$

Let $\Gamma^{\pm}$ be, say, semicircles, $\Gamma^{\pm} = \{z : |z - R| = R, \, z \in \mathbb{C}^{\pm}\}$. Then $\mathcal{V}_0(B, \Gamma^{\pm}) = \pi \beta^2 R$ and $d_0 = d_0(\Gamma^{\pm}) = R$. Thus, the solvability condition (8.11) reads now as $\beta^2 < \dfrac{R}{4\pi}$. This means that Theorem 3.1 guarantees the unique solvability of the basic equation (3.7) in any semidisc $\mathcal{S}_r^{\pm} \subset \mathbb{C}^{\pm}$ of radius r centered at the point $z = R$, with r satisfying the inequalities $r_{\min} \le r < r_{\max}$ where

$$r_{\min} = \frac{d_0}{2} - \sqrt{\frac{d_0^2}{4} - \mathcal{V}_0(B, \Gamma^{\pm})} = \frac{R}{2} - \sqrt{\frac{R^2}{4} - \pi\beta^2 R} < \frac{R}{2}$$
$$r_{\max} = d_0 - \sqrt{\mathcal{V}_0(B, \Gamma^{\pm})} = R - \sqrt{\pi\beta^2 R} > \frac{R}{2}.$$

For instance, if $\beta^2 = \dfrac{3}{16}\dfrac{R}{\pi}$ then $r_{\min} = \dfrac{R}{4}$ and $r_{\max} = R\left(1 - \dfrac{\sqrt{3}}{4}\right) \approx 0.6R$. In this case the solution (number) $H_1^{\pm}$ belongs to the semidisc $|z - R| \le \dfrac{R}{4}$, $z \in \mathbb{C}^{\pm}$, and no other solutions exist in the semidisc $|z - R| < R\left(1 - \dfrac{\sqrt{3}}{4}\right)$, $z \in \mathbb{C}^{\pm}$.

In fact the model (8.1)–(8.3) with $n = 1$, $0 < a < \infty$ and $b_1(\mu) \equiv \beta$, $\beta \in \mathbb{R}^+$, allows to calculate the function $V_1(z)$ in explicit form:

$$V_1(z) = \beta^2 \ln \frac{z}{z - a} \tag{8.12}$$

where the physical-sheet logarithm branch is chosen in such a way that

$$\left(\ln \frac{z}{z - a}\right)_{\text{phys}} = \ln|z| - \ln|z - a| \qquad \text{for } z > a.$$

The expression (8.12) gives an opportunity to treat the physical as well as unphysical sheets of the transfer function $M_1(z) = \lambda_1 - z + V_1(z)$ immediately.

In the case considered both values $V_1^{(0)}$ and $V_1^{(a)}$ are infinite. Thus the equation $M_1(z) = 0$ has two roots in the physical sheet (see Remark 8.2), say z_0, $z_0 < 0$, and z_a, $z_a > a$, representing eigenvalues of the operator $\mathbf{H}$. One can even calculate the main terms of their asymptotics as $\beta \to 0$:

$$z_0 \sim -a \exp(-\lambda_1/\beta^2), \qquad z_a \sim a\,\{1 + \exp[-(a - \lambda_1)/\beta^2]\}.$$

Riemann surface of the function M_1 coincides with that of V_1. We denote unphysical sheets of this surface by Π_k, $k = \pm 1, \pm 2, \ldots$, assuming

$$V_1(z)\Big|_{\Pi_k} = \beta^2 \left[\left(\ln \frac{z}{z-a} \right)_{\text{phys}} + 2\pi \mathrm{i}\, k \right]$$

and that $k = 0$ in this equation corresponds to the physical sheet Π_0.

Put as previously $a = 2R$, $\lambda_1 = R$ and take $\beta = \sqrt{\frac{R}{2}}\,\widetilde{\beta}$. We want to show that the equation $M_1(z)\Big|_{\Pi_k} = 0$ has at least one solution (resonance) in each unphysical sheet Π_k, $k = \pm 1, \pm 2, \ldots$, with no restrictions on $\widetilde{\beta}$ and R such that $0 < \widetilde{\beta} < +\infty$, $0 < R < +\infty$.

