A relation of Berezin-Toeplitz operators to Schrödinger operators and the probabilistic representation of Berezin-Toeplitz semigroups

Bernhard G. Bodmann

Department of Mathematics, University of Florida, FL 32611, USA

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Abstract:

A class of functions is specified which give rise to semibounded quadratic forms on weighted Bergman spaces and thus can be interpreted as symbols of self-adjoint Berezin-Toeplitz operators. A similar class admits a probabilistic expression of the sesqui-analytic integral kernel for the associated semigroups. Both results are the consequence of a relation of Berezin-Toeplitz operators to Schrödinger operators defined via certain quadratic forms. The probabilistic expression is derived in conjunction with the Feynman-Kac-Itô formula.

1 Introduction

The results presented here are inspired by a concept of Daubechies and Klauder [DK85, DK86] which provides a probabilistic expression for the unitary group generated by Hamiltonians arising from the so-called anti-Wick quantization prescription. Based on geometric considerations, several works [KO89, AKL93, Kla94, AK96] advocate natural generalizations of this expression which can be related [Kla88, Mar92] to Hamiltonians from the more universal Berezin-Toeplitz quantization scheme [Ber74].

The generalization presented in this paper evolved from a pattern behind the construction in [DK85, DK86], according to which certain Berezin-Toeplitz operators can be realized as limits of monotone families of Schrödinger operators. Hereby a Berezin-Toeplitz operator T_f is understood as the compression of a suitable multiplication operator $M_f : \psi \mapsto f\psi$ from a Hilbert space of square-integrable functions to a closed subspace of analytic functions. In this context the real-valued function f is called a symbol and can be interpreted as a classical observable corresponding to the operator T_f. In quantum mechanics Berezin-Toeplitz operators model a variety of systems with canonical or other degrees of freedom [Ber74, BMM96]. A Schrödinger operator H, on the other hand, can be thought of as a second-order differential operator on a subset Λ of d-dimensional Euclidean space $\mathbb{R}^d$, formally written $H = \sum_{k=1}^{d}(i\partial_k + a_k)^2 + v$ with the vector potential $a : \Lambda \to \mathbb{R}^d$ and the scalar potential $v : \Lambda \to \mathbb{R}$. Fairly general conditions have been worked out under which this formal expression characterizes H uniquely as a self-adjoint operator [LS81, CFKS87, Hin92, HS92, BHL00].

The first task in this paper is to find conditions on f which guarantee that T_f can be defined as a self-adjoint operator. Secondly we derive the analogue of the probabilistic expression in [DK85] for Berezin-Toeplitz semigroups in this general setting. Unlike the results [DK85, DKP87] derived in analogy of the anti-Wick situation and the strategies advocated for their generalization [KO89, AKL93, Kla94, AK96], we stay with a probabilistic representation which is based on standard Brownian motion [RY99]. This way one may benefit from the associated repertory of probabilistic techniques. A minor difference with the original [DK85] is that we are concerned with Berezin-Toeplitz semigroups, not with the unitary groups associated with quantum mechanical time evolution.

The structure and contents of this paper are as follows:

After fixing the notation, we start in Section 3 with a review of Hilbert spaces of analytic functions which are known as weighted Bergman spaces. In Sections 4 and 5 we construct Berezin-Toeplitz operators on these spaces and state sufficient conditions on their symbols which guarantee self-adjointness and semiboundedness. Hereby an essential tool is that certain Berezin-Toeplitz operators can be extended to self-adjoint Schrödinger operators which are defined via quadratic forms.

In Section 6 we continue the spirit of [DK85, DK86] and derive a probabilistic expression for Berezin-Toeplitz semigroups which uses a realization of Berezin-Toeplitz operators as limits of monotone families of Schrödinger operators and the probabilistic representation of Schrödinger semigroups with the help of the Feynman-Kac-Itô formula [Sim79a, Sim82, BHL00].

This paper is not entirely self-contained. For the relevant background information, the reader is referred to [Sim78] and [BHL00] which neatly comprise the essential building blocks for the results presented here.

2 Basic definitions

Let $\mathbb{R}$ denote the real numbers. By convention, $\mathbb{D}$ is always an open, simply connected set in the plane of complex numbers $\mathbb{C} := \{z = z_1 + iz_2 : z_{1,2} \in \mathbb{R}\}$. The boundary of a set $A \subset \mathbb{C}$ is written as ∂A, its complement in $\mathbb{C}$ as $\mathbb{C} \setminus A$ or A^c. We will denote the real and imaginary parts of a complex number z as z_1 and z_2, respectively, and the complex conjugate as $\bar{z} = z_1 - iz_2$.

