OPERATOR-NORM CONVERGENCE OF THE TROTTER PRODUCT FORMULA FOR HOLOMORPHIC SEMIGROUPS

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ABSTRACT. We study the error estimates in operator norm for the Trotter product formula. It is shown that some recent results in this direction can be extended to non-self-adjoint generators and to the case of a Banach space for a class of holomorphic semigroups.

KEYWORDS : Trotter product formula, operator-norm convergence, holomorphic semigroups.

AMS SUBJECT CLASSIFICATION: 47D03, 47B25, 35K22, 41A80.

1 Introduction

Recently the operator-norm convergence and the error bound estimates were obtained for the Trotter and the Trotter-Kato product formulae in the abstract context of semigroups in a Hilbert space with self-adjoint generators ([11] [6] [9] [10]), as well as for the Schrödinger semigroups ([3] [4] [5] [7]).

Let A and B be non-negative self-adjoint operators on the Hilbert space $\mathcal{H}$. Then some of these abstract results can be formulated as follows (cf [9]): if B is A-bounded with a relative bound $a < 1$ and $A \geq I$, then for the Trotter formula one gets the operator-norm estimate

$$\left\| \left(e^{-tA/n}e^{-tB/n}\right)^n - e^{-t(A+B)} \right\| \leq C\frac{\ln n}{n} \tag{1}$$

uniformly in t for any bounded interval $[0, T]$.

The aim of the present note is to extend the class of semigroups for which we have the operator-norm convergence of the Trotter product formula. This extension is accomplished in two directions. First, for the case of a Hilbert space $\mathcal{H}$ when B is an unbounded *not* self-adjoint operator, see section 3. Second, for a Banach space $\mathcal{B}$; our main result concerns the bounded operators B and it is stated in section 4.

For the both of our results the properties of holomorphic semigroups are essential. So we begin with some preliminary statements concerning bounded and holomorphic semigroups.

2 Preliminaries

As a starting point we recall the following theorem (see e.g. [12]), which relates a strongly continuous semigroup $Q(t)$ with the resolvent $R_\lambda(A)$ of its generator A.

Proposition 1 *If $Q(t)_{t\geq 0}$ is a strongly continuous semigroup on a Banach space $\mathcal{B}$, then: there exists constants $C_A \geq 1$ and $\gamma \in \mathbb{R}$ such that*

$$\forall t,\ 0 \leq t < \infty, \quad \|Q(t)\| \leq C_A e^{\gamma t}; \tag{2}$$

for $\lambda \in \mathbb{C},\ Re(\lambda) > \gamma$, the integral

$$R_\lambda(A)x = \int_0^{+\infty} e^{-\lambda t} Q(t) x dt, \qquad x \in \mathcal{B} \tag{3}$$

defines a bounded operator, with the range $\mathcal{D}(A)$ such that $R_\lambda(A) = (\lambda I + A)^{-1}$ and $\|R_\lambda(A)\| \leq C_A / Re(\lambda - \gamma)$, where the operator A is densely defined and closed.

Remark 1 If $\gamma \leq 0$, $Q(t)$ is called a bounded semigroup (otherwise, $Q(t)$ is called a quasi-bounded semigroup of type $\gamma > 0$). For any strongly continuous semigroup, we can construct a bounded semigroup by adding some constant $\eta \geq \gamma$ to its generator. Let $\tilde{A} = A + \eta I$, then for the semigroup $\tilde{Q}(t)$ generated by $\tilde{A}$, one has $\|\tilde{Q}(t)\| \leq C_A$, $t \geq 0$, and the open half-plane $\mathrm{Re}(\lambda) > \gamma - \eta$ is included into the resolvent set $\rho(\tilde{A})$ of $\tilde{A}$. So it is not restrictive to suppose that the considered semigroup $Q(t)$ is bounded and that $\{z \in \mathbb{C},\ \mathrm{Re}(z) \geq 0\} \subset \rho(A)$.
Remark 2 If $\|Q(t)\| \leq 1$, $t \geq 0$, the semigroup is called a contraction semigroup. It is clear that the method of the preceding remark does not permit to construct a contraction semigroup from a bounded semigroup.

The following observations will be usefull for below, see section 3.

Lemma 1 *Let $Q(t)$ be a bounded semigroup with boundedly invertible generator A, then for all $t \geq 0$, and for any $n \geq 0$, we have the estimate:*

$$\left\| \left(Q(t) - \sum_{k=0}^{n} \frac{(-tA)^k}{k!} \right) A^{-n-1} \right\| \leq C_A \frac{t^{n+1}}{(n+1)!}. \tag{4}$$

Proof.

Since $0 \in \rho(A)$, we can use (3) for $\lambda = 0$ and $x \in \mathcal{B}$ to obtain:

$$(Q(t) - I)A^{-1}x = -\int_0^t Q(\tau) x d\tau \tag{5}$$

Therefore, $\|Q(t)\| \leq C_A$ implies the estimates for $n = 0$. Suppose that (4) holds for an integer n. Then by the representation

$$\left(Q(t) - \sum_{k=0}^{n+1} \frac{(-tA)^k}{k!} \right) A^{-n-2} = -\int_0^t \left(Q(\tau) - \sum_{k=0}^{n} \frac{(-\tau A)^k}{k!} \right) A^{-n-1} d\tau \tag{6}$$

and (4) one extends the estimate to $n+1$. Hence by induction, the result holds for any integer n.

□

Similarly, we obtain a representation for a restricted development of $(I+A)^{-1}$.

Lemma 2 *Let A be as in Lemma 1. Then for any $n \geq 0$:*

$$(I + A)^{-1} A^{-n-1} = \left(\sum_{k=0}^{n} (-A)^k \right) A^{-n-1} + (-1)^{n+1} (I + A)^{-1} \tag{7}$$

Proof.

