Computer–Assisted Proofs for Fixed Point Problems in Sobolev Spaces

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Abstract. In this paper we extend the technique of computer–assisted proofs to fixed point problems in Sobolev spaces. Up to now the method was limited to the case of spaces of analytic functions. The possibility to work with Sobolev spaces is an important progress and opens up many new domains of applications. Our discussion is centered around a concrete problem that arises in the theory of critical phenomena and describes the phase transition in a hierarchical system of random resistors. For this problem we have implemented in particular the convolution product based on the Fast Fourier Transform (FFT) algorithm with rigorous error estimates.

Key words: computer–assisted proofs, constructive analysis in Sobolev spaces, phase transitions in random media, discrete convolutions are convolutions of splines.

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1. Introduction

In this paper we study certain aspects of a model that describes the conductivity in a disordered material. A disordered material is often modeled in statistical mechanics by what is known as a network of random resistors. One would like to be able to describe, for example, what happens in a d–dimensional cubic lattice where each link represents a resistor whose resistance is a random variable. One of the quantities of interest is the effective conductivity of such a network. This conductivity could be defined, for instance, by taking the limit as L goes to infinity of the conductivity $\sigma(L)$ measured on a cube of side length L.

More generally, a network of random resistors is defined by giving a graph and a sequence Σ_{ij} of random variables with $0 \leq \Sigma_{ij} \leq \infty$. Σ_{ij} describes the conductivity of the link in the graph that connects the vertices i and j. For simplicity one considers in general families of independent identically distributed random variables. Such models have been widely studied over the last decades, see for example [Z, K1, Be1, K2, Be2, BW, SW, BSW, K3, Be3, BO, SS] and for a purely probabilistic approach to the problem [Bl, EB, W1-2].

The model which we study in this paper belongs to a class of models where one permits the links to be perfect insulators, that is $q \equiv P(\Sigma_{ij} = 0) > 0$, but for which on the other hand $\Sigma_{ij} < \infty$. This situation is interesting, as it presents the phenomenon of percolation: For q close to one, the links that are conducting form disconnected finite sets and the effective conductivity of the network is trivially zero. This is not any longer the case for q close to zero, where the conductivity depends on the resistivity of the connecting links. As a consequence, a phase transition occurs in the effective conductivity for a certain critical value of the parameter q. The introduction of the parameter q permits therefore, through the mechanism of percolation, to obtain a model that models a critical phenomenon.

The study of phase transitions has progressed enormously with the arrival of renormalization group methods. Different types of renormalization schemes have been proposed to describe the phase transition on regular lattices of random resistors, in particular transformations of the Migdal–Kadanoff type [K3, BO, Bl], and so–called exact renormalization schemes on hierarchical lattices [SW, Be3, SS].

Some of the difficulties that have to be overcome in a renormalization group study disappear when one considers hierarchical models. Moreover, these models typically seem to provide good approximations to more complicated systems [BO].

In Section 1.1 we describe the hierarchical lattice of random resistors for which we have studied the phase transition in this paper. This lattice has been proposed by [SW] as an approximation to the square lattice in two dimensions. In Section 1.2, we state our main result.

1.1. The Model

The hierarchical network that we study in this paper is constructed recursively as indicated in Fig. 1.2.

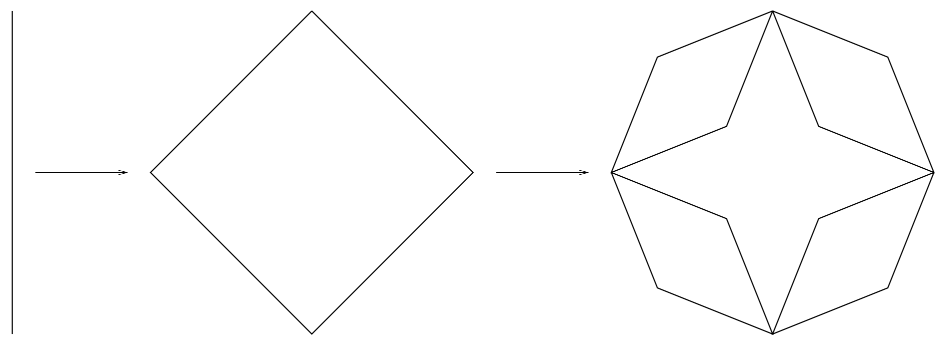

Fig. 1.1: The hierarchical lattice at order 2.

Consider the mapping on graphs that consists of replacing every link by two pairs of links. If we start with a graph consisting of one link connecting two sites, then after applying this operation we end up with four sites and after n applications we end up with a graph of 4^n links. For $n \geq 1$, the network of random resistors that we consider is obtained after n iterations of the procedure outlined above, and consists of resistors with conductivities described by 4^n independent copies $\Sigma_0^{(k)}$ of a random variable Σ_0. The random variables $\Sigma_0^{(k)}$ are therefore *i.i.d.*.

The choice of a hierarchical geometry permits to give a simple formulation of the effective conductivity of the network, when measured between the two vertices of the initial link, in terms of a map. Indeed, using the composition laws for conductors connected in series and in parallel, the conductivity of a circuit that consists of four resistors with conductivities σ_1, σ_2, σ_3, σ_4 arranged in a loop is

$$\sigma = D_c(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \equiv \frac{1}{\frac{1}{\sigma_1} + \frac{1}{\sigma_2}} + \frac{1}{\frac{1}{\sigma_3} + \frac{1}{\sigma_4}}. \tag{1.1}$$

If the conductivities σ_1, σ_2, σ_3, σ_4 are random, this is also the case for the effective conductivity σ. Therefore, by applying this nonlinear average to each of the loops of the hierarchical network of order n, we obtain the random variables that describe the conductivities of the links of the network at the level $n-1$. Since the random variables that were given on the network of order n were *i.i.d.*, the 4^{n-1} random variables that are obtained for the network of order $n-1$ are also *i.i.d.*. They are independent copies of the random variable given by

$$\Sigma_1 = D_c(\Sigma_0^{(1)}, \Sigma_0^{(2)}, \Sigma_0^{(3)}, \Sigma_0^{(4)}).$$

By applying successively the nonlinear average D_c, one can therefore go up the hierarchy to compute the effective conductivity Σ_n of the network of order n. We note that by applying one more time the average D_c to four independent copies of Σ_n, we obtain the conductivity Σ_{n+1} of the hierarchical network of order $n+1$ made up from resistors with conductivities given by 4^{n+1} independent copies of the random variable Σ_0.

We are interested in the limit as $n \to \infty$ of the sequence Σ_n that we have just defined. This limit corresponds to the effective conductivity of our hierarchical network in the infinite volume limit. It is not difficult to see that our model is an approximation to the renormalization on a simple square lattice in $d = 2$ dimensions. In [SW] a detailed discussion of this approximation can be found. See also [BO]. The questions that arise naturally are the following. First of all, a phenomenon of self–averaging should lead to a deterministic effective conductivity for the infinite network. Next, it is interesting to know in what way this conductivity depends on the conductivity of each link of the network, that is on the distribution of the initial random variable Σ_0. Finally, once the convergence of the effective conductivity is established, one can study the fluctuations of Σ_n around this limit.

A certain amount of information can easily be obtained by studying the parameter

$$p_n = P(\Sigma_n > 0). \tag{1.2}$$

Recall that we permit the links to be perfect insulators with a nonzero probability. Here, $p_0 \equiv p$ is the probability that a resistor of the original (infinite) network is not broken. From (1.1) it is not difficult to see that the conductivity of a diamond circuit is nonzero with probability

$$\tilde{p} = g(p) \equiv 1 - (1 - p^2)^2, \tag{1.3}$$

where p is the probability that each of the for resistors has a nonzero conductivity. The function g is characteristic for percolation problems [G]. The function g is increasing on the interval $[0, 1]$ and has, in addition to the fixed points at zero and one, a unique unstable fixed point p_c in the interval $(0, 1)$. The value of p_c is,

$$p_c = \frac{\sqrt{5} - 1}{2}. \tag{1.4}$$

There are therefore three cases. If $p < p_c$, it follows immediately that $p_n \to 0$ as $n \to \infty$. Hence, the effective conductivity of the network is zero with probability one in this case. For $p > p_c$, one has $\lim_{n\to\infty} p_n = 1$. This means that with probability one there is a path made from resistors with nonzero conductivity that connects the two sites of the lattice for $n = 0$. In other words, the percolation threshold of the network is given by p_c.

We note that this does not imply that the effective conductivity is nonzero. However, it has been proved in [W1] that for $p > p_c$ the sequence Σ_n converges with probability one to a constant $\sigma^*(p)$, and in [Sh] that this constant is strictly positive. Therefore, the percolation threshold corresponds exactly to the phase transition of the effective conductivity.

At the critical point $p = p_c$, one has $p_n = p_c$ for all n. This means that the probability $P(\Sigma_n = 0)$ is invariant. In the following we are going to be interested in the part of the distribution of Σ_n supported on $(0, \infty)$. An argument based on the study of the expectation of Σ_n shows, however, that the effective conductivity of the network in the infinite volume limit is still zero in this case. Indeed, if we denote by $E(X)$ the expectation of a random variable X, one has

$$E(\Sigma_{n+1}) = E\big(D_c(\Sigma_n^{(1)}, \dots, \Sigma_n^{(4)})\big) = E\Big(\frac{2}{\frac{1}{\Sigma_n^{(1)}} + \frac{1}{\Sigma_n^{(2)}}}\Big),$$

where we have used independence of the random variables. Since, in addition we have the following inequality between the arithmetic mean and the geometric mean,

$$\frac{2}{\frac{1}{x} + \frac{1}{y}} \leq \frac{x+y}{2},$$

and since the left hand side of this expression is equal to zero if x or y are zero, we can bound $E(\Sigma_{n+1})$ by the expectation of the average of $\Sigma_n^{(1)}$ and $\Sigma_n^{(2)}$ over the set where the random variables are strictly positive. Therefore, one obtains

$$E(\Sigma_{n+1}) \leq p_n E(\Sigma_n).$$

Since $p_n = p_c$ for all n, we have $E(\Sigma_n) \to 0$ as $n \to \infty$, and the sequence Σ_n converges to zero in probability.

The goal of this paper is to describe how the Σ_n converge to zero at the critical point, that is to describe the fluctuations of the Σ_n around their limiting value. A numerical study [SW] of the probability densities of the random variables Σ_n indicates that if one normalizes Σ_n with an appropriate factor μ_n that fixes the expectation of the random variable $\mu_n\Sigma_n$ at the expectation of Σ_0, the sequence $\mu_n\Sigma_n$ converges in distribution to a multiple of a (universal) random variable Σ_*, and

$$\lim_{n\to\infty} \frac{\mu_{n+1}}{\mu_n} = \lambda^* \approx 1.756,$$

independently of the choice of Σ_0. This means that the fluctuations of the effective conductivity of the network present a certain universality in the limit $n \to \infty$: the limiting probability densities for different initial random variables Σ_0 distinguish themselves only by a change of scale. Furthermore, the probability density for the positive values of Σ_* decays faster than exponentially at zero and infinity. Therefore, at the critical point, the behavior of the fluctuations distinguishes itself from the supercritical case for which a perturbative computation [SS] indicates that the sequence of properly normalized random variables Σ_n converges to a normal distribution (a proof of this fact will appear in [WW]). Also, since $\lambda^* < 2$, conductance fluctuations can be thought of as anomalously large compared to the supercritical case.

In this work we address the question of the existence of a positive real number λ^* and a random variable Σ_* such that

$$\Sigma_* = \lambda^* D_c(\Sigma_*^{(1)}, \ldots, \Sigma_*^{(4)}). \tag{1.5}$$

We note that the map D_c is homogeneous of degree one and the random variables $\lambda\Sigma_*$ are therefore solutions of (1.5) for all $\lambda > 0$. The number λ^* gives the dynamics of the renormalization group D_c at the critical point on the whole of the set $\lambda\Sigma_*$ and is related to the critical exponent t that describes the phase transition of the effective conductivity,

$$\sigma^*(p) \sim (p - p_c)^t, \quad p > p_c,$$

through the formula, cf. [SW],

$$t = \frac{\log \lambda^*}{\log g'(p_c)}\,.$$

1.2. Main Result

In order to study the fluctuations of the effective conductivity of our hierarchical network at the critical point, we work in the framework of functional analysis. The part of the distribution of the random variables Σ_n that interests us is the one that is supported on $(0, \infty)$. One assumes that this part of the distribution of Σ_0 is absolutely continuous with respect to Lebesgue measure and one derives the functional equation for the probability densities that correspond to the nonlinear average (1.1). It turns out to be simpler to work with the resistivities instead of the conductivities. One considers therefore the random variables Υ for the resistivity given by

$$\Upsilon = \frac{1}{\Sigma}.$$

If σ is the density of the random variable Σ, then the density ρ of Υ is given by

$$\rho(x) = T(\sigma)(x) \equiv \frac{1}{x^2}\sigma\Big(\frac{1}{x}\Big). \tag{1.6}$$

The average D_c for the conductivities can be rewritten for the resistivities as

$$D_r(r_1, \ldots, r_4) = \frac{1}{\frac{1}{r_1+r_2} + \frac{1}{r_3+r_4}}. \tag{1.7}$$

The functional equation for the density $\mathcal{D}_r(\rho)$ of the average (1.7) of four independent copies of a random variable Υ with density ρ is therefore given in terms of the map T and the convolution operator. Indeed, the probability density ρ of a sum of two random variables with densities ρ_1 and ρ_2 is given by the convolution of ρ_1 and ρ_2, that is by

$$\rho(x) = (\rho_1 * \rho_2)(x) = \int_0^x \rho_1(y)\rho_2(x-y)\,dy. \tag{1.8}$$

Therefore, it follows from (1.7) that

$$\mathcal{D}_r(\rho) = T\big(T(\rho*\rho) * T(\rho*\rho)\big). \tag{1.9}$$

One observes that, formally, the Dirac–densities $\delta(x-a)$ are fixed points of this transformation for all $a \geq 0$. They correspond to the limiting densities in the non–critical cases.

In order to obtain an equation for the probability densities with support on $(0,\infty)$ we determine the contributions of the four resistors to the finite value of $D_r(r_1,\ldots r_4)$. By inspecting (1.7), one observes that they are of two types: either $r_1+r_2=\infty$ and $r_3+r_4<\infty$ (with the corresponding symmetric case $r_1+r_2<\infty$ and $r_3+r_4=\infty$), or all of the four resistivities are finite. At the critical point p_c, one determines easily that the probability of the first case is given by

$$c_1 \equiv 2p_c(1-p_c^2) = 0.763..., \tag{1.10}$$

whereas the probability of the second case is

$$c_2 \equiv p_c^3 = 0.236.... \tag{1.11}$$

Obviously $c_1+c_2=1$. Therefore, the operator that acts on the probability densities and corresponds to the finite part of the map D_r on the random variables is given at the critical point by

$$\mathcal{D}(\rho) = c_1(\rho*\rho) + c_2 T\big(T(\rho*\rho)*T(\rho*\rho)\big). \tag{1.12}$$

In order to rewrite the fixed point problem (1.5) in terms of the probability densities, one uses that for $\lambda>0$ the probability density of a random variable Υ/λ is given by $S_\lambda\rho$, where ρ is the probability density of Υ and where S_λ is the operator that changes the scale,

$$S_\lambda f(x) \equiv \lambda f(\lambda x)\,, \quad \lambda>0. \tag{1.13}$$

Therefore the fixed point problem (1.5) can be rewritten as

$$\rho^* = S_{\lambda^*}\mathcal{D}(\rho^*). \tag{1.14}$$

The proof of the existence of a real $\lambda^*$and a function ρ^* satisfying (1.14) which we present here is constructive. In particular, we will be capable of providing explicit bounds on λ^* as well as an approximation to the function ρ^*. The graph of the approximation that we have obtained this way is represented in Fig. 1.2.

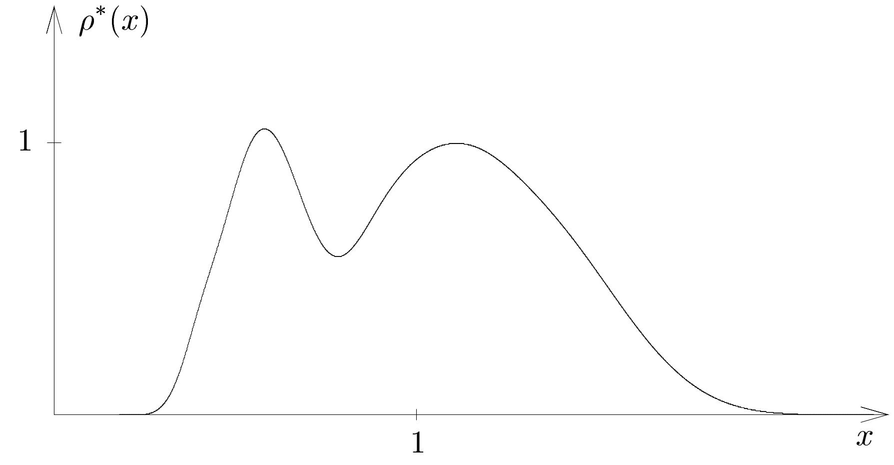

Fig. 1.2: The fixed point ρ^*.

The operator (1.12) was studied by mathematically rigorous, computer–assisted, constructive analysis. Before we state with precision the result that is proved in this paper we define the function spaces with which we work. For that we first define the notation that will be used later.

Notation. We denote by $\mathbb{R}_+$ the set of nonnegative real numbers. The set of positive real numbers will be denoted by $\mathbb{R}_+^*$. For an interval $I \subseteq \mathbb{R}$, we denote by $\mathcal{C}^n(I)$ the set of functions that are n times continuously differentiable on I. The derivative of a function f of one variable will be denoted by f'. For a positive function μ, we denote by $L^1(\mathbb{R}_+, \mu(x)\ dx)$ the space of functions defined on $\mathbb{R}_+$ and integrable with respect to the measure $\mu(x)\ dx$. Finally $W_1^1(\mathbb{R}_+, \mu(x)\ dx)$ is the Sobolev space of functions of $L^1(\mathbb{R}_+, \mu(x)\ dx)$ with one distributional derivative in $L^1(\mathbb{R}_+, \mu(x)\ dx)$. For $r \geq 0$ and x in a metric space, $B_r(x)$ will denote the open ball of radius r centered at x.

Definition 1.1. *For α, $\beta \geq 0$ and functions $w_{\alpha\beta}$ given by*

$$w_{\alpha\beta}(x) = \exp\Big(\frac{\alpha}{x} + \beta x\Big), \tag{1.15}$$

we define $\mathcal{B}_{\alpha\beta}$ to be the Banach space $L^1(\mathbb{R}_+, w_{\alpha\beta}(x)\ dx)$. We denote the norm of $f \in \mathcal{B}_{\alpha\beta}$ by $\|f\|_{\alpha\beta}$, that is,

$$\|f\|_{\alpha\beta} = \int_0^\infty w_{\alpha\beta}(x)\,|f(x)|\,dx. \tag{1.16}$$

We furthermore define

$$\mathcal{B} = \bigcap_{\substack{\alpha>0\\ \beta>0}} \mathcal{B}_{\alpha\beta}. \tag{1.17}$$

Remark 1.2. It is clear that we have the inclusion $\mathcal{B}_{\sigma\tau} \subseteq \mathcal{B}_{\alpha\beta}$ for $\sigma \geq \alpha$ and $\tau \geq \beta$. The inclusion is strict unless $\sigma = \alpha$ and $\tau = \beta$.

We will also need the following definitions of the mass and expectation.

Definition 1.3. *We define the mass $M(f)$ of a function $f \in L^1(\mathbb{R}_+)$ by*

$$M(f) = \int_0^\infty f(x)\, dx. \tag{1.18}$$

If $f \in L^1(\mathbb{R}_+, (1+|x|)\, dx)$, we define the expectation $E(f)$ by

$$E(f) = \int_0^\infty x f(x)\, dx. \tag{1.19}$$

We can now state the principal result of this paper.

Theorem 1.4. *There exists a real number λ^* and a function $f^* \in \mathcal{B}$ that satisfy the equation*

$$S_{\lambda^*}\mathcal{D}(f^*) = f^*.$$

In addition f^* has the following two properties

$$\begin{aligned}&(1)\ M(f^*) = 1,\\ &(2)\ f^* \in \mathcal{C}^\infty(\mathbb{R}^+).\end{aligned}$$

Note that this theorem does not imply that the fixed point f^* is a probability density, since f^* is not necessarily a positive function. While we see strong numerical evidences for positivity of the fixed point f^*, we have no proof of this fact.

Before terminating this section, we summarize in the following lemma some of the properties that are satisfied by the maps that are contained in (1.14).

Lemma 1.5. *The maps S_λ form a multiplicative group, i.e., $S_1 = I$, and $S_{\lambda_1}S_{\lambda_2} = S_{\lambda_1\lambda_2}$. Moreover, for f and g integrable functions one has*

$$S_\lambda(f * g) = S_\lambda f * S_\lambda g, \tag{1.20}$$

$$S_\lambda T = T S_{1/\lambda}. \tag{1.21}$$

If f, $g \in L^1(\mathbb{R}_+, (1+|x|)\, dx)$, then the mass and expectation satisfy the following identities

$$\begin{aligned}
M(f) &= M(S_\lambda f) = M(Tf)\,,\\
M(f*g) &= M(f)M(g)\,,\\
E(S_\lambda f) &= \frac{1}{\lambda}E(f)\,,\\
E(f*g) &= M(f)E(g) + E(f)M(g)\,.
\end{aligned} \tag{1.22}$$

Even though the proof of Theorem 1.4 needs in part a computer for its proof, the properties (1) and (2) follow directly from the existence of the fixed point. The regularity of the fixed point will be established later, cf. Proposition 2.3, whereas property (1) is proved in the following lemma.

Lemma 1.6. *Let $f \in L^1(\mathbb{R}_+)$ and let $\lambda > 0$ arbitrary. If f satisfies $S_\lambda \mathcal{D}(f) = f$, then $M(f) = 1$.*

Proof. Using the relations (1.22) one finds using $M(f) = M(S_\lambda \mathcal{D}(f))$

$$M(f) = c_1 M(f)^2 + c_2 M(f)^4 .$$

Using the monotonicity of the function $x \mapsto c_2 x^3 + c_1 x - 1$, one verifies that if $c_1 + c_2 = 1$, then the only zero is given by $x_0 = 1$. Therefore, $M(f) = 1$.

■

1.3. Computer–Assisted Proofs

The rest of the proof of Theorem 1.4 is the main part of this paper. It is based on a very large number of inequalities proved rigorously with the help of a computer. The use of a computer for proving theorems in analysis has become standard by now. This method, which allows to do constructive functional analysis on a computer, has been developed by O.E. Lanford in his seminal paper [L1], and has then been generalized by [EKW2]. This technique of proof has since then been applied to problems of various origin. See for example [BS, C, CC, dlL, EKW1-2, EW1-2, FL, FS, KP, KSW, KW1-7, L1-3, LR1-3, M, R, Se1-2, St].

The proofs constructed up to now have in common that they all deal with spaces of analytic functions. One important novelty of the work presented here is that a proof is constructed for function spaces of L^1–type. The basic ideas underlying the proof remain the same, but the generalization to L^1 spaces uses approximation methods which are typical in numerical analysis, and we will explain how to control discretization errors in this context.

A computer–assisted proof is complete once the program has come to an end without a "domain error".

2. Organization of the Proof

In order to prove existence of a fixed point f^* for the map $S_{\lambda^*}\mathcal{D}$, we will rely on the contraction mapping principle. The following argument shows, however, that $S_{\lambda^*}\mathcal{D}$ cannot even be hyperbolic due to the presence of a symmetry. Recall that the nonlinear average D_r is homogeneous of order 1. This causes the scaling operator S_λ to commute with $\mathcal{D}$ for every $\lambda > 0$. Hence, existence of a fixed point f^* implies existence of a one parameter family of fixed points $\{S_\lambda f^*\}_{\lambda>0}$. In the case $E(f^*) \neq 0$, this family can be parameterized by the expectation $E(S_\lambda f^*) = E(f^*)/\lambda$. We first remove this symmetry and make the fixed point problem hyperbolic by introducing the family of maps $\{\mathcal{N}_\lambda\}_{\lambda>0}$ defined by

$$\mathcal{N}_\lambda(f) = S_\lambda\Big(c_\lambda(f) f * f + c_2 T\big(T(f*f) * T(f*f)\big)\Big)\,, \tag{2.1}$$

where $c_\lambda(f)$ is such that

$$E\big(\mathcal{N}_\lambda(f)\big) = 1\,. \tag{2.2}$$

The expression inside the outer brackets on the RHS of (2.1) differs from the map $\mathcal{D}$ only by the coefficient $c_\lambda(f)$. The following remark relates the maps $S_\lambda\mathcal{D}$ and $\mathcal{N}_\lambda$: If $c_\lambda(f_\lambda) = c_1$ for some fixed $\lambda > 0$ and some f_λ, then $S_\lambda\mathcal{D}(f_\lambda) = \mathcal{N}_\lambda(f_\lambda)$ by definition of $\mathcal{D}$ and $\mathcal{N}_\lambda$. This leads to the following criterion for the existence of a fixed point of $S_{\lambda^*}\mathcal{D}$.

Lemma 2.1. *If there exist a real $\lambda^* > 0$ and a fixed point f_{λ^*} of $\mathcal{N}_{\lambda^*}$ with $c_{\lambda^*}(f_{\lambda^*}) = c_1$, then f_{λ^*} is solution of the functional equation $f = S_{\lambda^*}\mathcal{D}(f)$.*

To prove existence of λ^* and f_{λ^*}, we will study the family $\{\mathcal{N}_\lambda\}$ in a neighborhood (λ^-, λ^+) of our best numerical value for λ^*. For each value of λ in this neighborhood, the contraction mapping principle will be used to prove existence of a fixed point f_λ of $\mathcal{N}_\lambda$. The maps $\mathcal{N}_\lambda$ are hyperbolic but not contracting in the neighborhood of their fixed point, due to an unstable direction that, roughly speaking, crosses transversally the manifold of functions with total mass equal to one. To cope with this problem, we will adopt later a standard strategy that consists of applying a variant of Newton's method. Once the existence of a fixed point $f_\lambda = \mathcal{N}_\lambda(f_\lambda)$ is established for all $\lambda \in [\lambda^-, \lambda^+]$, we will show that $c_{\lambda^-}(f_{\lambda^-}) < c_1 < c_{\lambda^+}(f_{\lambda^+})$. A continuity argument will finally yield the existence of a $\lambda^* \in (\lambda^-, \lambda^+)$ and a function f_{λ^*} satisfying the hypothesis of Lemma 2.1.

Before entering into more details, we introduce some notation and state a few results concerning the domains of definition and target spaces of $\mathcal{N}_\lambda$. Using the commutation and distributivity properties (1.20) and (1.21), one rewrites $\mathcal{N}_\lambda$ as

$$\mathcal{N}_\lambda(f) = c_\lambda(f)\mathcal{N}_\lambda^1(f) + c_2\mathcal{N}_\lambda^2(f)\,, \tag{2.3}$$

where

$$\mathcal{N}_\lambda^1(f) = S_\lambda(f*f)\,, \tag{2.4}$$

$$\mathcal{N}_\lambda^2(f) = T\big(T\mathcal{N}_\lambda^1(f) * T\mathcal{N}_\lambda^1(f)\big)\,. \tag{2.5}$$

From the condition (2.2), the coefficient $c_\lambda(f)$ is expressed in terms of the expectation of $\mathcal{N}^1_\lambda(f)$ and $\mathcal{N}^2_\lambda(f)$. Since $E(\mathcal{N}^1_\lambda(f)) = 2M(f)E(f)/\lambda$ and

$$\frac{1}{\lambda}E(\mathcal{N}^2_\lambda(f)) = E\big(T\big(T(f*f)*T(f*f)\big)\big) \equiv E_2(f) \tag{2.6}$$

one gets

$$c_\lambda(f) = \frac{\lambda - c_2E_2(f)}{2M(f)E(f)}\,. \tag{2.7}$$

We will see that $E_2(f)$ is finite for $f \in \mathcal{B}_{\alpha\beta}$ with $\beta > 0$. However, $c_\lambda(f)$ is not defined for functions f such that $M(f)E(f) = 0$ and we will exclude functions of this type, i.e., functions in

$$\mathcal{H} = \{f \in L^1(\mathbb{R}^+, (1+|x|)\,dx) \mid M(f)E(f) = 0\}\,. \tag{2.8}$$

We now state a result about the domains of definition and target spaces of the maps $\mathcal{N}_\lambda$.