All functions appearing in this text are tacitly understood to be measurable, each one in its appropriate sense. The positive and negative parts f^+ and f^- of a real-valued function f are defined by $f^\pm := \max\{\pm f, 0\}$, such that $f = f^+ - f^-$. The indicator function of a set A is denoted as χ_A.

Several spaces of complex-valued functions are used in the text. The space of arbitrarily often differentiable functions with compact support inside $\mathbb{D}$ is referred to as $C_c^\infty(\mathbb{D})$. Concerning differentiability, $\mathbb{D}$ is hereby regarded as a subset of a real vector space $\mathbb{C} \cong \mathbb{R}^2$. The space of Lebesgue-essentially bounded functions on $\mathbb{D}$ is denoted as $L^\infty(\mathbb{D})$.

2.1. Definition. A complex-valued function $\phi : \mathbb{D} \to \mathbb{C}$ defined on $\mathbb{D}$ is called **analytic in** $\mathbb{D}$ if it is differentiable and satisfies the Cauchy-Riemann differential equations, which are stated as

$$(\partial_1 + i\partial_2)\,\phi(z_1 + iz_2) = 0, \tag{1}$$

with the partial derivatives $\partial_{1,2} := \dfrac{\partial}{\partial z_{1,2}}$.

2.2. Definition. A positive function $g : \mathbb{D} \to]0,\infty[$ is said to be **essentially bounded away from zero on compacts inside** $\mathbb{D}$ if for any given compact set $C \subset \mathbb{D}$ there exists a $\delta > 0$ such that

$$\operatorname*{ess\,inf}_{z\in C} g(z) := \sup_{A} \inf_{z\in A} g(z) \geq \delta\,. \tag{2}$$

Hereby the supremum is taken over all Lebesgue-measurable subsets $A \subset C$ such that the Lebesgue measure λ vanishes on the difference, $\lambda(C \setminus A) = 0$.

2.3. Definition. A real-valued function $f : \mathbb{C} \to \mathbb{R}$ belongs to the **Kato class** K **in two dimensions** [Kat73, Sim82] if the following condition is satisfied:

$$\lim_{r\searrow 0} \sup_{z\in\mathbb{C}} \int_{\{|y-z|<r\}} |f(y)| \ln|z-y| \; d\lambda(y) = 0\,. \tag{3}$$

Whenever this property only holds locally, that is, $\chi_C f \in K$ for all compact sets C in $\mathbb{C}$, we write $f \in K_{loc}$. It is useful to know that the local Kato property implies local integrability with respect to the Lebesgue measure, $K_{loc} \subset L^1_{loc}(\mathbb{C})$.

If a function satisfies $f^+ \in K_{loc}$ and $f_- \in K$ then it is called **Kato decomposable**, symbolized as $f \in K_\pm$.

In order to apply these notions to functions which are at first only defined on a subset $\mathbb{D} \subset \mathbb{C}$, they are by convention extended to be zero on $\mathbb{D}^c$.

3 Hilbert spaces of analytic functions

3.1. Definition. Given a positive function $g : \mathbb{D} \to]0,\infty[$ we associate with it the so-called **weighted Bergman space**

$$L^2_a(g\lambda) := \{\phi : \mathbb{D} \to \mathbb{C},\ \text{analytic in } \mathbb{D} \text{ and } (\phi,\phi) < \infty\}\,, \tag{4}$$

a vector space which is endowed with the inner product

$$(\phi,\psi) := \int_{\mathbb{D}} \overline{\phi(z)}\psi(z)\, g(z) d\lambda(z) \tag{5}$$

where λ stands for the Lebesgue measure on the complex plane.

3.2. Remarks. As a special case, if $\mathbb{D}$ is a bounded domain and the weight function is constant, $L^2_a(g\lambda)$ is the well-known Bergman space [Ber50, Mes62].

The inner product suggests that $L^2_a(g\lambda)$ can be identified with a vector-subspace of $L^2(g\lambda)$. To be precise, we recall that the latter consists of equivalence classes of $g\lambda$-square-integrable functions on $\mathbb{D}$ which differ from each other on a set of Lebesgue measure zero. Consequently, we identify each function in $L^2_a(g\lambda)$ with its equivalence class from $L^2(g\lambda)$.

