An approach to Mel'nikov theory in Celestial Mechanics

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Abstract

Using a completely analytic procedure – based on a suitable extension of a classical method – we discuss an approach to the Poincaré-Mel'nikov theory, which can be conveniently applied also to the case of non-hyperbolic critical points, and even if the critical point is located at the infinity. In this paper, we concentrate our attention on the latter case, and precisely on problems described by Kepler-like potentials in one or two degrees of freedom, in the presence of general time-dependent perturbations. We show that the appearance of chaos (possibly including Arnol'd diffusion) can be proved quite easily and in a direct way, without resorting to singular coordinate transformations, such as the McGehee or blowing-up transformations. Natural examples are provided by the classical Gyldén problem, originally proposed in celestial mechanics, but also of interest in different fields, and by the general 3-body problem in classical mechanics.

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In a previous paper[1], we discussed a completely analytic procedure – based on a suitable extension of a classical method[2] – to introduce the Poincaré-Melnikov theory[3–5] concerning the appearance of chaotic behaviour, which can be applied even to the case where the critical point is *not hyperbolic*. The case of nonhyperbolic stationary points has been already considered by several authors, although with quite different methods or in different contexts: we mention here Ref.s 6-11; some other references, more strictly related to our arguments, will be given in the following. In Ref. 1, we have also shown that our procedure may work even in some problems where the critical point is located *at the infinity* of the real line, as occurs in the case of the classical Sitnikov problem[12].

In this paper, we want first of all to reconsider and refine this approach, concentrating our attention precisely on problems described by Kepler-like potentials, in the presence of time-dependent perturbations of very general form. An example is provided by the classical Gyldén problem originally proposed in celestial mechanics, but also of interest in different fields (see Ref. 13). We shall then extend this procedure to problems with 2 degrees of freedom and in the presence of rotational symmetry, and show that the appearance of chaos (possibly including Arnol'd diffusion[14]) can be proved quite easily and in a direct way, without resorting to singular coordinate transformations, such as the McGehee transformations[15].

Natural examples are provided by the general planar 3-body problem in classical mechanics[16–18].

Let us remark that, although our procedure is quite general, we shall consider here – for the sake of definiteness and clarity – only the case of the Kepler potential $V(r) = 1/r$: from the following discussion it will appear completely clear that the method can be easily applied – with suitable minor adjustments – to different classes of problems, for instance to problems where the Kepler potential is replaced by another "long range" potential, as e.g. $V \sim 1/r^\beta$.

I. The Gyldén-like problems

We start considering a 2-degrees of freedom problem described by a Hamiltonian of the form

$$H = \frac{1}{2}\mathbf{p}^2 - \frac{1}{r} + \epsilon W(r,t) \tag{1}$$

where $\mathbf{p} \equiv (p_1, p_2)$, i.e a standard (planar) Kepler problem plus a smooth perturbation $\epsilon W(r,t)$. This problem can be easily reduced to a 2-dimensional dynamical system for the two variables $r,\ \dot{r}$, and for this reason we shall treat first this case, not only for better illustrating our procedure, but also in view of some direct and interesting applications, which include the classical Gyldén problem[13].

Consider the "parabolic" solution of the unperturbed ($\epsilon = 0$) problem (1), which plays here the role of the homoclinic solution corresponding to the critical point at the infinity, i.e. $r = \infty,\ \dot{r} = 0$: it satisfies the equations

$$\dot{r} = \pm\frac{\sqrt{2r^2 - k^2}}{r} \qquad\qquad \dot{\theta} = \frac{k}{r^2} \tag{2}$$

where $k \neq 0$ is the (constant) angular momentum, and the sign $-$ (resp. $+$) holds for $t < 0$ (resp. $t > 0$). From (2) one gets

$$\pm t = \frac{k^2 + r}{3}\sqrt{2r - k^2} + \text{const} \qquad\qquad \theta = \pm 2\arctan\frac{\sqrt{2r - k^2}}{k} + \text{const} \tag{3}$$