Proposition 2.2. *For $\alpha \geq 0$, $\beta > 0$ and $\lambda > 0$, $\mathcal{N}_\lambda$ is well defined as a map from $\mathcal{B}_{\alpha\beta}\backslash\mathcal{H}$ to $\mathcal{B}_{\sigma\tau}$ for all $\sigma \leq 4\alpha/\lambda$ and $\tau \leq \lambda\beta$.*

Proof. The operators S_λ, T and the convolution $f \mapsto f*f$ are well defined on $L^1(\mathbb{R}_+)$. Next, we show that the convolution product maps $\mathcal{B}_{\zeta\eta}\times\mathcal{B}_{\zeta\eta}$ into $\mathcal{B}_{(4\zeta)\eta}$. Using Fubini's theorem, we get the following inequalities

$$\begin{aligned}
\|f*g\|_{(4\zeta)\eta} &\leq \int_0^\infty dx\; w_{(4\zeta)\eta}(x)\int_0^x dy\;|f(y)||g(x-y)| \\
&= \int_0^\infty dy|f(y)|\int_0^\infty dx\; w_{(4\zeta)\eta}(x+y)|g(x)| \\
&\leq \sup_{\substack{x>0\\y>0}}\Big(\frac{w_{(4\zeta)\eta}(x+y)}{w_{\zeta\eta}(x)w_{\zeta\eta}(y)}\Big)\|f\|_{\zeta\eta}\|g\|_{\zeta\eta} \\
&= \|f\|_{\zeta\eta}\|g\|_{\zeta\eta}\,.
\end{aligned} \tag{2.9}$$

The last equality follows from

$$\sup_{\substack{x>0\\y>0}}\frac{w_{(4\zeta)\eta}(x+y)}{w_{\zeta\eta}(x)w_{\zeta\eta}(y)} = \sup_{\substack{x>0\\y>0}}\exp\big(-\zeta h(x,y)\big)\,,$$

and

$$h(x,y) = \frac{x+y}{xy} - \frac{4}{x+y} \geq 0$$

for all $x, y > 0$. Next, S_λ is bounded as a map from $\mathcal{B}_{\zeta\eta}$ to $\mathcal{B}_{(\zeta/\lambda)(\lambda\eta)}$:

$$\begin{aligned}
\|S_\lambda f\|_{(\zeta/\lambda)(\lambda\eta)} &= \int_0^\infty w_{(\zeta/\lambda)(\lambda\eta)}(x/\lambda)|f(x)|dx \\
&\leq \sup_{x>0}\Big(\frac{w_{(\zeta/\lambda)(\lambda\eta)}(x/\lambda)}{w_{\zeta\eta}(x)}\Big)\|f\|_{\zeta\eta} \\
&= \|f\|_{\zeta\eta}\,.
\end{aligned} \tag{2.10}$$

Therefore $\mathcal{N}_\lambda^1$ maps $\mathcal{B}_{\alpha\beta}$ into $\mathcal{B}_{(4\alpha/\lambda)(\lambda\beta)}$, and hence into $\mathcal{B}_{\sigma\tau}$ for $\sigma \le 4\alpha/\lambda$ and $\tau \le \lambda\beta$. In order to check that $\mathcal{N}_\lambda^2$ maps $\mathcal{B}_{\alpha\beta}$ into $\mathcal{B}_{(4\alpha/\lambda)(\lambda\beta)}$, we first note that T is obviously bounded as a map from $\mathcal{B}_{\zeta\eta}$ to $\mathcal{B}_{\eta\zeta}$, with

$$\|Tf\|_{\eta\zeta} = \|f\|_{\zeta\eta}\,, \tag{2.11}$$

since $w_{\zeta\eta}(1/x) = w_{\eta\zeta}(x)$ for all $x > 0$. Using (2.9), (2.11) and the bound on $\mathcal{N}_\lambda^1$, one concludes that $\mathcal{N}_\lambda^2(f) \in \mathcal{B}_{(4\alpha/\lambda)(4\lambda\beta)} \subset \mathcal{B}_{(4\alpha/\lambda)(\lambda\beta)}$ for $f \in \mathcal{B}_{\alpha\beta}$. Finally, the bounds (2.9), (2.11), and the fact that the expectation of a function in $\mathcal{B}_{\sigma\tau}$ is finite for $\tau > 0$ imply that $c_\lambda(f)$ is finite for $f \in \mathcal{B}_{\alpha\beta}\backslash\mathcal{H}$ and $\beta > 0$.

■

Proposition 2.2 together with Remark 1.2 immediately imply that every fixed point $f \in \mathcal{B}_{\alpha\beta}$ of $\mathcal{N}_\lambda$ with $\alpha, \beta > 0$ and $\lambda \in (1,4)$ satisfies $f \in \mathcal{B}$. Furthermore, using the regularization properties of the convolution, one can show that every such fixed point is a smooth function. More precisely, we have the following proposition, whose proof can be found in the appendix.

Proposition 2.3. *Let $\alpha, \beta > 0$, $\lambda \in (1,4)$, and let $f \in \mathcal{B}_{\alpha\beta}$ be a fixed point of $\mathcal{N}_\lambda$. Then $f \in \mathcal{B}$, f is of class $\mathcal{C}^\infty(\mathbb{R}_+)$, and $f' \in \mathcal{B}$.*

The following theorem implies Theorem 1.4.

Theorem 2.4. *Let $\lambda^- \equiv 1.7562035$ and $\lambda^+ \equiv 1.7562048$. Then,*

(a) *For some $\alpha, \beta > 0$, there is a continuous family $\{f_\lambda\}_{\lambda\in[\lambda^-,\lambda^+]}$ of functions in $\mathcal{B}_{\alpha\beta}\backslash\mathcal{H}$ such that $\mathcal{N}_\lambda(f_\lambda) = f_\lambda$ for all $\lambda \in [\lambda^-, \lambda^+]$,*

(b) $c_{\lambda^-}(f_{\lambda^-}) < c_1 < c_{\lambda^+}(f_{\lambda^+})\,.$ (2.12)

Our main result follows from Theorem 2.4 and Proposition 2.3.

Proof of Theorem 1.4. Assume first that the map $\lambda \mapsto c_\lambda(f_\lambda)$ is continuous. Then Theorem 2.4 implies the existence of a $\lambda^* \in (\lambda^-, \lambda^+)$ for which $c_{\lambda^*}(f_{\lambda^*}) = c_1$ and $f_{\lambda^*} = \mathcal{N}_{\lambda^*} f_{\lambda^*}$, which using Lemma 2.1 implies that $S_{\lambda^*}\mathcal{D}(f_{\lambda^*}) = f_{\lambda^*}$, and using Proposition 2.3 that $f_{\lambda^*} \in \mathcal{B} \cap \mathcal{C}^\infty(\mathbb{R}_+)$. It remains to be checked that the map $\lambda \mapsto c_\lambda(f_\lambda)$ is indeed continuous. One first observes that the linear functionals $f \mapsto M(f)$ and $f \mapsto E(f)$ are bounded as maps from $\mathcal{B}_{\sigma\tau}$ to $\mathbb{R}$, provided $\tau > 0$, and that

$$|M(f)| \le \sup_{x>0}\Bigl(\frac{1}{w_{\sigma\tau}(x)}\Bigr)\|f\|_{\sigma\tau}\,, \tag{2.13}$$

$$|E(f)| \le \sup_{x>0}\Bigl(\frac{x}{w_{\sigma\tau}(x)}\Bigr)\|f\|_{\sigma\tau}\,. \tag{2.14}$$

Hence, $f \mapsto E_2(f)$ is continuous as a map from $\mathcal{B}_{\alpha\beta}$ to $\mathbb{R}$ for every $\alpha, \beta > 0$, using in addition the bounds derived in the proof of Proposition 2.2. Therefore $f \mapsto c_\lambda(f)$

is continuous as a map from $\mathcal{B}_{\alpha\beta}\backslash\mathcal{H}$ to $\mathbb{R}$ for every $\alpha, \beta > 0$ and $\lambda \in \mathbb{R}$. Next, for each $f \in \mathcal{B}_{\alpha\beta}\backslash\mathcal{H}$ with $\alpha, \beta > 0$, the map $\lambda \mapsto c_\lambda(f)$ is continuous. The continuity of $\lambda \mapsto c_\lambda(f_\lambda)$ as a map from $[\lambda^-, \lambda^+]$ to $\mathbb{R}$ follows.

$\blacksquare$

The proof of Theorem 2.4 is in part computer–assisted. Once (a) is established, the verification of part (b) involves mainly an explicit calculation, that will be given in Section 5.3. The remainder of this section is devoted to the proof of part (a).

First, in order to simplify further our estimates, we introduce yet another family of operators, closely related to $\{\mathcal{N}_\lambda\}_{\lambda>0}$. This family is defined by

$$\mathcal{N}_{\lambda,\kappa} = S_\kappa \mathcal{N}_\lambda \ , \tag{2.15}$$

where $\lambda, \kappa > 0$.

Lemma 2.5. *Let $\alpha \geq 0$ and $\beta, \lambda, \kappa > 0$. Then $\mathcal{N}_{\lambda,\kappa}$ is well defined as a map from $\mathcal{B}_{\alpha\beta}\backslash\mathcal{H}$ to $\mathcal{B}_{\sigma\tau}$ for every $\sigma \leq 4\alpha/\kappa\lambda$ and $\tau \leq \kappa\lambda\beta$, and one has*

$$\begin{array}{ccc} \mathcal{B}_{\alpha\beta} & \xrightarrow{\mathcal{N}_{\lambda,\kappa}} & \mathcal{B}_{(\frac{4\alpha}{\kappa\lambda})(\kappa\lambda\beta)} \\ S_{1/\kappa}\Big\downarrow & & \Big\downarrow S_{1/\kappa} \\ \mathcal{B}_{(\kappa\alpha)(\frac{\beta}{\kappa})} & \xrightarrow{\mathcal{N}_{\kappa\lambda}} & \mathcal{B}_{(\frac{4\alpha}{\lambda})(\lambda\beta)} \end{array} \tag{2.16}$$

Proof. First, our previous result on the domains of definition and target spaces of $\mathcal{N}_\lambda$ implies that the operator $\mathcal{N}_{\lambda,\kappa}$ is well defined as a map from $\mathcal{B}_{\alpha\beta}\backslash\mathcal{H}$ to $\mathcal{B}_{(4\alpha/\kappa\lambda)(\kappa\lambda\beta)}$ whenever $\alpha \geq 0$ and $\beta, \lambda, \kappa > 0$. We now show that

$$c_\lambda(f) = c_{\kappa\lambda}(S_{1/\kappa} f) \, . \tag{2.17}$$

Using (2.7), we see that

$$c_{\kappa\lambda}(S_{1/\kappa} f) = \frac{\kappa\lambda - c_2 E_2(S_{1/\kappa} f)}{2M(S_{1/\kappa} f) E(S_{1/\kappa} f)} \, ,$$

and the relations $M(S_{1/\kappa} f) = M(f)$, $E(S_{1/\kappa} f) = \kappa E(f)$ and $E_2(S_{1/\kappa} f) = \kappa E_2(f)$ lead to (2.17). Using (2.17), we compute

$$\begin{aligned} \mathcal{N}_{\kappa\lambda}(S_{1/\kappa} f) &= c_{\kappa\lambda}(S_{1/\kappa} f) \mathcal{N}^1_{\kappa\lambda}(S_{1/\kappa} f) + c_2 \mathcal{N}^2_{\kappa\lambda}(S_{1/\kappa} f) \\ &= c_\lambda(f) \mathcal{N}^1_\lambda(f) + c_2 \mathcal{N}^2_\lambda(f) \\ &= \mathcal{N}_\lambda(f) \, , \end{aligned}$$

and conclude by observing that $\mathcal{N}_\lambda = S_{1/\kappa}\mathcal{N}_{\lambda,\kappa}$.

■

From Lemma 2.5 it follows that the fixed points of $\mathcal{N}_\lambda$ for $\lambda \in [\lambda^-, \lambda^+]$ are related to the fixed points of $\mathcal{N}_{\lambda^+,\kappa}$ for $\kappa \in [\lambda^-/\lambda^+, 1]$. Furthermore, the operators $\mathcal{N}_{\lambda,\kappa}$ are well defined as maps from $\mathcal{B}_{\alpha\beta}\backslash\mathcal{H}$ to $\mathcal{B}_{\alpha\beta}$ for λ and κ satisfying

$$\frac{1}{\lambda} \leq \kappa \leq \frac{4}{\lambda}. \tag{2.18}$$

This condition is easily seen to hold for $\kappa \in [\lambda^-/\lambda^+, 1]$ and $\lambda \in [\lambda^-, \lambda^+]$.

As mentioned earlier, the fixed points f_λ of the maps $\mathcal{N}_\lambda$ are not attractive. Numerically, the two largest eigenvalues of $D\mathcal{N}_\lambda(f_\lambda)$ are roughly 1.37 and 0.54 for $\lambda \approx \lambda^*$. In the context of computer–assisted proofs, the standard way of solving a hyperbolic fixed point problem is to turn it into a fixed point problem for a contraction by proceeding in the following way. We choose an invertible linear map M close to the inverse of $1 - D\mathcal{N}_{\lambda^*}(f_{\lambda^*})$ and define

$$\mathcal{M}_{\lambda,\kappa} = 1 + M(\mathcal{N}_{\lambda,\kappa} - 1). \tag{2.19}$$

In Section 7.2, we give a detailed description of M and establish its invertibility. Furthermore, we will see that M is bounded in $\mathcal{B}_{\alpha\beta}$ for all $\alpha, \beta \geq 0$. Hence, $\mathcal{M}_{\lambda,\kappa}$ is well defined as a map from $\mathcal{B}_{\alpha\beta}\backslash\mathcal{H}$ to $\mathcal{B}_{\alpha\beta}$ for all $\alpha \geq 0$, $\beta > 0$, and κ, λ satisfying (2.18).

The existence of the continuous family of fixed points $\{f_\lambda\}$ will follow from estimates on the contractions $\mathcal{M}_{\lambda^+,\kappa}$. These estimates are collected in the following proposition.

Proposition 2.6. *Let λ^+ and λ^- be defined as in Theorem 2.4. Then, for $\mu = 0.5$, $\nu = 0.9$ and $r = 9 \cdot 10^{-4}$, there is a function $f^0_{\lambda^+} \in \mathcal{B}_{\mu\nu}$ and two positive real numbers $q < 1$ and $\varepsilon < r(1-q)$ for which the following holds. For all $\kappa \in [\lambda^-/\lambda^+, 1]$, the operator $\mathcal{M}_{\lambda^+,\kappa}$ is well defined and continuously differentiable as a map from the closed ball $\overline{B_r(f^0_{\lambda^+})}$ to $\mathcal{B}_{\mu\nu}$ and satisfies for all $f \in \overline{B_r(f^0_{\lambda^+})}$*

$$\|\mathcal{M}_{\lambda^+,\kappa}(f^0_{\lambda^+}) - f^0_{\lambda^+}\|_{\mu\nu} \leq \varepsilon, \tag{2.20}$$

$$\|D\mathcal{M}_{\lambda^+,\kappa}(f)\| \leq q. \tag{2.21}$$

We now show that this result implies part (a) of Theorem 2.4.

Proof of Theorem 2.4 (a). By the contraction mapping principle, Proposition 2.6 implies the existence of a fixed point $f_{\lambda^+,\kappa} \in \mathcal{B}_{\mu\nu}$ of $\mathcal{M}_{\lambda^+,\kappa}$ for all $\kappa \in [\lambda^-/\lambda^+, 1]$. From the invertibility of the operator M, the functions $f_{\lambda^+,\kappa}$ are also fixed points of the operators $\mathcal{N}_{\lambda^+,\kappa}$. Hence, the conjugation relation (2.16) ensures the existence of the family $\{f_\lambda\}$ of fixed points of $\mathcal{N}_\lambda$ for $\lambda \in [\lambda^-, \lambda^+]$. These fixed points are given by $f_\lambda = S_{1/\kappa} f_{\lambda^+,\kappa}$, $\kappa = \lambda/\lambda^+$. Since $S_{1/\kappa} f_{\lambda^+,\kappa} \in \mathcal{B}_{(\lambda^- \mu/\lambda^+)\nu}$ for all $\kappa \in [\lambda^-/\lambda^+, 1]$, it follows that for all $\lambda \in [\lambda^-, \lambda^+]$, $f_\lambda \in \mathcal{B}_{\alpha\beta}$ for some $\alpha, \beta > 0$.

Finally, we prove the continuity of the family $\{f_\lambda\}$. From Proposition 2.3, it follows that $f_\lambda \in \mathcal{B} \cap \mathcal{C}^\infty(\mathbb{R}_+)$ with $f'_\lambda \in \mathcal{B}$. Since the functions $f_{\lambda^+,\kappa} = S_\kappa f_{\kappa\lambda^+}$ have the same properties, $\kappa \mapsto S_{1/\kappa} f_{\lambda^+,\kappa}$ is continuous as a map from $[\lambda^-/\lambda^+, 1]$ to $\mathcal{B}_{\alpha\beta}$ provided that the family $\{f_{\lambda^+,\kappa}\}_\kappa$ is continuous in $\mathcal{B}_{\mu\nu}$. In order to show that this is indeed the case, we check that $\{f_{\lambda^+,\kappa}\}_\kappa$ is continuous at $\kappa = \kappa_0$ for each $\kappa_0 \in [\lambda^-/\lambda^+, 1]$. Let us fix such a κ_0 and denote $f_0 \equiv f_{\lambda^+,\kappa_0}$. First, since the contraction mapping principle and Proposition 2.6 imply that f_0 belongs to the ball $B_{\tilde{r}}(f^0_{\lambda^+})$, where $\tilde{r} = \varepsilon/(1-q) < r$, then, for every $\tilde{\varepsilon} > 0$ satisfying $\tilde{\varepsilon} < r - \tilde{r}$, the ball $\overline{B_{\tilde{\varepsilon}}(f_0)}$ is contained in $B_r(f^0_{\lambda^+})$. Hence, by (2.21), the operators $\mathcal{M}_{\lambda^+,\kappa}$ are strict contractions there with rate q. Next, since $f_0 \in \mathcal{B} \cap \mathcal{C}^\infty(\mathbb{R}_+)$ with $f'_0 \in \mathcal{B}$, $\kappa \mapsto \mathcal{M}_{\lambda^+,\kappa}(f_0)$ is continuous as a map from $\mathbb{R}^*_+$ to $\mathcal{B}_{\alpha\beta}$ for all $\alpha, \beta > 0$. This implies that there is a $\delta = \delta(\tilde{\varepsilon})$ such that $\mathcal{M}_{\lambda^+,\kappa}$ maps the ball $\overline{B_{\tilde{\varepsilon}}(f_0)}$ into itself for each κ with $|\kappa - \kappa_0| < \delta$: for $f \in \overline{B_{\tilde{\varepsilon}}(f_0)}$, it follows from the continuity of the map $\kappa \mapsto \mathcal{M}_{\lambda^+,\kappa}(f_0)$ and $q < 1$ that

$$\begin{aligned}
\|\mathcal{M}_{\lambda^+,\kappa}(f) - f_0\|_{\mu\nu} &\leq \|\mathcal{M}_{\lambda^+,\kappa}(f) - \mathcal{M}_{\lambda^+,\kappa}(f_0)\|_{\mu\nu} + \|\mathcal{M}_{\lambda^+,\kappa}(f_0) - f_0\|_{\mu\nu} \\
&\leq q\tilde{\varepsilon} + \|\mathcal{M}_{\lambda^+,\kappa}(f_0) - \mathcal{M}_{\lambda^+,\kappa_0}(f_0)\|_{\mu\nu} \\
&< \tilde{\varepsilon}\,.
\end{aligned}$$

Therefore, the contraction mapping principle implies the existence of a fixed point of $\mathcal{M}_{\lambda^+,\kappa}$ in the ball $B_{\tilde{\varepsilon}}(f_0)$ whenever $|\kappa - \kappa_0| < \delta$. By uniqueness of the fixed points $f_{\lambda^+,\kappa}$, one concludes that $\|f_{\lambda^+,\kappa} - f_{\lambda^+,\kappa_0}\|_{\mu\nu} < \tilde{\varepsilon}$ for all κ satisfying $|\kappa - \kappa_0| < \delta$. ∎

Proposition 2.6 reduces the proof of Theorem 1.4 to the verification of the estimates (2.12), (2.20) and (2.21). This verification is computer–assisted, and yields $q \approx 0.85$ and $\varepsilon \approx 1.15 \cdot 10^{-4}$. The function $f^0_{\lambda^+}$ has been numerically determined to be a very good approximation of the fixed point f_{λ^+}. It is given by the linear interpolation of 2^{17} positive numbers at well chosen points. Regarding the computation of the norm of the tangent map $D\mathcal{M}_{\lambda^+,\kappa}(f)$, we will take advantage of the fact that $D\mathcal{N}_\lambda(f)$ has very good contraction properties on certain subspaces of finite codimension provided f has some regularity. In particular, the nontrivial action of the operator M can be restricted to a finite dimensional subspace, and the computation of the norm of $D\mathcal{M}_{\lambda^+,\kappa}(f)$ essentially requires to explicitly evaluate $D\mathcal{M}_{\lambda^+,\kappa}(f)$ on finitely many basis vectors.

The remainder of this paper is devoted to the proof of Proposition 2.6 and the verification of inequality (2.12). In Section 3, we review the basic approach of computer–assisted proofs, and extend it to function spaces of L^1–type. In Sections 4 and 5, we give a detailed account of the rigorous implementation on a computer of the maps $\mathcal{N}_\lambda$ and of the computation of bound (2.20) and inequality (2.12). Section 6 is devoted to the tangent maps $D\mathcal{N}_\lambda$ and their contraction properties, whereas Section 7 deals with $D\mathcal{M}_{\lambda+,\kappa}$ and the computation of the bound (2.21). `Section.8` is available as a supplement to this paper, and contains the source code of the program (`proof.f`) and two input data files (`fpoint.lp` and `fpoint.lm`). The program has been written in Fortran 77 [†] and consists of a (short) main body and several subroutines ordered in a "bottom–up" hierarchy, accordingly to the organization of the paper. Except for Section 3, references to the program are collected in remarks at the end of each section. For a description of the input data files, see Sections 5.2 and 5.3.

3. Constructive Analysis in $\mathcal{B}_{\alpha\beta}$

Computer–assisted proofs rely on the ability, first, to discretize the problem under study in terms of objects that are representable on a computer, and, second, to have a rigorous control on the errors arising from the discretization. We note that in this respect, arithmetic operations are special, since controlling them rigorously requires an explicit knowledge of how rounding is performed by the computer. Nevertheless, we emphasize that the main difficulties related to discretization are usually concerned with the specific transformations involved in the functional equation under study, the control of numerical rounding being typically of no particular relevance.

To address discretization issues, one introduces the notions of bounds and standard sets. Denoting, for any set Σ, by $\mathcal{P}(\Sigma)$ the set of all subsets of Σ, we start by defining what we call a bound in the context of computer–assisted proofs.

Definition 3.1. *Let φ be a map from $D_\varphi \subseteq \Sigma$ to Σ'. Denote by f the set map obtained by lifting φ in the canonical way, that is, $D_f = \mathcal{P}(D_\varphi)$ and $f(S) = \{s' \in \Sigma' \mid s' = \varphi(s) \text{ for some } s \in S\}$ for every $S \in D_f$. We say that a set map $g : \mathcal{P}(\Sigma) \supseteq D_g \to \mathcal{P}(\Sigma')$ is a bound on φ if $D_f \supseteq D_g$ and if $f(S) \subseteq g(S)$ for all $S \in D_g$.*

We remark that g being a bound on φ means $\varphi(s) \in g(S)$ whenever $s \in S \in D_g$. Bounds of this type have the following two properties. First, they make it possible to estimate complicated maps in terms of simpler ones, since the composition of two bounds, if well defined, provides a bound on the composition. Second, they can be implemented on a computer. Indeed, given a set Σ, we start by specifying a finite collection of sets $\mathrm{std}(\Sigma) \subset \mathcal{P}(\Sigma)$ that can be represented on the computer with a given data type. The elements of $\mathrm{std}(\Sigma)$ are referred to as the *standard sets* of Σ. Next, given a map φ: $\Sigma \supseteq D_\varphi \to \Sigma'$, we construct a bound on φ within the class of maps g: $\mathrm{std}(\Sigma) \supseteq D_g \to \mathrm{std}(\Sigma')$. Finally, it is in general possible to characterize the images of g in $\mathrm{std}(\Sigma')$ constructively and implement this map on the computer. We note that one can usually choose $\mathrm{std}(\Sigma)$ and $\mathrm{std}(\Sigma')$ specifically adapted to the map φ in order to improve the bounds g that can be constructed.

[†] Mostly standard Fortran 77. Some extensions that can be found on most popular compilers (SUN(TM), Microsoft(TM)PowerFortran,...) are used for convenience. In particular, we make use of the double precision complex data type `complex*16`.

Unless specified otherwise, the standard sets for a Cartesian product $\Sigma \times \Sigma'$ will be defined by setting $\mathrm{std}(\Sigma \times \Sigma') = \mathrm{std}(\Sigma) \times \mathrm{std}(\Sigma')$.

3.1. Operations Involving Real and Complex Numbers

In our application of the above mentioned procedure to the case of real numbers, we have followed the approach of [KSW] which is based on the 64 bit IEEE standard for floating point arithmetics. This standard specifies two things: a format for floating point numbers (IEEE numbers) and rules concerning rounding after the operations $+, -, *, /$ and $\sqrt{\ }$. We will not discuss the detail of the implementation, but refer the interested reader to the corresponding section of [KSW]. We first choose a subset $\mathcal{S}$ of IEEE numbers, the "safe range", for which no underflows nor overflows can occur. The standard sets of $\mathbb{R}$ and $\mathbb{R}_+^*$ are defined as follows.

Definition 3.2. *We define* $\mathrm{std}(\mathbb{R})$ *as the collection of all (closed) intervals* $[a, b]$ *with* $a \le b$ *elements of* $\mathcal{S}$. *We define* $\mathrm{std}(\mathbb{R}_+^*)$ *as the subset of* $\mathrm{std}(\mathbb{R})$ *made of intervals* $[a, b]$ *with* $a > 0$.

To represent an interval in $\mathrm{std}(\mathbb{R})$ on the computer, we use for convenience the data type for complex numbers available in Fortran. Given $a \le b \in \mathcal{S}$, the procedure `sbound` returns the interval $[a, b]$, whereas, given the interval $[a, b] \in \mathrm{std}(\mathbb{R})$, `rl` and `ru`, respectively, returns a and b. We add two more functions, `siconst` and `srconst`, which, given $r \in \mathcal{S}$ an integral constant, and, respectively, $r \in \mathcal{S}$ an IEEE number, return the (unique) singleton in $\mathrm{std}(\mathbb{R})$ containing r.