First, we note that for z lying in the line $\operatorname{Re} z = R$, $z = R\,(1 + \mathrm{i}\,\tan\varphi)$ where $0 \le \varphi < \frac{\pi}{2}$ or $\frac{3\pi}{2} < \varphi \le 2\pi$, this equation can be transformed into the following equations for the argument φ:

$$\tan\varphi = \widetilde{\beta}^2 \left(\varphi - \frac{\pi}{2} + \pi k \right), \quad k \in \mathbb{Z}, \qquad \varphi \in \left[0, \frac{\pi}{2} \right) \tag{8.13}$$

and

$$\tan\varphi = \widetilde{\beta}^2 \left(\varphi - \frac{3\pi}{2} + \pi k \right), \quad k \in \mathbb{Z}, \qquad \varphi \in \left(\frac{3\pi}{2}, 2\pi \right]. \tag{8.14}$$

Considering the graphs $y = \tan\varphi$ and $y = \widetilde{\beta}^2 \left(\varphi - \frac{\pi}{2} + \pi k \right)$ for $0 \le \varphi < \frac{\pi}{2}$ one immediately checks that Eq. (8.13) has no solutions for entire $k \le 0$ while it necessarily gets a single root φ_k for any entire positive k, corresponding to a resonance $z_k = R\,(1 + \mathrm{i}\,\tan\varphi_k)$ belonging to the upper halfplane of the unphysical sheet Π_k with $k = 1, 2, 3, \ldots$. At the same time, considering the graphs $y = \tan\varphi$ and $y = \widetilde{\beta}^2 \left(\varphi - \frac{3\pi}{2} + \pi k \right)$ for $\frac{3\pi}{2} < \varphi \le 2\pi$ one finds that Eq. (8.14) has no solutions for entire $k \ge 0$ and necessarily gets a single root φ_k for any entire negative k, corresponding to a resonance $z_k = R\,(1 + \mathrm{i}\,\tan\varphi_k)$ belonging to the lower halfplane of Π_k with $k = -1, -2, -3, \ldots$. (Also one observes that the resonances z_k and z_{-k} are situated symmetrically with respect to the real axis.) Therefore we have proved that the resonance set of the transfer function $M_1(z)$ in the model considered is indeed nonempty in every unphysical sheet.

APPENDIX A: THE NORM OF AN OPERATOR WITH RESPECT TO A SPECTRAL MEASURE

Let $\mathcal{H}', \mathcal{H}''$ be separable Hilbert spaces, not necessarily distinct, and $T \in \mathbf{B}(\mathcal{H}', \mathcal{H}'')$. Let E be the spectral measure of a self-adjoint operator in $\mathcal{H}'$ with the support $\sigma = \operatorname{supp} E$, $\sigma \subset \mathbb{R}$. By the E-norm of the operator T we understand a number $\|T\|_E$ defined as

$$\|T\|_E^2 = \sup_{\{\delta_k\}} \sum_k \|T E(\delta_k) T^*\| \tag{A.1}$$

where $\{\delta_k\}$ stands for a finite or countable complete system of pairwise nonintersecting subsets of the set σ measurable with respect to E, i. e., δ_k are Borel subsets of σ, with $\delta_k \cap \delta_l = \emptyset$ if $k \neq l$ and $\bigcup_k \delta_k = \sigma$. For $S \in \mathbf{B}(\mathcal{H}'', \mathcal{H}')$ we define $\|S\|_E \stackrel{\mathrm{def}}{=} \|S^*\|_E$.

One can easily check that

$$\|T\| \leq \|T\|_E \,. \tag{A.2}$$

Indeed, $\|T\|^2 = \big\||T|\big\|^2 = \|TT^*\|$. Since $\sum\limits_k E(\delta_k) = I'$ with I' being the identity operator in $\mathcal{H}'$, we conclude that

$$\|TT^*\| = \|T \sum_k E(\delta_k) T^*\| \leq \sum_k \|TE(\delta_k)T^*\| \,.$$

From this we immediately obtain (A.2). The equality $\|T\| = \|T\|_E$ is attained if the support σ of the measure E consists of a single point and, thus, T is a multiple of the identity.