Let us denote by

$$R = R(t) \qquad \text{and} \qquad \Theta = \Theta(t) \tag{4}$$

the expressions giving the dependence of r and of θ on the time t which are obtained "inverting" equations (3) with the conditions $R(0) = r_{min} = k^2/2$ and $\Theta(0) = \pi$ (let us emphasize that it is *not* necessary, for our purposes, to have the explicit form of these functions). It will be useful only to remark that $R(t)$ is an even function and $\Theta(t)$ is a odd function of time. The choice $\Theta(0) = \pi$ corresponds to select the solution describing the parabola with axis coinciding with the x_1 axis and going to infinity when $x_1 \to +\infty$; the whole Hamiltonian (1), including the perturbation $W(r,t)$, is rotationally invariant, and therefore this choice is clearly not restrictive. We refer to the next section, where we shall consider symmetry-breaking perturbations, for some comment on this aspect.

The first point is now to find the negatively and positively asymptotic sets to the critical point $r=\infty,\ \dot{r}=0$ in the presence of the perturbation $\epsilon W(r,t)$; notice that this point would in fact correspond to the critical point $x=y=0$ under the McGehee's singular coordinate transformation[12,13,15]

$$r=\frac{1}{x^2} \qquad \dot{r}=y \qquad \mathrm{d}t=\frac{1}{x^3}\,\mathrm{d}s \tag{5}$$

but – instead of using this transformation (which usually requires quite cumbersome calculations), we introduce a direct method similar to the classical one used in Ref. 2 (see also Ref. 1). Precisely, we want to show that some natural assumptions on the perturbation $W(r,t)$ may guarantee not only the existence of smooth solutions approaching $r=\infty,\ \dot{r}=0$ for $t\to\pm\infty$, playing in this context the role of stable and unstable manifolds of the critical point at the infinity, but also the possible presence of infinitely many intersections of these asymptotic sets on the Poincaré sections.

It is important to remark that the critical point at the infinity is clearly not a hyperbolic point, and therefore the standard results of perturbation theory valid for hyperbolic points cannot be applied here. In particular, in order to be granted that the perturbation will preserve the property of the point at the infinity of being a critical equilibrium, we have to impose that the perturbation vanishes at the infinity. We shall give the precise rate of this vanishing in the following (see eq. (18) below).

Let us write problem (1) in the form

$$\ddot{r}=\frac{k^2}{r^3}-\frac{1}{r^2}-\epsilon\,\frac{\partial W(r,t)}{\partial r} \tag{6}$$

and let us look for solutions $r(t)$ of the perturbed problem (6) "near" the family of the homoclinic orbits $R(t-t_0)$: we then put (see Ref. 2)

$$r=R(t-t_0)+z(t-t_0) \tag{7}$$

Inserting into (6) (with the time shift $t - t_0 \to t$), we obtain

$$\ddot{z} + \Big(\frac{3k^2}{R^4} - \frac{2}{R^3}\Big) z = G(z(t), t + t_0) \tag{8}$$

where the r.h.s. (which we will shortly denote by $G(t, t_0)$) is given by

$$G(t, t_0) \equiv -\epsilon \, \frac{\partial W(R(t), t + t_0)}{\partial r} + \text{higher order terms in } z(t) \tag{9}$$

Consider the homogeneous equation obtained putting $\epsilon = 0$ in (8): one solution is clearly $\dot{R}(t)$, another independent solution $\psi(t)$ can be constructed with standard methods (see e.g. Ref. 19) or by direct substitution: these two solutions have a different behaviour for $t \to \pm\infty$, precisely:

$$\dot{R}(t) \sim \ |t|^{-1/3} \ \to 0 \qquad \text{and} \qquad \psi(t) \sim \ |t|^{4/3} \to \infty \tag{10}$$

As well known, the general solution $z(t)$ of the complete (nonhomogeneous) equation (8) can be written in the following integral form, with A, B arbitrary constants and t_1 arbitrarily fixed[2,19]:

$$z(t) = A\dot{R}(t) + B\psi(t) - \dot{R}(t) \int_{t_1}^{t} \psi(s) G(s, t_0) \, \mathrm{d}s + \psi(t) \int_{t_1}^{t} \dot{R}(s) G(s, t_0) \, \mathrm{d}s \tag{11}$$