For $n = 0$, it follows from the resolvent formula:

$$(I + A)^{-1} - A^{-1} = -(I + A)^{-1}A^{-1}. \tag{8}$$

Suppose that (7) holds for an integer n, then:

$$(I + A)^{-1}A^{-n-2} = \left(\sum_{k=0}^{n}(-A)^k\right) A^{-n-2} + (-1)^{n+1}(I + A)^{-1}A^{-1} \tag{9}$$

Applying (8) to the last term we get the representation (7) for $n + 1$, and thus for any n by induction. □

Lemma 3 *If $Q(t)$ is a bounded semigroup with boundedly invertible generator A then :*

$$\left\| \frac{1}{t^2} \left((I + tA)^{-1} - Q(t)\right) A^{-2} \right\| \leq 3C_A/2, \quad \forall t > 0 \tag{10}$$

Proof.

By Lemma 1 one gets

$$\|(Q(t) - I + tA)\frac{1}{t^2}A^{-2}\| \leq \frac{C_A}{2} \tag{11}$$

On the other hand by Lemma 2, we have

$$\left\| \left((I + tA)^{-1} - I + tA\right) \frac{1}{t^2}A^{-2} \right\| = \|(I + tA)^{-1}\| \leq C_A \tag{12}$$

Here the last estimate follows from $(I + tA)^{-1} = (1/t)R_{1/t}(A)$ and $\|R_\lambda(A)\| \leq C_A/(\lambda + \delta)$, $\delta > 0$, for bounded semigroups with boundedly invertible generators, see Remark 1. Hence (10) follows from (11) and (12). □

Now let $U(z)$ be a family of operators with z taking complex values in the sector:

$$S_\omega = \{z \in \mathbb{C}, \ \mathrm{Re}(z) > 0 \text{ and } |\arg(z)| < \omega\} \tag{13}$$

where $0 < \omega \leq \pi/2$.

Definition 1 (cf [2]) *We define a bounded holomorphic semigroup of angle ω on a Banach space $\mathcal{B}$ to be a family of bounded operators $U(z)$ where $z \in S_\omega$, and satisfying the following conditions:*

(i) $U(z_1)U(z_2) = U(z_1 + z_2)$ for all $z_1, z_2 \in S_\omega$.
(ii) If $0 < \epsilon < \omega$, then, $\|U(z)\| \leq M_\epsilon$ for all $z \in S_{\omega-\epsilon}$ and some $M_\epsilon < \infty$.
(iii) $U(z)$ is an analytic function of $z \in S_\omega$.
(iv) If $x \in \mathcal{B}$, and $0 < \epsilon < \omega$, then $\lim_{z\to 0} U(z)x = x$ provided $z \in S_{\omega-\epsilon}$.

Let $\sigma(A) = \mathbb{C} \setminus \rho(A)$ denote the spectrum of A. We mention the following characterisation of generators of holomorphic semigroups (see e.g. [2, Theorems 2.33 and 2.34]):

Proposition 2 *Operator A on a Banach space $\mathcal{B}$ is the generator of a bounded holomorphic semigroup of angle $\omega \leq \pi/2$ if and only if A is a closed operator with a dense domain such that:*

$$\exists 0 < \omega \leq \frac{\pi}{2},\ \sigma(A) \subset \left\{ w \in \mathbb{C},\ |\arg(w)| \leq \frac{\pi}{2} - \omega \right\} \tag{14}$$

and

$$\forall \epsilon > 0,\ \exists N_\epsilon \geq 0,\ \forall w \in S_{\omega-\epsilon+\pi/2},\ \|(w+A)^{-1}\| \leq N_\epsilon |w|^{-1} \tag{15}$$

We shall use below the following property which is an alternative characterisation of these semigroups, see e.g. [2, Theorems 2.38 and 2.39]:

Proposition 3 *If $U(z)$ is a bounded holomorphic semigroup of angle ω with generator A, then for all $z \in S_\omega$, $U(z)\mathcal{B} \subseteq \mathcal{D}(A)$, and for $0 < \epsilon < \omega$, there is a constant $C_\epsilon > 0$ such that:*

$$\|AU(z)\| \ \leq \ C_\epsilon |z|^{-1} \tag{16}$$

for all $z \in S_{\omega-\epsilon}$.

In fact, the constant can be determined explicitly, by $C_\epsilon = M_{\epsilon/2}/\sin(\omega - \epsilon/2 - |\arg(z)|)$.

Remark 3 Similar to strongly continuous semigroups, a family $U(z)$, $z \in S_\omega$ is called a quasi-bounded holomorphic semigroup of angle ω if there exists a constant $\gamma > 0$ such that $e^{-\gamma Re(z)} U(z)$ is a bounded holomorphic semigroup of angle ω.

We are also interested in perturbations of these semigroups. If B is bounded, the answer is simple: if A is generator of a contraction semigroup, then $A+B+\eta I$, $\eta \geq \|B\|$, is also generator of a contraction semigroup; if A is generator of a quasi-bounded holomorphic semigroup of angle ω, the same is true for $A+B$. If B is only relatively bounded with respect to A, we cite the following result ([8, Ch. IX]):

Proposition 4 *For any A, generator of a bounded holomorphic semigroup in S_ω, and $0 < \epsilon < \omega$, there exists a positive constant δ_ϵ with the following property. If B is relatively bounded with respect to A so that:*

$$\forall x \in \mathcal{D}(A) \subset \mathcal{D}(B), \quad \|Bx\| \leq a\|Ax\| \tag{17}$$

with $a < \delta_\epsilon$, then $A+B$ is the generator of a bounded holomorphic semigroup in $S_{\omega-\epsilon}$.