LEMMA A.1 *The following equalities are valid:*

$$\|T\|_E^2 = \sup_{\{\delta_k\}} \sum_k \|TE(\delta_k)\|^2 = \sup_{\{\delta_k\}} \sum_k \|E(\delta_k)T^*\|^2 \,. \tag{A.3}$$

P r o o f. To begin with we note that for any E-measurable set δ

$$\|TE(\delta)T^*\| = \|TE(\delta) \cdot E(\delta)T^*\| \leq \|TE(\delta)\| \cdot \|E(\delta)T^*\|.$$

Since $[TE(\delta)]^* = E(\delta)T^*$, we have $\|[TE(\delta)]^*\| = \|E(\delta)T^*\|$. Therefore,

$$\|TE(\delta)T^*\| \leq \|TE(\delta)\|^2 = \|E(\delta)T^*\|^2. \tag{A.4}$$

On the other hand, for any $f \in \mathcal{H}''$

$$\|E(\delta)T^* f\|^2 = \langle E(\delta)T^* f, E(\delta)T^* f\rangle = \langle TE(\delta)T^* f, f\rangle \leq \|TE(\delta)T^*\| \, \|f\|^2$$

and this means

$$\|TE(\delta)\|^2 = \|E(\delta)T^*\|^2 \leq \|TE(\delta)T^*\|. \tag{A.5}$$

It follows from (A.4) and (A.5) that, in fact,

$$\|TE(\delta)T^*\| = \|TE(\delta)\|^2 = \|E(\delta)T^*\|^2 \,. \tag{A.6}$$

Further, we can take $\delta = \delta_k$ and sum in (A.6) over k. Then we can take for resulting sums the exact upper bounds. Finally, one finds that Eqs. (A.3) are indeed valid. The proof is complete. □

Obviously, $\|\alpha T\|_E = |\alpha| \, \|T\|_E$.

At the same time, if the operators $T_1, T_2 : \mathcal{H}' \to \mathcal{H}''$ have finite E-norms then their sum $T_1 + T_2$ has a finite E-norm and

$$\|T_1 + T_2\|_E \leq \|T_1\|_E + \|T_2\|_E \,. \tag{A.7}$$

Indeed,

$$\left(\sum_k \|(T_1+T_2)E(\delta_k)\|^2\right)^{1/2} \leq \left(\sum_k (\|T_1E(\delta_k)\| + \|T_2E(\delta_k)\|)^2\right)^{1/2}$$
$$\leq \left(\sum_k \|T_1E(\delta_k)\|^2\right)^{1/2} + \left(\sum_k \|T_2E(\delta_k)\|^2\right)^{1/2}.$$

Using the statement of Lemma A.1, we come immediately to (A.7).

The condition $T = 0$, if $\|T\|_E = 0$, follows from the inequality (A.2).

Hence, the function $\|\cdot\|_E$ is indeed a norm. Due to (A.2), the limit of each Cauchy sequence of operators from $\mathbf{B}(\mathcal{H}', \mathcal{H}'')$ having finite E-norms and converging with respect to the norm $\|\cdot\|_E$ is automatically an element of $\mathbf{B}(\mathcal{H}', \mathcal{H}'')$ having finite E-norm, too. Therefore, the operators from $\mathbf{B}(\mathcal{H}', \mathcal{H}'')$, having finite E-norms, constitute a Banach space.