Let us now look for solutions $z^{(-)}(t)$ (and resp. $z^{(+)}(t)$) of (11) with the property of being bounded for $t \to -\infty$ (resp. $t \to +\infty$); these will provide solutions

$$r^{(\pm)}(t) = R + z^{(\pm)}$$

of (6) which belong, by definition, to the unstable and stable manifolds of the point $r = \ \infty, \ \dot{r} = 0$. From (11), we have then to require that the two quantities

$$\psi(t) \Big(B + \int_{t_1}^{t} \dot{R}(s) G(s, t_0) \, \mathrm{d}s \Big) \qquad \text{and} \qquad \dot{R}(t) \Big(A - \int_{t_1}^{t} \psi(s) G(s, t_0) \, \mathrm{d}s \Big) \tag{12}$$

remain bounded as $t \to -\infty$ when looking for the $r^{(-)}$ solutions, and resp. as $t \to +\infty$ for the $r^{(+)}$ solutions.

Consider now the linearization of the problem (11) around the solution $z(t) \equiv 0$: this amounts in particular to deleting the higher-order terms in the expression of G in (9). Taking also into account the different behaviour given in (10) of the two fundamental solutions $\dot{R}(t)$ and $\psi(t)$, it is easy to see that the above conditions on the quantities (12) are simultaneously satisfied both at $t = -\infty$ and at $t = +\infty$ if for some t_0 the following Mel'nikov-type condition

$$\int_{-\infty}^{+\infty} \dot{R}(t) \frac{\partial W(R(t), t+t_0)}{\partial r} \equiv M(t_0) = 0 \tag{13}$$

is fulfilled, together with the additional one, i.e. that the quantity

$$\dot{R}(t) \int_{t_1}^{t} \psi(s) \frac{\partial W(R(s), s+t_0)}{\partial r} \, \mathrm{d}s \tag{14}$$

is bounded as $t \to \pm\infty$. Once these conditions are satisfied, one can conclude that, as a consequence of the implicit-function theorem[2] (or also thanks to a suitable version of the Lyapunov-Schmidt procedure, see e.g. Ref. 20), there exists a smooth and bounded solution of (8).

Let us now discuss the two above conditions (13-14).

The first one is identical to the usual Mel'nikov condition obtained under the standard assumption that the critical point is hyperbolic[3−5]: eq.(13) can therefore be viewed as an extension of the classical Mel'nikov formula to the present "degenerate" case, in which the critical point is at the infinity. Let us now assume that the perturbation $W(r,t)$ is a smooth function, periodic in the time t, with arbitrary period T (it is not restrictive to assume $T = 2\pi$) and zero mean-valued:

$$\int_0^{2\pi} W(r,t)\mathrm{d}t = 0 \tag{15}$$

Then one can consider its Fourier expansion

$$W(r,t) = \sum_{n=1}^{\infty} \left(A_n(r) \cos nt + B_n(r) \sin nt \right) \tag{16}$$

and (thanks to the parity of the function $R(t)$) write down the Mel'nikov function $M(t_0)$ defined in (13) in the form:

$$M(t_0) = \sum_{n=1}^{\infty} \left(\alpha_n \cos nt_0 + \beta_n \sin nt_0 \right) \tag{17}$$

where

$$\alpha_n = \int_{-\infty}^{+\infty} \dot{R}(t) \frac{\mathrm{d}A_n(R(t))}{\mathrm{d}r} \sin nt \; \mathrm{d}t \quad , \quad \beta_n = \int_{-\infty}^{+\infty} \dot{R}(t) \frac{\mathrm{d}B_n(R(t))}{\mathrm{d}r} \sin nt \; \mathrm{d}t \tag{17'}$$

¿From this expression, one can immediately conclude that the function $M(t_0)$, being a smooth periodic function with vanishing mean value, must certainly possess zeroes, thus fulfilling condition (13).

For what concerns the second condition, which requires the boundedness of (14) and which appears here to compensate the lack of the "exponential dichotomy" peculiar of the hyperbolic case, a simple estimate of the behaviour as $|t| \to \infty$ of the integral in (14), using (10) and (16), and recalling from (3) that $R(t) \sim |t|^{2/3}$ as $|t| \to \infty$, shows that it is sufficient to assume that the quantities $A_n(r), \; B_n(r)$ in the expansion (16) vanish as $r \to \infty$ according to

$$A_n(r) \sim \frac{a_n}{r^\delta} \qquad , \qquad B_n(r) \sim \frac{b_n}{r^\delta} \qquad \text{with} \qquad \delta > 1/2 \tag{18}$$

in order to be granted that the above condition on (14) is satisfied. It can be noticed that the same condition (18) would also guarantee that under the McGehee transformation[12,13,15] (5), the perturbation is not singular at $x = 0$.