For the perturbation of contraction semigroups we have (see e.g. [2, Theorem 3.7 and Corollary 3.8]):

Proposition 5 *Let A and B be generators of contraction semigroups. If B is relatively bounded with respect to A with A-bound $a < 1$, then $A+B$ is generator of a contraction semigroup.*

Finally, the class of semigroups we shall consider below is:

Definition 2 *We say that $U(z)$, $z \in S_\omega$, is a holomorphic contraction semigroup of angle ω if it is a bounded holomorphic semigroup of angle ω, and its restriction on $\mathbb{R}^+$ is a contraction semigroup.*

Notice that this class of semigroups is not empty:

(i) Let $U(t)$, $t \in \mathbb{R}$, be a contraction semigroup with generator A on a Banach space $\mathcal{B}$, such that $U(t)\mathcal{B} \subseteq \mathcal{D}(A)$ for $t > 0$. If $\|AU(t)\| \leq ct^{-1}$ for some $c > 0$ and all $t > 0$, then there exists $\omega > 0$ such that $U(t)$ may be analytically continued to a bounded holomorphic semigroup of angle ω, see [2, Theorem 2.39].

(ii) Let A be a sectorial operator in a Hilbert space $\mathcal{H}$, i.e. its numerical range $\{(u, Au), u \in \mathcal{D}(A),$ and $\|u\| = 1\} \subset S_{\pi/2-\omega}$, $0 < \omega \leq \pi/2$. If A is closed, then it is the generator of a holomorphic contraction semigroup of angle ω, see [8, Ch. IX, Theorem 1.24].

3 Trotter product formula in a Hilbert space

The aim of this section is to prove the operator-norm convergence of the Trotter product formula in a separable Hilbert space $\mathcal{H}$ when the perturbation B is not self-adjoint. Without loss of generality (see Remarks 1-3) we suppose that generator A is a self-adjoint operator, $A \geq I$, which implies by Proposition 2 that e^{-tA} is a holomorphic contraction semigroup of angle $\pi/2$. For the operator B, we require that:

- B is the generator of a contraction semigroup (H1),
- there is a real $\alpha \in [0, 1)$ such that $\mathcal{D}(A^\alpha) \subset \mathcal{D}(B)$ and $\mathcal{D}(A^\alpha) \subset \mathcal{D}(B^*)$ (H2).

Remark 4 We note that the last assumption (introduced by Ichinose and Tamura in [6]) implies that B and B^* are relatively bounded with respect to A with the relative bound zero. Indeed, for $\eta > 0$ by $\mathcal{D}(A + \eta I) \subset \mathcal{D}(A^\alpha) \subset \mathcal{D}(B)$ and the spectral theorem one gets:

$$\|B(A + \eta I)^{-1}\| \leq \|BA^{-\alpha}\|\|A^\alpha(A + \eta I)^{-1}\| \leq \alpha^\alpha(1 + \eta)^{-(1-\alpha)}\|BA^{-\alpha}\| \qquad (18)$$

Since the operators A^α and B are closed, the inclusions (H2) are equivalent to A^α-boundedness of B and B^*. In particular, $\|BA^{-\alpha}\| \leq d$ and $\|B^*A^{-\alpha}\| \leq d'$ for some $d, d' > 0$. Therefore, for any $\epsilon > 0$, there exists $\eta > 0$ such that:

$$\forall x \in \mathcal{D}(\tilde{A}) \subset \mathcal{D}(B), \ \|Bx\| \leq \epsilon\|\tilde{A}x\| \qquad (19)$$

where we set $\tilde{A} = A + \eta I$. By Proposition 4 this implies that for η sufficiently large, $\tilde{A} + B$ is the generator of a bounded holomorphic semigroup. Therefore by Remark 3, $A + B$ is the generator of a quasi-bounded holomorphic semigroup. Furthermore since B is the generator of a contraction semigroup, by Proposition 5 the semigroup generated by $\tilde{A} + B$ is also a contraction.

The proof of the main theorem of this section involves three lemmata. For the two first we suppose only that B and B^* are A-bounded: there exist positive constants a and b such that:

$$\forall x \in \mathcal{D}(A) \subset \mathcal{D}(B), \ \|Bx\| \leq a\|Ax\| + b\|x\| \qquad (20)$$

$$\forall \phi \in \mathcal{D}(A^*) \subset \mathcal{D}(B^*), \ \|B^*\phi\| \leq a\|A^*\phi\| + b\|\phi\| \qquad (21)$$

If A is boundedly invertible, then we can set $b = 0$ with another constant a.

Lemma 4 *Let A (invertible) and B be generators of contraction semigroups. Let B and B^* be A-bounded as in (20), (21) with $a < 1$ and $b\|A^{-1}\| + a < 1$. Then operator $H = A + B$ with $\mathcal{D}(H) = \mathcal{D}(A)$ is the boundedly invertible generator of a contraction semigroup, and there exists a constant L_1 such that for all $\tau \geq 0$:*

$$\begin{aligned} \left\| A^{-1}\left(e^{-\tau B}e^{-\tau A} - e^{-\tau(A+B)}\right)\right\| &\leq L_1\tau \quad (22) \\ \left\| \left(e^{-\tau B}e^{-\tau A} - e^{-\tau(A+B)}\right)A^{-1}\right\| &\leq L_1\tau \quad (23) \end{aligned}$$

Proof.

The first part of the statement follows from Proposition 5 and the standard Kato stability theorems about bounded invertibility [8].