If the spectral measure E corresponds to a self-adjoint operator having a simple pure discrete spectrum, then, evidently, $\|T\|_E$ coincides with the Hilbert-Schmidt norm $\|T\|_2$, $\|T\|_2 = \sum_{n=1}^{\infty} \|Te_n\|^2$, where $\{e_n\}$ is an arbitrary orthonormal basis of $\mathcal{H}'$. In general we have only the inequalities

$$\|T\| \leq \|T\|_E \leq \|T\|_2 . \tag{A.8}$$

Along with the E-norm (A.1) one may consider as well a whole family of operator norms defined with respect to a spectral measure E:

$$\|T\|_{p,E} = \left(\sup_{\{\delta_k\}} \sum_k \|TE(\delta_k)\|^p\right)^{1/p} = \left(\sup_{\{\delta_k\}} \sum_k \|E(\delta_k)T^*\|^p\right)^{1/p}, \quad p \geq 1. \tag{A.9}$$

The norm (A.1) is a particular case of these norms for $p = 2$. Many properties of the norms (A.9) are similar to those for the respective norms $\|\cdot\|_p$ on classes of compact operators. Note, in particular, that if, for $T_1 : \mathcal{H}' \to \mathcal{H}''$, $T_2 : \mathcal{H}'' \to \mathcal{H}'$ and $1/p + 1/q = 1$, the norms $\|T_1\|_{p,E}$ and $\|T_2\|_{q,E}$ are finite, then

$$\sup_{\{\delta_k\}} \sum_k \|T_1E(\delta_k)T_2\| \leq \|T_1\|_{p,E} \cdot \|T_2\|_{q,E} .$$

Note also that for any $1 \leq p < \infty$

$$\|T\| \leq \|T\|_{p,E} \leq \|T\|_p .$$

We do not describe properties of $\|\cdot\|_{p,E}$ for arbitarary p, since, in this work, we use only the norm $\|\cdot\|_E \equiv \|\cdot\|_{2,E}$.

APPENDIX B: THE INTEGRAL OF AN OPERATOR-VALUED FUNCTION OVER A SPECTRAL MEASURE

To avoid confusion with the measure E_j, we shall denote the spectral function of the self-adjoint operator A_j, $j = 0, 1$, by $E^j(\mu)$ (i.e., with superscript): $E^j(\mu) = E_j\Big((-\infty, \mu)\Big)$, $\mu \in \mathbb{R}$. Recall that $E^j(\mu)$ is a projection-valued function satisfying the conditions of monotonicity, $E^j(\mu_1) \leq E^j(\mu_2)$

for $\mu_1 < \mu_2$, and completeness, $s\text{-}\lim\limits_{\mu\to-\infty} E^j(\mu) = 0$, $s\text{-}\lim\limits_{\mu\to+\infty} E^j(\mu) = I_j$. In addition, this function is continuous from the left, $s\text{-}\lim\limits_{\mu'\uparrow\mu} E^j(\mu') = E^j(\mu)$.

Let $F(\mu)$ be a function defined on an interval $[a,b]$, $-\infty < a < b < +\infty$, whose values are bounded operators acting from $\mathcal{H}_j$ to $\mathcal{H}_i$, that is, $F : [a,b] \to \mathbf{B}(\mathcal{H}_j, \mathcal{H}_i)$, $i,j = 0,1$ and where it is not necessary that $i \neq j$. Following [3], we say the function F is uniformly (strongly, weakly) integrable from the right over the spectral measure E_j on $[a,b)$ if the limit

$$\int\limits_a^b F(\mu)\, dE^j(\mu) \ \stackrel{\text{def}}{=} \lim_{\max\limits_{k=1}^{n}|\delta_k^{(n)}|\to 0} \sum_{k=1}^{n} F(\xi_k)\, E_j(\delta_k^{(n)}) \tag{B.1}$$

exists considered in the sense of the uniform (strong, weak) operator topology. Here, $\delta_k^{(n)} = [\mu_{k-1}, \mu_k)$ and $|\delta_k^{(n)}| = \mu_k - \mu_{k-1}$, $k = 1,2,\ldots,n$, where $\mu_0, \mu_1, \ldots, \mu_n$ is any subsequence of numbers from the interval $[a,b]$ satisfying the conditions $a = \mu_0 < \mu_1 < \ldots < \mu_n = b$. By ξ_k we understand an arbitrary point of $\delta_k^{(n)}$. The limit value (B.1), if it exists, is called the right integral of the function F over the measure E_j on $[a,b)$ in the sense of Riemann-Stieltjes.