Changing now the point of view, and considering the Poincaré sections of the $r^{(-)}$ and $r^{(+)}$ solutions, the above arguments show that, once conditions (13-14) are satisfied, there occurs a crossing of the negatively and positively asymptotic sets on the Poincaré section. One usually imposes at this point that the zeroes of the Mel'nikov function (13) are simple zeroes, i.e. that

$$\frac{\partial M}{\partial t_0} \neq 0 \tag{19}$$

which ensures the transversality of the intersections, recalling that the function $M(t_0)$ expresses the signed distance between the intersecting manifolds[4,5]. Actually, it can be noted that it is not strictly necessary to impose this condition, indeed – according to an interesting and useful result by Burns and Weiss[21] – it is sufficient that the crossing is "topological", i.e., roughly, that there is really a "crossing", from one side to the other. But in our case this is certainly satisfied, because $M(t_0)$, being a smooth periodic and zero mean-valued function, must necessarily change sign (see also Ref. 22 for a careful discussion on non-transversal crossings). Using then standard arguments, which are not based on hyperbolicity, thanks to the periodicity of the perturbation, one immediately deduces[2,4,5,21] that there is an infinite sequence of intersections, leading to a situation similar to the usual chain of homoclinic intersections typical of the homoclinic chaos.

The presence of such infinitely many intersections is clearly reminiscent of the chaotic behaviour expressed by the Birkhoff-Smale theorem in terms of the equivalence to a symbolic dynamics described by the Smale horseshoes. Actually, this theorem cannot be directly used in the present context because its standard proof is intrinsically based on hyperbolicity properties[4,5]. However, several arguments can be invoked even in the present "degenerate" situation. First of all, for the case of degenerate critical points at the infinity, we can refer to the classical arguments used in Ref. 12, and reconsidered by many others (see e.g. Ref.s 13,16,23). More specifically, see Ref. 24, where an equivalence to a "nonhyperbolic horseshoe" has been proved, in which the contracting and expanding actions are not exponential but "polynomial" in time. Let us also notice, incidentally, that the presence of Smale horseshoes and of a positive topological entropy has been proved by means of a quite general geometrical or "topological" procedure[21] which holds, in the presence of area-preserving perturbations, even in cases of nonhyperbolic equilibrium points (i.e. not only in the case of degenerate critical points at the infinity, see Ref.s 1,25). Alternatively, in the general situation, one may possibly resort to the method of "blowing-up", devised to investigate the properties of nonhyperbolic singularities by means of suitable changes of coordinates[26,27].

Finally, for what concerns the regularity of the solutions and of the asymptotic sets, see in particular Ref.s 23-25,28.

Summarizing, we can state the following:

Proposition 1. *Consider a Kepler-like problem as in (1), and assume that the perturbation $W(r,t)$ is a smooth time-periodic function with zero mean value (16). Assume that it vanishes with $r \to \infty$ in such a way that (18) are* *satisfied. Then there is a chaotic behaviour of the solution, induced by a chain of infinitely many intersections in the Poincaré section of the negatively and positively asymptotic sets of the critical point at the infinity.*

Let us conclude this section with the obvious remark that the case of the Gyldén problem, for which the perturbation is given by

$$W(r,t) = \frac{\mu(t)}{r} \tag{20}$$

where $\mu(t)$ is a periodic function, satisfies all the above assumption and therefore exhibits chaotic behaviour[13]. The above discussion then generalizes this result to a larger class of problems and under weaker assumptions.