In virtue of the identity

$$\begin{aligned} A^{-1}\left(e^{-\tau B}e^{-\tau A} - e^{-\tau(A+B)}\right) &= A^{-1}BB^{-1}\left(e^{-\tau B} - I\right)e^{-\tau A} \\ &+ A^{-1}\left(e^{-\tau A} - I\right) + A^{-1}HH^{-1}\left(I - e^{-\tau H}\right) \end{aligned}$$

and Lemma 1 we get (22):

$$\begin{aligned} \left\| A^{-1}\left(e^{-\tau B}e^{-\tau A} - e^{-\tau H}\right)\right\| &\leq \|A^{-1}B\| \left\| B^{-1}\left(e^{-\tau B} - I\right)\right\| \\ &+ \left\| A^{-1}\left(e^{-\tau A} - I\right)\right\| + \|A^{-1}H\| \left\| H^{-1}\left(I - e^{-\tau H}\right)\right\| \\ &\leq \|A^{-1}B\|\tau + \tau + \|A^{-1}H\|\tau \\ &\leq 2(1 + a + b\|A^{-1}\|)\tau \end{aligned}$$

In the last inequality we used that (21) implies the boundedness of the closed operator $A^{-1}B$, and that $\|A^{-1}H\| \leq \|I + A^{-1}B\| \leq 1 + a + b\|A^{-1}\|$. To prove (23) one has to use (20), and finally to put $L_1 = 2(1 + a + b\|A^{-1}\|)$. In fact one can relax the demand of contraction in this lemma. This changes only coefficients in the estimates above. □

Lemma 5 *Let A, B and $H = A + B$ be the same as in Lemma 4. Then there exists a constant L_2 such that for all $\tau \geq 0$:*

$$\begin{aligned} \left\| A^{-1}\left(e^{-\tau B}e^{-\tau A} - e^{-\tau(A+B)}\right)A^{-1}\right\| &\leq L_2\tau^2 \quad (24) \\ \left\| A^{-1}\left(e^{-\tau A}e^{-\tau B} - e^{-\tau(A+B)}\right)A^{-1}\right\| &\leq L_2\tau^2 \quad (25) \end{aligned}$$

Proof.

In virtue of

$$\begin{aligned} e^{-\tau B}e^{-\tau A} - e^{-\tau H} &= \left(I - e^{-\tau B}\right)\left(I - e^{-\tau A}\right) \\ &+ \left(e^{-\tau A} - (I + \tau A)^{-1}\right) + \left(e^{-\tau B} - (I + \tau B)^{-1}\right) + \left((I + \tau H)^{-1} - e^{-\tau H}\right) \\ &+ \tau H(I + \tau H)^{-1} - \tau A(I + \tau A)^{-1} - \tau B(I + \tau B)^{-1} \end{aligned}$$

and the identity

$$\begin{aligned} &A^{-1}\left(\tau H(I + \tau H)^{-1} - \tau A(I + \tau A)^{-1} - \tau B(I + \tau B)^{-1}\right)A^{-1} \\ = \; &\tau^2\left((I + \tau A)^{-1} + A^{-1}B(I + \tau B)^{-1}BA^{-1} - A^{-1}H(I + \tau H)^{-1}HA^{-1}\right) \end{aligned}$$

we get the representation

$$
\begin{aligned}
&A^{-1}(e^{-\tau B}e^{-\tau A}-e^{-\tau H})A^{-1}= \\
&\quad A^{-1}BB^{-1}(I-e^{-\tau B})(I-e^{-\tau A})A^{-1} \\
&\quad +\left(e^{-\tau A}-(I+\tau A)^{-1}\right)A^{-2}+A^{-1}B\left(e^{-\tau B}-(I+\tau B)^{-1}\right)B^{-2}BA^{-1} \\
&\quad +A^{-1}H\left((I+\tau H)^{-1}-e^{-\tau H}\right)H^{-2}HA^{-1}+\tau^2(I+\tau A)^{-1} \\
&\quad +\tau^2A^{-1}B(I+\tau B)^{-1}BA^{-1}-\tau^2A^{-1}H(I+\tau H)^{-1}HA^{-1}
\end{aligned}
$$

which gives the following estimate:

$$
\begin{aligned}
&\frac{1}{\tau^2}\left\|A^{-1}(e^{-\tau B}e^{-\tau A}-e^{-\tau H})A^{-1}\right\| \leq \\
&\qquad \frac{1}{\tau}\|A^{-1}B\|\left\|(I-e^{-\tau B})B^{-1}\right\|\frac{1}{\tau}\left\|(I-e^{-\tau A})A^{-1}\right\| \\
&\qquad +\frac{1}{\tau^2}\left\|\left(e^{-\tau A}-(I+\tau A)^{-1}\right)A^{-2}\right\| \\
&\qquad +\frac{1}{\tau^2}\|A^{-1}B\|\left\|\left(e^{-\tau B}-(I+\tau B)^{-1}\right)B^{-2}\right\|\|BA^{-1}\| \\
&\qquad +\frac{1}{\tau^2}\|A^{-1}H\|\left\|\left((I+\tau H)^{-1}-e^{-\tau H}\right)H^{-2}\right\|\|HA^{-1}\| \\
&\qquad +1+\|A^{-1}B\|\|BA^{-1}\|+\|A^{-1}H\|\|HA^{-1}\|
\end{aligned}
$$

Now Lemmata 1 and 3 together with (20), (21) imply (24): we can take $L_2=\frac{a'}{2}+\frac{a'}{2}+\frac{3}{2}+\frac{3}{2}a'^2+\frac{3}{2}a'^2+1+a'^2+(1+a')^2$ with $a'=a+b\|A^{-1}\|$. Similarly one gets (25).

□

In the next lemma we suppose the Ichinose-Tamura conditions (H2) and without loss of generality, see Remarks 1-3, we put $A\geq I$.