By using the IEEE specifications related to the rounding occurring after the operations $+, -, *, /$ and $\sqrt{\ }$, one first writes two functions, `rup` and `rdown`, which, given r_c the rounded result of any of these operations, compute an upper bound and a lower bound, respectively, on the exact result r. If these bounds do not belong to $\mathcal{S}$, a flag is raised and the program stops. In a straightforward manner, one next constructs bounds in $\mathrm{std}(\mathbb{R})$ (in the sense of Definition 3.1) on the maps $x \mapsto -x$ (`sneg`), $|x|$ (`sabs`), $1/x$ (`sinv`), x^2 (`spower2`), $\sqrt{x}$ (`ssqrt`), and $(x, y) \mapsto x + y$ (`ssum`), $x - y$ (`sdiff`), $x * y$ (`sprod`) and x/y (`squot`). We will also need a bound on the function $x \mapsto \exp(x)$. The precision with which this function is evaluated is not specified by the IEEE standard. Hence, we make use of the bounds constructed so far and compose them in the following way. First, we use $\exp(nx) = \exp(x)^n$ to restrict the Taylor expansion of $\exp(x)$ to cases where $|x| < 0.03$. Next, we compute the first three terms in the expansion and bound the tail by a geometrical series. This bound is implemented in the function `sexp`. We note that it is only involved in the computation of the weight $w_{\alpha\beta}$ and is not required to be of great accuracy. Finally, we will need to evaluate for x close to zero and $n = 0, \ldots, 3$,

$$\mathrm{Log}_n(x) \equiv -(-x)^{-n} \sum_{k=n+1}^{\infty} \frac{(-x)^k}{k}. \tag{3.1}$$

Note that the second factor is just the tail of the Taylor expansion of $\log(1+x)$. In particular, $\mathrm{Log}_0(x) = \log(1+x)$. One easily checks that the inequalities

$$-\sum_{k=1}^{m} \frac{(-x)^k}{k+n} - \left|\frac{x^{m+1}}{m+n+1}\right| \leq \mathrm{Log}_n(x) \leq -\sum_{k=1}^{m} \frac{(-x)^k}{k+n} + \left|\frac{x^{m+1}}{m+n+1}\right|$$

are valid for all $m \geq 1$. With $m = 4$, a sufficiently accurate bound is constructed in `selogne` from the previous inequalities.

We end this section with the discussion of a bound on the discrete convolution. For $r = (r_0, \ldots, r_{n-1})$ and $s = (s_0, \ldots, s_{n-1}) \in \mathbb{R}^n$, the discrete convolution $r * s$ is the element of $\mathbb{R}^{2n-1}$ given by

$$(r * s)_k = \sum_{i+j=k} r_i s_j, \quad k = 0, \ldots, 2n-2. \tag{3.2}$$

Computing $r * s$ according to (3.2) involves $\mathcal{O}(n^2)$ operations, and becomes impractical for large n. The standard strategy is to go into Fourier space where the convolution becomes the pointwise product of vectors. The gain in computational time is due to the fact that the discrete Fourier transform can be implemented with an $\mathcal{O}(n \log_2 n)$ algorithm, known as the Fast Fourier Transform (FFT). More precisely, the discrete Fourier transform is a map from $\mathbb{C}^n$ to $\mathbb{C}^n$ defined by

$$\big(\mathcal{F}(z)\big)_k = \sum_{j=0}^{n-1} \exp\Big(i\frac{2\pi kj}{n}\Big) z_j, \quad k = 0, \ldots, n-1, \tag{3.3}$$

for $z = (z_0, \ldots, z_{n-1}) \in \mathbb{C}^n$. The inverse Fourier transform $\mathcal{F}^{-1}$ is given by

$$\big(\mathcal{F}^{-1}(z)\big)_k = \frac{1}{n}\sum_{j=0}^{n-1} \exp\Big(-i\frac{2\pi kj}{n}\Big) z_j, \quad k = 0, \ldots, n-1. \tag{3.4}$$

For $r, s \in \mathbb{R}^n$ as above, one has the well known relation

$$(r * s)_k = (\mathcal{F}^{-1}(\mathcal{F}(\tilde{r}) \cdot \mathcal{F}(\tilde{s})))_k, \quad k = 0, \ldots, 2n-2, \tag{3.5}$$

where $\tilde{r} = (r_0, \ldots, r_{n-1}, 0, \ldots, 0) \in \mathbb{R}^{2n}$ and $\tilde{s} = (s_0, \ldots, s_{n-1}, 0, \ldots, 0) \in \mathbb{R}^{2n}$. An efficient implementation of (3.3) and (3.4) follows from the observation that if n is odd, then $\mathcal{F}(z)$ can be decomposed into the sum of the Fourier transform of two vectors in $\mathbb{C}^{n/2}$. Hence, for n a power of 2, one can repeat this decomposition $\log_2 n$ times until only the Fourier transform of a single complex number remains to be computed. $\mathcal{F}(z)$ is finally obtained by summing the intermediate Fourier transforms, which needs $\mathcal{O}(n)$ operations.

In order to implement a bound on the discrete Fourier transform, we first need to choose the standard sets of $\mathbb{C}$. For our purpose, it is sufficient to define them in terms of $\mathrm{std}(\mathbb{R})$ as follows.

Definition 3.3. *We define* $\mathrm{std}(\mathbb{C})$ *to be the collection of all sets* $R + i * I$ *of the form*

$$R + i * I = \{x + i \cdot y \in \mathbb{C} \mid x \in R,\, y \in I\} \tag{3.6}$$

with R *and* I *elements of* $\mathrm{std}(\mathbb{R})$.

The only operations in $\mathbb{C}$ involved in the FFT algorithm are the addition and product, bounds on which are readily implemented from our bounds acting on $\mathrm{std}(\mathbb{R})$. One also needs bounds on the trigonometric factors appearing in (3.3) and (3.4). From the periodicity properties of the functions sin and cos, one first notes that it is sufficient to construct a bound on the maps $(l, m) \mapsto \sin(l\pi/m), \cos(l\pi/m)$ where $m \geq 4$ is a power of 2 and l ranges in $\{0, \ldots, m/4\}$. The case $l = 0$ is trivial. For $l = 1$ and $m = 4, 8, \ldots$, one evaluates sin and cos recursively: For $m = 4$ one has $\cos(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$, and for $m > 4$ a power of 2 one uses the half angle formulas

$$\cos(x/2) = \sqrt{\frac{1 + \cos(x)}{2}}, \quad \sin(x/2) = \frac{1}{2}\frac{\sin(x)}{\cos(x/2)}.$$

Finally, for $l > 1$, one applies the double angle formulas.

Remark. Bounds on the Fourier transform and inverse Fourier transform are implemented in the procedure `fft` according to the FFT algorithm. We do not enter into the details of this algorithm, and refer the reader to [PFTV], from which the code has been adapted to interval analysis using the bounds described above. Adapting again to interval analysis a code from [PFTV], a bound on the discrete convolution $r \mapsto r * r$ is implemented in the procedure `fastconvolution1`, while the general case $(r, s) \mapsto r * s$ is implemented in `fastconvolution2`. Those bounds are restricted to vectors whose dimension is a power of 2. In the sequel, we actually compute the discrete convolution of vectors of the form $r = (0, r_1, \ldots, r_n, 0)$, n a power of 2. The first two elements and last two elements of such convolution are trivially zero and are updated directly in the procedures `fastconvolution1` and `fastconvolution2`. For convenience later on, see Section 4.4, we also add at the beginning and the end of the result one element zero.

3.2. Standard Sets of $\mathcal{B}_{\alpha\beta}$

We now describe the standard sets of the Banach spaces $\mathcal{B}_{\alpha\beta}$. As mentioned above, the choice of these sets should be adapted to the problem in order to optimize the bounds that one needs to construct. Although functions in $\mathcal{B}_{\alpha\beta}$ are in general irregular, the fixed points of the maps $\mathcal{N}_\lambda$ are smooth. Furthermore, these maps are continuous and preserve the regularity. Therefore, we will take for our standard sets of $\mathcal{B}_{\alpha\beta}$ balls centered at regular functions. To represent a regular function on the computer, we will rely on the approximation scheme of spline interpolation.

A spline function of order n is a function in $\mathcal{C}^{n-1}$ which is piecewise polynomial, each of the polynomials being of degree n. For our purpose, it is sufficient to consider splines of order one as the centers of our standard sets, i.e., continuous piecewise affine functions. This choice is a compromise between the quality of the approximation and the simplicity of the bounds that we will have to construct. Note that increasing the order of the interpolation does not lead in general to better approximations. Indeed, for a function $f \in \mathcal{C}^\infty([a,b])$ and a typical partition of $[a,b]$ with mesh size $\varepsilon > 0$, the associated interpolation of f by a spline g of order $n-1$ satisfies for a norm of L^1–type

$$\|f - g\| \approx \varepsilon^n \|f^{(n)}\|.$$

Hence, depending upon the behavior of $f^{(n)}$, it can become better to consider finer partitions of $[a,b]$ rather than to increase the order of the interpolation.

We now introduce a few objects that will be used to define the standard sets of $\mathcal{B}_{\alpha\beta}$.

Definition 3.4. *For $n \geq 2$, we denote by $\mathcal{P}_n$ the set of all partitions p of $\mathbb{R}_+^*$ of the form*

$$p = \{0 < x_0 < x_1 < \ldots < x_n < \infty\}. \tag{3.7}$$

Furthermore, we denote by $\mathcal{P}_n^u$ the subset of $\mathcal{P}_n$ made of uniform partitions, i.e., partitions $p = \{x_i\}_{i=0}^n \in \mathcal{P}_n$ satisfying $x_i - x_{i-1} = \varepsilon$, $i = 1, \ldots, n$, for some $\varepsilon > 0$.

The uniform partitions have been introduced in order to simplify the implementation of a bound on the convolution operator. For $p = \{x_i\}_{i=0}^n \in \mathcal{P}_n$ and $\lambda > 0$, we adopt the convention to denote by λp the partition $\{\lambda x_i\}_{i=0}^n$. Next, we describe more precisely the piecewise affine functions we will work with.

Definition 3.5. *We define $\mathcal{A}$ to be the set of all functions $\rho \in \mathcal{C}^0(\mathbb{R}_+)$ for which there is an $n \geq 2$ and a partition $p = \{x_i\}_{i=0}^n \in \mathcal{P}_n$ such that ρ is affine on $[x_{i-1}, x_i]$ for $i = 1, \ldots, n$ and $\rho(x) = 0$ for $x \notin (x_0, x_n)$. Furthermore, $\mathcal{A}^u$ denotes the subset of $\mathcal{A}$ consisting of those functions for which p can be chosen uniform.*

We note that $\mathcal{A}, \mathcal{A}^u \subset \mathcal{B}_{\alpha\beta}$ for all $\alpha, \beta \geq 0$. Given a partition $p = \{x_i\}_{i=0}^n \in \mathcal{P}_n$ and a set of values $v = \{v_i\}_{i=0}^n \in \mathbb{R}^{n+1}$ satisfying $v_0 = v_n = 0$, we denote by $\mathcal{T}_1(p, v)$

the linear interpolation of (p,v) in $\mathcal{A}$, i.e.,

$$\mathcal{T}_1(p,v)(x) = \begin{cases} v_i + \frac{v_{i+1}-v_i}{x_{i+1}-x_i}(x-x_i) \text{ for } x\in[x_i,x_{i+1}] \text{ and } i\in\{0,\ldots,n-1\}, \\ 0 \qquad \text{otherwise.} \end{cases} \tag{3.8}$$

Conversely, associated with every function $\rho\in\mathcal{A}$, there is a pair (p,v) in $\mathcal{P}_n\times\mathbb{R}^{n+1}$, for some $n\geq 2$, satisfying $\mathcal{T}_1(p,v)=\rho$. If $\rho\not\equiv 0$ and if one imposes a minimality condition on n, then the associated pair (p,v) is unique.

Definition 3.6. *For $\rho\not\equiv 0$ a function in $\mathcal{A}$, let*

$$n(\rho)\equiv\min\{n\geq 2 \mid \exists\ (p,v)\in\mathcal{P}_n\times\mathbb{R}^{n+1} \text{ such that } \mathcal{T}_1(p,v)=\rho\},$$

and define $\pi(\rho)$ to be the (unique) element of $\mathcal{P}_{n(\rho)}\times\mathbb{R}^{n(\rho)+1}$ satisfying $\mathcal{T}_1\big(\pi(\rho)\big)=\rho$.

Note that by definition of $\mathcal{A}$, one has always $\pi(\rho)=(\cdot,\{\rho_i\}_{i=0}^{n(\rho)})$ with $\rho_0=\rho_{n(\rho)}=0$. In order to define the standard sets of $\mathcal{A}$ and $\mathcal{A}^u$, we need to choose the standard sets of $\mathcal{P}_n$ and $\mathcal{P}_n^u$.

Definition 3.7. *For $n\geq 2$, we define $\mathrm{std}(\mathcal{P}_n)$ to be the collection of all sets $(X_0,\ldots,X_n)$ of the form*

$$(X_0,\ldots,X_n)=\big\{\{x_i\}_{i=0}^n\in\mathcal{P}_n \mid x_0\in X_0,\ldots,x_n\in X_n\big\}, \tag{3.9}$$

with $X_0,\ldots,X_n$ any increasing sequence of $n+1$ pairwise disjoint elements of $\mathrm{std}(\mathbb{R}_+^*)$. Similarly, we define $\mathrm{std}(\mathcal{P}_n^u)$ as the collection of all sets (A,E) of the form

$$(A,E)=\big\{p\in\mathcal{P}_n^u \mid p=\{a+i\varepsilon\}_{i=0}^n, a\in A \text{ and } \varepsilon\in E\big\}, \tag{3.10}$$

with $A,E\in\mathrm{std}(\mathbb{R}_+^*)$.

Note that $\mathrm{std}(\mathcal{P}_n^u)$ is not a subset of $\mathrm{std}(\mathcal{P}_n)$. Indeed, the sets (A,E) contain only uniform partitions, whereas there are always non–uniform partitions in each set $(X_0,\ldots,X_n)$ which is not a singleton. The standard sets of $\mathcal{A}$ and $\mathcal{A}^u$ are defined in terms of $\mathrm{std}(\mathcal{P}_n)$ and $\mathrm{std}(\mathcal{P}_n^u)$ as follows.

Definition 3.8. *Let $N=2^{20}$. We define $\mathrm{std}(\mathcal{A})$, respectively $\mathrm{std}(\mathcal{A}^u)$, to be the collection of all sets (P,V) of the form*

$$(P,V)=\big\{\rho\in\mathcal{A} \mid \rho=\mathcal{T}_1(p,v), p\in P \text{ and } v\in V\big\}, \tag{3.11}$$

with $P\in\mathrm{std}(\mathcal{P}_n)$, respectively $P\in\mathrm{std}(\mathcal{P}_n^u)$, $V\in\mathrm{std}(\mathbb{R}^{n+1})$ and $2\leq n\leq N$.

Finally, we introduce the standard sets of $\mathcal{B}_{\alpha\beta}$. They will be of two types, denoted by $\mathrm{std}(\mathcal{B}_{\alpha\beta})$ and $\mathrm{std}(\mathcal{B}_{\alpha\beta})^u$.

Definition 3.9. *Let $\alpha \geq 0$ and $\beta \geq 0$. We define* $\mathrm{std}(\mathcal{B}_{\alpha\beta})$*, respectively* $\mathrm{std}(\mathcal{B}_{\alpha\beta})^u$*, to be the collection of all sets (P, V, G) of the form*

$$(P, V, G) = \big\{f \in \mathcal{B}_{\alpha\beta} \mid f = \rho + g,\ \rho \in (P, V),\ g \in \mathcal{B}_{\alpha\beta} \text{ and } \|g\|_{\alpha\beta} \leq G\big\}, \tag{3.12}$$

with $(P, V) \in \mathrm{std}(\mathcal{A})$, respectively $(P, V) \in \mathrm{std}(\mathcal{A}^u)$, and $G \in \mathcal{S}$, $G \geq 0$.

Hence, a set (P, V, G) is the union of all balls of radius G that are centered at piecewise affine functions belonging to (P, V).

Remark. In our program, the data type with which a set (P, V, G) is represented, with $P \in \mathrm{std}(\mathcal{P}_n)$ and $V \in \mathrm{std}(\mathbb{R}^{n+1})$, is a $2 \times (n + 2)$ matrix, say `f`, with entries of complex data type. (Recall that we use the Fortran data type for complex numbers to represent the elements of $\mathrm{std}(\mathbb{R})$ and $\mathrm{std}(\mathbb{R}_+^*)$.) The entries `f(1,0)` up to `f(1,`n`)` contain V, and `f(1,`$n+1$`)` contains the interval $[0, G]$. The entries `f(0,0)` up to `f(0,`n`)` contain the partition P. If $P = (A, E) \in \mathrm{std}(\mathcal{P}_n^u)$, then `f(0,0)`$= A$ and `f(0,`$n + 1$`)`$=$ E. If $P \notin \mathrm{std}(\mathcal{P}_n^u)$, then `f(0,`$n + 1$`)`$=$ $[0, 0]$. Given an integer $n \geq 2$ and $a, \varepsilon \in \mathcal{S}$, $a, \varepsilon > 0$, the procedure `fzero` returns a standard set $(P, V, G) \in \mathrm{std}(\mathcal{B}_{\alpha\beta})^u$ where $G = 0$, $V = ([0, 0], \ldots, [0, 0])$ and P contains the partition $\{a + i\varepsilon\}_{i=0}^n$. Finally, given $(P, V, G) \in \mathrm{std}(\mathcal{B}_{\alpha\beta})$ and an integer i, the procedure `get_f_on_i` returns two elements of $\mathrm{std}(\mathbb{R}_+^*)$ and two elements of $\mathrm{std}(\mathbb{R})$ containing respectively x_{i-1}, x_i, ρ_{i-1} and ρ_i for all $\rho \in (P, V)$, $\pi(\rho) = (\{x_j\}, \{\rho_j\})_{j=0}^n$.

We end this section with a few comments about the strategy that we will adopt when constructing bounds on the various maps that enter the definition of $\mathcal{N}_\lambda$. Some of these maps are linear and preserve $\mathcal{A}$. Let $\mathcal{L}$ be such a map. Then, for $f = \rho + g$ with $\rho \in \mathcal{A}$, the piecewise affine part ρ and the general term g can be treated separately, and since the piecewise affine parts carry the relevant information, it is natural to describe the affine part of $\mathcal{L}(f)$ by $\mathcal{L}(\rho)$ and its general term by $\mathcal{L}(g)$. Moreover, the choice of the standard set image in $\mathrm{std}(\mathcal{A})$ containing $\mathcal{L}(\rho)$ is straightforward. For instance, the product of a function $f \in \mathcal{B}_{\alpha\beta}$ by a scalar $\lambda \in \mathbb{R}$ is bounded using $\pi(\lambda\rho) = (p, \lambda v)$, where $(p, v) = \pi(\rho)$, and $\|\lambda g\|_{\alpha\beta} = |\lambda|\, \|g\|_{\alpha\beta}$. This bound is implemented in the procedure `fmult`. In general, however, the maps that will be considered do not preserve $\mathcal{A}$. Let $\mathcal{U} : \mathcal{B}_{\alpha\beta} \to \mathcal{B}_{\zeta\eta}$ be such a transformation. For $f = \rho + g$ with $\rho \in \mathcal{A}$ and $g \in \mathcal{B}_{\alpha\beta}$, we write

$$\mathcal{U}(\rho + g) = \mathcal{U}(\rho) + \mathcal{W}(\rho, g).$$

Since $\mathcal{U}(\rho) \notin \mathcal{A}$, we will consider the linear interpolation $\tilde{\rho} \in \mathcal{A}$ of $\mathcal{U}(\rho)$ at well chosen points. This choice will usually be a compromise between the quality of the approximation and the simplicity of the implementation. One then has

$$\mathcal{U}(\rho + g) = \tilde{\rho} + \mathcal{W}(\rho, g) + \big(\mathcal{U}(\rho) - \tilde{\rho}\big),$$

and the last two terms on the RHS correspond to the general term $\tilde{g}$ of $\mathcal{U}(\rho + g)$. They will be bounded using first

$$\|\tilde{g}\|_{\zeta\eta} \leq \|\mathcal{W}(\rho, g)\|_{\zeta\eta} + \|\mathcal{U}(\rho) - \tilde{\rho}\|_{\zeta\eta}.$$

Next, explicit formulas involving ρ, g and the values of $\mathcal{U}(\rho)$ at the chosen interpolation points, together with the use of interval analysis, will lead to a rigorous upper bound on the previous expression, and hence to the representable $G \in \mathcal{S}$ entering Definition 3.9. We note that the elements of $\mathrm{std}(\mathbb{R})$ defining the piecewise affine function $\tilde{\rho}$ consist in general of intervals of non zero length. Nevertheless, since the bound G has been computed for all reals in those intervals, one can "close" each of them by picking arbitrarily one of the representable numbers it contains. This will prevent the standard sets containing the piecewise affine part from "opening up" substantially when bounds are composed, in particular when evaluating convolution products.

4. Operations Involving Functions

In this section, we construct bounds (in the sense of Section 3) on the various maps that enter the definition of the transformations $\mathcal{N}_\lambda$. Most of the bounds given here follow from direct calculations and are easy to prove. We have grouped some of these calculations in the appendix.

In the following, we will usually consider $f \in \mathcal{B}_{\alpha\beta}$ of the form $f = \rho + g$, where $\rho \in \mathcal{A}$ will always stand for the piecewise affine part of f and $g \in \mathcal{B}_{\alpha\beta}$ for the general term of f. Furthermore, when not explicitly mentioned otherwise, $n(\rho)$ and $\pi(\rho) = (p, v)$ will be denoted by n and $(\{x_i\}_{i=0}^n, \{\rho_i\}_{i=0}^n)$, respectively. Finally, we denote the interval $[x_{i-1}, x_i]$ by I_i, $i = 1, \ldots, n$.

4.1. Elementary Operations

We start with the map $f \mapsto \|f\|_{\alpha\beta}$, a bound on which is constructed from $\mathrm{std}(\mathcal{B}_{\alpha\beta})$ to $\mathrm{std}(\mathbb{R}_+)$ using the triangle inequality and, for the piecewise affine part, the estimate

$$\begin{aligned}\|\rho\|_{\alpha\beta} &= \sum_{i=1}^{n} \int_{I_i} w_{\alpha\beta}(x)\, |\rho(x)|\, dx \\ &\leq \sum_{i=1}^{n} \sup_{x\in I_i} \big(w_{\alpha\beta}(x)\big)(x_i - x_{i-1}) \frac{|\rho_{i-1}| + |\rho_i|}{2}. \end{aligned} \tag{4.1}$$

The convexity of $w_{\alpha\beta}$ leads to

$$\sup_{x\in I_i} w_{\alpha\beta}(x) = \max\{w_{\alpha\beta}(x_{i-1}), w_{\alpha\beta}(x_i)\}. \tag{4.2}$$

We next consider the mass $M(f)$ and the expectation $E(f)$ of a function $f \in \mathcal{B}_{\alpha\beta}$, for which it will be sufficient to construct bounds from $\mathrm{std}(\mathcal{B}_{\alpha\beta})^u$ to $\mathrm{std}(\mathbb{R})$. By linearity, we can first treat separately the affine part ρ, and by using $\rho_0 = \rho_n = 0$, a direct calculation yields

$$M(\rho) = \varepsilon \sum_{i=1}^{n-1} \rho_i, \tag{4.3}$$

$$E(\rho) = \varepsilon \sum_{i=1}^{n-1} \rho_i x_i, \tag{4.4}$$

where $\varepsilon = x_1 - x_0$. Using $M(\rho) - |M(g)| \leq M(f) \leq M(\rho) + |M(g)|$ and the corresponding inequality for $E(f)$, we get the desired bounds by estimating the mass and expectation of the general term g with (2.13) and (2.14). The supremum of $1/w_{\alpha\beta}$ appearing in (2.13) is taken at $x_c = \sqrt{\alpha/\beta}$, which leads to

$$\sup_{x>0} \Big(\frac{1}{w_{\alpha\beta}(x)} \Big) = \exp(-2\sqrt{\alpha\beta}). \tag{4.5}$$

Similarly, one computes

$$\sup_{x>0} \Big(\frac{x}{w_{\alpha\beta}(x)} \Big) = \frac{1 + \sqrt{1 + 4\alpha\beta}}{2\beta} \exp\big(-\sqrt{1 + 4\alpha\beta}\big). \tag{4.6}$$

We end this section with the discussion of a bound on the addition of two functions f_1, $f_2 \in \mathcal{B}_{\alpha\beta}$. Due to the linearity of this map and the fact that the addition of two functions in $\mathcal{A}$ is again in $\mathcal{A}$, it is natural to choose for the general term of $f_1 + f_2$ the addition of the general terms of those two functions, whose norm is bounded by using the triangle inequality. It then remains to construct a bound on the map $+ : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$. Let $\rho_1, \rho_2 \in \mathcal{A}$ such that $\pi(\rho_1) = (p_1, v_1) \in \mathcal{P}_n \times \mathbb{R}^{n+1}$ and $\pi(\rho_2) = (p_2, v_2) \in \mathcal{P}_m \times \mathbb{R}^{m+1}$. If ρ_1 and ρ_2 have no common nodes, then $(p, w) \equiv \pi(\rho_1 + \rho_2) \in \mathcal{P}_{n+m+1} \times \mathbb{R}^{n+m+2}$, with p being the refined partition made of the ordered union of p_1 and p_2. The last is valid only if ρ_1 and ρ_2 have no common nodes and we shall construct a bound whose domain is restricted to such cases. Hence, denoting $p = \{y_i\}_{i=0}^{n+m+1}$ and $w = \{w_i\}_{i=0}^{n+m+1}$, there is, for each $i = 0, \ldots, n+m+1$, a unique j such that

$$w_i \equiv (\rho_1 + \rho_2)(y_i) = \begin{cases} (v_1)_j + \rho_2((p_1)_j) & \text{if } y_i \in p \cap p_1, \\ (v_2)_j + \rho_1((p_2)_j) & \text{if } y_i \in p \cap p_2. \end{cases}$$

To implement this bound with interval analysis, we must check first that the nodes in $\mathrm{std}(\mathbb{R}_+^*)$ of the standard sets $P_1 \in \mathrm{std}(\mathcal{P}_n)$ and $P_2 \in \mathrm{std}(\mathcal{P}_m)$ containing the partition p_1 and p_2 are pairwise disjoint intervals. This implies that every function in (P_1, V_1) is linear on each node of P_2, and vice versa. This in turn implies that a bound on the evaluation $\rho_2((p_1)_j)$ and $\rho_1((p_2)_j)$ is readily obtained from (3.8) using interval analysis. We note finally that when the standard set containing ρ_1 (ρ_2) is in $\mathrm{std}(\mathcal{A}^u)$, i.e., with P_1 (P_2) of the form (A, E), we first proceed to the evaluation of the nodes in terms of A and E.

Finally, one constructs a bound on the difference of two functions by composing the previous bound with a bound on the unary minus $\mathcal{B}_{\alpha\beta} \ni f \mapsto -f$ obtained from $||-g||_{\alpha\beta} = ||g||_{\alpha\beta}$ and $\pi(-\rho) = (p, -v)$, where $(p, v) = \pi(\rho)$.

Remark. A bound on the weight $w_{\alpha\beta}$ is computed in the procedure `sw`. The inequality (4.1) is implemented in `snorm_pl`, whereas (4.2) and (4.6) are implemented in `ssup_of_w` and `ssup_of_x_over_w`, respectively. For several intervals $I \subset \mathbb{R}_+^*$, we will need to evaluate later the quantities $\sup_{x\in I} 1/w_{\alpha\beta}(x)$, $\int_I w$, and $|I|^{-1}\int_I w$, where $|I|$ denotes the length of I. By using the convexity of w, a bound on the first quantity is computed in `ssup_of_winverse`, whereas the other two quantities are bounded in `sint_of_w`. The other bounds described in this section are implemented in the procedures `snorm`, `smass`, `sexpectation`, `fadd` and `fdiff`.