II. Problems with 2 degrees of freedom.

We now consider the case of planar Kepler-like problems as in (1) but in the presence of perturbations of the more general form $W = W(r,\theta,t)$

$$H = \frac{1}{2}\mathbf{p}^2 - \frac{1}{r} + \epsilon W(r,\theta,t) \equiv H_0 + \epsilon W(r,\theta,t) \tag{21}$$

In this case the reduction of the problem as performed in sect. **I** is no longer possible, and we have to handle the four variables x_1, p_1, x_2, p_2 (or $r,\ \dot r,\ \theta,\ \dot\theta$). The first point to be remarked is that the degeneracy of the critical point at the infinity, $r=\infty,\ \dot r = 0$, appears now even worse than before, indeed we have here a "continuous family of points at the infinity", due to the arbitrarity of the angle θ; more precisely, the homoclinic manifold, i.e. the set of solutions of the unperturbed equation which are doubly asymptotic to $r=\infty,\ \dot r=0$, is given here, for each fixed value $k\neq 0$ of the angular momentum, by the 2-dimensional manifold described by the family of parabolas of equations $R = R(t-t_0),\ \Theta(t-t_0)+\theta_0$, where $R(t),\ \Theta(t)$ have been defined in sect. **I** (see (3-4)), with arbitrary t_0, θ_0, or – in cartesian coordinates $u \equiv (x_1,p_1,x_2,p_2)$ – by

$$\chi \equiv \chi(\theta_0, t-t_0) \equiv \tag{22}$$

$$\begin{aligned}&\Big(R(t-t_0)\cos\left(\Theta(t-t_0)+\theta_0\right),\\&\dot R(t-t_0)\cos\left(\Theta(t-t_0)+\theta_0\right)-R(t-t_0)\dot\Theta(t-t_0)\sin\left(\Theta(t-t_0)+\theta_0\right),\\&R(t-t_0)\sin\left(\Theta(t-t_0)+\theta_0\right),\\&\dot R(t-t_0)\sin\left(\Theta(t-t_0)+\theta_0\right)+R(t-t_0)\dot\Theta(t-t_0)\cos\left(\Theta(t-t_0)+\theta_0\right)\Big)\end{aligned}$$

In order to find conditions ensuring the occurrence of intersections of stable and unstable manifolds for the perturbed problem, we follow a similar (suitably extended) procedure as in sect. **I**. We first look for smooth solutions near the homoclinic manifold (see sect. **I**; here, clearly, $z \equiv (z_1, z_2, z_3, z_4)$)

$$u = \chi(\theta_0, t-t_0) + z(\theta_0, t-t_0) \tag{23}$$

of the problem, which we now write in the form

$$\dot u = J\nabla_u H \equiv F(u) + \epsilon J\nabla_u W \tag{24}$$

(J being the standard symplectic matrix). Linearizing the problem along an arbitrarily fixed solution $\chi(\theta_0, t-t_0)$ in the family (22), we get the following equation for $z(t)$

$$\dot z = A(\theta_0, t)z + G(\theta_0, t, t_0) \tag{25}$$

where

$$A(\theta_0, t) = (\nabla_u F)(\chi(\theta_0, t)) \tag{26}$$

and

$$G(\theta_0, t, t_0) = \epsilon J(\nabla_u W)(R(t), \Theta(t)+\theta_0, t+t_0) \tag{27}$$

All the solutions $z(t)$ of (25) are are given by (cf. Ref. 19; when not essential, the dependence on θ_0, t_0 will be sometimes dropped, for notational simplicity)

$$z(t) = z_h(t) + \Phi(t)\int_{t_1}^{t} \Phi^{-1}(s)G(s)\mathrm{d}s \tag{28}$$

where $z_h(t)$ is any solution of the homogeneous linear problem

$$\dot z = A(\theta_0, t)z \tag{29}$$

and Φ is a fundamental matrix of solutions of (29). There are certainly two solutions of (29), which are bounded for any $t \in \mathbf{R}$ (and vanish for $t \to \pm\infty$), namely

$$\frac{\partial \chi}{\partial t} \qquad \text{and} \qquad \frac{\partial \chi}{\partial \theta} \tag{30}$$