Now we will state conditions which guarantee that $L_a^2(g\lambda)$ forms a Hilbert-subspace of $L^2(g\lambda)$.

3.3. Proposition. If g is essentially bounded away from zero on compacts inside $\mathbb{D}$ then $L_a^2(g\lambda)$ is complete with respect to the norm-topology induced by the inner product.

Proof. Let $(\psi_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $L_a^2(g\lambda)$. First we show uniform convergence of ψ_n on compacts inside $\mathbb{D}$. Consider a compact subset $C \subset \mathbb{D}$ and a safety radius $r < \inf_{y\in\partial\mathbb{D}} |z-y|$ for all $z \in C$. By assumption there is a lower bound $\delta > 0$ such that (2) is satisfied. Using the mean value property for analytic functions, Jensen's inequality in conjunction with the convex square-modulus function, and the lower bound for g we estimate

$$\begin{aligned}
\sup_{z'\in C} |\psi_n(z') - \psi_m(z')|^2 &= \sup_{z'\in C} \left| \frac{1}{\pi r^2} \int_{B(r,z')} (\psi_n(z) - \psi_m(z))\, d\lambda(z) \right|^2 && (6)\\
&\le \frac{1}{\pi r^2} \sup_{z'\in C} \int_{B(r,z')} |\psi_n(z) - \psi_m(z)|^2\, d\lambda(z) && (7)\\
&\le \frac{1}{\pi r^2 \delta} \int_{B(r,z')} |\psi_n(z) - \psi_m(z)|^2\, g(z) d\lambda(z) && (8)\\
&\le \frac{1}{\pi r^2 \delta} \|\psi_n - \psi_m\|^2 . && (9)
\end{aligned}$$

The right-hand side can be made arbitrarily small and thus the sequence (ψ_n) converges uniformly on C. We can therefore conclude that the pointwise limit defines a function $\psi : \psi(z) = \lim_{n\to\infty} \psi_n(z)$ which is also analytic in $\mathbb{D}$.

It remains to show that the convergence $\psi_n \to \psi$ is also in the sense of the norm. Due to pointwise convergence and Fatou's lemma the inequality

$$\|\psi - \psi_m\| \le \liminf_{n\to\infty} \|\psi_n - \psi_m\| \tag{10}$$

follows, therefore the Cauchy property entails norm convergence. □

3.4. Lemma. Under the same assumption on g as in the preceding lemma, the **point-evaluation functionals** F_z parameterized by $z \in \mathbb{D}$ which are defined according to

$$\begin{aligned}
F_z : L_a^2(g\lambda) &\longrightarrow \mathbb{C} && (11)\\
\psi &\longmapsto \psi(z) && (12)
\end{aligned}$$

are continuous linear mappings.

Proof. Let C be a compact neighborhood of z and select a convergent sequence $(\psi_n)_{n\in\mathbb{N}}$ in $L_a^2(g\lambda)$. Since the sequence has the Cauchy property, we can use the chain of inequalities (6) to show that $\psi_n(z)$ is also Cauchy, and therefore convergent. □

3.5. Proposition. If g is essentially bounded away from zero on compacts inside $\mathbb{D}$, the weighted Bergman space $L^2_a(g\lambda)$ possesses a **reproducing kernel**. More explicitly, there is a kernel $\kappa : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ such that any function ψ in $L^2_a(g\lambda)$ satisfies the integral equation

$$\psi(z') = \int_{\mathbb{D}} \kappa(z', z)\psi(z)\, g(z) d\lambda(z)\,. \tag{13}$$

Proof. To see this, we observe that due to the continuity of the functional F_z and the completeness of $L^2_a(g\lambda)$ the Riesz representation theorem implies that there is a vector κ_z in $L^2_a(g\lambda)$ such that

$$(\kappa_z, \psi) = \psi(z) \text{ for all } \psi \in L^2_a(g\lambda). \tag{14}$$

Inserting the definition of the inner product (5) yields the desired integral equation (13) with the claimed kernel given by $\kappa(z, z') = (\kappa_z, \kappa_{z'})$. □

The last proposition ensures that all bounded operators have integral kernels.