4.2. The Scaling Operator

It will be sufficient for our purpose to construct a bound on

$$\begin{aligned} S_\lambda : \mathcal{B}_{\alpha\beta} &\to \mathcal{B}_{\frac{\alpha}{4}\gamma} \\ f(x) &\mapsto h(x) = \lambda f(\lambda x), \end{aligned} \tag{4.7}$$

acting from $\mathrm{std}(\mathcal{B}_{\alpha\beta})^u$ to $\mathrm{std}(\mathcal{B}_{(\alpha/4)\gamma})^u$. We recall that the scaling operators are bounded under constraints which translate in this particular case into $\lambda \le 4$ (if $\alpha > 0$) and $\gamma \le \lambda\beta$. It will be checked by the program that these inequalities are satisfied for the values of λ, α, β and γ we will use. Since S_λ is linear and preserves $\mathcal{A}^u$, we can treat separately the piecewise affine part and the general term. A bound on $S_\lambda : \mathcal{A}^u \to \mathcal{A}^u$ is obtained from the relation $\pi(S_\lambda\rho) = (p/\lambda, \lambda v)$, where $(p, v) = \pi(\rho)$. For the general term we estimate using (4.5),

$$\begin{aligned} \|S_\lambda g\|_{\frac{\alpha}{4}\gamma} &\le \sup_{x>0} \frac{w_{\frac{\alpha}{4}\gamma}(x/\lambda)}{w_{\alpha\beta}(x)} \|g\|_{\alpha\beta} \\ &= \exp\Big(-2\sqrt{\alpha(1-\lambda/4)(\beta-\gamma/\lambda)}\Big) \|g\|_{\alpha\beta}, \end{aligned} \tag{4.8}$$

the last equality being valid under the conditions on λ, α, β and γ mentioned above. Note that the scaling operator (4.7) is a strict contraction for $\gamma/\beta < \lambda < 4$. This will be used to improve the bound on $\mathcal{N}_\lambda^1(f) = S_\lambda f * S_\lambda f$ that we shall construct later. In (4.7), taking a larger target space, i.e., $\mathcal{B}_{(\alpha/\sigma)\gamma}$ with $\sigma > 4$, would lead to a better contraction. Nevertheless, $\sigma = 4$ is the largest value for which $\mathcal{N}_\lambda^1$ maps $\mathcal{B}_{\alpha\beta}$ into $\mathcal{B}_{\alpha\gamma}$, cf. (2.9).

Remark. A bound on $S_\lambda : \mathcal{A}^u \to \mathcal{A}^u$ is implemented in the procedure `fscale_pl`, and the bound (4.8) in `fscale_gen`. Those two procedures are called in `fscale` to build the desired bound on the operator (4.7).

4.3. The Operator T

We now construct a bound on the operator

$$\begin{aligned} T : \mathcal{B}_{\alpha\beta} &\to \mathcal{B}_{\beta\alpha} \\ f(x) &\mapsto h(x) = \frac{1}{x^2} f(\frac{1}{x}), \end{aligned} \tag{4.9}$$

acting from $\mathrm{std}(\mathcal{B}_{\alpha\beta})^u$ to $\mathrm{std}(\mathcal{B}_{\beta\alpha})$. For $f = \rho + g$ with $\rho \in \mathcal{A}^u$ and $g \in \mathcal{B}_{\alpha\beta}$, one has $Tf = T\rho + Tg$. Since $T\rho$ is not piecewise linear, we must first choose a function $\tilde{\rho} \in \mathcal{A}$ which approximates $T\rho$. Denoting again $\pi(\rho) = (\{x_i\}_{i=0}^n, \{\rho_i\}_{i=0}^n)$, we consider for $\tilde{\rho}$ the linear interpolation of $T\rho$ at the nodes

$$\tilde{x}_i = 1/x_{n-i}, \quad i = 0, \ldots, n. \tag{4.10}$$

Therefore, we define $\tilde{\rho}$ to be

$$\tilde{\rho} = \mathcal{T}_1(\tilde{p}, \tilde{v}), \tag{4.11}$$

where $\tilde{p} = \{\tilde{x}_i\}_{i=0}^n$, and $\tilde{v} = \{\tilde{\rho}_i\}_{i=0}^n$ with

$$\tilde{\rho}_i \equiv (T\rho)(\tilde{x}_i) = x_{n-i}^2 \rho_{n-i}, \quad i = 0, \ldots, n. \tag{4.12}$$

Next, the general term $\tilde{g}$ of Tf is given by $\tilde{g} = T\rho - \tilde{\rho} + Tg$, and we use (2.11) to estimate

$$\|\tilde{g}\|_{\beta\alpha} \le \|T\rho - \tilde{\rho}\|_{\beta\alpha} + \|g\|_{\alpha\beta}. \tag{4.13}$$

In order to bound the first term on the RHS of (4.13), we use again (2.11) together with the linearity of T and the fact that it is an involution. This leads to

$$\begin{aligned} \|T\rho - \tilde{\rho}\|_{\beta\alpha} &= \|\rho - T\tilde{\rho}\|_{\alpha\beta} \\ &\le \sum_{i=1}^n \sup_{x \in I_i} w_{\alpha\beta}(x) \int_{I_i} \left|(\rho - T\tilde{\rho})(x)\right| dx. \end{aligned} \tag{4.14}$$

Finally, an explicit bound on the integral appearing in the previous expression follows from a direct calculation and is given in the next lemma.

Lemma 4.1. *Let $\rho \in \mathcal{A}^u$, and $\tilde{\rho}$ be defined as in (4.11). With $\pi(\rho) = (\{x_i\}_{i=0}^n, \{\rho_i\}_{i=0}^n)$, $\varepsilon = x_1 - x_0$ and $I_i = [x_{i-1}, x_i]$, one has*

$$\begin{aligned} \int_{I_i} |(\rho - T\tilde{\rho})(x)|\, dx \le \frac{\varepsilon}{4}\Big(\log\Big(1 + \frac{\varepsilon}{x_{i-1}}\Big)|\rho_i - \rho_{i-1}| \\ + \varepsilon\Big|\frac{\rho_i}{x_{i-1}} - \frac{\rho_{i-1}}{x_i}\Big| + \varepsilon(x_i - \varepsilon/2)\Big|\frac{\rho_i}{x_{i-1}^2} - \frac{\rho_{i-1}}{x_i^2}\Big|\Big). \end{aligned} \tag{4.15}$$

Remark. A bound on $T : \mathcal{B}_{\alpha\beta} \to \mathcal{B}_{\beta\alpha}$ is implemented as described here in the procedure `ft`.

4.4. The Convolution

For $\alpha, \beta > 0$ and $\gamma \in [\alpha, 4\alpha]$, we consider in this section the operator

$$\begin{aligned} C : \mathcal{B}_{\alpha\beta} \times \mathcal{B}_{\alpha\beta} &\to \mathcal{B}_{\gamma\beta} \\ (f, h) &\mapsto f * h. \end{aligned} \tag{4.16}$$

As mentioned before, we specifically introduced standard sets of piecewise affine functions defined on uniform partitions in order to simplify the construction of a bound on the convolution. Hence, our bound will act from $\mathrm{std}(\mathcal{B}_{\alpha\beta})^u \times \mathrm{std}(\mathcal{B}_{\alpha\beta})^u$ to $\mathrm{std}(\mathcal{B}_{\gamma\beta})^u$. To simplify further the explicit expressions which we shall derive below, we restrict its domain to pairs (F_1, F_2) for which the standard sets (A_1, E_1) and (A_2, E_2) containing the partitions associated with the affine functions in F_1 and F_2 satisfy $E_1 = E_2$ and both E_1 and E_2 are singletons. This ensures that all affine functions in F_1 and F_2 are defined on (uniform) partitions with identical mesh size.

Let $f = \rho + g_f$ and $h = \sigma + g_h$, with $\rho, \sigma \in \mathcal{A}^u$ and $g_f, g_h \in \mathcal{B}_{\alpha\beta}$. Then, one has

$$f * h = \rho * \sigma + \rho * g_h + g_f * h. \tag{4.17}$$

The relevant information is carried by the term $\rho * \sigma$. Since it does not belong to $\mathcal{A}$, we will proceed as in the previous section and approximate it by a function $\tilde{\rho} \in \mathcal{A}^u$. The general term of $f * h$ will be given by

$$\tilde{g} = (\rho * \sigma - \tilde{\rho}) + \rho * g_h + g_f * h. \tag{4.18}$$

One can estimate the last two terms on the RHS of the previous expression using the bound (2.9). However, estimating the norm of the first term requires an explicit expression for $(\rho * \sigma)(x)$, $x > 0$. We now derive this expression, which will be used also to specify $\tilde{\rho}$. We first state an intermediate result whose proof can be found in the appendix.

Lemma 4.2. *If $\rho, \sigma \in \mathcal{A}^u$ have uniform partitions with identical mesh size ε, then $(\rho * \sigma)'' \in \mathcal{A}^u$. Furthermore, assume $n(\rho) = n(\sigma) \equiv n$, denote $\pi(\rho) = (\{x_i\}, \{\rho_i\})_{i=0}^n$, $\pi(\sigma) = (\{y_i\}, \{\sigma_i\})_{i=0}^n$, and define $\{S_k\}_{k=0}^{2n}$ to be the discrete convolution of $\{\rho_i\}_{i=0}^n$ and $\{\sigma_i\}_{i=0}^n$, i.e.,*

$$S_k = \sum_{i+j=k} \rho_i \sigma_j. \tag{4.19}$$

Then $n((\rho * \sigma)'') = 2n$, and $(\rho * \sigma)'' = \mathcal{T}_1\big(\{z_k\}_{k=0}^{2n}, \{v_k\}_{k=0}^{2n}\big)$ where

$$z_k = x_0 + y_0 + k\varepsilon, \tag{4.20}$$

$$v_k = \frac{1}{\varepsilon}\big(S_{k+1} - 2S_k + S_{k-1}\big), \tag{4.21}$$

with the convention $S_{-1} = S_{2n+1} = 0$.

We now specify the nature of $\rho * \sigma$.

Lemma 4.3. *Let $\rho, \sigma \in \mathcal{A}^u$ as in Lemma 4.2 and define p to be the partition $\{z_k\}_{k=0}^{2n}$ where z_k is given by (4.20). Then $\rho * \sigma$ is in $\mathcal{C}^2(\mathbb{R}^+)$, has support in $[z_0, z_{2n}]$ and is there identical to its natural cubic spline approximation φ at the nodes of p, i.e., with the boundary conditions $\varphi''(z_0) = \varphi''(z_{2n}) = 0$.*

Proof. First, it follows from $\operatorname{supp}(\rho) \subseteq (x_0, x_n)$ and $\operatorname{supp}(\sigma) \subseteq (y_0, y_n)$ that the support of $\rho * \sigma$ is in $(x_0 + y_0, x_n + y_n)$. Then, the regularity properties of the convolution imply $\rho * \sigma \in \mathcal{C}^2(\mathbb{R}^+)$. To see that $\rho * \sigma$ is equal to its natural cubic spline approximation at the nodes $a \equiv z_0 < z_1 < \ldots < z_{2n} \equiv b$, one shows that it minimizes the quantity

$$\int_a^b \varphi''(x)^2 dx$$

over all $\varphi \in \mathcal{C}^2[a,b]$ satisfying $\varphi(z_k) = (\rho * \sigma)(z_k)$, $k = 0, \ldots, 2n$. Let φ be such a function. Setting $\varphi_0 \equiv \rho * \sigma$, one has

$$\begin{aligned}\int_a^b \varphi''(x)^2 dx &= \int_a^b (\varphi_0''(x) + (\varphi - \varphi_0)''(x))^2 dx \\ &= \int_a^b \varphi_0''(x)^2 dx + 2\int_a^b \varphi_0''(x)(\varphi - \varphi_0)''(x) dx + \int_a^b (\varphi - \varphi_0)''(x)^2 dx. \quad (4.22)\end{aligned}$$

We will see that the second term on the RHS is zero, yielding

$$\int_a^b \varphi''(x)^2 dx \geq \int_a^b \varphi_0''(x)^2 dx.$$

The conclusion then follows from a well known result in spline theory, see for instance [N], which ensures that such a minimization problem has a unique solution given by the natural cubic spline interpolation of the data points entering the constraints of the minimization problem. It remains to see that the second term on the RHS of (4.22) is zero, i.e., that φ_0'' and $(\varphi - \varphi_0)''$ are orthogonal in $L^2[a,b]$. From Lemma 4.2, we have $\varphi_0'' \in \mathcal{A}_p$, where $\mathcal{A}_p$ denotes the subspace of $L^2[a,b]$ consisting of all functions $\tau \in \mathcal{A}$ with $\pi(\tau) = (p_\tau, \cdot)$ and p_τ a subpartition of p. Next, we observe that every $\psi \in \mathcal{C}^2[a,b]$ with $\psi(z_k) = 0$, $k = 0, \ldots, 2n$, satisfies $\psi'' \in \mathcal{A}_p^\perp$: a base of $\mathcal{A}_p$ is given by $\{\tau_k\}_{k=1}^{2n-1}$, τ_k being the "hat" function centered at z_k, i.e., with χ_I the characteristic function of the interval I,

$$\tau_k(x) = \chi_{[z_{k-1}, z_k]}(x)(x - z_{k-1}) + \chi_{(z_k, z_{k+1}]}(x)(z_{k+1} - x),$$

and a simple calculation using integration by parts leads to

$$\int_a^b \tau_k(x)\psi''(x) dx = 0,$$

$k = 1, \ldots, 2n-1$. We conclude the proof by noting that the conditions of the minimization problem are $(\varphi - \varphi_0)(z_k) = 0$, $k = 0, \ldots, 2n$.

■

As a consequence, it follows that $\rho * \sigma$ is given on each interval $[z_k, z_{k+1}]$, $k = 0, \ldots, 2n-1$, by the cubic polynomial

$$(\rho * \sigma)(z_k + \theta) = C_0(k) + C_1(k)\theta + C_2(k)\theta^2 + C_3(k)\theta^3\,, \tag{4.23}$$

$\theta \in [0, \varepsilon]$, where the coefficients $C_i(k)$ take the form

$$\begin{aligned}
C_0(k) &= \frac{\varepsilon}{6}\big(S_{k+1} + 4S_k + S_{k-1}\big),\\
C_1(k) &= \frac{1}{2}\big(S_{k+1} - S_{k-1}\big),\\
C_2(k) &= \frac{1}{2\varepsilon}\big(S_{k+1} - 2S_k + S_{k-1}\big),\\
C_3(k) &= \frac{1}{6\varepsilon^2}\big(S_{k+2} - 3S_{k+1} + 3S_k - S_{k-1}\big),
\end{aligned} \tag{4.24}$$

using again the convention $S_{-1} = S_{2n+1} = 0$. Indeed, $C_2(k)$ is just $(\rho * \sigma)''(z_k)/2$ and has been directly computed in Lemma 4.2. Using $(\rho * \sigma)(z_0) = (\rho * \sigma)(z_{2n}) = 0$, the remaining coefficients are obtained from $C_2(k)$ and the formula for natural cubic spline interpolation, see for instance [ANW].

A natural choice for the affine part $\tilde{\rho}$ of $\rho * \sigma$ would be to consider the linear interpolation of the values of $\rho * \sigma$ at the points z_k, $k = 0, \ldots, 2n$. However, that would amount to double the number of parameters and would lead eventually to memory problems when estimating the precision of the approximate fixed point $f^0_{\lambda+}$. Hence, we choose here to define $\tilde{\rho}$ in terms of the same number of parameters as ρ and σ, and we consider

$$\tilde{\rho} = \mathcal{T}_1\big(\{z_{2l}\}_{l=0}^n, \{C_0(2l)\}_{l=0}^n\big), \tag{4.25}$$

Note that the nodes z_{2l} generate a uniform partition, so that $\tilde{\rho} \in \mathcal{A}^u$.

To conclude this section, we come back to the general term $\tilde{g}$ of $f * h$ given by (4.18). We first use the triangle inequality and (2.9) to get

$$||\tilde{g}||_{\gamma\beta} \leq ||\rho * \sigma - \tilde{\rho}||_{\gamma\beta} + ||\rho||_{\alpha\beta}||g_h||_{\alpha\beta} + ||g_f||_{\alpha\beta}||h||_{\alpha\beta}. \tag{4.26}$$

The bound on the map $f \mapsto ||f||_{\alpha\beta}$ described earlier allows us to estimate the last two terms on the RHS of (4.26). A bound on the first term is obtained by a direct calculation using (4.25) and the explicit expression (4.23). The result is formulated in the next lemma.

Lemma 4.4. *Let ρ, σ and $\tilde{\rho}$ defined as above. Then*

$$||\rho * \sigma - \tilde{\rho}||_{\gamma\beta} \leq \varepsilon^3 \sum_{l=0}^{n-1} \sup_{x \in I_l} w_{\gamma\beta}(x)\Big(\frac{4}{3}|C_2(2l+1)| + \frac{3\varepsilon}{4}\big(|C_3(2l)| + |C_3(2l+1)|\big)\Big). \tag{4.27}$$

where $I_l = [z_{2l}, z_{2l+2}]$.

Remarks.

- By definition of $\mathcal{A}^u$, the first and last two elements of the discrete convolution (4.19) are trivially zero, so that only the convolution of $\{\rho_i\}_{i=1}^{n-1}$ and $\{\sigma_i\}_{i=1}^{n-1}$ needs to be computed. Furthermore, recall that in order to simplify the implementation of the bound on the discrete convolution, we have restricted its domain to the standard sets of $\mathrm{std}(\mathbb{R}^n)$ for which n is a power of 2. Hence, our bound on the convolution (4.16) is defined only on elements of $\mathrm{std}(\mathcal{B}_{\alpha\beta})^u$ with partitions in $\mathrm{std}(\mathcal{P}_n^u)$ such that $n-1$ is a power of 2.
- Bounds on the quantities $\varepsilon^{i-1}C_i(k)$, $i = 0,\dots,3$ respectively, are computed in the procedure `cubic_spline_coeff` and saved in the vectors `st0`, `st1`, `st2` and `st3`. Note that the interpolation (4.25) and the bound (4.27) provide a bound on the convolution from $\mathrm{std}(\mathcal{A}^u) \times \mathrm{std}(\mathcal{A}^u)$ to $\mathrm{std}(\mathcal{B}_{\gamma\beta})$. It is implemented in the procedure `fcubic_to_pwlinear`. Finally, `fconvolute2` computes the desired bound on (4.16), making first use of `fastconvolution2` to get the discrete convolution (4.19), whereas `fconvolute1` is adapted to the special case $f = h$. Those two subroutines have a call to `sexp_of_tconv`, which has been introduced to compute an accurate bound on the expectation of $\mathcal{N}_\lambda^2(f)$, cf. (2.6), and will be explained in Section 5.1.

4.5. The Identity

Another operator we need to consider is the identity. Indeed, we recall that ultimately we want to compose the bounds constructed so far in order to get bounds on the maps of interest. However, the bounds constructed so far do not have always matching range and domain, and cannot in general be composed as such. In particular, the bound on the operator T applies in $\mathrm{std}(\mathcal{B}_{\alpha\beta})$ whereas the convolution is defined only on $\mathrm{std}(\mathcal{B}_{\alpha\beta})^u \times \mathrm{std}(\mathcal{B}_{\alpha\beta})^u$. Furthermore, the bound on the convolution is defined for pairs whose affine parts satisfy constraints on the mesh of their partitions. Hence, we need a bound on the identity map $I : \mathcal{B}_{\alpha\beta} \to \mathcal{B}_{\alpha\beta}$ defined from $\mathrm{std}(\mathcal{B}_{\alpha\beta})$ to $\mathrm{std}(\mathcal{B}_{\alpha\beta})^u$ such that the affine part of all functions in every standard set image is ensured to possess a given partition.

Let $p = (x_0,\dots,x_n) \in \mathcal{P}_n^u$ a fixed but arbitrary uniform partition, and $f = \rho + g$ with $\rho \in \mathcal{A}$ and $g \in \mathcal{B}_{\alpha\beta}$. For the new affine part $\tilde\rho$ of f with partition p, we would like to consider the linear spline interpolation of ρ at the nodes of p. However, in order for $\tilde\rho$ to be in $\mathcal{A}^u$, one must ensure $\tilde\rho$ to be continuous, so that we define

$$\tilde\rho = \mathcal{T}_1\big(p, \{0, \rho(x_1), \dots, \rho(x_{n-1}), 0\}\big). \tag{4.28}$$

Then, from

$$f = \tilde\rho + \big(\rho - \tilde\rho\big) + g,$$

the new general term $\tilde g$ reads $(\rho - \tilde\rho) + g$ and its norm is simply bounded by

$$\|\rho - \tilde\rho + g\|_{\alpha\beta} \le \|\rho - \tilde\rho\|_{\alpha\beta} + \|g\|_{\alpha\beta}. \tag{4.29}$$

The first term on the RHS is bounded using the bounds constructed previously on the norm in $\mathcal{B}_{\alpha\beta}$ and the difference of two functions.

For every uniform partition p, the previous construction leads to a specific bound on the identity map. This bound can be optimized from case to case by adapting p to the function ρ so that $\|\rho - \tilde{\rho}\|_{\alpha\beta}$ is minimal. Again, our approach is to seek for a compromise between accuracy and simplicity of the implementation. First, we choose not to increase the number of parameters from ρ to $\tilde{\rho}$, so that $n \le n(\rho)$. Hence, the only free parameters for p are the first node x_0 and last node x_n. Given a $\tau > 0$, the interval (x_0, x_n) is chosen to be the smallest interval such that $|\rho(x)| < \tau$ for $x \notin (x_0, x_n)$. This interval might be fairly different from $\operatorname{supp}(\rho)$, leading to a mesh ε smaller than $|\operatorname{supp}(\rho)|/n$ and hence a better approximation of ρ on regions where the information is more relevant. The cutoff τ may vary from place to place in the proof and has been determined empirically.

Remark. Given an integer $n \ge 2$, and two positive representable numbers x_0 and s, the procedure `fidentity` constructs a standard set in $\operatorname{std}(\mathcal{P}_n^u)$ containing the uniform partition p with $\operatorname{supp}(p) = (x_0, x_0 + s)$, and computes a bound on the identity map as described above. The representable numbers x_0 and s which describe the support adapted to a given function are determined in the procedure `rsupport`.

5. The Maps $\mathcal{N}_\lambda$

The goal of this section is to explain how bound (2.20) of Proposition 2.6 is computed and how inequality (2.12) of Theorem 2.4 is checked. A major step is to compute $\mathcal{N}_{\lambda^+}$ and $\mathcal{N}_{\lambda^-}$ on various functions of $\mathcal{B}_{\mu\nu}$, with μ, ν, λ^- and λ^+ as in Proposition 2.6. We will see in Section 5.2 that these maps must be estimated from $\mathcal{B}_{\mu\nu}$ to $\mathcal{B}_{\mu(\lambda^+\nu/\lambda^-)}$, a space slightly smaller than $\mathcal{B}_{\mu\nu}$, since $\lambda^+/\lambda^- \approx 1 + 7 \cdot 10^{-7}$. In the sequel we denote $\delta = \lambda^-/\lambda^+$, and begin in the next section by describing the construction of a bound on the maps $\mathcal{N}_\lambda : \mathcal{B}_{\mu\nu} \to \mathcal{B}_{\mu(\nu/\delta)}$.

5.1. A Bound on $\mathcal{N}_\lambda$

We recall that for $\lambda > 0$, $\mathcal{N}_\lambda : \mathcal{B}_{\mu\nu}/\mathcal{H} \to \mathcal{B}_{\mu(\nu/\delta)}$ is well defined provided $1/\delta \le \lambda \le 4$, cf. Proposition 2.2, and is given by

$$\mathcal{N}_\lambda(f) = c_\lambda(f)\mathcal{N}_\lambda^1(f) + c_2\mathcal{N}_\lambda^2(f), \tag{5.1}$$

where

$$\mathcal{N}_\lambda^1(f) = S_\lambda(f * f), \tag{5.2}$$

$$\mathcal{N}_\lambda^2(f) = T\big(T\mathcal{N}_\lambda^1(f) * T\mathcal{N}_\lambda^1(f)\big), \tag{5.3}$$

$$c_\lambda(f) = \frac{\lambda}{2}\frac{1 - c_2E\big(\mathcal{N}_\lambda^2(f)\big)}{M(f)E(f)}. \tag{5.4}$$

The expression (5.4) is more convenient for our present purpose than (2.7). In principle, one readily gets a bound on $\mathcal{N}_\lambda$ by composing the bounds constructed in the previous section. However, one can without too much effort improve this bound in two ways.

First, the distributivity and commutativity properties of the operators involved in (5.1) give us the freedom to choose the order in which the bounds are composed. The order can affect the estimates, since in general these properties are not shared by the bounds. Regarding $\mathcal{N}_\lambda^1$, the fact that S_λ preserves $\mathcal{A}^u$ yields slightly better estimates by using

$$\mathcal{N}_\lambda^1(f) = S_\lambda f * S_\lambda f, \tag{5.5}$$

instead of (5.2). Furthermore, in order to get as much contraction as possible from the scaling operator, cf. Section 4.2, one chooses the sequence of spaces

$$\mathcal{N}_\lambda^1 : \mathcal{B}_{\mu\nu} \xrightarrow{S_\lambda} \mathcal{B}_{\frac{\mu}{4}(\nu/\delta)} \xrightarrow{*} \mathcal{B}_{\mu(\nu/\delta)}. \tag{5.6}$$

A bound on (5.6) follows by composing the bounds of Section 4. We now turn to $\mathcal{N}_\lambda^2$ and, as above, let S_λ act first, considering (5.3) with $\mathcal{N}_\lambda^1$ as in (5.5). Regarding the choice of spaces, we note that one could exploit the operators T and the outer convolution to estimate $\mathcal{N}_\lambda^2$ in the smaller space $\mathcal{B}_{\mu(\nu/\delta)}$, as needed. That would permit us to consider ν instead of ν/δ in (5.6) for which S_λ is a better contraction. Nevertheless, δ is so close to one that it does not lead to any significant improvement, and for convenience one simply constructs a bound on $\mathcal{N}_\lambda^2$ by composing the previous bound on $\mathcal{N}_\lambda^1$ and a bound on the map

$$\mathcal{B}_{\mu(\nu/\delta)} \xrightarrow{T} \mathcal{B}_{(\nu/\delta)\mu} \xrightarrow{*} \mathcal{B}_{(\nu/\delta)\mu} \xrightarrow{T} \mathcal{B}_{\mu(\nu/\delta)}. \tag{5.7}$$

The target space of the convolution above is chosen in order to minimize the norm of the general term arising from the convolution of the piecewise affine part.

The second improvement concerns the computation of the coefficient $c_\lambda(f)$. An estimation of the quantity $E(\mathcal{N}_\lambda^2(f))$ entering (5.4) is of poor quality if obtained by composing the bound on $\mathcal{N}_\lambda^2$ described above and the bound on the expectation as given in the previous section. Exploiting the structure of $\mathcal{N}_\lambda^2$ and the fact that it maps into a smaller space, due to the outer convolution, leads to a substantial improvement. Defining

$$\begin{aligned} \mathcal{E} : \mathcal{B}_{\alpha\beta} \times \mathcal{B}_{\alpha\beta} &\to \mathbb{R} \\ (f, h) &\mapsto E(T(f * h)), \end{aligned} \tag{5.8}$$

one has $E(\mathcal{N}_\lambda^2(f)) = \mathcal{E}(T\mathcal{N}_\lambda^1(f), T\mathcal{N}_\lambda^1(f))$. In order to construct a bound on $\mathcal{E}$ acting from $\mathrm{std}(\mathcal{B}_{\alpha\beta})^u \times \mathrm{std}(\mathcal{B}_{\alpha\beta})^u$ to $\mathrm{std}(\mathbb{R})$, we first observe that for $g \in \mathcal{B}_{\zeta\eta}$,

$$|E(Tg)| \le \int_0^\infty \frac{1}{x}|g(x)|\, dx \le \sup_{x>0} \frac{x}{w_{\eta\zeta}(x)} \|g\|_{\zeta\eta}. \tag{5.9}$$

Next, for $f = \rho + g_f$ and $h = \sigma + g_h$, with $\rho, \sigma \in \mathcal{A}^u$ and $g_f, g_h \in \mathcal{B}_{\alpha\beta}$, one has

$$\mathcal{E}(f,h) = E\big(T(\rho * \sigma)\big) + E\big(T(\rho * g_h + g_f * h)\big), \tag{5.10}$$

and, since $\rho * g_h + g_f * h \in \mathcal{B}_{(4\alpha)\beta}$, one obtains from (5.9) the following estimate on the second term in the RHS of (5.10),

$$|E\big(T(\rho * g_h + g_f * h)\big)| \le \sup_{x>0} \frac{x}{w_{\beta(4\alpha)}(x)} (\|\rho\|_{\alpha\beta}\|g_h\|_{\alpha\beta} + \|g_f\|_{\alpha\beta}\|h\|_{\alpha\beta}). \tag{5.11}$$

Finally, the first term on the RHS of (5.10) can be computed explicitly. We use the same notation as in Section 4.4. Then, $\rho * \sigma$ is given on each interval $[z_k, z_{k+1}]$, $k = 0, \ldots, 2n-1$, by the cubic polynomial

$$(\rho * \sigma)(z_k + \theta) = C_0(k) + C_1(k)\theta + C_2(k)\theta^2 + C_3(k)\theta^3,$$

where the coefficients $C_i(k)$ are given by (4.24). Hence,

$$\begin{aligned} E\big(T(\rho * \sigma)\big) &= \int_0^\infty \frac{1}{x}(\rho * \sigma)(x)\, dx \\ &= \sum_{k=0}^{2n-1} \int_0^1 \frac{1}{\theta + z_k/\varepsilon}\big(C_0 + \varepsilon\theta C_1(k) + \varepsilon^2\theta^2 C_2(k) + \varepsilon^3\theta^3 C_3(k)\big)\, d\theta, \end{aligned} \tag{5.12}$$

and using

$$\int_0^1 \frac{x^n}{x + 1/a} dx = a^{-n} \int_0^a \frac{x^n}{x+1} dx = \mathrm{Log}_n(a),$$

where Log_n is defined in (3.1), one can integrate each term in (5.12) and gets finally

$$E\big(T(\rho * \sigma)\big) = \sum_{k=0}^{2n-1} \sum_{m=0}^{3} \varepsilon^m C_m(k)\, \mathrm{Log}_m\Big(\frac{\varepsilon}{z_k}\Big). \tag{5.13}$$

Remark. A bound on (5.8) is implemented in `sexp_of_tconv`. Since the quantities entering (5.13) are computed during the estimation of the convolution, this subroutine is called in `fconvolute1` and `fconvolute2`. A bound on $\mathcal{N}_\lambda$ is implemented in `fN`. This subroutine also returns a standard set containing the coefficient $c_\lambda(f)$ that will be used to check (2.12), treating separately the special case where the value of $E(f)$ is known exactly, cf. Section 5.3.