as one may easily verify (this also follows from general arguments[29,30]). As seen in sect. **I**, due to the degeneracy of the critical point, instead of the exponential dichotomy, typical of the standard hyperbolic case, we now get a power behaviour $|t|^\sigma$ of the solutions, but the general arguments for controlling the behaviour for large $|t|$ of the solutions $z(t)$ in (28) can still be used. Precisely (cf. Ref. 29), observing also that, for Hamiltonian problems, the matrix $\Phi^{-1}J$ is the transposed of a fundamental matrix of solutions of the same problem (29), one deduces from (28) and (30) that there exist bounded solutions of (25), both for $t \to +\infty$ and for $t \to -\infty$, if the two following conditions are satisfied (cf. Ref.s 29,30)

$$\int_{-\infty}^{+\infty} \Big(\frac{\partial \chi(\theta_0, t)}{\partial t}\ , \nabla_u W(R(t), \Theta(t) + \theta_0, t + t_0)\Big) \mathrm{d}t \equiv M_1(\theta_0, t_0) = 0 \tag{31}$$

and

$$\int_{-\infty}^{+\infty} \Big(\frac{\partial \chi(\theta_0, t)}{\partial \theta}\ , \nabla_u W(R(t), \Theta(t) + \theta_0, t + t_0)\Big) \mathrm{d}t \equiv M_2(\theta_0, t_0) = 0 \tag{32}$$

where $\big(\ \cdot\ ,\ \cdot\ \big)$ stands for the scalar product in $\mathbf{R}^4$. Proceeding just as in sect. **I**, we assume that the perturbation $W(r, \theta, t)$ is a smooth time-periodic function, and we still assume to hold a condition analogous to (18) in order to to guarantee the boundedness of $z(t)$ at $t = \pm\infty$, as already discussed. The above conditions (31-32) can be more conveniently rewritten in the following form, using (22),

$$M_1(\theta_0, t_0) = \tag{33}$$
$$\int_{-\infty}^{+\infty} \Big[\dot{R}(t) \frac{\partial W(R(t), \Theta(t) + \theta_0, t + t_0)}{\partial r} + \dot{\Theta}(t) \frac{\partial W(R(t), \Theta(t) + \theta_0, t + t_0)}{\partial \theta}\Big] \mathrm{d}t = 0$$

$$M_2(\theta_0, t_0) = \int_{-\infty}^{+\infty} \frac{\partial W(R(t), \Theta(t) + \theta_0, t + t_0)}{\partial \theta} = 0 \tag{34}$$

It can be significant to remark here that it is easy to verify that these conditions are identical to the Mel'nikov conditions for the appearance of homoclinic intersections given e.g. in Ref.s 5,31 in the standard hyperbolic case and deduced by means of a different procedure, namely

$$\int_{-\infty}^{+\infty}\big(\nabla_u H_0, J\nabla_u W\big)(\chi(\theta_0,t),t+t_0)\ \mathrm{d}t = \int_{-\infty}^{+\infty}\{H_0,W\}(\chi(\ldots))\ \mathrm{d}t = 0 \quad (35)$$

and

$$\int_{-\infty}^{+\infty}\big(\nabla_u K, J\nabla_u W\big)(\chi(\theta_0,t),t+t_0)\ \mathrm{d}t = \int_{-\infty}^{+\infty}\{K,W\}(\chi(\ldots))\ \mathrm{d}t = 0 \quad (36)$$

where H_0 is the unperturbed Hamiltonian, and $K = x_2p_1 - x_1p_2$ is the angular momentum (which is indeed a constant of the motion for H_0). We can then say that, again and apart from the additional condition (18), just as in sect. **I**, our procedure provides an extension of these formulas to the nonhyperbolic degenerate case we are considering here.

Notice also that, as a consequence of the vanishing of the perturbation W at $t = \pm\infty$, the first Mel'nikov condition (33) can be written in the simpler form

$$M_1(\theta_0,t_0) = \int_{-\infty}^{+\infty}\frac{\partial W(R(t),\Theta(t)+\theta_0,t+t_0)}{\partial t}\mathrm{d}t = 0 \qquad (33')$$

where clearly the derivative $\partial/\partial t$ must be performed only with respect to the explicit time-dependence of W.

It is also clear that, when the perturbation is independent of θ, as in the cases considered in sect. **I**, one consistently gets that the first condition (33) becomes just (13), whereas the second one (34) is identically satisfied.