3.6. Corollary. If g is essentially bounded away from zero on compacts inside $\mathbb{D}$, then any bounded operator B on $L^2_a(g\lambda)$ possesses an integral kernel given by $B(z, z') = (\kappa_z, B\kappa_{z'})$, which means that the image of $\psi \in L^2_a(g\lambda)$ is expressed as

$$B\psi(z) = \int_{\mathbb{D}} d\lambda(z')g(z')\, B(z, z')\psi(z')\,. \tag{15}$$

Proof. That $B(z, z')$ is indeed an integral kernel results from the reproducing property (13) and Fubini's theorem. The sesqui-analyticity follows from the point-evaluation property (14) and the analyticity of the functions in $L^2_a(g\lambda)$. □

3.7. Remark. Since (15) makes sense even for $\psi \in L^2(g\lambda)$, any bounded operator extends naturally via its integral kernel to $L^2(g\lambda)$. From this point of view, κ is the integral kernel of an **orthogonal projection operator**, henceforth called $\mathcal{K}$, which maps $L^2(g\lambda)$ onto $L^2_a(g\lambda)$.

3.8. Examples. Various examples of Lie group representations are realized on specific weighted Bergman spaces. We list a few groups and their unitary irreducible action which is in all but the last example related to Moebius transformations on the associated representation spaces. For more details, see [Per86] or [Nee96]. Unless otherwise stated we set $\mathbb{D} = \mathbb{C}$.

1. *Heisenberg-Weyl group.* Hereby the Hilbert space is specified by the weight function $g(z) = \frac{1}{\pi}e^{-|z|^2}$. The reproducing kernel is $\kappa(z, z') = e^{z\bar{z}'}$. This space is also known as Fock-Bargmann space [Bar61]. The group representation $D(\alpha, \beta)$ with parameters $\alpha \in [0, 2\pi[, \beta \in \mathbb{C}$ acts on a vector ψ by

$$D(\alpha, \beta)\psi(z) = e^{i\alpha}\, e^{-|\beta|^2}\, e^{\beta z}\, \psi(z - \bar{\beta})\,. \tag{16}$$

2. $SU(2)$ *group.* For each integer or half-integer $j \in \frac{1}{2}\mathbb{N}$ a $(2j+1)$-dimensional space is defined by setting $g(z) = \frac{2j+1}{\pi}(1+|z|^2)^{-2j-2}$. The reproducing kernel is $\kappa(z,z') = (1+z\bar{z}')^{2j}$. The group parameterized by $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$ acts as

$$D(\alpha,\beta)\psi(z) = (\beta z + \bar{\alpha})^{2j}\psi\Big(\frac{\alpha z - \bar{\beta}}{\beta z + \bar{\alpha}}\Big)\,. \tag{17}$$

3. $SU(1,1)$ *group.* There are two well-known ways to represent this group on weighted Bergman spaces.

   a) The first one is described in [Per86]. Unlike the previous examples, here $\mathbb{D}$ is not the whole complex plane, but the unit disc $\mathbb{D} = \{z : |z| < 1\}$, and $g(z) = \frac{2k-1}{\pi}(1 - z\bar{z})^{2k-2}$ with a fixed number $k \in \{1, 3/2, 2, \dots\}$. The reproducing kernel is $\kappa(z,z') = (1 - z\bar{z}')^{-2k}$. The group action is given with the parameters $\alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1$ as

$$D(\alpha,\beta)\psi(z) = (\beta z + \bar{\alpha})^{-2k}\psi\Big(\frac{\alpha z + \bar{\beta}}{\beta z + \bar{\alpha}}\Big)\,. \tag{18}$$

   b) An alternative to [Per86] comes from the so-called Barut-Girardello representation [BG71] of the $SU(1,1)$ group. We set $\mathbb{D} = \mathbb{C}$ again, select k as above and choose $g(z) = \frac{2|z|^{2k-1}}{\pi\Gamma(2k)}K_{2k-1}(2|z|)$ with the Gamma-function Γ and the modified Bessel function $K_\sigma(r) = \frac{1}{2}(r/2)^\sigma \int_0^\infty t^{-\sigma-1}\exp(-t - r^2/4t)\,dt$ for $r > 0, \sigma \in \mathbb{R}$ [GR95, 8.432(6)]. The kernel is given as the confluent hypergeometric limit function $\kappa(z,z') = {}_0F_1(2k, z\bar{z}') = \sum_{n=0}^\infty (z\bar{z}')^n/(2k)_n\Gamma(n+1)$. Hereby $(2k)_n$ denotes the Pochhammer symbol, defined by $(2k)_0 := 1$ and the recursion $(2k)_n = (2k)_{n-1}(2k+n-1)$. The group action, however, does not seem to be related to Moebius transformations and will be omitted here.