Lemma 6 *Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$, with $A\geq I$. If B is the boundedly invertible generator of a contraction semigroup and there exists $\alpha\geq 0$ such that $\mathcal{D}(A^\alpha)\subset\mathcal{D}(B)$, then for any $k\geq 1$ and $\tau>0$:*

$$\left\|\left(e^{-\tau B}e^{-\tau A}\right)^k A\right\| \leq \frac{L_3}{\tau^\alpha}+\frac{1}{ek\tau}\ ,\ \alpha>0 \tag{26}$$

$$\left\|\left(e^{-\tau B}e^{-\tau A}\right)^k A\right\| \leq \tilde{L}_3(1+\ln k)+\frac{1}{ek\tau}\ ,\ \alpha=0 \tag{27}$$

Proof.

$$
\begin{aligned}
\left\|\left(e^{-\tau B}e^{-\tau A}\right)^k A\right\| &\leq \left\|\left(\left(e^{-\tau B}e^{-\tau A}\right)^k-e^{-k\tau A}\right)A\right\|+\left\|e^{-k\tau A}A\right\| \\
&\leq \left\|\sum_{j=0}^{k-1}\left(e^{-\tau B}e^{-\tau A}\right)^{k-1-j}\left(e^{-\tau B}-I\right)e^{-\tau A}e^{-j\tau A}A\right\|+\left\|e^{-k\tau A}A\right\| \\
&\leq \left\|\left(e^{-\tau B}-I\right)B^{-1}\right\|\sum_{j=0}^{k-1}\|BA^{-\alpha}\|\left\|A^\alpha e^{-(j+1)\tau A}A\right\|+\left\|e^{-k\tau A}A\right\|
\end{aligned}
$$

¿From the hypothesis $\mathcal{D}(A^\alpha)\subset\mathcal{D}(B)$ we deduce (cf Remark 4) that $\|BA^{-\alpha}\|\leq d$. By the spectral theorem for self-adjoint operators we get:

$$\left\|A^{1+\alpha}e^{-(j+1)\tau A}\right\| \leq \left(\frac{1+\alpha}{\tau e(j+1)}\right)^{1+\alpha} \quad \text{and} \quad \left\|e^{-k\tau A}A\right\| \leq \frac{1}{ek\tau}$$

Therefore, using Lemma 1, we conclude:

$$\left\|\left(e^{-\tau B}e^{-\tau A}\right)^k A\right\| \leq \tau d \frac{(1+\alpha)^{1+\alpha}}{(e\tau)^{1+\alpha}} \sum_{j=0}^{k-1} \frac{1}{(j+1)^{1+\alpha}} + \frac{1}{ek\tau}.$$

Since $\alpha > 0$, this gives the annonced result (26) with

$$L_3 = d(1+\alpha)^{1+\alpha} e^{-(1+\alpha)} \sum_{j=1}^{\infty} (1/j)^{1+\alpha},$$

and (27) for $\alpha = 0$ with $\tilde{L}_3 = \|B\|/e$.

□

Theorem 1 *Let A be a self-adjoint bounded from below operator on a Hilbert space $\mathcal{H}$. If B is the generator of a contraction semigroup, and there exists $\alpha \in [0,1)$ such that B and B^* are A^α-bounded, then there are constants $M_1, M_2, \eta > 0$, such that for any $t \geq 0$ and $n > 2$:*

$$\left\|\left(e^{-tB/n}e^{-tA/n}\right)^n - e^{-t(A+B)}\right\| \leq (M_1 + M_2 t^{1-\alpha}) e^{\eta t} \frac{\ln n}{n^{1-\alpha}}, \quad \alpha > 0 \qquad (28)$$

$$\left\|\left(e^{-tB/n}e^{-tA/n}\right)^n - e^{-t(A+B)}\right\| \leq (M_1 + \tilde{M}_2 t) e^{\eta t} \frac{2(\ln n)^2}{n}, \quad \alpha = 0 \qquad (29)$$

Proof.

Let $\tilde{A} = A + \eta_1 I$, $\tilde{B} = B + \eta_2 I$ and $\tilde{H} = \tilde{A} + \tilde{B}$. By Remarks 1 and 2, $\tilde{B}$ is boundedly invertible for any $\eta_2 > 0$. Now if η_1 is sufficiently large, $\tilde{A} \geq I$ and $\tilde{H}$ is boundedly invertible as well as $\|\tilde{B}\tilde{A}^{-1}\| \leq a < 1$, see Remark 4. Then $e^{-t\tilde{A}}$ is a holomorphic contraction semigroup of angle $\pi/2$, and by Propositions 4 and 5 $e^{-t\tilde{H}}$ is a holomorphic contraction semigroup of angle $\pi/2 - \epsilon$ for some $\epsilon < \pi/2$. Notice that $\|\tilde{A}\tilde{H}^{-1}\| \leq \frac{1}{1-a}$.

We put $\tau = t/n$, $\tilde{U}(t) = e^{-t\tilde{H}}$, and $\tilde{T}(\tau) = e^{-\tau\tilde{B}}e^{-\tau\tilde{A}}$. To estimate the left-hand side of (28) we use

$$\left(e^{-tB/n}e^{-tA/n}\right)^n - e^{-t(A+B)} = (\tilde{T}^n(\tau) - \tilde{U}^n(\tau))e^{t(\eta_1+\eta_2)}$$

and identity:

$$\begin{aligned}
\tilde{T}(\tau)^n - \tilde{U}(\tau)^n &= \sum_{m=0}^{n-1} \tilde{T}(\tau)^{n-m-1}(\tilde{T}(\tau) - \tilde{U}(\tau))\tilde{U}(\tau)^m \\
&= \tilde{T}(\tau)^{n-1}\tilde{A}\tilde{A}^{-1}(\tilde{T}(\tau) - \tilde{U}(\tau)) \\
&\quad + (\tilde{T}(\tau) - \tilde{U}(\tau))\tilde{A}^{-1}\tilde{A}\tilde{H}^{-1}\tilde{H}\tilde{U}(\tau)^{n-1} \\
&\quad + \sum_{m=1}^{n-2} \tilde{T}(\tau)^{n-m-1}\tilde{A}\tilde{A}^{-1}(\tilde{T}(\tau) - \tilde{U}(\tau))\tilde{A}^{-1}\tilde{A}\tilde{H}^{-1}\tilde{H}\tilde{U}(\tau)^m
\end{aligned}$$

which implies :

$$\left\|\tilde{T}(\tau)^n - \tilde{U}(\tau)^n\right\| \leq \|\tilde{T}(\tau)^{n-1}\tilde{A}\|\|\tilde{A}^{-1}(\tilde{T}(\tau) - \tilde{U}(\tau))\|$$
$$+\|(\tilde{T}(\tau) - \tilde{U}(\tau))\tilde{A}^{-1}\|\|\tilde{A}\tilde{H}^{-1}\|\|\tilde{H}\tilde{U}(\tau)^{n-1}\|$$
$$+\sum_{m=1}^{n-2} \|\tilde{T}(\tau)^{n-m-1}\tilde{A}\|\|\tilde{A}^{-1}(\tilde{T}(\tau) - \tilde{U}(\tau))\tilde{A}^{-1}\|\|\tilde{A}\tilde{H}^{-1}\|\|\tilde{H}\tilde{U}(\tau)^m\|$$