Similarly, we say the function $G : [a,b] \to \mathbf{B}(\mathcal{H}_i, \mathcal{H}_j)$, $i,j = 0,1$ is uniformly (strongly, weakly) integrable from the left over the spectral measure E_j on $[a,b)$, $-\infty < a < b < +\infty$, if it exists the limit

$$\int\limits_a^b dE^j(\mu)\, G(\mu) \ \stackrel{\text{def}}{=} \lim_{\max\limits_{k=1}^{n}|\delta_k^{(n)}|\to 0} \sum_{k=1}^{n} E_j(\delta_k^{(n)}) G(\xi_k) \tag{B.2}$$

considered in the sense of the uniform (strong, weak) operator topology. The limit value (B.2), if it exists, is called the left integral of the function G over measure E_j on $[a,b)$ in the sense of Riemann-Stieltjes.

Since the Banach spaces $\mathbf{B}(\mathcal{H}_i, \mathcal{H}_j)$ and $\mathbf{B}(\mathcal{H}_j, \mathcal{H}_i)$ are closed with respect not only to the uniform (u) but also to the strong (s) and weak (w) convergence of operators, the integrals (B.1) and (B.2), if they exist in some sense, determine certain bounded operators belonging, respectively, to $\mathbf{B}(\mathcal{H}_i, \mathcal{H}_j)$ and $\mathbf{B}(\mathcal{H}_j, \mathcal{H}_i)$.

Evidently, if the integrals (B.1) and (B.2) exist in the sense of the strong operator topology, they exist as well in the sense of the weak operator topology. In turn, existence of these integrals in the sense of the uniform operator topology implies their existence in the sense of the strong as well as weak operator topology. The following simple statement holds.

Lemma B.1 *The function $F(\mu)$, $F : [a,b] \to \mathbf{B}(\mathcal{H}_j, \mathcal{H}_i)$, is integrable in the sense of the uniform operator topology over the measure E_j in $[a,b)$ from the left iff the function $[F(\mu)]^*$ is integrable over the same measure from the right, and also in the sense of the uniform operator topology. The same statement is true with respect to the simultaneous integrability of these functions with respect to the weak operator topology. In general it only follows from the existence of one of the integrals $\int\limits_a^b F(\mu)\, dE^j(\mu)$ and $\int\limits_a^b dE^j(\mu)\, [F(\mu)]^*$ in the sense of the strong operator topology that the other one exists with respect to the weak operator topology. In all cases the integrability of $F(\mu)$ and $[F(\mu)]^*$ implies the equality*

$$\left[\int\limits_a^b F(\mu)\, dE^j(\mu)\right]^* = \int\limits_a^b dE^j(\mu)\, [F(\mu)]^*. \tag{B.3}$$

P r o o f. Note that
$$\left(\sum_{k=1}^{n} F(\xi_k)\, E_j(\delta_k^{(n)})\right)^* = \sum_{k=1}^{n} E_j(\delta_k^{(n)})\, [F(\xi_k)]^* .$$
Therefore, the validity of the first and second statements of the lemma follows from continuity of the involution $T \to T^*$ with respect to u- and w-convergence in $\mathbf{B}(\mathcal{H}_j, \mathcal{H}_i)$. The last statement follows from the fact that strong convergence of a sequence of operators in $\mathbf{B}(\mathcal{H}_j, \mathcal{H}_i)$ implies also weak convergence of this sequence. However, one can not claim that the sequence of respective adjoint operators converges strongly, since the involution $T \to T^*$ is not continuous with respect to s-convergence (see, e. g., [6], § 5 of Chapter 2).

The proof of the lemma is complete. □

Some sufficient conditions for the integrability of an operator-valued function $F(\mu)$ in the sense of the uniform operator topology are given in the following statement.