Let us now introduce the "Mel'nikov potential" $\mathcal{W} = \mathcal{W}(\theta_0,t_0)$ (cf. Ref. 32), corresponding to the perturbation $W(r,\theta,t)$:

$$\mathcal{W}(\theta_0,t_0) = \int_{-\infty}^{+\infty} W(R(t),\Theta(t)+\theta_0,t+t_0)\mathrm{d}t \qquad (37)$$

then one gets from this definition and from (33′-34)

$$M_1(\theta_0,t_0) = \frac{\partial\mathcal{W}}{\partial t_0} = 0 \qquad M_2(\theta_0,t_0) = \frac{\partial\mathcal{W}}{\partial\theta_0} = 0 \qquad (38)$$

In other words, the two Mel'nikov conditions are equivalent to the existence of stationary points for the Mel'nikov potential $\mathcal{W}(\theta_0, t_0)$. On the other hand, $\mathcal{W}$ is a smooth doubly-periodic function, and such a function certainly possesses points $\overline{\theta_0}, \overline{t_0}$ where the two partial derivatives in (38) vanish, and this implies that the two conditions (33′-34) are certainly satisfied.

Now, exactly the same arguments (and with analogous remarks) presented in sect. **I** show that the vanishing of the Mel'nikov functions entails the presence of a complicated dynamics, produced by the chain of the infinitely many intersections of the asymptotic sets: see, e.g., Ref.s 5,28 for a detailed description of this "multidimensional" case.

In this context, we can also consider the appearance of Arnol'd diffusion[14,33]. Indeed, the integrals of the Poisson brackets appearing in the Mel'nikov condition (35-36) give precisely the total amount of the "variations" produced by the perturbation, from $t = -\infty$ to $t = +\infty$, to the quantities H_0 and K along the homoclinic solution χ. Proceeding in a similar way as in Ref. 14, we can now look for the intersections of the asymptotic sets corresponding respectively for $t \to -\infty$ and for $t \to +\infty$ to *different* values k_1 and k_2 of the angular momentum K. Observing that for both these solutions the energy H_0 is the same, $H_0 = 0$, then – following a by now classical idea[14] – the above Mel'nikov conditions must be replaced by conditions of the form

$$M_1(\theta_0, t_0) = 0 \qquad\qquad M_2(\theta_0, t_0) + k_1 - k_2 = 0 \tag{39}$$

and, exactly as in Ref. 14, the conclusion is that, for $|k_1 - k_2|$ small enough, intersections occur even from homoclinic solutions corresponding to different values of K. This argument is then consistent with the occurence of Arnol'd diffusion: actually, a complete argument would necessitate a consideration of the role played, in the present "degenerate" situation, by the usual "non-resonance" conditions: anyway, we point out that, in the same situation, i.e. in the case of general 3-body problem, but using a different approach, the occurrence of Arnol'd diffusion has been discussed in great detail by Xia[17−18]. On the other hand, it is also known that the phenomenon of Arnol'd diffusion requires a quite delicate treatment: see Ref. 34 for a careful and updated discussion on this point.

Let us remark incidentally that the introduction of the above Mel'nikov potential $\mathcal{W}$ would be in general impossible if the perturbation is not Hamiltonian, i.e. if the problem (24) has the form of a general dynamical system

$$\dot{u} = J\nabla_u H_0 + \epsilon g(u,t) \equiv F(u) + \epsilon g(u,t) \tag{40}$$

where the perturbing term $g(u,t)$ (or $g(r,\dot{r},\theta,\dot{\theta},t)$) is "generic". Then in this case the above arguments cannot be repeated, and in particular the two Mel'nikov conditions, which can now be written in the general form

$$M_1(\theta_0,t_0) = \int_{-\infty}^{+\infty} \Big(\nabla_u H_0, g\Big)(\chi(\theta_0,t), t+t_0)\ \mathrm{d}t = 0 \tag{41}$$

$$M_2(\theta_0,t_0) = \int_{-\infty}^{+\infty} \Big(\nabla_u K, g\Big)(\chi(\theta_0,t), t+t_0)\ \mathrm{d}t = 0 \tag{42}$$

give two "unrelated" restrictions on θ_0, t_0, and one then remains with the problem of discovering if there are or not some $\overline{\theta_0}, \overline{t_0}$ which satisfy simultaneously both these conditions.