Hence by Lemmata 4, 5, and 6, and Proposition 3 we obtain the estimate :

$$\begin{aligned}
\|\tilde{T}(\tau)^n - \tilde{U}(\tau)^n\| \leq & \left(\frac{L_3}{\tau^\alpha} + \frac{1}{(n-1)e\tau}\right) L_1\tau + \frac{L_1}{1-a}\frac{C_\epsilon}{n-1} \\
& + \sum_{m=1}^{n-2}\left(L_3\tau^{1-\alpha} + \frac{1/e}{n-m-1}\right)\frac{L_2}{1-a}\frac{C_\epsilon}{m} \\
\leq & \; L_3L_1\frac{t^{1-\alpha}}{n^{1-\alpha}} + \frac{L_1}{n-1}\left(1/e + \frac{C_\epsilon}{1-a}\right) + \frac{L_3L_2C_\epsilon}{1-a}\frac{t^{1-\alpha}}{n^{1-\alpha}}\sum_{m=1}^{n-2}\frac{1}{m} \\
& + \frac{L_2C_\epsilon/e}{1-a}\sum_{m=1}^{n-2}\frac{1}{n-m-1}\frac{1}{m} \\
\leq & \; L_3L_1\frac{t^{1-\alpha}}{n^{1-\alpha}} + \frac{L_1}{n-1}\left(1/e + \frac{C_\epsilon}{1-a}\right) \\
& + 2\frac{L_3L_2C_\epsilon}{1-a}t^{1-\alpha}\frac{\ln n}{n^{1-\alpha}} + 4\frac{L_2C_\epsilon/e}{1-a}\frac{\ln n}{n}
\end{aligned}$$

Here we used:

$$\sum_{m=1}^{n-1}\frac{1}{(n-m)m} = \frac{2}{n}\sum_{m=1}^{n-1}\frac{1}{m} \leq \frac{2}{n}(1+\ln(n-1)) \leq 4\frac{\ln n}{n} \tag{30}$$

This estimate implies the annonced result (28) for $\alpha > 0$, with $M_1 = 4L_1\left(1/e + \frac{C_\epsilon}{1-a}\right) + 4\frac{L_2C_\epsilon/e}{1-a}$ and $M_2 = 2L_3L_1 + 2\frac{L_3L_2C_\epsilon}{1-a}$ for $\eta = \eta_1 + \eta_2$. In a similar way for $\alpha = 0$ one gets:

$$\begin{aligned}
\|\tilde{T}(\tau)^n - \tilde{U}(\tau)^n\| \;\leq\; & \tilde{L}_3(1+\ln(n-1))L_1\frac{t}{n} + \frac{L_1}{e(n-1)} + \frac{L_1C_\epsilon}{1-a}\frac{1}{n-1} \\
& + \sum_{m=1}^{n-2}\left(\tilde{L}_3\frac{t}{n}(1+\ln(n-m-1)) + \frac{1/e}{n-m-1}\right)\frac{L_2}{1-a}\frac{C_\epsilon}{m}
\end{aligned}$$

This estimate gives (29) with $\tilde{M}_2 = 2\tilde{L}_3L_1 + 2\frac{\tilde{L}_3L_2C_\epsilon}{1-a}$.

□

Corollary 1 *Under the assumptions of Theorem 1 we get*

$$\left\|\left(e^{-tA/n}e^{-tB/n}\right)^n - e^{-t(A+B)}\right\| \;\leq\; (M_3 + M_4t^{1-\alpha})e^{\eta t}\frac{\ln n}{n^{1-\alpha}}\;,\;\; \alpha > 0 \tag{31}$$
$$\left\|\left(e^{-tA/n}e^{-tB/n}\right)^n - e^{-t(A+B)}\right\| \;\leq\; (M_3 + \tilde{M}_4t)e^{\eta t}\frac{2(\ln n)^2}{n}\;,\;\; \alpha = 0 \tag{32}$$

for any $t \geq 0$ and $n > 2$.

Proof.

Let $\tilde{F}(\tau) = e^{-\tau\tilde{A}}e^{-\tau\tilde{B}}$, Then by the same arguments as above

$$\begin{aligned}
\tilde{U}(\tau)^n - \tilde{F}(\tau)^n &= \sum_{m=0}^{n-1} \tilde{U}(\tau)^{n-m-1}(\tilde{U}(\tau) - \tilde{F}(\tau))\tilde{F}(\tau)^m \\
&= \tilde{U}(\tau)^{n-1}\tilde{H}\tilde{H}^{-1}\tilde{A}\tilde{A}^{-1}(\tilde{U}(\tau) - \tilde{F}(\tau)) \\
&\quad +(\tilde{U}(\tau) - \tilde{F}(\tau))\tilde{A}^{-1}\tilde{A}\tilde{F}(\tau)^{n-1} \\
&\quad + \sum_{m=1}^{n-2} \tilde{U}(\tau)^{n-m-1}\tilde{H}\tilde{H}^{-1}\tilde{A}\tilde{A}^{-1}(\tilde{U}(\tau) - \tilde{F}(\tau))\tilde{A}^{-1}\tilde{A}\tilde{F}(\tau)^m
\end{aligned}$$