LEMMA B.2 *Any operator function F, $F : [a, b] \to \mathbf{B}(\mathcal{H}_i, \mathcal{H}_j)$, which satisfies the Lipschitz condition*

$$\|F(\mu_2) - F(\mu_1)\| \le C_F\, |\mu_2 - \mu_1| \qquad \forall \mu_1, \mu_2 \in [a, b] \tag{B.4}$$

with some constant $C_F > 0$, is right-integrable with respect to E_j in the sense of the operator norm topology.

A proof of this statement can be found in Ref. [3] (see in [3] Lemma 7.2 and Remark 7.3).

The integrals $\int\limits_a^b F(\mu)\, dE^j(\mu)$ and $\int\limits_a^b dE^j(\mu)\, G(\mu)$ with $a = -\infty$ or $b = +\infty$ are understood as respective limits, if they exist, of integrals with finite bounds; for example,

$$\int\limits_a^b dE^j(\mu)\, G(\mu) \stackrel{\text{def}}{=} \lim_{a' \downarrow a,\, b' \uparrow b} \int\limits_{a'}^{b'} dE^j(\mu)\, G(\mu).$$

Also, we define

$$\int\limits_{\sigma(A_j)} dE_j(\mu)\, G(\mu) \stackrel{\text{def}}{=} \int\limits_a^b dE^j(\mu)\, G(\mu), \qquad \int\limits_{\sigma(A_j)} F(\mu)\, dE_j(\mu) \stackrel{\text{def}}{=} \int\limits_a^b F(\mu)\, dE^j(\mu) \tag{B.5}$$

where (a, b) is an arbitrary open interval entirely containing the set $\sigma(A_j)$. This definition is correct, since the support of the spectral measure E_j is just the spectrum $\sigma(A_j)$.

LEMMA B.3 *Let a function $X : \sigma(A_j) \to \mathbf{B}(\mathcal{H}_i, \mathcal{H}_i)$ be bounded, $\|X\|_\infty = \sup\limits_{\mu \in \sigma(A_j)} \|X(\mu)\| < \infty$, and satisfy the Lipschitz condition* (B.4). *Then, if the E_j-norm $\|B_{ij}\|_{E_j}$ of the operator B_{ij} is finite, then the integrals*

$$\int\limits_{\sigma(A_j)} dE_j(\mu)\, B_{ji}\, X(\mu) \quad \textit{and} \qquad \int\limits_{\sigma(A_j)} X(\mu)\, B_{ij}\, dE_j(\mu)$$

exist[7] *in the sense of the operator norm topology, and the following estimates are valid for their norms:*

$$\left\| \int\limits_{\sigma(A_j)} dE_j(\mu)\, B_{ji}\, X(\mu) \right\| \le \|B_{ji}\|_{E_j} \cdot \|X\|_\infty \quad \textit{and} \quad \left\| \int\limits_{\sigma(A_j)} X(\mu)\, B_{ij}\, dE_j(\mu) \right\| \le \|B_{ji}\|_{E_j} \cdot \|X\|_\infty .$$

P r o o f . The proof will be given for the case of the integral $\int\limits_{\sigma(A_j)} dE_j(\mu)\, B_{ji}\, X(\mu)$. To this end let us consider a partition $\left\{\delta_k^{(n)}\right\}_{k=1}^n$ of an interval $[a', b')$ for finite a', b' and the respective integral sum. We have for this sum:

$$\left\| \sum_{k=1}^n E_j(\delta_k^{(n)}) B_{ji} X(\xi_k) f \right\|^2 = \langle \sum_{k=1}^n E_j(\delta_k^{(n)}) B_{ji} X(\xi_k) f, \sum_{m=1}^n E_j(\delta_m^{(n)}) B_{ji} X(\xi_m) f \rangle$$

where $\left\{\xi_k \in \delta_k^{(n)}\right\}_{k=1}^n$ is an arbitrary set of points belonging to the intervals $\delta_k^{(n)}$. Since $E_j(\delta_k^{(n)}) E_j(\delta_m^{(n)}) = 0$ for $\delta_k^{(n)} \cap \delta_m^{(n)} = \emptyset$, we find