5.2. Existence of the Family of Fixed Points: First Estimate

We now explain how the quantity ε entering inequality (2.20) of Proposition 2.6 is computed. Recall that it consists in an upper bound on

$$\|\mathcal{M}_{\lambda^+,\kappa}(f^0_{\lambda^+}) - f^0_{\lambda^+}\|_{\mu\nu}, \tag{5.14}$$

uniform in $\kappa \in [\delta, 1]$, $\delta = \lambda^-/\lambda^+$, where $\lambda^- < \lambda^+$, μ and ν are given in Proposition 2.6, and $f^0_{\lambda^+}$ is an approximate fixed point in $\mathcal{A}^u$. From the definition of $\mathcal{N}_{\lambda^+,\kappa}$ and $\mathcal{M}_{\lambda^+,\kappa}$, cf. (2.15) and (2.19), it follows that

$$\|\mathcal{M}_{\lambda^+,\kappa}(f^0_{\lambda^+}) - f^0_{\lambda^+}\|_{\mu\nu} \leq \|M\|\, \|\mathcal{N}_{\lambda^+,\kappa}(f^0_{\lambda^+}) - f^0_{\lambda^+}\|_{\mu\nu}, \tag{5.15}$$

and from (2.10) one obtains

$$\begin{aligned}\|\mathcal{N}_{\lambda^+,\kappa}(f^0_{\lambda^+}) - f^0_{\lambda^+}\|_{\mu\nu} &\leq \|S_\kappa(\mathcal{N}_{\lambda^+}(f^0_{\lambda^+}) - f^0_{\lambda^+})\|_{\mu\nu} + \|S_\kappa f^0_{\lambda^+} - f^0_{\lambda^+}\|_{\mu\nu} \\ &\leq \|\mathcal{N}_{\lambda^+}(f^0_{\lambda^+}) - f^0_{\lambda^+}\|_{\mu(\nu/\delta)} + \|(S_\kappa - 1) f^0_{\lambda^+}\|_{\mu\nu}. \end{aligned} \tag{5.16}$$

The last inequality is valid since $f^0_{\lambda^+}$ and $\mathcal{N}_{\lambda^+}(f^0_{\lambda^+})$ belong to $\mathcal{B}_{\mu(\nu/\delta)}$. Indeed, $f^0_{\lambda^+}$ has compact support in $(0,\infty)$, and $\mathcal{N}_{\lambda^+}$ preserves this property. Therefore, $\mathcal{N}_{\lambda^+}(f^0_{\lambda^+}) \in \mathcal{B}_{\alpha\beta}$ and $f^0_{\lambda^+} \in \mathcal{B}_{\alpha\beta}$ for all $\alpha, \beta \geq 0$. By composing the bounds constructed in the previous sections, one gets an estimate for the first term on the RHS of (5.16). At this point, the only dependence on the parameter κ lies in the second term of (5.16), which one bounds uniformly using the following result.

Lemma 5.1. *Let $0 < \kappa \leq 1$ and $f \in W^1_1(\mathbb{R}^+, w_{\alpha\beta}(x)dx)$. If $f' \in \mathcal{B}_{\alpha\gamma}$ for some $\gamma > \beta/\kappa$, then*

$$\|(S_\kappa - 1) f\|_{\alpha\beta} \leq (1-\kappa)\big(\|f\|_{\alpha\beta} + \|x f'(x)\|_{\alpha(\beta/\kappa)}\big). \tag{5.17}$$

By definition, the function $f^0_{\lambda^+} \in \mathcal{A}^u$ satisfies the hypothesis of Lemma 5.1 for all $\alpha, \beta \geq 0$. Furthermore, the bound (5.17) is decreasing in κ. Hence, one has for all $\kappa \in [\delta, 1]$,

$$\|(S_\kappa - 1) f^0_{\lambda^+}\|_{\mu\nu} \leq (1-\delta)\big(\|f^0_{\lambda^+}\|_{\mu\nu} + \|x (f^0_{\lambda^+})'(x)\|_{\mu(\nu/\delta)}\big). \tag{5.18}$$

Collecting the inequalities (5.15), (5.16) and (5.18) yields the desired bound ε on (5.14), uniform in $\kappa \in [\delta, 1]$. The only missing information is the norm of the operator M appearing in (5.15), which will be given in Section 7.2.

We end this section with the

Proof of Lemma 5.1. For $f \in W^1_1(\mathbb{R}^+, dx)$, one can rewrite

$$S_\kappa f(x) - f(x) = (\kappa - 1) f(x) + \kappa \int_x^{\kappa x} f'(y)dy.$$

Hence,

$$\|(S_\kappa - 1) f\|_{\alpha\beta} \leq (1-\kappa)\|f\|_{\alpha\beta} + \kappa \int_0^\infty dx\, w_{\alpha\beta}(x) \int_{\kappa x}^x |f'(y)|dy. \tag{5.19}$$

Furthermore,

$$\begin{aligned}\int_0^\infty dx\, w_{\alpha\beta}(x) \int_{\kappa x}^{x} |f'(y)| dy &= \int_0^\infty dy\, |f'(y)| \int_y^{y/\kappa} w_{\alpha\beta}(x) dx \\ &\leq \frac{1-\kappa}{\kappa} \int_0^\infty y|f'(y)| \max\{w_{\alpha\beta}(y), w_{\alpha\beta}(y/\kappa)\} dy \\ &\leq \frac{1-\kappa}{\kappa} \sup_{y\geq 0} \frac{\max\{w_{\alpha\beta}(y), w_{\alpha\beta}(y/\kappa)\}}{w_{\alpha(\beta/\kappa)}(y)} \|yf'(y)\|_{\alpha(\beta/\kappa)}.\end{aligned}$$

Finally, one checks that the supremum in the previous expression is bounded by one for $\kappa \leq 1$, which leads to (5.17).

■

Remark. The quantity ε is computed in the procedure `compute_residual`. This procedure also returns a bound on the first term in the RHS of (5.16), a quantity that will be used in Section 5.3. The bound (5.17) is implemented in the procedure `snorm_of_Skappam1`, where the second term on the RHS of (5.17) is estimated using, for $\rho \in \mathcal{A}$ with $\pi(\rho) = (\{x_i\}, \{\rho_i\})_{i=0}^n$,

$$\|x\rho'(x)\|_{\alpha(\beta/\kappa)} \leq \frac{1}{2} \sum_{i=1}^{n} \sup_{x\in I_i} w_{\alpha(\beta/\kappa)}(x) |\rho_i - \rho_{i-1}| (x_i + x_{i-1}). \tag{5.20}$$

Finally, the (positive) representable numbers that describe the approximate fixed point $f^0_{\lambda^+} \in \mathcal{A}^u$ are contained in the file `fpoint.lp`. The first two numbers in this file are the boundary points of $\mathrm{supp}(f^0_{\lambda^+})$. They determine the partition $p \in \mathcal{P}^n_u$, $n = 2^{17} + 1$, satisfying $\pi(f^0_{\lambda^+}) = (p, \cdot)$. The last 2^{17} numbers are the (nonzero) entries of the vector v, where $\pi(f^0_{\lambda^+}) = (\cdot, v)$. Given a nonnegative $G \in \mathcal{S}$, the subroutine `read_fp` reads the file `fpoint.lp` and constructs a standard set (P, V, G) with $(P, V) \in \mathrm{std}(\mathcal{A}^u)$ containing $f^0_{\lambda^+}$.

5.3. Existence of the Fixed Point f_{λ^*}

Recall that once the existence of the continuous family $\{f_\lambda\}$ of fixed points of $\mathcal{N}_\lambda$ is established for $\lambda \in [\lambda^-, \lambda^+]$, our main result, namely the existence of a $\lambda^* \in [\lambda^-, \lambda^+]$ and a function f^* satisfying $S_{\lambda^*}\mathcal{D}(f^*) = f^*$, follows from

$$c_{\lambda^-}(f_{\lambda^-}) < c_1 < c_{\lambda^+}(f_{\lambda^+}),$$

where c_1 is given by (1.10) and $c_\lambda(f)$ by (5.4). Checking this inequality amounts to computing for each of the three quantities involved a standard set in $\mathrm{std}(\mathbb{R})$.

Let us start with $c_{\lambda^+}(f_{\lambda^+})$. Suppose that one has a standard set, say in $\mathrm{std}(\mathcal{B}_{\mu\nu})$, containing the fixed point f_{λ^+}. Then one readily gets a standard set in $\mathrm{std}(\mathbb{R})$ containing $c_{\lambda^+}(f_{\lambda^+})$ by composing our bounds to compute

$$c_{\lambda^+}(f_{\lambda^+}) = \frac{\lambda^+}{2}\frac{1 - c_2 E\big(\mathcal{N}^2_{\lambda^+}(f_{\lambda^+})\big)}{M(f_{\lambda^+})}. \tag{5.21}$$

The previous expression follows from (5.4) and $E(f_{\lambda^+}) = 1$, a property satisfied by definition of the maps $\mathcal{N}_\lambda$, cf. (2.2). In order to check that $c_1 < c_{\lambda^+}(f_{\lambda^+})$, the size of the standard set obtained from (5.21) must be small enough, which ultimately requires to localize well enough the fixed point f_{λ^+}. In particular, Proposition 2.6 implies only that $f_{\lambda^+} \in B_{\tilde{r}}(f^0_{\lambda^+})$ with $\tilde{r} = \varepsilon/(1-q)$. This cannot be used to construct a suitable standard set containing f_{λ^+}, since the ball $B_{\tilde{r}}(f^0_{\lambda^+})$ also contains the fixed point $f_{\lambda^+,\delta} = S_\delta f_{\lambda^-}$ for which $c_{\lambda^+}(S_\delta f_{\lambda^-}) = c_{\lambda^-}(f_{\lambda^-}) < c_1$. In order to get a suitable standard set, we first use that the approximate fixed point $f^0_{\lambda^+}$ has been numerically determined as a very good approximation of f_{λ^+}, and exploit our bounds to compute

$$\begin{aligned}\|\mathcal{M}_{\lambda^+}(f^0_{\lambda^+}) - f^0_{\lambda^+}\|_{\mu\nu} &\le \|M\|\,\|\mathcal{N}_{\lambda^+}(f^0_{\lambda^+}) - f^0_{\lambda^+}\|_{\mu\nu}\\ &\le \varepsilon',\end{aligned} \tag{5.22}$$

with $\varepsilon' \approx 4.97\cdot 10^{-7}$ (to be compared with $\varepsilon \approx 1.15\cdot 10^{-4}$). Next, since by Proposition 2.6, $\mathcal{M}_{\lambda^+} = \mathcal{M}_{\lambda^+,1}$ is a contraction on the ball $B_r(f^0_{\lambda^+}) \in \mathcal{B}_{\mu\nu}$ with rate $q < 1$, one infers from (5.22) and the contraction mapping principle that

$$\|f_{\lambda^+} - f^0_{\lambda^+}\|_{\mu\nu} \le \frac{\varepsilon'}{1-q}.$$

Finally, one constructs in $\mathrm{std}(\mathcal{B}_{\mu\nu})$ the standard set whose affine part is given by the singleton $\{f^0_{\lambda^+}\}$ and whose general term has norm $\varepsilon'/(1-q)$. This set contains f_{λ^+} and allows to check that $c_1 < c_{\lambda^+}(f_{\lambda^+})$.

We now consider $c_{\lambda^-}(f_{\lambda^-})$, setting again $\delta = \lambda^-/\lambda^+$. For convenience, we work with the fixed point $f_{\lambda^+,\delta}$ of $\mathcal{M}_{\lambda^+,\delta}$ whose existence is guaranteed in $B_r(f^0_{\lambda^+}) \in \mathcal{B}_{\mu\nu}$ by Proposition 2.6. Lemma 2.5 implies that $f_{\lambda^-} = S_{1/\delta} f_{\lambda^+,\delta}$ and identity (2.17) leads to

$$c_{\lambda^-}(f_{\lambda^-}) = c_{\lambda^+}(f_{\lambda^+,\delta}) = \delta\frac{\lambda^+}{2}\frac{1 - c_2 E\big(\mathcal{N}^2_{\lambda^+}(f_{\lambda^+,\delta})\big)}{M(f_{\lambda^+,\delta})}, \tag{5.23}$$

where $E(f_{\lambda^+,\delta}) = 1/\delta$ has been used. In order to check that $c_{\lambda^-}(f_{\lambda^-}) < c_1$ using the previous relation, we must localize $f_{\lambda^+,\delta}$ closely enough. For this purpose, we have determined a very good approximation $f^0_{\lambda^-}$ to the fixed point f_{λ^-}. As $f^0_{\lambda^+}$, it is given by the linear interpolation of 2^{17} positive values at well chosen points. First, we check using our bounds that $f^0_{\lambda^-}$ satisfies

$$\|S_\delta f^0_{\lambda^-} - f^0_{\lambda^+}\|_{\mu\nu} < r, \tag{5.24}$$

with r as in Proposition 2.6, and,

$$\begin{aligned}
\|\mathcal{M}_{\lambda^+,\delta}(S_\delta f^0_{\lambda^-}) - S_\delta f^0_{\lambda^-}\|_{\mu\nu} &\le \|M\|\,\|\mathcal{N}_{\lambda^+,\delta}(S_\delta f^0_{\lambda^-}) - S_\delta f^0_{\lambda^-}\|_{\mu\nu} \\
&= \|M\|\,\|S_\delta \mathcal{N}_{\delta\lambda^+}(f^0_{\lambda^-}) - S_\delta f^0_{\lambda^-}\|_{\mu\nu} \\
&\le \|M\|\,\|\mathcal{N}_{\lambda^-}(f^0_{\lambda^-}) - f^0_{\lambda^-}\|_{\mu(\nu/\delta)} \\
&\le \varepsilon'', \qquad (5.25)
\end{aligned}$$

with $\varepsilon'' \approx 4.97 \cdot 10^{-7}$. Inequality (5.24) ensures that $S_\delta f^0_{\lambda^-} \in B_r(f^0_{\lambda^+})$. Hence, Proposition 2.6 and inequality (5.25) imply by the contraction mapping principle that

$$\|f_{\lambda^+,\delta} - S_\delta f^0_{\lambda^-}\|_{\mu\nu} \le \frac{\varepsilon''}{1-q}.$$

As above, this leads to the construction of a suitable standard set in $\mathrm{std}(\mathcal{B}_{\mu\nu})$ containing the fixed point $f_{\lambda^+,\delta}$.

To conclude, we emphasize that the accuracy of the bounds on (5.21) and (5.23) is crucial, since it determines how close to λ^* one can take λ^- and λ^+, and since, on the other hand, the size of the interval $[\lambda^-, \lambda^+]$ must be small enough in order to prove the existence of the family $\{f_\lambda\}_{\lambda\in[\lambda^-,\lambda^+]}$.

Remark. The bounds ε' and ε'' are computed in the procedure `compute_residual` introduced earlier. The computations of $c_{\lambda^+}(f_{\lambda^+})$ and $c_{\lambda^-}(f_{\lambda^-})/\delta$ are carried out in the procedure `fN`, in which a bound on the maps $\mathcal{N}_\lambda$ is implemented as explained in Section 5.1. The (positive) representable numbers that describe $f^0_{\lambda^-}$ are contained in the file `fpoint.lm`. This file is organized in the same way as `fpoint.lp`, and a standard set containing $f^0_{\lambda^-}$ is constructed by the subroutine `read_fp` described in Section 5.2.

6. Contractivity Properties of $D\mathcal{N}_\lambda$

As mentioned in Section 2, the tangent map $D\mathcal{N}_\lambda(f)$ is a contraction on certain subspaces of $\mathcal{B}_{\alpha\beta}$ with finite codimension. The main goal of this section is to describe these subspaces and compute the contraction factors. Those will be used in Section 7 to estimate the norm of the tangent map of $\mathcal{M}_{\lambda^+,\kappa}$. We first introduce some notations and check that $\mathcal{N}_\lambda$ is $\mathcal{C}^1$ on its domain of definition. The (Fréchet) derivative of $\mathcal{N}_\lambda$ at $f \in \mathcal{B}_{\alpha\beta}$ is explicitly given by

$$D\mathcal{N}_\lambda(f)h = S_\lambda\Big(2c_\lambda(f)\,f * h + 4c_2 T\big(T(f*f) * T(f*h)\big) + \delta_\lambda(f,h)\,f*f\Big), \qquad (6.1)$$

where the variation $\delta_\lambda(f,h)$ of $c_\lambda(f)$ is such that

$$E\big(D\mathcal{N}_\lambda(f)h\big) = 0. \qquad (6.2)$$

Indeed, since all functions in the range of $\mathcal{N}_\lambda$ have the same expectation, the tangent space contains only functions with expectation zero. Defining $\mathcal{N}_\lambda^1$ and $\mathcal{N}_\lambda^2$ as in (2.4) and (2.5), we rewrite (6.1) as

$$D\mathcal{N}_\lambda(f)h = c_\lambda(f)D\mathcal{N}_\lambda^1(f)h + c_2 D\mathcal{N}_\lambda^2(f)h + \delta_\lambda(f,h)\,\mathcal{N}_\lambda^1(f), \tag{6.3}$$

where

$$D\mathcal{N}_\lambda^1(f)h = 2S_\lambda(f * h), \tag{6.4}$$
$$D\mathcal{N}_\lambda^2(f)h = 2T\big(T\mathcal{N}_\lambda^1(f) * TD\mathcal{N}_\lambda^1(f)h\big). \tag{6.5}$$

From the condition (6.2), δ_λ is expressed in terms of the expectation of the three terms on the RHS of (6.3). Using the relations (1.22), one gets

$$\delta_\lambda(f,h) = -c_\lambda(f)\Big(\frac{M(h)}{M(f)} + \frac{E(h)}{E(f)}\Big) - \lambda c_2 \frac{E\big(D\mathcal{N}_\lambda^2(f)h\big)}{2M(f)E(f)}. \tag{6.6}$$

Now, for $\alpha \geq 0$ and $\beta, \lambda > 0$, one easily checks that the estimates of Proposition 2.2 imply that whenever $\mathcal{N}_\lambda : \mathcal{B}_{\alpha\beta}/\mathcal{H} \to \mathcal{B}_{\sigma\tau}$ is well defined, i.e., $\sigma \leq 4\alpha/\lambda$, $\tau \leq \lambda\beta$, it is continuously differentiable on $\mathcal{B}_{\alpha\beta}/\mathcal{H}$.

We will need later to estimate $D\mathcal{N}_\lambda(f)$ for $\mathcal{N}_\lambda : \mathcal{B}_{\alpha\beta} \to \mathcal{B}_{\alpha\gamma}$ with γ slightly larger than β. In this case, the conditions for $D\mathcal{N}_\lambda(f)$ to be bounded become $\gamma/\beta \leq \lambda \leq 4$. We will see below that under the stronger conditions $\gamma/\beta < \lambda < 4$, the tangent map $D\mathcal{N}_\lambda(f)$ is actually compact provided f is sufficiently regular, i.e., there is a sequence of subspaces of finite codimension on which $D\mathcal{N}_\lambda(f)$ converges to zero. These subspaces are defined as follows.

Definition 6.1. *For $p = \{x_0, \ldots, x_n\}$ a partition in $\mathcal{P}_n$ and $a, b > 0$, we define the following subspaces of $\mathcal{B}_{\alpha\beta}$,*

$$\mathcal{L}_\alpha^a = \{h \in \bigcup_{\eta \geq 0} \mathcal{B}_{\alpha\eta} \mid \mathrm{supp}(h) \subseteq (0,a)\},$$
$$\mathcal{C}_p = \{h \in L^1(\mathbb{R}_+) \mid \mathrm{supp}(h) \subseteq (x_0, x_n),\ \int_{x_{i-1}}^{x_i} h(x)\,dx = 0 \text{ for } i = 1, \ldots, n\},$$
$$\mathcal{R}_\beta^b = \{h \in \bigcup_{\zeta \geq 0} \mathcal{B}_{\zeta\beta} \mid \mathrm{supp}(h) \subseteq (b, \infty)\}.$$

Furthermore, we denote by $\mathcal{B}_{\alpha\beta}^p$ the following subspace of $\mathcal{B}_{\alpha\beta}$,

$$\mathcal{B}_{\alpha\beta}^p = \mathcal{L}_\alpha^{x_0} \oplus \mathcal{C}_p \oplus \mathcal{R}_\beta^{x_n}. \tag{6.7}$$

For a small enough and b large enough, it turns out that when restricted to $\mathcal{L}_\alpha^a$, $\mathcal{R}_\beta^b$, and $\mathcal{C}_p$ respectively, the tangent map $D\mathcal{N}_\lambda(f)$ has norms of value $\mathcal{O}(e^{-1/a}||f||_{\alpha\beta})$,

$\mathcal{O}(e^{-b}||f||_{\alpha\beta})$, and $\mathcal{O}(|p|\,||f'||_{\alpha\beta})$, where $|p|$ denotes the mesh size of the partition p. More generally, if $\mathcal{C}_p$ consists of functions whose $n-1$ first moments vanish on every interval of the partition p, the norm of the tangent map restricted to $\mathcal{C}_p$ is $\mathcal{O}(|p|^n||f^{(n)}||_{\alpha\beta})$. For our purpose, it is sufficient to consider $n = 1$.

Before deriving explicitly the contraction factors, we remark that we will need to evaluate later $D\mathcal{N}_\lambda(f)$ on the complement of $\mathcal{B}^p_{\alpha\beta}$ in $\mathcal{B}_{\alpha\beta}$. It is easily seen that for a given partition p, every $h \in \mathcal{B}_{\alpha\beta}$ can be uniquely decomposed into a sum $h = g + \tau$ where $g \in \mathcal{B}^p_{\alpha\beta}$ and where τ is constant on each interval of the partition p and satisfies $\mathrm{supp}(\tau) = \mathrm{supp}(p)$. More precisely, with $p = \{x_0, \ldots, x_n\}$ and χ_I the characteristic function of the interval I, one has

$$\mathcal{B}_{\alpha\beta} = \mathcal{B}^p_{\alpha\beta} \oplus \mathcal{V}^p, \tag{6.8}$$

where $\mathcal{V}^p$ is the n-dimensional vector space defined by

$$\mathcal{V}^p = \{\tau\,|\tau = \sum_{i=1}^n \lambda_i \chi_{[x_{i-1},x_i]}, \lambda_i \in \mathbb{R}\}. \tag{6.9}$$

Section 6.4 is devoted to the construction of a bound on the map $D\mathcal{N}_\lambda(f) : \mathcal{V}^p \to \mathcal{B}_{\alpha\gamma}$.

Remark. In (6.3), there are factors that depend only on the base function f. These factors, namely $\mathcal{N}^1_\lambda(f)$, $T\mathcal{N}^1_\lambda(f)$ and their norms, together with $c_\lambda(f)$, $E(f)$ and $M(f)$, are computed once and for all in the subroutine `compute_constant_terms` using the bounds of Section 4. According to Proposition 2.6, f is represented by the standard set in $\mathrm{std}(\mathcal{B}_{\mu\nu})^u$ whose affine part is the singleton $\{f^0_{\lambda+}\}$ and whose general term g satisfies $||g||_{\mu\nu} \leq 9 \cdot 10^{-4}$. Finally, for given standard sets containing $M(h), E(h)$ and $E(D\mathcal{N}^2_\lambda(f)h)$, a bound on $\delta_\lambda(f, h)$ is computed in the procedure `sdelta1` using (6.6).

6.1. Oscillatory Functions

We derive now an upper bound on the norm of the operator $D\mathcal{N}_\lambda(f) : \mathcal{C}_p \to \mathcal{B}_{\alpha\gamma}$, with $\mathcal{C}_p$ as in Definition 6.1 and with $f = \rho + g \in \mathcal{B}_{\alpha\beta}/\mathcal{H}$, $\rho \in \mathcal{A}^u$. For the first two terms in (6.3), and for $||g||_{\alpha\beta}$ small, the contraction factor will come from the convolution in $D\mathcal{N}^1_\lambda$. Hence, we first use the bounds obtained in Proposition 2.2 and get in full generality

$$||D\mathcal{N}_\lambda(f)h||_{\alpha\gamma} \leq (\,|c_\lambda(f)| + 2c_2||\mathcal{N}^1_\lambda(f)||_{\alpha\gamma})\,||D\mathcal{N}^1_\lambda(f)h||_{\alpha\gamma} + |\delta_\lambda(f,h)|\,||\mathcal{N}^1_\lambda(f)||_{\alpha\gamma}. \tag{6.10}$$

In the previous expression, only the quantities that depend on h remain to be estimated.

Let us begin with $\delta_\lambda(f, h)$. For $h \in \mathcal{C}_p$, one has $M(h) = 0$ and the first term in (6.6) vanishes. Next, $E(h)$ is expressed in term of the largest interval in the partition p.

Denoting $p = \{x_0, \ldots, x_n\}$ and $I_i = [x_{i-1}, x_i]$, $i = 1, \ldots, n$, the identity $\int_{I_i} h(x)dx = 0$ implies

$$\Big|\int_{I_i} xh(x)\,dx\Big| = \Big|\int_{I_i} \big(x - \frac{x_i + x_{i-1}}{2}\big)h(x)\,dx\Big| \le \frac{1}{2}(x_i - x_{i-1}) \int_{I_i} |h(x)|\,dx,$$

which in turn yields

$$|E(h)| \le \frac{1}{2} \max_{i=1,\ldots,n} \{x_i - x_{i-1}\} \sup_{x>0}\Big(\frac{1}{w_{\alpha\beta}(x)}\Big)\|h\|_{\alpha\beta}. \tag{6.11}$$

Finally, since $D\mathcal{N}^2_\lambda(f)h \in \mathcal{B}_{\alpha(4\gamma)}$, it follows from (2.14) that

$$|E\big(D\mathcal{N}^2_\lambda(f)h\big)| \le 2\sup_{x>0}\Big(\frac{x}{w_{\alpha(4\gamma)}(x)}\Big)\|\mathcal{N}^1_\lambda(f)\|_{\alpha\gamma}\|D\mathcal{N}^1_\lambda(f)h\|_{\alpha\gamma}. \tag{6.12}$$

Inserting (6.11) and (6.12) into (6.6) leads to an estimate for the second term on the RHS of (6.10). In order to bound the RHS of (6.12) and the first term on the RHS of (6.10), it then remains to estimate $\|D\mathcal{N}^1_\lambda(f)h\|_{\alpha\gamma}$.