Notice that the Lemmata 4 and 5 hold for $\tilde{F}(\tau)$. By a simple modification of Lemma 6 we get:

$$\begin{aligned}
\left\|\tilde{A}\left(e^{-\tau\tilde{A}}e^{-\tau\tilde{B}}\right)^k\right\| &\leq \frac{L_4}{\tau^\alpha} + \frac{1}{ek\tau}\ , \ \alpha > 0 \\
\left\|\tilde{A}\left(e^{-\tau\tilde{A}}e^{-\tau\tilde{B}}\right)^k\right\| &\leq \tilde{L}_4(1 + \ln k) + \frac{1}{ek\tau}\ , \ \alpha = 0
\end{aligned}$$

These ingredients ensure the estimates (31) and (32). □

Corollary 2 *Under the same conditions as above, we have the operator-norm convergence of a symmetrized Trotter formula, i.e. there exists $M_5, M_6, \eta > 0$, such that for any $t \geq 0$ and $n > 2$:*

$$\left\|\left(e^{-tA/2n}e^{-tB/n}e^{-tA/2n}\right)^n - e^{-t(A+B)}\right\| \leq (M_5 + M_6 t^{1-\alpha})e^{\eta t}\frac{\ln n}{n^{1-\alpha}}\ , \ \alpha > 0 \quad (33)$$

$$\left\|\left(e^{-tA/2n}e^{-tB/n}e^{-tA/2n}\right)^n - e^{-t(A+B)}\right\| \leq (M_5 + \tilde{M}_6 t)e^{\eta t}\frac{2(\ln n)^2}{n}\ , \ \alpha = 0 \quad (34)$$

Proof.

Since Lemmata 4, 5, and 6 can be easily extend to the symmetrized product $e^{-\tau A/2}e^{-\tau B}e^{-\tau A/2}$, the proof of the Theorem 1 carries through verbatim to obtain (33) and (34). □

4 Trotter product formula in a Banach space

In the present section we study the convergence of Trotter product formula in the operator-norm topology for holomorphic semigroups on a Banach space. As above we suppose that A is the generator of a holomorphic contraction semigroup, but our method is efficacious only for a bounded perturbation B. Thus $A+B$ is the generator of a quasi-bounded holomorphic semigroup. Our main result is the estimate:

$$\left\|\left(e^{-tB/n}e^{-tA/n}\right)^n - e^{-t(A+B)}\right\| \leq O\left(\frac{(\ln n)^2}{n}\right),\ n \geq 2 \qquad (35)$$

uniformly in t for any compact subset of $[0, +\infty)$. The boundedness of B is essential for the next statement which is an adaptation of Lemma 6 to a Banach space.

Lemma 7 *If A is the generator of a holomorphic contraction semigroup and B is bounded, then for any $\tau > 0$ and $k \geq 1$:*

$$e^{-k\tau\|B\|} \left\| \left(e^{-\tau B} e^{-\tau A}\right)^k A \right\| \leq 2\|B\| C_\epsilon (1 + \ln(k)) + \frac{C_\epsilon}{k\tau} \tag{36}$$

Proof.

Notice that for $\tilde{B} = B + \|B\|$, $\|e^{-\tau \tilde{B}}\| \leq 1$ and

$$\begin{aligned}
\left\| \left(e^{-\tau \tilde{B}} e^{-\tau A}\right)^k A \right\| &\leq \left\| \left(\left(e^{-\tau \tilde{B}} e^{-\tau A}\right)^k - e^{-k\tau A} \right) A \right\| + \left\| e^{-k\tau A} A \right\| \\
&\leq \left\| \sum_{j=0}^{k-1} \left(e^{-\tau \tilde{B}} e^{-\tau A}\right)^{k-1-j} \left(e^{-\tau \tilde{B}} - I\right) e^{-\tau A} e^{-j\tau A} A \right\| + \left\| e^{-k\tau A} A \right\| \\
&\leq \left\| e^{-\tau \tilde{B}} - I \right\| \sum_{j=0}^{k-1} \left\| e^{-(j+1)\tau A} A \right\| + \left\| e^{-k\tau A} A \right\|
\end{aligned}$$

The rest is due to the Lemma 1 for a bounded $\tilde{B}$ and Proposition 3 for a bounded holomorphic semigroup:

$$\begin{aligned}
e^{-k\tau\|B\|} \| \left(e^{-\tau B} e^{-\tau A}\right)^k A \| &\leq \|\tilde{B}\| \sum_{j=0}^{k-1} \frac{C_\epsilon}{j+1} + \frac{C_\epsilon}{k\tau} \\
&\leq 2\|B\| C_\epsilon (1 + \ln(k)) + \frac{C_\epsilon}{k\tau}
\end{aligned}$$

□

Theorem 2 *Let A be the generator of a holomorphic contraction semigroup on a Banach space $\mathcal{B}$. If B is bounded, then there exists $N_1, N_2, \eta > 0$ such that for $t \geq 0$ and $n > 1$ we have the estimate:*

$$\left\| \left(e^{-tB/n} e^{-tA/n}\right)^n - e^{-t(A+B)} \right\| \leq (N_1 + N_2 t) e^{\eta t} \frac{(\ln n)^2}{n} \tag{37}$$

Proof.