$$\begin{aligned} \left\| \sum_{k=1}^n E_j(\delta_k^{(n)}) B_{ji} X(\xi_k) f \right\|^2 &= \langle \sum_{k=1}^n [X(\xi_k)]^* B_{ij} E_j(\delta_k^{(n)}) B_{ji} X(\xi_k) f, f \rangle \\ &\le \sum_{k=1}^n \|B_{ij} E_j(\delta_k^{(n)}) B_{ji}\| \cdot \|X(\xi_k)\|^2 \cdot \|f\|^2 \\ &\le \mathcal{V}_j(B)\big|_{[a',b')} \|X\|_\infty^2 \|f\|^2 . \end{aligned}$$

This means

$$\left\| \sum_{k=1}^n E_j(\delta_k^{(n)}) B_{ji} X(\xi_k) \right\| \le \sqrt{\mathcal{V}_j(B)\big|_{[a',b')}} \cdot \|X\|_\infty .$$

Since the total variation $\mathcal{V}_j(B) = \|B_{ji}\|_{E_j}^2$ is supposed to be finite, we have

$$\left\| \int\limits_{a'}^{b'} dE^j(\mu)\, B_{ji}\, X(\mu) \right\| \underset{\substack{a' \to +\infty \\ (b' > a')}}{\longrightarrow} 0,$$

This means that the integral $\int\limits_a^{+\infty} dE^j(\mu)\, B_{ji}\, X(\mu)$ with a finite lower bound a converges with respect to the uniform operator topology. Existence of the integral $\int\limits_{-\infty}^{b} dE^j(\mu)\, B_{ji}\, X(\mu)$ with a finite b can

[7]The function X can be extended outside the set $\sigma(A_j)$ in an arbitrary way when the definitions (B.1), (B.2) and (B.5) are used, retaining only the Lipschitz condition (B.4).

be proved in the same way. The existence of the integral $\int\limits_{\sigma'(A_j)} dE_j(\mu)\, B_{ji}\, X(\mu)$ with respect to the operator norm topology as well as its norm estimate follow immediately from these results.
□

In the same way as the for the previous integrals one can define and treat the integral

$$\int\limits_{\sigma(A_j)} X(\mu) B_{ij} dE_j(\mu) B_{ji} Y(\mu) \tag{B.6}$$

with X, Y like the function X in Lemma B.3 and the same $B_{ij} = B_{ji}^*$. First, we extend X and Y outside the set $\sigma(A_j)$ retaining the Lipschitz condition (B.4) and introduce for $-\infty < a < b < +\infty$ the value

$$\int\limits_a^b X(\mu) B_{ij} dE^j(\mu) B_{ji} Y(\mu) \;\stackrel{\text{def}}{=}\; \lim_{\max\limits_{k=1}^{n} |\delta_k^{(n)}| \to 0} \; \sum_{k=1}^{n} X(\xi_k) B_{ij} E_j(\delta_k^{(n)}) B_{ji} Y(\xi_k) \tag{B.7}$$

with $\delta_k^{(n)}$ and $\xi_k^{(n)}$ taken as in (B.1). Then we consider the limits $a \to -\infty$ and/or $b \to +\infty$ if necessary. As a result one has the following

LEMMA B.4 *Let the functions* $X, Y : \sigma(A_j) \to \mathbf{B}(\mathcal{H}_i, \mathcal{H}_i)$ *be bounded,* $\|X\|_\infty < \infty$, $\|Y\|_\infty < \infty$, *and satisfy the Lipschitz condition* (B.4) *and let* $\|B_{ij}\|_{E_j} < \infty$, *also. Then the integral* (B.6) *exists in the sense of the operator norm topology and*

$$\left\| \int\limits_{\sigma(A_j)} X(\mu) B_{ij} dE_j(\mu) B_{ji} Y(\mu) \right\| \le \|B_{ij}\|_{E_j}^2 \cdot \|X\|_\infty \cdot \|Y\|_\infty \,.$$

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