In order to treat $D\mathcal{N}^1_\lambda(f)$, one has the possibility to exploit, as in the previous section, the distributivity of the scaling operator S_λ with respect to the convolution. It turns out that the order is not crucial and we consider for simplicity

$$D\mathcal{N}^1_\lambda : \mathcal{B}_{\alpha\beta} \times \mathcal{C}_p \xrightarrow{\;*\;} \mathcal{B}_{(4\alpha)\beta} \xrightarrow{2S_\lambda} \mathcal{B}_{\alpha\gamma}. \tag{6.13}$$

We begin with the convolution and use the following result.

Lemma 6.2. *Let $f \in W^1_1(\mathbb{R}_+, w_{\alpha\beta}(x)\,dx)$ and $h \in \mathcal{C}_p$ with $p = \{x_0, \ldots, x_n\}$. Then,*

$$\|f * h\|_{(4\alpha)\beta} \le \frac{1}{2} c_{\alpha\beta}(p)\|f'\|_{\alpha\beta}\|h\|_{\alpha\beta}, \tag{6.14}$$

where, denoting $I_i = [x_{i-1}, x_i]$,

$$c_{\alpha\beta}(p) = \max_{i=1,\ldots,n}\Big\{\sup_{x\in I_i} \frac{1}{w_{\alpha\beta}(x)} \int_{I_i} w_{\alpha\beta}(x)\,dx\Big\}. \tag{6.15}$$

Since for $\rho \in \mathcal{A}$ one has by definition $\rho \in W^1_1(\mathbb{R}_+, w_{\alpha\beta}(x)dx)$, the previous lemma together with Proposition 2.2 imply, with $f = \rho + g$, $\rho \in \mathcal{A}^u$, $g \in \mathcal{B}_{\alpha\beta}$, and $h \in \mathcal{C}_p$,

$$\begin{aligned}\|f * h\|_{(4\alpha)\beta} &\le \|\rho * h\|_{(4\alpha)\beta} + \|g\|_{\alpha\beta}\|h\|_{\alpha\beta} \\ &\le \Big(\frac{1}{2}c_{\alpha\beta}(p)\|\rho'\|_{\alpha\beta} + \|g\|_{\alpha\beta}\Big)\|h\|_{\alpha\beta}.\end{aligned} \tag{6.16}$$

Next, since $\gamma/\beta \le \lambda \le 4$, inequality (4.8) applies (with α replaced by 4α), and we finally obtain

$$\|D\mathcal{N}_\lambda^1(f)h\|_{\alpha\gamma} \le 2e^{-A}\Big(\frac{1}{2}c_{\alpha\beta}(p)\|\rho'\|_{\alpha\beta} + \|g\|_{\alpha\beta}\Big)\|h\|_{\alpha\beta}, \tag{6.17}$$

where $A = 2\sqrt{\alpha(4-\lambda)(\beta-\gamma/\lambda)}$.

A few comments are in order. In (6.17), the contraction factor is not only given by $c_{\alpha\beta}(p)$ but also by how close in $\mathcal{B}_{\alpha\beta}$ the base function f is to a regular function together with the norm of that function in $W_1^1(\mathbb{R}_+, w_{\alpha\beta}(x)dx)$. The fact that the fixed point whose existence we want to prove is smooth plays an important role here. To make a connection with Proposition 2.6, the quantity $\|g\|_{\alpha\beta}$ in (6.17) is the radius of the ball on which the tangent maps $D\mathcal{M}_{\lambda^+,\kappa}$ need to be contractions. All the other terms can be made as small as we wish by letting the size of the largest interval in p go to zero, cf. (6.11) and (6.15). Note that $c_{\alpha\beta}(p)$ depends sensitively on α and β, and optimizing this factor requires to consider a partition p with smaller intervals where the weight $w_{\alpha\beta}$ varies strongly. We will encounter later other optimization criteria for p. We shall denote by p_r the partition p which we will eventually choose, cf. Section 7.1.

We end this section with the

Proof of Lemma 6.2. Define the function h_1 by

$$h_1(x) = \int_{x_0}^{x} h(\xi)\,d\xi,$$

for $x \in (x_0, x_n)$, and $h_1(x) = 0$ otherwise. Note that $h_1' = h$ and, by definition of h, $h_1(x_i) = 0$ for $i = 0, \ldots, n$. Hence, integration by parts leads to

$$\|f * h\|_{(4\alpha)\beta} = \|f' * h_1\|_{(4\alpha)\beta} \le \|f'\|_{\alpha\beta}\|h_1\|_{\alpha\beta}.$$

It remains to estimate the norm of h_1 in term of h. For $i = 1, \ldots, n$ and $x \in [x_{i-1}, x_i]$, one has

$$|h_1(x)| = \frac{1}{2}\Big(\Big|\int_{x_{i-1}}^{x} h(\xi)\,d\xi\Big| + \Big|\int_{x}^{x_i} h(\xi)\,d\xi\Big|\Big) \le \frac{1}{2}\int_{I_i} |h(\xi)|\,d\xi,$$

which in turn yields

$$\begin{aligned}
\|h_1\|_{\alpha\beta} &= \sum_{i=1}^{n}\int_{I_i} w_{\alpha\beta}(x)\,|h_1(x)|\,dx, \\
&\le \frac{1}{2}\sum_{i=1}^{n}\int_{I_i} w_{\alpha\beta}(x)\int_{I_i}|h(\xi)|\,d\xi\,dx \\
&\le \frac{1}{2}\max_{i=1,\ldots,n}\Big\{\sup_{x\in I_i}\frac{1}{w_{\alpha\beta}(x)}\int_{I_i} w_{\alpha\beta}(x)\,dx\Big\}\|h\|_{\alpha\beta}.
\end{aligned}$$

■

Remark. The quantity $c_{\alpha\beta}(p_r)$ is computed in the subroutine `swsupint`. The estimates (6.11), (6.12) and (6.17) are implemented in `fDN_center` to compute (6.10), with a call to the subroutine `sdelta1` to get $\delta_\lambda(f,h)$. The quantity $\|\rho'\|_{\alpha\beta}$ entering (6.17) is bounded in `snorm_of_der_pl` by

$$\|\rho'\|_{\alpha\beta} \leq \sum_{j=1}^{m} \sup_{x\in I_j} w_{\alpha\beta}(x)|\rho_j - \rho_{j-1}|, \tag{6.18}$$

where $\pi(\rho) = (\{y_j\},\{\rho_j\})_{j=0}^m$ and $I_j = [y_{j-1}, y_j]$.

6.2. Functions with Support Near the Origin

In this section, we consider $D\mathcal{N}_\lambda(f)$ acting on functions $h \in \mathcal{L}_\alpha^a$ for a small enough. As in the previous section, but for different reasons, the contractivity properties of $D\mathcal{N}_\lambda(f)$ are entirely due to the term $D\mathcal{N}_\lambda^1(f)$. Indeed, since the functions f which will be considered have in general a support given by $\mathbb{R}_+$, the support of $D\mathcal{N}_\lambda^1(f)\,h$ for $h \in \mathcal{L}_\alpha^a$ is also equal to $\mathbb{R}_+$ due to the convolution. Hence, the size of $D\mathcal{N}_\lambda^2(f)h$ is essentially given by the size of $D\mathcal{N}_\lambda^1(f)\,h$, and we proceed as before starting with the bound (6.10) on $\|D\mathcal{N}_\lambda(f)h\|_{\alpha\gamma}$.

The last term in the expression (6.6) for $\delta_\lambda(f,h)$ is again bounded using (6.12). The h-dependent coefficients of the first two terms in (6.6) are given by $M(h)$ and $E(h)$, which are bounded using

$$|M(h)| \leq \frac{1}{w_{\alpha\beta}(a)}\|h\|_{\alpha\beta}, \tag{6.19}$$

$$|E(h)| \leq \frac{a}{w_{\alpha\beta}(a)}\|h\|_{\alpha\beta}, \tag{6.20}$$

provided $a \leq \sqrt{\alpha/\beta}$ for the first inequality, and $a \leq \bigl(1+\sqrt{1+4\alpha\beta}\bigr)/2\beta$ for the second inequality, cf. the discussion of (4.5) and (4.6).

It remains to bound $D\mathcal{N}_\lambda^1(f)h$ in $\mathcal{B}_{\alpha\gamma}$. We consider

$$D\mathcal{N}_\lambda^1 : \mathcal{B}_{\alpha\beta} \times \mathcal{L}_\alpha^a \xrightarrow{\;*\;} \mathcal{B}_{(\eta\alpha)\beta} \xrightarrow{\;2S_\lambda\;} \mathcal{B}_{\alpha\gamma}, \tag{6.21}$$

with $\eta \in [\lambda, 4]$ a parameter to be chosen later. For the convolution, we use the

Lemma 6.3. *Let $f \in \mathcal{B}_{\alpha\beta}$ and $h \in \mathcal{L}_\alpha^a$. Then, for $1 \leq \eta \leq 4$,*

$$\|f*h\|_{(\eta\alpha)\beta} \leq \exp\Bigl(-\frac{\alpha\sqrt{\eta}(2-\sqrt{\eta})}{a}\Bigr)\|f\|_{\alpha\beta}\|h\|_{\alpha\beta}. \tag{6.22}$$

Proof. Exploiting $\mathrm{supp}(h) \subseteq (0,a)$, we proceed as in Proposition 2.2 and get

$$\|f*h\|_{(\eta\alpha)\beta} \le \sup_{\substack{x>0\\ a>y>0}} \exp\bigl(-\alpha g(x,y)\bigr)\|f\|_{\alpha\beta}\|h\|_{\alpha\beta},$$

where

$$g(x,y) = \frac{x+y}{xy} - \frac{\eta}{x+y}.$$

Since $g(x,y) \ge 0$ for $\eta \le 4$, one has $\sup\exp(-\alpha g) = \exp(-\alpha \inf g)$ and, using $\eta \ge 1$, we compute

$$\inf_{\substack{x>0\\ a>y>0}} g(x,y) = \inf_{a>y>0} \frac{\sqrt{\eta}(2-\sqrt{\eta})}{y} = \frac{\sqrt{\eta}(2-\sqrt{\eta})}{a}.$$

■

We now turn to the scaling operator. Since $\mathrm{supp}(f*h) = \mathbb{R}_+$ for functions f that will be considered, the following general bound is optimal,

$$\|S_\lambda g\|_{\alpha\gamma} \le \exp\bigl(-2\sqrt{\alpha(1-\lambda/\eta)(\beta-\gamma/\lambda)}\,\bigr)\|g\|_{(\eta\alpha)\beta}, \tag{6.23}$$

which is valid provided $\gamma/\beta \le \lambda \le \eta$. From (6.23) and (6.22), we get a bound on $\|D\mathcal{N}^1_\lambda(f)h\|_{\alpha\gamma}$. We now optimize the parameter η. Since ultimately we will get the needed contraction factor by choosing a small enough, and since (6.23) does not depend on a, we consider (6.22) only. For $\lambda \ge 1$, the maximum of $\sqrt{\eta}(2-\sqrt{\eta})$ on $[\lambda,4]$ is taken at $\eta = \lambda$, and one gets finally

$$\|D\mathcal{N}^1_\lambda(f)h\|_{\alpha\gamma} \le 2\exp\Bigl(-\frac{\alpha\sqrt{\lambda}(2-\sqrt{\lambda})}{a}\Bigr)\|f\|_{\alpha\beta}\|h\|_{\alpha\beta}. \tag{6.24}$$

Recall that (6.24) is valid provided $\gamma/\beta \le \lambda \le 4$. Furthermore, it leads for a small enough to a strict contraction only if $\lambda < 4$: this is the first compactness condition.

Before ending this section, let us comment on the optimization of the contraction factor. Instead of (6.21), one can consider $D\mathcal{N}^1_\lambda(f)h = 2(S_\lambda f * S_\lambda h)$ with $S_\lambda : \mathcal{B}_{\alpha\beta} \to \mathcal{B}_{(\alpha/\eta)\gamma}$ and $\eta \in [\lambda,4]$ a parameter to be optimized. Since $S_\lambda h \in \mathcal{L}^{a/\lambda}_{(\alpha/\eta)\gamma}$ is of order $\mathcal{O}(e^{-1/a})$ if $\eta > \lambda$, one gets a second a-dependent contraction factor from the convolution. However, optimizing η leads to the same bound as (6.24), and we use (6.21) for convenience of implementation.

Remark. The bounds (6.19), (6.20) and (6.24) are implemented in the procedure `fDN_left` to compute (6.10). The conditions on a under which (6.19) and (6.20) are valid are first checked, namely $a \le \sqrt{\alpha/\beta}$ and $a \le \bigl(1+\sqrt{1+4\alpha\beta}\bigr)/2\beta$. An explicit check of $\gamma/\beta \le \lambda \le 4$ is also necessary. Up to now, this inequality was implicitly verified when bounds were computed, as in (6.17) for instance.

6.3. Functions with Support Near Infinity

We now consider functions $h \in \mathcal{R}^b_\beta$ with b large enough. Here, the situation differs from the previous cases in the sense that the term $D\mathcal{N}^2_\lambda(f)h$ is small independently of the size of $D\mathcal{N}^1_\lambda(f)h$. Indeed, the property of h to have support away from the origin is preserved by $D\mathcal{N}^1_\lambda(f)$. After applying the transformation T, one obtains a function whose support is near the origin, and the result from the previous section related to the convolution yields a second exponentially small factor. Hence, we simply start with the triangle inequality to get from (6.3)

$$\|D\mathcal{N}_\lambda(f)h\|_{\alpha\gamma} \le |c_\lambda(f)|\|D\mathcal{N}^1_\lambda(f)h\|_{\alpha\gamma} + c_2\|D\mathcal{N}^2_\lambda(f)h\|_{\alpha\gamma} + |\delta_\lambda(f,h)|\,\|\mathcal{N}^1_\lambda(f)\|_{\alpha\gamma}. \tag{6.25}$$

Let us begin with the first term. The main contraction factor is here entirely due to the scaling operator acting on $\mathcal{R}^b_\beta$. Furthermore, for $f \in \mathcal{B}_{\alpha\beta}$, the map $h \mapsto f * h$ preserves $\mathcal{R}^b_\beta$. Hence, one has the choice of the order in which the scaling and the convolution are composed. By letting the scaling act first, one gains a (b-independent) contraction factor when applying this operator to the function f. Recall that $S_\lambda : \mathcal{B}_{\alpha\beta} \to \mathcal{B}_{(\alpha/4)\gamma}$ is a strict contraction for $\gamma/\beta < \lambda < 4$. One can improve this factor by considering $\mathcal{B}_{0\gamma}$ for the target space of S_λ. Hence, we consider finally

$$D\mathcal{N}^1_\lambda/2 : \mathcal{B}_{\alpha\beta} \times \mathcal{R}^b_\beta \xrightarrow{\ S_\lambda\ } \mathcal{B}_{0\gamma} \times \mathcal{R}^{b/\lambda}_\gamma \xrightarrow{\ \ *\ \ } \mathcal{B}_{\alpha\gamma}. \tag{6.26}$$

Provided $\gamma/\beta \le \lambda$, the scaling operator in (6.26) is bounded, and, since $S_\lambda h$ has again support away from the origin, the convolution above is well defined even for $\alpha > 0$. For $f \in \mathcal{B}_{\alpha\beta}$, one estimates as usual

$$\begin{aligned}\|S_\lambda f\|_{0\gamma} &\le \sup_{x>0} \frac{w_{0\gamma}(x/\lambda)}{w_{\alpha\beta}(x)}\,\|f\|_{\alpha\beta} \\ &= \exp\Big(-2\sqrt{\alpha(\beta-\gamma/\lambda)}\Big)\,\|f\|_{\alpha\beta},\end{aligned} \tag{6.27}$$

and for $h \in \mathcal{R}^b_\beta$, one uses the knowledge about the support of h to get

$$\begin{aligned}\|S_\lambda h\|_{0\gamma} &\le \sup_{x>b} \frac{w_{0\beta}(x/\lambda)}{w_{\alpha\gamma}(x)}\|h\|_{\alpha\beta} \\ &= \exp\big(-\alpha/b - b(\beta-\gamma/\lambda)\big)\|h\|_{\alpha\beta},\end{aligned} \tag{6.28}$$

the last equality being valid if $b \ge \sqrt{\alpha/(\beta-\gamma/\lambda)}$. Next, we consider the convolution in (6.26). For $f \in \mathcal{B}_{0\gamma}$ and $h \in \mathcal{R}^{b/\lambda}_\gamma$, we proceed as in Proposition 2.2 and get

$$\begin{aligned}\|f * h\|_{\alpha\gamma} &\le \sup_{\substack{x>0\\ y>b/\lambda}} \Big(\frac{w_{\alpha\gamma}(x+y)}{w_{0\gamma}(x)w_{0\gamma}(y)}\Big)\|f\|_{0\gamma}\|h\|_{0\gamma} \\ &= \sup_{\substack{x>0\\ y>b/\lambda}} \exp\Big(\frac{\alpha}{x+y}\Big)\|f\|_{0\gamma}\|h\|_{0\gamma} \\ &= \exp(\alpha\lambda/b)\|f\|_{0\gamma}\|h\|_{0\gamma}.\end{aligned} \tag{6.29}$$

Finally, (6.27), (6.28) and (6.29) lead to

$$\|D\mathcal{N}^1_\lambda(f)h\|_{\alpha\gamma} \le 2e^{-A}\exp\bigl(-b(\beta-\gamma/\lambda)\bigr)\|h\|_{\alpha\beta}\|f\|_{\alpha\beta}, \tag{6.30}$$

where $A = 2\sqrt{\alpha(\beta-\gamma/\lambda)} - \alpha(\lambda-1)/b$. Although the convolution deteriorates the b-independent factor given by the scaling, (6.26) is still a good choice due to the large values of b that will be considered. Proceeding in this way is not crucial, but allows to take smaller values for b, thereby saving about 10 percent of the computation time devoted to the evaluation of $D\mathcal{N}_\lambda(f)$ on $\mathcal{V}^p$, the space of piecewise constant functions. We conclude by observing that (6.30) yields a bound which is exponentially small in b only if $\gamma/\beta < \lambda$: this is the second compactness condition.

Next, we consider the second term in (6.25). One has

$$\|D\mathcal{N}^2_\lambda(f)h\|_{\alpha\gamma} \le 2\|T\mathcal{N}^1_\lambda(f) * TD\mathcal{N}^1_\lambda(f)h\|_{\gamma\alpha}.$$

From $D\mathcal{N}^1_\lambda(f)h \in \mathcal{R}^{b/\lambda}_\gamma$ it follows that $TD\mathcal{N}^1_\lambda(f)h \in \mathcal{L}^{\lambda/b}_\gamma$, and applying Lemma 6.3 with $\eta = 1$ leads to

$$\|D\mathcal{N}^2_\lambda(f)h\|_{\alpha\gamma} \le 2\exp\Bigl(-\frac{\gamma b}{\lambda}\Bigr)\|\mathcal{N}^1_\lambda(f)\|_{\alpha\gamma}\|D\mathcal{N}^1_\lambda(f)h\|_{\alpha\gamma}. \tag{6.31}$$

It remains to estimate $\delta_\lambda(f,h)$. The expectation of $D\mathcal{N}^2_\lambda(f)h$ is simply bounded by

$$|E(D\mathcal{N}^2_\lambda(f)h)| \le \sup_{x>0}\Bigl(\frac{x}{w_{\alpha\gamma}(x)}\Bigr)\|D\mathcal{N}^2_\lambda(f)h\|_{\alpha\gamma}. \tag{6.32}$$

Note that in the previous cases, we used the properties of the convolution near the origin to bound this quantity according to (6.12). Here, these properties have been used already in the bound (6.31) to extract a second exponentially small factor in b. Therefore, inserting (6.31) into (6.32) leads to a better estimate than (6.12). Finally, for $h \in \mathcal{R}^b_\beta$ and b large, one has the following bounds on $M(h)$ and $E(h)$

$$|M(h)| \le \frac{1}{w_{\alpha\beta}(b)}\|h\|_{\alpha\beta}, \qquad |E(h)| \le \frac{b}{w_{\alpha\beta}(b)}\|h\|_{\alpha\beta}, \tag{6.33}$$

provided $b \ge \sqrt{\alpha/\beta}$ for the first inequality, and $b \ge (1+\sqrt{1+4\alpha\beta})/2\beta$ for the second inequality.

Remark. The bounds (6.30),(6.31), (6.32) and (6.33) are implemented in `fDN_right` to estimate (6.25). The validity conditions of (6.28) and (6.33), namely $b \ge \sqrt{\alpha/(\beta-\gamma/\lambda)}$ ($\ge \sqrt{\alpha/\beta}$) and $b \ge (1+\sqrt{1+4\alpha\beta})/2\beta$, are explicitly checked.

6.4. Piecewise Constant Functions

Finally, we consider the case of functions h in $\mathcal{V}^p$. On this space, the tangent map $D\mathcal{N}_\lambda(f)$ is not a contraction and the relevant information is contained in the images $D\mathcal{N}_\lambda(f)h$ of the basis vectors h of $\mathcal{V}^p$. Therefore, in order to keep track of this information, we need to construct a bound on the tangent map in the sense of Section 3. For $p = \{x_0, \dots, x_n\}$ and $I_i = (x_i, x_{i-1})$, a basis of $\mathcal{V}^p$ is given by $\{\chi_{I_i}\}_{i=1}^n$. Hence, we introduce the following set $\mathcal{X}$ of characteristic functions,

$$\mathcal{X} = \{c\chi_{[a,a+\delta]} \,|\, c \in \mathbb{R}, a > 0, \delta > 0\},$$

and we construct a bound on $D\mathcal{N}_\lambda : \mathcal{B}_{\alpha\beta} \times \mathcal{X} \to \mathcal{B}_{\alpha\gamma}$ acting from $\mathrm{std}(\mathcal{B}_{\alpha\beta})^u \times \mathrm{std}(\mathcal{X})$ to $\mathrm{std}(\mathcal{B}_{\alpha\gamma})$, where we define $\mathrm{std}(\mathcal{X})$ to be the collection of all sets of the form

$$(A, B, C) = \{h \in \mathcal{X} \mid h = c\chi_{[a,a+\delta]} \text{ with } a \in A, \delta \in B, c \in C\} \tag{6.34}$$

for $C \in \mathrm{std}(\mathbb{R})$ and $A, B \in \mathrm{std}(\mathbb{R}_+^*)$.

Note that, once a bound on $D\mathcal{N}_\lambda^1 : \mathcal{B}_{\alpha\beta} \times \mathcal{X} \to \mathcal{B}_{\alpha\gamma}$ has been obtained, composing it with the bounds of Section 4 readily yields bounds on the first two terms of $D\mathcal{N}_\lambda(f)h$, cf. (6.3) and (6.5). To compute the coefficient $\delta_\lambda(f,h)$ in the third term of (6.3), the only missing quantities are the mass and the expectation of $h \in \mathcal{X}$. Those are obtained from the equalities

$$M(\chi_{[a,a+\delta]}) = \delta \ , \qquad E(\chi_{[a,a+\delta]}) = \delta(a + \delta/2). \tag{6.35}$$

It remains to construct a bound on $D\mathcal{N}_\lambda^1$. We consider

$$D\mathcal{N}_\lambda^1/2 : \mathcal{B}_{\alpha\beta} \times \mathcal{X} \xrightarrow{S_\lambda} \mathcal{B}_{\frac{\alpha}{4}\gamma} \times \mathcal{X} \xrightarrow{*} \mathcal{B}_{\alpha\gamma}. \tag{6.36}$$

The reason for this choice is as follows. Some of the functions h will have support close to the origin or far away from the origin. In such cases, we know from the previous sections that the scaling in (6.36) is a very good contraction. Hence, considering (6.36) will automatically yield an extra contraction factor and improve the bound on the convolution between $S_\lambda h$ and the general term of $S_\lambda f$.

A bound on $S_\lambda : \mathcal{X} \to \mathcal{X}$ is easily obtained from

$$S_\lambda \chi_{[a,a+\delta]} = \lambda \chi_{[a/\lambda,(a+\delta)/\lambda]}.$$

Next, we construct a bound on the convolution defined from $\mathrm{std}(\mathcal{B}_{\zeta\eta})^u \times \mathrm{std}(\mathcal{X})$ to $\mathrm{std}(\mathcal{B}_{(\gamma\eta)})^u$, with $\gamma \in [\zeta, 4\zeta]$. Let $f = \rho + g$, $\rho \in \mathcal{A}^u$ and $g \in \mathcal{B}_{\zeta\eta}$. One has $f * h = \rho * h + g * h$, and the second term of this equality will be a part of the general term $\tilde{g}$ of $f * h$ and will be treated as usual. The first term contains the relevant information and needs to be computed explicitly. In the sequel, we consider for simplicity $h = \chi_{[a,a+\delta]}$.

Let $\pi(\rho) = (\{x_i\}, \{\rho_i\})_{i=0}^n$ and denote by ε the mesh of the uniform partition associated with ρ. If $\varepsilon \geq \delta$, the function $\rho * h$ takes a simpler form than in the case $\varepsilon < \delta$, and we restrict the domain of our bound to such cases in order to simplify the implementation. Define

$$y_k = a + x_0 + k\varepsilon, \qquad k = 0, \ldots, n+1, \tag{6.37}$$

and $I_k = [y_k, y_{k+1}]$, $k = 0, \ldots, n$. It is clear from the properties of the convolution that $\rho * h$ is continuous and has a support equal to $(y_0, y_n + \delta)$. Next, a short computation shows that provided $\varepsilon \geq \delta$, $\rho * h$ is given on the interval I_k by

$$(\rho * h)(y_k + \theta) = \begin{cases} \delta(\rho_k - \delta\rho'_{k-1}/2) + \theta\delta\rho'_{k-1} + \theta^2(\rho'_k - \rho'_{k-1})/2\,, & 0 \leq \theta \leq \delta, \\ \delta(\rho_k - \delta\rho'_k/2) + \theta\delta\rho'_k\,, & \delta \leq \theta \leq \varepsilon, \end{cases} \tag{6.38}$$

with the convention that $\rho_{-1} = \rho_{n+1} = 0$, and where

$$\rho'_k = \frac{\rho_{k+1} - \rho_k}{\varepsilon}.$$

Indeed, one has

$$(\rho * h)(y_k + \theta) = \int_a^b \rho(y_k + \theta - x)\, dx = \int_0^\delta \rho(x_k + \theta - \xi)\, d\xi. \tag{6.39}$$

Two cases arise: if $\theta \geq \delta$, the function ρ in the above integral is given by

$$\rho(x) = \rho_k + (x - x_k)\rho'_k. \tag{6.40}$$

Inserting (6.40) into (6.39) and integrating lead to the second part of (6.38). For $\theta < \delta$, we rewrite (6.39) as

$$(\rho * h)(y_k + \theta) = \int_0^\theta \rho(x_k + \theta - \xi)\, d\xi + \int_\theta^\delta \rho(x_k + \theta - \xi)\, d\xi. \tag{6.41}$$

In the first term, ρ is again given by (6.40), whereas in the second term one has

$$\rho(x) = \rho_{k-1} + (x - x_{k-1})\rho'_{k-1}. \tag{6.42}$$

Inserting (6.40) and (6.42) into (6.41) yields the first part of (6.38). Next, we define the affine part $\tilde{\rho}$ of $f * h$ to be the linear interpolation of $\rho * h$ at the nodes $\{y_k\}$. More precisely, we consider

$$\tilde{\rho} = \mathcal{T}_1\big(\{y_i\}_{i=0}^{n+1}, \{\tilde{\rho}_i\}_{i=0}^{n+1}\big), \tag{6.43}$$

where

$$\tilde{\rho}_k = \delta(\rho_k - \delta\rho'_{k-1}/2), \qquad k = 0, \ldots, n+1. \tag{6.44}$$

Note that $\tilde{\rho} \in \mathcal{A}^u$. Finally, the general term of $f * h$ is given by $\tilde{g} = \rho * h - \tilde{\rho} + g * h$ and one gets

$$\|\tilde{g}\|_{\gamma\eta} \leq \|\rho * h - \tilde{\rho}\|_{\gamma\eta} + \|g\|_{\zeta\eta}\|h\|_{\zeta\eta}. \tag{6.45}$$

For $h = \chi_{[a,a+\delta]}$, one simply uses that

$$||h||_{\zeta\eta} \leq \delta \sup_{x \in [a,a+\delta]} w_{\zeta\eta}(x).$$

To bound the first term on the RHS of (6.45), we first note that on the interval I_k, $k = 0, \ldots, n$,

$$\begin{aligned}\tilde{\rho}(y) &= \tilde{\rho}_k + (y - y_k)\frac{\tilde{\rho}_{k+1} - \tilde{\rho}_k}{\varepsilon} \\ &= \delta(\rho_k - \delta\rho'_{k-1}/2) + \delta(y - y_k)(\rho'_k - \delta(\rho'_k - \rho'_{k-1})/2\varepsilon).\end{aligned}$$

From this formula and the expression (6.38) for $\rho * h$, one computes for $\theta \in [0, \delta]$,

$$|(\rho * h - \tilde{\rho})(y_k + \theta)| = \theta\Big(\delta - \frac{\theta}{2} - \frac{\delta^2}{2\varepsilon}\Big)|\rho'_k - \rho'_{k-1}|, \tag{6.46}$$

and for $\theta \in [\delta, \varepsilon]$,

$$|(\rho * h - \tilde{\rho})(y_k + \theta)| = \frac{\delta^2}{2}\Big(1 - \frac{\theta}{\varepsilon}\Big)|\rho'_k - \rho'_{k-1}|. \tag{6.47}$$

Therefore, integrating (6.46) and (6.47) leads to

$$\begin{aligned}\|\rho * h - \tilde{\rho}\|_{\gamma\eta} &\leq \sum_{k=0}^{n} \sup_{x \in I_k} w_{\gamma\eta}(x) \int_{I_k} |(\rho * \chi - \tilde{\rho})(y)|\, dy \\ &= \delta^2\Big(\frac{1}{4} - \frac{\delta}{6\varepsilon}\Big) \sum_{k=0}^{n} \sup_{x \in I_k} w_{\zeta\eta}(x)|\rho_{k+1} - 2\rho_k + \rho_{k-1}|\,,\end{aligned} \tag{6.48}$$

with the convention $\rho_{-1} = \rho_{n+1} = 0$.