Let again $\tilde{A} = A + \eta_1 I$, $\tilde{B} = B + \eta_2 I$, and $\tilde{H} = \tilde{A} + \tilde{B}$. As above we can choose $\eta_1, \eta_2 \geq 0$ such that: $\tilde{A}$, $\tilde{B}$, and $\tilde{H}$ are boundedly invertible, with $\|\tilde{B}\tilde{A}^{-1}\| = a < 1$, and $e^{-t\tilde{B}}$, $e^{-t\tilde{H}}$ become contractions, e.g. for $\eta_2 \geq \|B\|$. Let $\tau = t/n$, $\tilde{U}(t) = e^{-t\tilde{H}}$, and $\tilde{T}(\tau) = e^{-\tau\tilde{B}} e^{-\tau\tilde{A}}$. The starting point is the following identity:

$$\begin{aligned}
\tilde{T}(\tau)^n - \tilde{U}(\tau)^n &= \sum_{m=0}^{n-1} \tilde{T}(\tau)^{n-m-1} (\tilde{T}(\tau) - \tilde{U}(\tau)) \tilde{U}(\tau)^m \\
&= \tilde{T}(\tau)^{n-1} \tilde{A}\tilde{A}^{-1} (\tilde{T}(\tau) - \tilde{U}(\tau)) \\
&\quad + (\tilde{T}(\tau) - \tilde{U}(\tau)) \tilde{A}^{-1} \tilde{A} \tilde{H}^{-1} \tilde{H} \tilde{U}(\tau)^{n-1} \\
&\quad + \sum_{m=1}^{n-2} \tilde{T}(\tau)^{n-m-1} \tilde{A}\tilde{A}^{-1} (\tilde{T}(\tau) - \tilde{U}(\tau)) \tilde{A}^{-1} \tilde{A} \tilde{H}^{-1} \tilde{H} \tilde{U}(\tau)^m
\end{aligned}$$

Since $a < 1$, one gets $\|\tilde{A}\tilde{H}^{-1}\| = \|(I + \tilde{B}\tilde{A}^{-1})^{-1}\| \leq \frac{1}{1-a}$. Then by Lemmata 4, 5, 7 and Proposition 3 we obtain:

$$\begin{aligned}\|\tilde{T}(\tau)^n - \tilde{U}(\tau)^n\| &\leq \|\tilde{B}\|C_\epsilon(1+\ln(n-1))L_1\frac{t}{n} + \frac{L_1 C_\epsilon}{n-1} + \frac{L_1 C'_\epsilon}{1-a}\frac{1}{n-1} \\ &+ \sum_{m=1}^{n-2}\left(\|\tilde{B}\|C_\epsilon\frac{t}{n}(1+\ln(n-m-1)) + \frac{C_\epsilon}{n-m-1}\right)\frac{L_2}{1-a}\frac{C'_\epsilon}{m}\end{aligned}$$

By (30) we get

$$\begin{aligned}\|\tilde{T}(\tau)^n - \tilde{U}(\tau)^n\| &\leq 2tL_1C_\epsilon\|\tilde{B}\|\frac{\ln(n)}{n} + \left(C_\epsilon + \frac{C'_\epsilon}{1-a}\right)\frac{2L_1}{n} \\ &+ \frac{4L_2C_\epsilon C'_\epsilon t}{1-a}\|\tilde{B}\|\frac{\ln^2 n}{n} + \frac{4L_2C_\epsilon C'_\epsilon}{1-a}\frac{\ln(n)}{n} \\ &\leq (N_1+N_2t)\frac{\ln^2 n}{n},\end{aligned}$$

where we set $N_1 = \left(C_\epsilon + \frac{C'_\epsilon}{1-a}\right)8L_1 + \frac{8L_2C_\epsilon C'_\epsilon}{1-a}$ and $N_2 = 4\|\tilde{B}\|C_\epsilon\left(L_1 + \frac{L_2C'_\epsilon}{1-a}\right)$. Finally, due to identity

$$\left(e^{-tB/n}e^{-tA/n}\right)^n - e^{-t(A+B)} = (\tilde{T}^n(\tau) - \tilde{U}^n(\tau))e^{t(\eta_1+\eta_2)} \tag{38}$$

we obtain the annonced result for $\eta = \eta_1 + \eta_2$.

□

Corollary 3 *Under the same conditions as in Theorem 2, there exists $N_3, N_4, \eta > 0$ such that for $t \geq 0$ and $n > 1$:*

$$\left\|\left(e^{-tA/n}e^{-tB/n}\right)^n - e^{-t(A+B)}\right\| \leq (N_3+N_4t)e^{\eta t}\frac{(\ln n)^2}{n} \tag{39}$$

Proof.

The proof follows the same line of reasoning as above with modifications explained in Corollary 1. □

Corollary 4 *The operator-norm convergence of the symmetrized version of the Trotter product formula takes place under the same hypothesis and with the same rate:*

$$\left\|\left(e^{-tA/2n}e^{-tB/n}e^{-tA/2n}\right)^n - e^{-t(A+B)}\right\| \leq (N_5+N_6t)e^{\eta' t}\frac{(\ln n)^2}{n} \tag{40}$$

5 Conclusion

Up to our knowledge, all the preceding results concerning operator-norm convergence of the Trotter or Trotter-Kato product formulae require non-negative self-adjoint generators A and B on a Hilbert space $\mathcal{H}$. This means holomorphic contraction semigroups of angle $\omega = \pi/2$. The evident advantage for estimates is that in this case one can widely exploit the spectral theorem, see [9]-[11], and e.g. Lemma 6. It seems however that such an assumption is not essential: the main result of section 3 shows that we get similar estimate for holomorphic contraction semigroups if one relaxes self-adjointness of B but demands its A^α-smallness for $\alpha \in [0, 1)$, see (H2). In a Banach space we generalize the Trotter-Chernoff classical results [13], [1] to the operator-norm convergence only for bounded B and holomorphic contraction semigroups. The hypothesis of boundedness of B is restrictive, but in the case of se! ! lf-adjoint operators on $\mathcal{H}$, no better estimation is known when B is bounded, cf [9], [10]. Finally we would like to remark that demand of contraction is not as superfluous as one could suppose. For demonstration we address the reader to an instructive example by Trotter [13] where it is called "the norm condition".

The results of the section 3 could find some applications in quantum mechanics and in the theory of Schrödinger semigroups [3-6]: $H = -\Delta + \lambda V$ with a real potential V and a complex coupling constant λ. If V relatively bounded with respect to $(-\Delta)^\alpha$ for some $\alpha \in [0, 1)$, then these operators satisfy the hypothesis of the Theorem 1.

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