We can then summarize the above discussion in the following form.

Proposition 2. *Let us consider a perturbed Kepler problem with Hamiltonian (21) where W is a smooth, time-periodic function, vanishing at the infinity according to (18). Then, there is a chaotic behaviour (possibly giving rise also to Arnol'd diffusion), induced by an infinite sequence of intersections in the Poincaré section of the negatively and positively asymptotic sets of the critical point at the infinity. In the case where the perturbed problem has the form (40) with a non-Hamiltonian perturbation $g(u,t)$, the same result is true if there are some $\overline{\theta_0}, \overline{t_0}$ satisfying simultaneously the two conditions (41-42).*

III. Some applications and final remarks

We shortly consider here some applications of the discussion presented in sect. **II**.

As a first particular case, assume that the perturbation $W(r,\theta,t)$ is of the form

$$W = W(r,\ \alpha\theta + bt) \tag{43}$$

where α, b are arbitrary constants (the constant α should be clearly an integer, and $b \neq 0$), then the two conditions ($33'$-34) actually coincide; observing on the other hand that the Fourier expansion of such a W is a series as in (16) in terms of the single variable $\alpha\theta + bt$, then these two conditions take the same form as in (17), and the Mel'nikov function is a smooth periodic function of $\alpha\theta_0 + bt_0$, leading thus directly to the same conclusions obtained above for what concerns the existence of zeroes, and of their properties as well (it is really not a restriction to assume that W is zero mean-valued, cf. Ref. 13).

A specially important example of this situation is provided by the restricted circular 3-body problem, in this case indeed the perturbation is given by[16]

$$W = \frac{1}{r} - \frac{\cos(\theta - t)}{r^2} - \frac{1}{\sqrt{1 + r^2 + 2r\cos(\theta - t)}} \tag{44}$$

and therefore just one condition has to be considered. The presence of the chaos produced by the chain of intersections of the asymptotic sets then is automatically granted by our discussion. Notice that the above expression (44) is actually the first-order expansion of the full potential in terms of the parameter ϵ (which in this case is given by the mass ratio μ between two celestial bodies), but also the exact expression of this potential, as given in Ref. 16, is in fact a function of $\theta - t$ only.

It is completely clear that, in the presence of a more general perturbation, e.g. of the form (just to give an example)

$$W(r,\theta,t) = W_{(1)}(r,\theta - t) + W_{(2)}(r,\theta + t)$$

one now obtains two different Mel'nikov conditions, and the presence of a chaotic behaviour follows from the existence of simultaneous solutions $\overline{\theta_0}, \overline{t_0}$, which is ensured by the arguments shown in sect. **II**.

The above results hold essentially unchanged if the perturbation depends on two (or more) parameters $\epsilon_1, \epsilon_2, \ldots$, i.e. if one assumes that W may be written in the form (cf. Ref. 29)

$$W = \epsilon_1 W_{(1)}(r,\theta,t) + \epsilon_2 W_{(2)}(r,\theta,t) + \ldots \tag{45}$$

A natural example is provided by the planar 3-body problem, where one has to deal, in the more general elliptic case, with a quite complicated expression of the perturbation containing three parameters (different masses and eccentricity[17−18]). Let us remark, however, that, at least in the simpler case of restricted elliptic problem, in which one has two parameters ($\epsilon_1 = \mu$ is one mass ratio and $\epsilon_2 = e$ the eccentricity), the perturbation cannot be written as in the above "first-order" form (45), but rather it takes the form[17]

$$W = \mu(W_{(1)}(r,\theta,t) + e\ W_{(2)}(r,\theta,t)) \tag{46}$$

in which the eccentricity plays the role of a "second-order" perturbation. On the basis of our previous arguments, we can just say that the presence of zeroes of the Mel'nikov functions, as obtained above for the circular case $\epsilon_2 = e = 0$, cannot be destroyed by the higher-order perturbation due to the eccentricity, and therefore chaos should be expected to persist in the elliptic case, whereas Arnol'd diffusion should appear as a second-order effect. A complete study of the general 3-body problem, and a full discussion of its chaotic properties, including the appearance of Arnol'd diffusion, together with several other dynamical features, is given in Ref.s 16-18.

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