Remark. A set $(A, B, C) \in \mathrm{std}(\mathcal{X})$ is represented on the computer by a vector, say `fb`, with `fb(1)`$=A$, `fb(2)`$=B$, and `fb(3)`$=C$. The scaling $S_\lambda : \mathcal{X} \to \mathcal{X}$ is implemented in `fscale_chi`. A bound on the convolution in (6.36) is implemented in the procedure `fconv_chi` from (6.43), (6.45) and (6.48), where we first check the condition $\varepsilon \geq \delta$. We note that for the purpose of the proof of Proposition 2.6, the base function f is always represented by the same standard set in $\mathrm{std}(\mathcal{B}_{\alpha\beta})^u$. Hence, the only quantity in (6.44) that may change from basis vector to basis vector is δ. By choice of the partition p_r, see Section 7.1, most of the basis vectors have equal δ, and the computation of the $\tilde{\rho}_k$'s is carried out only once for such basis vectors. Finally, the bounds on the scaling and on the convolution in (6.36) and the bounds from Section 4 are composed in the subroutine `fDN_chi` to implement a bound on $D\mathcal{N}_\lambda(f) : \mathcal{X} \to \mathcal{B}_{\alpha\gamma}$, $f \in \mathcal{B}_{\alpha\beta}$.

7. The Tangent Maps $D\mathcal{M}_{\lambda,\kappa}$

In this section we explain how a uniform upper bound on the contraction rate of the operators $\mathcal{M}_{\lambda^+,\kappa}$ in a neighborhood of the fixed point f^* is obtained for all $\kappa \in [\lambda^-/\lambda^+, 1]$. This will complete the proof of Proposition 2.6. We recall that the operators $\mathcal{M}_{\lambda^+,\kappa}$ are given in terms of the original maps $\mathcal{N}_{\lambda^+,\kappa} = S_\kappa \mathcal{N}_{\lambda^+}$ by

$$\mathcal{M}_{\lambda^+,\kappa} = 1 + M(\mathcal{N}_{\lambda^+,\kappa} - 1), \tag{7.1}$$

where M is some fixed invertible linear map close to the inverse of $1 - D\mathcal{N}_{\lambda^*}(f_{\lambda^*})$. Since $\mathcal{N}_{\lambda^+,\kappa}$ is already a good contraction on the subspaces $\mathcal{B}^p_{\alpha\beta}$ for certain partitions p, we need M to be different from the identity only on the finite dimensional subspace $\mathcal{V}^p$, cf. (6.8). In Section 7.1, we introduce some notation and express the norm of a linear map in $\mathcal{B}_{\alpha\beta}$ in terms of its norms when restricted to $\mathcal{B}^p_{\alpha\beta}$ and $\mathcal{V}^p$. The description of M is given in Section 7.2. The last section is devoted to the final estimate needed to prove Proposition 2.6.

7.1. Decomposition of the Operator Norm

Let $p = \{x_0, \ldots, x_n\}$ be a partition in $\mathcal{P}_n$. In order to express the projector on $\mathcal{V}^p$, we introduce two maps associated with p: the finite rank operator $\mathcal{I}_p : \mathcal{B}_{\alpha\beta} \to \mathbb{R}^n$ defined by

$$\mathcal{I}_p f = \left\{ \frac{1}{|I_i|} \int_{I_i} f(x)dx \right\}_{i=1,\ldots,n}, \tag{7.2}$$

and $\mathcal{J}_p : \mathbb{R}^n \to \mathcal{V}^p$ defined by

$$\mathcal{J}_p \{f_i\}_{i=1}^n = \sum_{i=1}^n f_i \chi_{I_i}, \tag{7.3}$$

where $I_i = (x_{i-1}, x_i]$ and $|I|$ is the Lebesgue measure of $I \subset \mathbb{R}$. With this notation, the projector $\mathcal{Q}_p$ on $\mathcal{V}^p$ may be written as

$$\mathcal{Q}_p = \mathcal{J}_p \mathcal{I}_p. \tag{7.4}$$

Let A be a bounded linear map in $\mathcal{B}_{\alpha\beta}$. One has

$$\|Af\|_{\alpha\beta} \le \|f\|_p \max\{ \|A|_{\mathcal{V}^p}\|, \|A|_{\mathcal{B}^p_{\alpha\beta}}\| \}, \tag{7.5}$$

where $\|\cdot\|_p$ is the norm in $\mathcal{B}_{\alpha\beta}$ given by

$$\|f\|_p = \|\mathcal{Q}_p f\|_{\alpha\beta} + \|(1 - \mathcal{Q}_p) f\|_{\alpha\beta}. \tag{7.6}$$

The norms $\|\cdot\|_p$ and $\|\cdot\|_{\alpha\beta}$ are equivalent, with

$$\|f\|_{\alpha\beta} \leq \|f\|_p \leq K_p^{\alpha\beta} \|f\|_{\alpha\beta} \tag{7.7}$$

for some constant $K_p^{\alpha\beta}$. From the definition of $\mathcal{B}^p_{\alpha\beta}$ and its subspaces $\mathcal{L}^a_\alpha$, $\mathcal{C}_p$, $\mathcal{R}^b_\beta$, it follows that

$$\|A|_{\mathcal{B}^p_{\alpha\beta}}\| = \max\bigl\{\|A|_{\mathcal{L}^{x_0}_\alpha}\|, \|A|_{\mathcal{C}_p}\|, \|A|_{\mathcal{R}^{x_n}_\beta}\|\bigr\}. \tag{7.8}$$

Furthermore, one has

$$\|A|_{\mathcal{V}^p}\| = \max_{i=1,\ldots,n} \|A\eta_i\|_{\alpha\beta}, \tag{7.9}$$

with η_i the characteristic function of I_i normalized in $\mathcal{B}_{\alpha\beta}$, i.e.,

$$\eta_i = \Bigl(\int_{I_i} w_{\alpha\beta}(x)\, dx\Bigr)^{-1} \chi_{I_i}. \tag{7.10}$$

Inserting (7.7), (7.8) and (7.9) into (7.5), one gets

$$\|A\| \leq K_p^{\alpha\beta} \max\bigl\{\{\|A\eta_i\|_{\alpha\beta}\}_{i=1}^n, \|A|_{\mathcal{L}^{x_0}_\alpha}\|, \|A|_{\mathcal{C}_p}\|, \|A|_{\mathcal{R}^{x_n}_\beta}\|\bigr\}. \tag{7.11}$$

For $A = D\mathcal{M}_{\lambda^+,\kappa}(f)$, evaluating the quantities in the RHS of this expression will yield the desired bound on the norm of the tangent map of $\mathcal{M}_{\lambda^+,\kappa}$. The bounds obtained in the previous section will allow us to estimate each of the last three quantities in one step, by evaluating in turn $\|D\mathcal{M}_{\lambda^+,\kappa}(f)h\|_{\alpha\beta}$ for all h in the unit balls of $\mathcal{L}^{x_0}_\alpha$, $\mathcal{C}_p$ and $\mathcal{R}^{x_n}_\beta$. In contrast, the contractivity of $\mathcal{M}_{\lambda^+,\kappa}$ on $\mathcal{V}^p$ follows from the specific choice of the operator M, and an explicit computation of the n quantities $\|A\eta_i\|_{\alpha\beta}$ is required. This accounts for most of the computation time of the proof.

This leads us to the problem of optimizing the partition p in (7.11) with respect to $A = D\mathcal{M}_{\lambda^+,\kappa}(f)$. Roughly speaking, the size of the intervals in $p = (x_0, \ldots, x_n)$ is determined by the contraction rate of A on $\mathcal{C}_p$ that we need to obtain. Hence, the number of intervals n is fixed by x_0 and x_n. In order to minimize n, we want to maximize x_0 and minimize x_n. These two parameters determine the contraction rate of A on $\mathcal{L}^{x_0}_\alpha$ and $\mathcal{R}^{x_n}_\beta$. Increasing α and β improves the contraction and allows to consider larger x_0 and smaller x_n. However, large values of α and β deteriorate the estimate (2.20) of Proposition 2.6, i.e., the precision of the approximate fixed point. Good values for α and β have been found empirically to be $\alpha = 0.5$ and $\beta = 0.9$, for which $x_0 = 0.065$, $x_n = 11.83$ and a (non–uniform) partition of 5050 intervals give the desired bound (2.21). In the sequel, we will refer to this partition as p_r and denote $n_r = 5050$.

We end this section with the computation of the equivalence constant $K_p^{\alpha\beta}$. First, we estimate $\|\mathcal{Q}_p f\|_{\alpha\beta}$. From

$$|(\mathcal{I}_p f)_i| = \Bigl|\frac{1}{|I_i|}\int_{I_i} f(x)\, dx\Bigr| \leq \frac{1}{|I_i|} \sup_{x\in I_i}\Bigl(\frac{1}{w_{\alpha\beta}(x)}\Bigr) \int_{I_i} w_{\alpha\beta}(x)\, |f(x)|\, dx, \tag{7.12}$$

it follows

$$
\begin{aligned}
\|\mathcal{Q}_p f\|_{\alpha\beta} &= \sum_{i=1}^{n} |(\mathcal{I}_p f)_i| \int_{I_i} w_{\alpha\beta}(x)\, dx \\
&\le \max_{i=1,\dots,n} \Big(\frac{1}{|I_i|} \sup_{x\in I_i} \frac{1}{w_{\alpha\beta}(x)} \int_{I_i} w_{\alpha\beta}(x)\, dx \Big) \|f\|_{\alpha\beta}.
\end{aligned}
$$

Hence, the following inequality

$$
\|f\|_p = \|\mathcal{Q}_p f\|_{\alpha\beta} + \|(1-\mathcal{Q}_p)f\|_{\alpha\beta} \le \|f\|_{\alpha\beta} + 2\|\mathcal{Q}_p f\|_{\alpha\beta},
$$

implies

$$
K_p^{\alpha\beta} \le 1 + 2 \max_{i=1,\dots,n} \Big(\frac{1}{|I_i|} \sup_{x\in I_i} \frac{1}{w_{\alpha\beta}(x)} \int_{I_i} w_{\alpha\beta}(x)\, dx \Big). \tag{7.13}
$$

Note that the previous upper bound tends to 3 from above when n increases and when the size of each interval goes to zero. Also, the weight contributes to this bound by its largest variation on the intervals $\{I_i\}$. We have already encountered a similar situation, cf. (6.15), and we chose to consider a non–uniform partition with a higher density of nodes where the weight varies strongly. For the partition p_r introduced above, one has $K_{p_r}^{\mu\nu} < 3.15$.

Remark. An upper bound on the equivalence constant $K_{p_r}^{\mu\nu}$ is computed in the procedure `compute_equiv_const`, using `swsupint` to estimate the second term in the RHS of (7.13). The first and last points in the partition p_r are $x_0 = 0.065$ and $x_{n_r} = 11.83$, respectively. The first 100 (`npr1`) intervals are uniform with mesh $\varepsilon_{r_1} = (x_{n_r} - x_0)10^{-4}$ (`sepspr1`), whereas the remaining 4950 (`npr2`) intervals are uniform with mesh $\varepsilon_{r_2} = 2\,\varepsilon_{r_1}$ (`sepspr2`).

7.2. The Operator M

As mentioned earlier, M should be a good approximation to the inverse of $1 - D\mathcal{N}_{\lambda^*}(f_{\lambda^*})$, and needs to be different from the identity on the finite dimensional space $\mathcal{V}^{p_r}$ only. Hence, for a certain partition $p \in \mathcal{P}_m$ to be chosen later, we write

$$
M = \big(1 - \mathcal{Q}_p D\mathcal{N}_{\lambda^+}(f^0_{\lambda^+})\mathcal{Q}_p\big)^{-1}, \tag{7.14}
$$

where $f^0_{\lambda^+}$ is the explicit approximate fixed point of $\mathcal{N}_{\lambda^+}$ entering the statement of Proposition 2.6. The previous expression involves the $m \times m$ matrix

$$
A = \mathcal{I}_p D\mathcal{N}_{\lambda^+}(f^0_{\lambda^+})\mathcal{J}_p, \tag{7.15}
$$

and can be rewritten as

$$
M = (1 - \mathcal{J}_p A \mathcal{I}_p)^{-1} = 1 + \mathcal{J}_p A (1-A)^{-1} \mathcal{I}_p. \tag{7.16}
$$

Since we look only for an approximation, the operations involved in the computation of the matrices A and $A(1-A)^{-1}$ need not to be exact. Hence, the use of interval analysis is not required here and we will rely on numerics only. The result of this operation will be denoted by B, i.e.,

$$B \approx A(1-A)^{-1}. \tag{7.17}$$

With the notation

$$C_p = \mathcal{J}_p C \mathcal{I}_p,$$

for C an $m \times m$ real matrix, M is finally defined by

$$M = 1 + B_p. \tag{7.18}$$

We note that the numerical invertibility of $1 - A$ does not imply the invertibility of M. Since this property is required in order for the fixed points of $\mathcal{N}_{\lambda^+,\kappa}$ and of $1 + M(\mathcal{N}_{\lambda^+,\kappa} - 1)$ to be in correspondence, we must check that M is indeed invertible. We exhibit a matrix C for which $(1 + B)C$ is invertible. This implies that the matrix $1 + B$ is invertible, which in turn ensures the invertibility of M. For C, we consider the matrix $1 - A$ that has been previously numerically determined. Then, we check rigorously with interval analysis that the matrix X given by

$$X = (1 + B)C - 1, \tag{7.19}$$

satisfies

$$\|X\| < 1, \tag{7.20}$$

for some norm on $\mathbb{R}^m$. From this inequality, it then follows that $1 + X$ is invertible. The norm on $\mathbb{R}^m$ we use in the program is $\|x\| = \max_{i=1,\ldots,m} |x_i|$, that is, for C a real matrix with coefficients $\{c_{ij}\}$,

$$\|C\| = \max_{i=1,\ldots,m} \sum_{j=1}^{n} |c_{ij}|. \tag{7.21}$$

We now discuss the choice of the partition p used in the definition of M. This partition will be denoted by p_s. Since the decomposition $\mathcal{B}^{p_r}_{\alpha\beta} \oplus \mathcal{V}^{p_r}$ has been introduced in order to isolate the subspace $\mathcal{B}^{p_r}_{\alpha\beta}$ on which $\mathcal{N}_{\lambda^+,\kappa}$ is a contraction and since the non trivial action of M should turn $\mathcal{M}_{\lambda^+,\kappa}$ into a contraction on $\mathcal{V}^{p_r}$, it is natural to require

$$\mathcal{B}^{p_r}_{\alpha\beta} \subseteq \mathrm{Ker}(B_{p_s}).$$

This is in particular true if p_s is a subpartition of p_r, i.e.,

$$p_s \subseteq p_r. \tag{7.22}$$

There is no need for p_s to be equal to p_r. In particular, p_s could have fewer nodes than p_r, which would improve performance with respect to memory and computation time.

By trial and error, we have determined a small partition which satisfies (7.22) and leads to a contraction on $\mathcal{V}^{p_r}$. This (uniform) partition contains $m_s = 500$ intervals. Hence, B is a 500×500 matrix with entries in $\mathcal{S}$, the set of (safe) representable real numbers.

For technical reason, the matrix A is not computed according to (7.15) with $p = p_s$. This would amount to computing the matrix elements $a_{ij} = (\mathcal{I}_{p_s} D\mathcal{N}_{\lambda^+}(f^0_{\lambda^+})\mathcal{J}_{p_s}\hat{x}_j)_i$, where $\{\hat{x}_j\}$ is the canonical base of $\mathbb{R}^{m_s}$. To avoid the writing of special procedures, we want to use our bound on $D\mathcal{N}_{\lambda^+}$ acting on $\mathcal{X}$, even though interval analysis is not required. Hence, the support of the characteristic function $\mathcal{J}_{p_s}\hat{x}_j$ must be small enough in order for $\mathcal{J}_{p_s}\hat{x}_j$ to be in the domain of that bound. (Recall that the convolution between a characteristic and a piecewise linear function was restricted to such cases.) For the partition p_s introduced above, the bound does not apply for any of the $\mathcal{J}_{p_s}\hat{x}_j$. Hence, we proceed to the evaluation of a different matrix A close to (7.15). First, we choose a partition $p_t \in \mathcal{P}_{n_t}$ of which p_s is a subpartition, all the intervals in p_t being small enough. Then we compute

$$a_{ij} = (\mathcal{I}_{p_s} D\mathcal{N}_{\lambda^+}(f^0_{\lambda^+})\tilde{\mathcal{J}}_{p_t}\hat{y}_{k(j)})_i, \tag{7.23}$$

where $\{\hat{y}_k\}$ is the canonical basis of $\mathbb{R}^{n_t}$ and $\tilde{\mathcal{J}}_{p_t} : \mathbb{R}^{n_t} \to \mathcal{V}^{p_t}$ satisfies $\mathcal{I}_{p_s}\tilde{\mathcal{J}}_{p_t}\hat{y}_{k(j)} = \hat{x}_j$.

We recall that in (7.23), the function $D\mathcal{N}_{\lambda^+}(f^0_{\lambda^+})\tilde{\mathcal{J}}_{p_t}\hat{y}_{k(j)}$ is given by our bound as a sum $\rho + g$, with $\rho \in \mathcal{A}$ and g a general term. For the purpose of computing A, g is discarded and it remains to discuss the map $\mathcal{I}_{p_s} : \mathcal{A} \to \mathbb{R}^{m_s}$. We will need to evaluate later this map rigorously and we now describe how to bound it. Let $p \in \mathcal{P}_m$ and $\pi(\rho) = (p_\rho, \cdot)$. Define $\tilde{p} \equiv p \cup p_\rho = \{y_j\}_{j=0}^N$ and $\tilde{\rho}_j = \rho(y_j)$. Then, writing $\tilde{I}_j = (y_{j-1}, y_j)$, one has for $i = 1, \ldots, m$,

$$\Big(\mathcal{I}_p\rho\Big)_i = \frac{1}{|I_i|}\sum_{\substack{j\\ \tilde{I}_j\subseteq I_i}}\int_{\tilde{I}_j}\tilde{\rho}(x)\,dx = \frac{1}{|I_i|}\sum_{\substack{j\\ \tilde{I}_j\subseteq I_i}}|\tilde{I}_j|\frac{\tilde{\rho}_j+\tilde{\rho}_{j-1}}{2}. \tag{7.24}$$

We restrict the domain of this bound to those ρ's for which the support of the partition p contains the support of p_ρ. By proceeding so, we ensure that no information is lost when projecting on $\mathcal{V}^p$.

We end this section by deriving an expression for the operator norm of M in $\mathcal{B}_{\alpha\beta}$. Recall that this quantity was needed in Section 5, cf. (5.15), (5.22) and (5.25). We start with the trivial estimate

$$\|M\| \le 1 + \|B_{p_s}\|, \tag{7.25}$$

and express the norm of the finite rank operator B_{p_s} in terms of the partition p_s and the matrix elements of B. Let $p = \{x_0, \ldots, x_n\} \in \mathcal{P}_n$, $I_i = (x_{i-1}, x_i)$ and let C be an $n \times n$ matrix with real entries $\{c_{ij}\}$. For $f \in \mathcal{B}_{\alpha\beta}$, one estimates

$$\begin{aligned}\|\mathcal{J}_p C\mathcal{I}_p f\|_{\alpha\beta} &= \sum_{i=1}^n\Big|\sum_{j=1}^n c_{ij}(\mathcal{I}_p f)_j\Big|\int_{I_i} w_{\alpha\beta}(x)\,dx\\ &\le \sum_{j=1}^n|(\mathcal{I}_p f)_j|\sum_{i=1}^n|c_{ij}|\int_{I_i} w_{\alpha\beta}(x)\,dx,\end{aligned}$$

and, using our previous bound (7.12) on $|(\mathcal{I}_p f)_j|$, one gets

$$\|C_p\| \leq N_p^{\alpha\beta}(C), \tag{7.26}$$

where

$$N_p^{\alpha\beta}(C) = \max_{j=1,\ldots,n} \Big(\frac{1}{|I_j|} \sup_{x \in I_j} \Big(\frac{1}{w_{\alpha\beta}(x)} \Big) \sum_{i=1}^{n} |c_{ij}| \int_{I_i} w_{\alpha\beta}(x)\, dx \Big). \tag{7.27}$$

Remark. A bound on the map $\mathcal{I}_p : \mathcal{A} \to \mathbb{R}^n$ is implemented in the procedure `projection`, checking first the condition on its domain of definition and using `fadd` to compute $\tilde{p}$ and $\{\tilde{\rho}_j\}$. In the procedure `compute_matrix`, the matrix B (`bm`) is computed using (7.17) and (7.23). A call to `show_invertibility` verifies that $1 + B$ is invertible. For the numerical inversion of $1 - A$, we use the standard algorithm of Gauss elimination, implemented in the subroutine `gaussj`. The operator norms of M and B_{p_s} are computed in `compute_matrix_norm`. Finally, the partition p_s satisfies $\mathrm{supp}(p_s) = \mathrm{supp}(p_r)$ and is uniform with mesh $\varepsilon_s = 10\, \varepsilon_{r_2}$ (`sepsps`). Hence, it contains 500 (`mps`) intervals.

7.3. Existence of the Family of Fixed Points: Second Estimate

In this section, we derive a uniform bound on the norm of the tangent maps $D\mathcal{M}_{\lambda^+,\kappa}(f)$ for κ in $[\lambda^-/\lambda^+, 1]$, and $f \in B_r(f^0_{\lambda^+}) \subset \mathcal{B}_{\mu\nu}$ with $\mu = 0.5$, $\nu = 0.9$ and $r = 9 \cdot 10^{-4}$. In the sequel, we set $\delta = \lambda^-/\lambda^+$. By definition, one has

$$D\mathcal{M}_{\lambda^+,\kappa}(f) = 1 + M\big(S_\kappa D\mathcal{N}_{\lambda^+}(f) - 1\big). \tag{7.28}$$

For $f \in \mathcal{B}_{\alpha\beta}/\mathcal{H}$, $D\mathcal{N}_\lambda(f)$ is bounded from $\mathcal{B}_{\alpha\beta}$ to $\mathcal{B}_{\alpha\gamma}$ provided $\gamma/\beta \leq \lambda \leq 4$. Hence, with $\beta = \nu$ and $\gamma = \nu/\kappa$, $S_\kappa D\mathcal{N}_{\lambda^+}(f)$ is bounded as a map from $\mathcal{B}_{\mu\nu}$ to $\mathcal{B}_{\mu\nu}$ provided $1/\kappa \leq \lambda^+ \leq 4$. One concludes that for all $\kappa \in [\delta, 1]$, $D\mathcal{M}_{\lambda^+,\kappa}(f)$ is bounded as a map from $\mathcal{B}_{\mu\nu}$ to $\mathcal{B}_{\mu\nu}$ provided $1/\delta \leq \lambda^+ \leq 4$. For the values of λ^+ and λ^- as given in the statement of Proposition 2.6, the previous condition is satisfied.

To estimate the norm of $D\mathcal{M}_{\lambda^+,\kappa}(f)$ in $\mathcal{B}_{\mu\nu}$, we proceed as outlined in Section 7.1 and bound each term on the RHS of (7.11). We start with the simple case $h \in \mathcal{B}^{p_r}_{\mu\nu}$. The property $p_s \subseteq p_r$ implies $B_{p_s} h = 0$, so that $Mh = h$. Hence,

$$\begin{aligned} \|D\mathcal{M}_{\lambda^+,\kappa}(f)h\|_{\mu\nu} &\leq \|M\|\, \|S_\kappa D\mathcal{N}_{\lambda^+}(f)h\|_{\mu\nu} \\ &\leq \|M\|\, \|D\mathcal{N}_{\lambda^+}(f)h\|_{(\kappa\mu)(\nu/\kappa)} \\ &\leq \|M\|\, \|D\mathcal{N}_{\lambda^+}(f)h\|_{\mu(\nu/\delta)}, \end{aligned} \tag{7.29}$$

for all $\kappa \in [\delta, 1]$. An upper bound on $\|M\|$ was described in the previous section. Representing f by the standard set in $\mathrm{std}(\mathcal{B}_{\mu\nu})^u$ whose affine part is the singleton $\{f^0_{\lambda^+}\}$ and whose general term has norm r, the bounds of Section 6 yield 0.85 as an upper bound on the RHS of (7.29) for all h in the unit ball of $\mathcal{L}^{x_0}_\mu$, $\mathcal{C}_{p_r}$, and $\mathcal{R}^{x_n}_\nu$.

Next we consider the more delicate case of $h \in \mathcal{V}^{p_r}$. According to (7.11), one has to estimate the n_r quantities $\|D\mathcal{M}_{\lambda^+,\kappa}(f)\eta_i\|_{\mu\nu}$ where η_i is the normalized characteristic function of the i^{th} interval I_i in the partition p_r. Recalling that $M = 1 + B_{p_s}$, we get from (7.28)

$$D\mathcal{M}_{\lambda^+,\kappa}(f)\eta_i = S_\kappa D\mathcal{N}_{\lambda^+}(f)\eta_i + B_{p_s}(S_\kappa D\mathcal{N}_{\lambda^+}(f) - 1)\eta_i. \tag{7.30}$$

The bound on the map $D\mathcal{N}_\lambda(f) : \mathcal{X} \to \mathcal{B}_{\alpha\gamma}$ previously constructed yields the function $D\mathcal{N}_{\lambda^+}(f)\eta_i$ represented by a standard set in $\mathrm{std}(\mathcal{A})$ and a general term. We denote the former by ρ_i and the latter by g_i, i.e.,

$$D\mathcal{N}_{\lambda^+}(f)\eta_i = \rho_i + g_i,$$

and rewrite (7.30) as

$$\begin{aligned} D\mathcal{M}_{\lambda^+,\kappa}(f)\eta_i &= S_\kappa(\rho_i + g_i) + B_{p_s}\big(S_\kappa(\rho_i + g_i) - \eta_i\big) \\ &= MS_\kappa g_i + S_\kappa\rho_i + B_{p_s}(S_\kappa\rho_i - \eta_i). \end{aligned} \tag{7.31}$$

The norm of the first term is bounded as before for all $\kappa \in [\delta, 1]$ by

$$\|MS_\kappa g_i\|_{\mu\nu} \le \|M\|\,\|g_i\|_{\mu(\nu/\delta)}. \tag{7.32}$$

To treat the remaining terms in (7.31), we express them as

$$\begin{aligned} S_\kappa\rho_i + B_{p_s}(S_\kappa\rho_i - \eta_i) = {}& S_\kappa\big(\rho_i + B_{p_s}(\rho_i - \eta_i)\big) \\ &+ B_{p_s}(S_\kappa - 1)\rho_i \\ &+ (1 - S_\kappa)B_{p_s}(\rho_i - \eta_i). \end{aligned} \tag{7.33}$$

As we shall see below, the last two terms are of order $1-\kappa$, and the first term is small due to the cancellations arising by construction from the action of M. This term, therefore, needs to be computed explicitly.

Let us begin with this term first. One starts by factorizing the action of S_κ by using again

$$\|S_\kappa\big(\rho_i + B_{p_s}(\rho_i - \eta_i)\big)\|_{\mu\nu} \le \|\rho_i + B_{p_s}(\rho_i - \eta_i)\|_{\mu(\nu/\delta)}, \tag{7.34}$$

which is valid for all $\kappa \in [\delta, 1]$. Next, since $p_s \subseteq p_r$, it is enough to construct a bound on the map

$$\mathcal{A} \times \mathcal{X} \ni (\rho, \chi_I) \mapsto \|\rho + \mathcal{J}_p C \mathcal{I}_p(\rho - \chi_I)\|_{\alpha\gamma}, \tag{7.35}$$

for $p \in \mathcal{P}_n$ and C an $n \times n$ matrix, restricted to intervals I satisfying $I \subseteq I_i$ for some interval I_i in the partition p. A bound on $\rho \mapsto \mathcal{I}_p\rho$ has already been discussed in the previous section. Denoting by I_i the i^{th} interval in the partition p, one has $(\mathcal{I}_p\chi_I)_i = 0$ if $I \cap I_i = \emptyset$, and otherwise

$$(\mathcal{I}_p\chi_I)_i = |I|/|I_i|. \tag{7.36}$$

A bound on the map $C : \mathbb{R}^n \to \mathbb{R}^n$ is readily implemented with interval analysis, and it only remains to consider the map $(\rho, v) \mapsto \|\rho + \mathcal{J}_p v\|_{\alpha\gamma}$, $(\rho, v) \in \mathcal{A} \times \mathbb{R}^n$. Let us denote $\pi(\rho) = (p_\rho, \cdot)$, $\tilde{p} = p \cup p_\rho = \{y_j\}_{j=0}^N$ and $\tilde{\rho}_j = \rho(y_j)$. Imposing the restriction $\operatorname{supp}(p_\rho) \subseteq \operatorname{supp}(p)$, one obtains

$$\|\rho + \mathcal{J}_p v\|_{\alpha\gamma} \leq \sum_{i=1}^{n} \sum_{\substack{j \\ \tilde{I}_j \subseteq I_i}} \sup_{x \in \tilde{I}_j} w_{\alpha\gamma}(x) |\tilde{I}_j| \frac{|\tilde{\rho}_j + v_i| + |\tilde{\rho}_{j-1} + v_i|}{2}, \tag{7.37}$$

where $\tilde{I}_j$ stands for the j^{th} interval in $\tilde{p}$. This finishes the construction of a bound on the map (7.35), which, given standard sets containing ρ_i and η_i, provides an estimate on the RHS of (7.34).

Next, the second term in (7.33) is simply bounded by

$$\|B_{p_s}(S_\kappa - 1)\rho_i\|_{\mu\nu} \leq \|B_{p_s}\| \, \|(S_\kappa - 1)\rho_i\|_{\mu\nu}. \tag{7.38}$$

The operator norm of B_{p_s} has been determined in the previous section, and Lemma 5.1 provides a bound uniform in κ for the second factor, namely

$$\|(S_\kappa - 1)\rho_i\|_{\mu\nu} \leq (1 - \delta)\big(\|\rho_i\|_{\mu\nu} + \|x\rho_i'\|_{\mu(\nu/\delta)}\big).$$

To treat the last term in (7.33), we use the

Lemma 7.1. *Let $p = \{x_0, \ldots, x_n\} \in \mathcal{P}_n$, C an $n \times n$ matrix with coefficients $\{c_{ij}\}$, and $0 < \kappa \leq 1$. Then the operator norm of $(1 - S_\kappa)C_p$ in $\mathcal{B}_{\alpha\beta}$ satisfies*

$$\|(1 - S_\kappa)C_p\| \leq R_p^{\alpha\beta}(\kappa) N_p^{\alpha\beta}(C), \tag{7.39}$$

where $N_p^{\alpha\beta}(C)$ is given by (7.27) and

$$R_p^{\alpha\beta}(\kappa) = (1 - \kappa) + \max_{i=1,\ldots,n} \frac{1}{\int_{I_i} w_{\alpha\beta}(x) dx} \bigg(\int_{x_{i-1}}^{x_{i-1}/\kappa} w_{\alpha\beta}(x)\, dx + \int_{x_i}^{x_i/\kappa} w_{\alpha\beta}(x)\, dx \bigg).$$

The only dependence on κ in (7.39) is in the factor $R_p^{\alpha\beta}(\kappa)$. Furthermore, $R_p^{\alpha\beta}(\kappa)$ is decreasing in κ. Hence, one obtains

$$\|(1 - S_\kappa)B_{p_s}(\rho_i - \eta_i)\|_{\mu\nu} \leq R_{p_s}^{\mu\nu}(\delta) N_{p_s}^{\mu\nu}(B)(\|\rho_i\|_{\mu\nu} + 1), \tag{7.40}$$

for all $\kappa \in [\delta, 1]$.

Finally, a bound on $\|D\mathcal{M}_{\lambda+,\kappa}(f)\eta_i\|_{\mu\nu}$ follows from (7.32), (7.34), (7.38) and (7.40). As mentioned earlier, computing this bound for the $n_r = 5050$ basis vectors η_i of $\mathcal{V}^{p_r}$ accounts for most of the computation time. In the terms involving explicitly η_i, one can, using the linearity, factorize the value of η_i, that is $(\int_{I_i} w_{\mu\nu})^{-1}$. Therefore, one only needs to compute an upper bound on this quantity. Furthermore, by proceeding like this one can take advantage of the fact that the value of χ_{I_i} can be represented by the standard set containing only the representable number one. This leads to a standard set containing $\rho_i + g_i$ which is more localized and improves the quality of the final bound.

We end this section with the

Proof of Lemma 7.1. For $f \in \mathcal{B}_{\alpha\beta}$, one has

$$\begin{aligned} \|(1-S_\kappa)C_p f\|_{\alpha\beta} &= \int_0^\infty w_{\alpha\beta}(x) \Big| (1-S_\kappa) \sum_{i=1}^n \sum_{j=1}^n c_{ij} (\mathcal{I}_p f)_j \chi_{I_i}(x) \Big| \, dx \\ &\leq \sum_{j=1}^n |(\mathcal{I}_p f)_j| \sum_{i=1}^n |c_{ij}| \int_0^\infty w_{\alpha\beta}(x) |(1-S_\kappa)\chi_{I_i}(x)| \, dx. \end{aligned} \tag{7.41}$$

Furthermore, one has

$$\int_0^\infty w_{\alpha\beta} |(1-S_\kappa)\chi_{I_i}| \leq \int_{x_{i-1}}^{x_{i-1}/\kappa} w_{\alpha\beta} + (1-\kappa) \int_{I_i} w_{\alpha\beta} + \kappa \int_{x_i}^{x_i/\kappa} w_{\alpha\beta}.$$

Factorizing $\int_{I_i} w_{\alpha\beta}$ in the previous expression and inserting the bound (7.12) on $|(\mathcal{I}_p f)_i|$ into (7.41) finally leads to (7.39).

■

Remark. A bound on the map $(\rho, v) \mapsto \|\rho + \mathcal{J}_{p_s} v\|_{\alpha\gamma}$ is implemented in the procedure `snorm_add`, and the product Cv is implemented in `linear_app`. A uniform bound on the norm of $(1-S_\kappa)B_{p_s}$ is computed in `compute_norm_of_1mSB`. Given $i \in \{1, \ldots, n_r\}$, the subroutine `init_chi` returns both a standard set in $\mathrm{std}(\mathcal{X})$ containing χ_{I_i} and the value of η_i, whereas the subroutine `fDM_chi` computes a bound on $\|D\mathcal{M}_{\lambda+,\kappa}(f)\eta_i\|_{\mu\nu}$. Finally, for all $f \in B_r(f^0_{\lambda+})$ and $\kappa \in [\delta, 1]$, a uniform bound on the norm of the tangent maps $D\mathcal{M}_{\lambda+,\kappa}(f)$ is implemented according to (7.11) in `compute_norm_of_DM`.

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Appendix

Proof of Proposition 2.3.

If for some fixed $\lambda \in (1,4)$ and $\alpha, \beta > 0$, f_λ is a fixed point of $\mathcal{N}_\lambda$ and belongs to $\mathcal{B}_{\alpha\beta}$, then Remark 1.2 and Proposition 2.2 imply that $f_\lambda \in \mathcal{B}$. We now prove that, in addition, f_λ is at least once differentiable, with $f'_\lambda \in \mathcal{B}$. The regularization properties of the convolution imply then immediately that f_λ is of class $\mathcal{C}^\infty(\mathbb{R}_+)$. For $\zeta, \eta \geq 0$, let $\mathcal{B}^1_{\zeta\eta}$ denote the Sobolev space of functions in $\mathcal{B}_{\zeta\eta}$ with one (distributional) derivative in $\mathcal{B}_{\zeta\eta}$, i.e.,

$$\mathcal{B}^1_{\zeta\eta} = \{ f \in \mathcal{B}_{\zeta\eta} \mid f' \in \mathcal{B}_{\zeta\eta} \},$$

with the norm

$$\|f\|^1_{\zeta\eta} = \|f\|_{\zeta\eta} + \|f'\|_{\zeta\eta}\,.$$

In the sequel, we adopt the shorter notation $\mathcal{B}_\zeta = \mathcal{B}_{\zeta\zeta}$ and $\mathcal{B}^1_\zeta = \mathcal{B}^1_{\zeta\zeta}$. One shows that the fixed point f_λ belongs to $\mathcal{B}^1_\zeta$ for all $\zeta > 0$ by the following argument. One exhibits an $h \in \mathcal{B}^1_\zeta$ and two sequences $\{f_n\}_{n\geq 0}$ and $\{g_n\}_{n\geq 0}$ satisfying $f_\lambda = h + f_n + g_n$ for all $n \geq 0$, such that $\{f_n\}_{n\geq 0}$ is Cauchy in $\mathcal{B}^1_\zeta$ and $\{g_n\}_{n\geq 0}$ converges to zero in $\mathcal{B}_\zeta$. Hence, f_λ is equal in $\mathcal{B}_\zeta$ to a function belonging to $\mathcal{B}^1_\zeta$. Since $\mathcal{N}_\lambda$ preserves the regularity, this function is also a fixed point of $\mathcal{N}_\lambda$. Therefore, it is equal to f_λ in $\mathcal{B}^1_\zeta$.

We first construct recursively the sequences $\{f_n\}_{n\geq 0}$ and $\{g_n\}_{n\geq 0}$. Since f_λ belongs to $\mathcal{B}_\zeta$ for all $\zeta > 0$, and since $\mathcal{C}^\infty_0(\mathbb{R}_+)$ is dense in $\mathcal{B}_\zeta$, there exist for every $\delta_0 > 0$ an $h \in \mathcal{B}^1_\zeta$ and a $g_0 \in \mathcal{B}_\zeta$ satisfying

$$f_\lambda = h + g_0, \tag{A.1}$$

with

$$\|g_0\|_\zeta \leq \delta_0. \tag{A.2}$$

Moreover, one defines

$$f_0 \equiv 0. \tag{A.3}$$

Denoting $c_\lambda(f_\lambda) = \bar{c}_\lambda$ and $\overline{\mathcal{N}}_\lambda = \bar{c}_\lambda \mathcal{N}^1_\lambda + c_2 \mathcal{N}^2_\lambda$, we now define for all $n \geq 0$,

$$\begin{aligned} f_{n+1} &= \overline{\mathcal{N}}_\lambda(h + f_n) - h + \mathcal{C}_\lambda(f_n, g_n), \\ g_{n+1} &= \overline{\mathcal{N}}_\lambda(g_n), \end{aligned} \tag{A.4}$$

where

$$\mathcal{C}_\lambda(f, g) = \overline{\mathcal{N}}_\lambda(h + f + g) - \overline{\mathcal{N}}_\lambda(h + f) - \overline{\mathcal{N}}_\lambda(g).$$

Note that $\mathcal{C}_\lambda(f, g)$ contains only cross terms between $h + f$ and g. We now check that the sequences $\{f_n\}_{n\geq 0}$ and $\{g_n\}_{n\geq 0}$ have the desired properties, i.e., $f_\lambda = h + f_n + g_n$, $\{g_n\}_{n\geq 0}$ converges to zero in $\mathcal{B}_\zeta$, and $\{f_n\}_{n\geq 0}$ is Cauchy in $\mathcal{B}^1_\zeta$. Since f_λ is a fixed point of $\overline{\mathcal{N}}_\lambda$, it first follows from (A.1) and (A.4) that

$$f_\lambda = h + f_n + g_n,$$

for all $n \geq 0$. Furthermore, $\zeta > 0$ and $\lambda \in (1,4)$ together with Proposition 2.2 imply that $\{g_n\}_{n\geq 0}$ converges to zero in $\mathcal{B}_\zeta$. Indeed, $\overline{\mathcal{N}}_\lambda$ is well defined as a map from $\mathcal{B}_\zeta$ to $\mathcal{B}_\zeta$, and the bounds obtained in the proof of Proposition 2.2 lead to

$$\|g_n\|_\zeta \leq \bar{c}_\lambda \|g_{n-1}\|_\zeta^2 + c_2 \|g_{n-1}\|_\zeta^4.$$

Applying this inequality recursively and using $\|g_0\|_\zeta \leq \delta_0 < 1$, one gets for all $n \geq 1$

$$\|g_n\|_\zeta \leq \delta^{2^n}, \tag{A.5}$$

where

$$\delta = (\bar{c}_\lambda + c_2)\delta_0 < 1$$

for δ_0 small enough. Note that (A.1), (A.2) and (A.5) imply, for δ_0 small enough, the uniform bound

$$\begin{aligned} \|f_n\|_\zeta &\leq \|f_\lambda - h\|_\zeta + \|g_n\|_\zeta \\ &\leq 2\delta_0. \end{aligned} \tag{A.6}$$

Next, in order to show that $f_n \in \mathcal{B}^1_\zeta$ for all $n \geq 0$, one proceeds as in Proposition 2.2 and studies the maps which enter the definition of $\overline{\mathcal{N}}_\lambda$ and $\mathcal{C}_\lambda$, i.e., S_λ, T and the convolution operator. From (2.9) and $(f * g)' = f' * g$, it follows that

$$\|f * g\|^1_{(4\sigma)\tau} \leq \|f\|^1_{\sigma\tau} \|g\|_{\sigma\tau}, \tag{A.7}$$

whereas (2.10) together with $\lambda > 1$ and $(S_\lambda f)' = \lambda S_\lambda f'$ leads to

$$\begin{aligned} \|S_\lambda f\|^1_{(\sigma/\lambda)(\lambda\tau)} &\leq \|f\|_{\sigma\tau} + \lambda \|f'\|_{\sigma\tau} \\ &\leq \lambda\, \|f\|^1_{\sigma\tau}. \end{aligned} \tag{A.8}$$

(A.7) and (A.8) imply in particular

$$\|S_\lambda (f * g)\|^1_{(4\zeta/\lambda)(\lambda\zeta)} \leq \lambda \|f\|^1_\zeta \|g\|_\zeta. \tag{A.9}$$

We now show that for all $\tau > \tau'$, T is a bounded operator from $\mathcal{B}^1_{\sigma\tau}$ to $\mathcal{B}^1_{\tau'\sigma}$. One has $\|Tf\|_{\tau'\sigma} = \|f\|_{\sigma\tau'} \leq \|f\|_{\sigma\tau}$, and using

$$(Tf)'(x) = -\frac{1}{x^2}\Big(2x(Tf)(x) + (Tf')(x)\Big),$$

one gets

$$\begin{aligned} \|(Tf)'\|_{\tau'\sigma} &\leq 2\int_0^\infty \frac{1}{x}\, w_{\tau'\sigma}(x)\, |Tf|(x)\, dx + \int_0^\infty \frac{1}{x^2}\, w_{\tau'\sigma}(x)\, |Tf'|(x)\, dx \\ &= 2\int_0^\infty x\, w_{\sigma\tau'}(x)\, |f|(x)\, dx + \int_0^\infty x^2\, w_{\sigma\tau'}(x)\, |f'|(x)\, dx \\ &\leq 2 \sup_{x>0} \frac{(1+x^2) w_{\sigma\tau'}(x)}{w_{\sigma\tau}(x)} \|f\|^1_{\sigma\tau} \\ &\leq C_{\tau\tau'} \|f\|^1_{\sigma\tau}, \end{aligned} \tag{A.10}$$

where $C_{\tau\tau'}$ is finite as long as $\tau > \tau'$. In particular, since $\lambda > 1$, (A.9) and (A.10) imply

$$\|TS_\lambda(f * g)\|^1_{\zeta(4\zeta/\lambda)} \leq C\|f\|^1_\zeta \|g\|_\zeta,$$

which in turn, with $\lambda < 4$, leads to

$$\|T\big(TS_\lambda(f * g) * TS_\lambda(\bar{g} * \bar{\bar{g}})\big)\|^1_\zeta \leq C\|f\|^1_\zeta \|g\|_\zeta \|\bar{g}\|_\zeta \|\bar{\bar{g}}\|_\zeta. \tag{A.11}$$

Therefore, $\overline{\mathcal{N}}_\lambda$ is well defined as a map from $\mathcal{B}^1_\zeta$ to $\mathcal{B}^1_\zeta$ for $\zeta > 0$ and $\lambda \in (1, 4)$. Assume now that $f_n \in \mathcal{B}^1_\zeta$ and that δ_0 is small enough. Then, (A.6), (A.9), and (A.11) lead to

$$\begin{aligned}
\|\overline{\mathcal{N}}_\lambda(h + f_n)\|^1_\zeta &\leq \|\overline{\mathcal{N}}_\lambda(h)\|^1_\zeta + C\|f_n\|_\zeta(\|h\|^1_\zeta + \|f_n\|^1_\zeta) \\
&\leq C_1 + \frac{1}{2}\bar{\delta}\|f_n\|^1_\zeta,
\end{aligned} \tag{A.12}$$

for some positive $\bar{\delta} < 1$. Similarly, using (A.5) and (A.6), one gets

$$\|\mathcal{C}_\lambda(f_n, g_n)\|^1_\zeta \leq C\|g_n\|_\zeta\big(\|f_n\|^1_\zeta + \|h\|^1_\zeta\big). \tag{A.13}$$

Therefore, (A.12) and (A.13) lead, together with (A.5), to

$$\|f_{n+1}\|^1_\zeta \leq C_2 + \bar{\delta}\|f_n\|^1_\zeta.$$

Using this bound recursively, one obtains $f_n \in \mathcal{B}^1_\zeta$ for all $n \geq 0$, together with the uniform estimate

$$\|f_n\|^1_\zeta \leq C_2 \sum_{k=0}^{n} \bar{\delta}^k \leq K. \tag{A.14}$$

Finally, we check that the sequence $\{f_n\}_{n\geq 0}$ is Cauchy in $\mathcal{B}^1_\zeta$. Since (A.5), (A.13) and (A.14) imply that

$$\lim_{n\to\infty} \|\mathcal{C}_\lambda(f_n, g_n)\|^1_\zeta = 0,$$

it only remains to show that $\{\overline{\mathcal{N}}_\lambda(h + f_n)\}_{n\geq 0}$ is Cauchy in $\mathcal{B}^1_\zeta$. We first verify that the sequence $\{h_n\}_{n\geq 0}$, with

$$h_n \equiv \mathcal{N}^1_\lambda(h + f_n),$$

is Cauchy in $\mathcal{B}^1_{(4\zeta/\lambda)(\lambda\zeta)}$. Since $\lambda \in (1, 4)$, this implies in particular the convergence of $\{\mathcal{N}^1_\lambda(h + f_n)\}_{n\geq 0}$ in $\mathcal{B}^1_\zeta$. Defining

$$\tilde{\mathcal{N}}^1_\lambda(f, g) = S_\lambda(f * g),$$

one has

$$\begin{aligned}
\|h_n - h_m\|^1_{\sigma\tau} &\leq 2\|\tilde{\mathcal{N}}^1_\lambda(h, f_n) - \tilde{\mathcal{N}}^1_\lambda(h, f_m)\|^1_{\sigma\tau} + \|\mathcal{N}^1_\lambda(f_n) - \mathcal{N}^1_\lambda(f_m)\|^1_{\sigma\tau} \\
&\leq 2\|\tilde{\mathcal{N}}^1_\lambda(h, f_n - f_m)\|^1_{\sigma\tau} + \|\tilde{\mathcal{N}}^1_\lambda(f_n, f_n - f_m)\|^1_{\sigma\tau} + \|\tilde{\mathcal{N}}^1_\lambda(f_m, f_m - f_n)\|^1_{\sigma\tau},
\end{aligned}$$

which leads, with (A.9) and (A.14), to

$$\begin{aligned}\|h_n - h_m\|^1_{(4\zeta/\lambda)(\lambda\zeta)} &\le C\big(2\|h\|^1_\zeta + \|f_n\|^1_\zeta + \|f_m\|^1_\zeta\big)\|f_n - f_m\|_\zeta \\ &\le K\|g_n - g_m\|_\zeta.\end{aligned}$$

Therefore, the convergence of $\{h_n\}$ in $\mathcal{B}^1_{(4\zeta/\lambda)(\lambda\zeta)}$ follows from the convergence of $\{g_n\}$ in $\mathcal{B}_\zeta$. Finally, in order to see that $\{\mathcal{N}^2_\lambda(h+f_n)\}$ is Cauchy in $\mathcal{B}^1_\zeta$, one observes that

$$\mathcal{N}^2_\lambda(h+f_n) = F(h_n),$$

where

$$F(f) = T\big(Tf * Tf\big),$$

and that the bounds obtained above imply the continuity of F as a map from $\mathcal{B}^1_{(4\zeta/\lambda)(\lambda\zeta)}$ to $\mathcal{B}^1_\zeta$ for $\zeta > 0$ and $\lambda \in (1,4)$. Hence, the convergence of $\{\mathcal{N}^2_\lambda(h+f_n)\}$ in $\mathcal{B}^1_\zeta$ follows from the convergence of $\{h_n\}$ in $\mathcal{B}^1_{(4\zeta/\lambda)(\lambda\zeta)}$.

■

Proof of Lemma 4.1.

For $x \in I_i$, the functions ρ and $T\tilde\rho$ are given by

$$\rho(x) = \frac{1}{\varepsilon}\Big(\rho_i(x - x_{i-1}) + \rho_{i-1}(x_i - x)\Big), \tag{A.15}$$

$$\begin{aligned}\tilde\rho(1/x) &= \Big(\frac{1}{x_{i-1}} - \frac{1}{x_i}\Big)^{-1}\Big(x^2_{i-1}\rho_{i-1}\big(\frac{1}{x} - \frac{1}{x_i}\big) + x_i^2\rho_i\big(\frac{1}{x_{i-1}} - \frac{1}{x}\big)\Big) \\ &= \frac{1}{\varepsilon x}\Big(x^3_{i-1}\rho_{i-1}(x_i - x) + x_i^3\rho_i(x - x_{i-1})\Big),\end{aligned} \tag{A.16}$$

and one computes

$$\begin{aligned}|\rho(x) - T\tilde\rho(x)| &= |\rho(x) - \tilde\rho(1/x)/x^2| \\ &= \frac{1}{\varepsilon}\Big|\rho_{i-1}(x_i - x)\big(1 - \frac{x^3_{i-1}}{x^3}\big) - \rho_i(x - x_{i-1})\big(\frac{x_i^3}{x^3} - 1\big)\Big| \\ &= \frac{(x - x_{i-1})(x_i - x)}{\varepsilon x^3}\big|\rho_{i-1}(x - x_{i-1})^2 - \rho_i(x_i - x)^2 - 3x(x_i\rho_i - x_{i-1}\rho_{i-1})\big| \\ &\le \frac{\varepsilon}{4}\Big(\frac{|\rho_i - \rho_{i-1}|}{x} + \frac{|x_i\rho_i - x_{i-1}\rho_{i-1}|}{x^2} + \frac{|x_i^2\rho_i - x^2_{i-1}\rho_{i-1}|}{x^3}\Big).\end{aligned} \tag{A.17}$$

Integrating the expression on the RHS of (A.17) leads to the stated result.

■

Proof of Lemma 4.2.

By definition of $\mathcal{A}$, it is clear that $(\rho * \sigma)'' \in \mathcal{A}$. Furthermore, because ρ and σ have a uniform partition with identical mesh size ε, $(\rho * \sigma)''$ is also defined on a uniform partition. It is given by $\{z_k\}_{k=0}^{2n}$ where z_k is defined in (4.20). It remains to compute $v_k \equiv (\rho * \sigma)''(z_k)$. With $\rho'(x) = \sum_{i=1}^{n} \rho_i' \chi_{I_i}(x)$ and $\sigma'(x) = \sum_{j=1}^{n} \sigma_j' \chi_{J_j}(x)$, where $\rho_i' = (\rho_i - \rho_{i-1})/\varepsilon$ and $\sigma_j' = (\sigma_j - \sigma_{j-1})/\varepsilon$, one gets

$$v_k = \varepsilon \sum_{i+j=k+1} \rho_i' \sigma_j'.$$

Expressing the RHS of the previous equality in terms of the coefficients (4.19) finally leads to the relation (4.21).

■

Proof of Lemma 4.4.

For $k = 2l$, $l = 0, \ldots, n-1$, and $\theta \in (0, 2\varepsilon)$, one has

$$\tilde{\rho}(z_k + \theta) = C_0(k) + \frac{\theta}{2\varepsilon}\Big(C_0(k+2) - C_0(k)\Big).$$

The continuity properties of $\rho * \sigma$ imply

$$\begin{aligned}
C_0(k+1) &= C_0(k) + \varepsilon C_1(k) + \varepsilon^2 C_2(k) + \varepsilon^3 C_3(k),\\
C_1(k+1) &= C_1(k) + 2\varepsilon C_2(k) + 3\varepsilon^2 C_3(k),\\
C_2(k+1) &= C_2(k) + 3\varepsilon C_3(k),
\end{aligned}$$

$k = 0, \ldots, 2n-1$. For $\theta \in (0, \varepsilon)$, these relations allow us to write

$$\tilde{\rho}(z_k + \theta) = C_0(k) + \theta C_1(k) + \varepsilon\theta\Big(2C_2(k+1) - \frac{5\varepsilon}{2} C_3(k) + \frac{\varepsilon}{2} C_3(k+1)\Big),$$

and, from (4.23),

$$(\rho * \sigma)(z_k + \theta) = C_0(k) + \theta C_1(k) + \theta^2\Big(C_2(k+1) - (3\varepsilon - \theta) C_3(k)\Big).$$

Hence,

$$(\tilde{\rho} - \rho * \sigma)(z_k + \theta) = \theta(2\varepsilon - \theta) C_2(k+1) + \theta\Big(\theta(3\varepsilon - \theta) - \frac{5\varepsilon^2}{2}\Big) C_3(k) + \frac{\varepsilon^2\theta}{2} C_3(k+1),$$

which leads to the estimate

$$\int_{z_k}^{z_{k+1}} |(\tilde{\rho} - \rho * \sigma)(y)| dy \le \varepsilon^3 \Big(\frac{2}{3}|C_2(k+1)| + \frac{\varepsilon}{2}|C_3(k)| + \frac{\varepsilon}{4}|C_3(k+1)|\Big).$$

For $\theta \in (\varepsilon, 2\varepsilon)$, one proceeds similarly and obtains

$$\int_{z_{k+1}}^{z_{k+2}} |(\tilde{\rho} - \rho * \sigma)(y)| dy \leq \varepsilon^3 \Big(\frac{2}{3} |C_2(k+1)| + \frac{\varepsilon}{4} |C_3(k)| + \frac{\varepsilon}{2} |C_3(k+1)| \Big),$$

and the bound (4.27) follows immediately.

■

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