3.9. Remarks. In all these examples the kernel can be constructed from the weight function with the help of the Gram-Schmidt orthogonalization procedure. Due to the rotational symmetry $g(z) = g(|z|)$, monomials $p_n : z \mapsto z^n$ with differing degree $n \in \{0,1,2,\dots\}$ are orthogonal, so the kernel is diagonalized in terms of the basis functions as $\kappa(z,z') = \sum_n c_n p_n(z)\overline{p_n(z')}$ with suitable normalization constants c_n. Here the summation runs for all the examples over the non-negative integers, except for the $SU(2)$ case, because there, only monomials with maximal degree $2j$ are square integrable.

In Section 6 we will present a probabilistic approach to construct the reproducing kernel, which does not rely on special symmetries of g.

4 Self-adjoint Berezin-Toeplitz operators defined by quadratic forms

In the remaining text we assume that $L^2_a(g\lambda)$ is complete.

4.1. Definition. Given the Hilbert space $L_a^2(g\lambda)$ and a real-valued function $f : \mathbb{D} \to \mathbb{R}$, we consider the **sesquilinear form**

$$\begin{aligned} \mathfrak{t}_f : \mathcal{Q}(\mathfrak{t}_f) \times \mathcal{Q}(\mathfrak{t}_f) &\longrightarrow \mathbb{C} && (19) \\ (\psi, \phi) &\longmapsto \int_{\mathbb{D}} f \overline{\psi} \phi \, g d\lambda && (20) \end{aligned}$$

with form domain

$$\mathcal{Q}(\mathfrak{t}_f) := \{\psi \in L_a^2(g\lambda) : \int_{\mathbb{D}} \left| f\psi^2 \right| g d\lambda < \infty\} . \tag{21}$$

When it is interpreted as quadratic form, $\mathfrak{t}_f$ is written as $\mathfrak{t}_f(\psi) := \mathfrak{t}_f(\psi, \psi)$.

The next concern is a condition to guarantee that $\mathfrak{t}_f$ is closed and semi-bounded, which in turn ensures that there is a self-adjoint operator associated with $\mathfrak{t}_f$ via the Friedrichs representation theorem.

4.2. Lemma. The sesquilinear form belonging to a non-negative function $f \geq 0$ is closed.

Proof. We need to show that $\mathcal{Q}(\mathfrak{t}_f)$, equipped with the form-norm $\|\bullet\|_{\mathfrak{t}_f}$ defined by

$$\|\psi\|_{\mathfrak{t}_f} := (\mathfrak{t}_f(\psi) + \|\psi\|^2)^{1/2} \text{ for } \psi \in \mathcal{Q}(\mathfrak{t}_f) , \tag{22}$$

is complete.

Suppose $(\psi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to the form-norm. Due to the estimate $\|\psi\| \leq \|\psi\|_{\mathfrak{t}_f}$ the sequence is convergent in $L_a^2(g\lambda)$, $\psi_n \to \psi$. Using pointwise convergence and Fatou's lemma, we obtain $\|\psi - \psi_n\|_{\mathfrak{t}_f} \leq \liminf_{m\to\infty} \|\psi_m - \psi_n\|_{\mathfrak{t}_f}$ and therefore the sequence (ψ_n) converges with respect to the form-norm. ☐

4.3. Theorem. If the form $\mathfrak{t}_{f^+}$ belonging to the positive part $f^+ \geq 0$ of a function $f = f^+ - f^-$ is densely defined and the negative part can be incorporated in $\mathfrak{t}_f$ as a form-bounded perturbation

$$\mathfrak{t}_{f^-}(\psi) \leq c_1 \, \mathfrak{t}_{f^+}(\psi) + c_2 \, \|\psi\|^2 \tag{23}$$

with relative form bound $c_1 < 1$ and a constant $c_2 \geq 0$, then $\mathfrak{t}_f$ is closed on $\mathcal{Q}(\mathfrak{t}_f) = \mathcal{Q}(\mathfrak{t}_{f^+})$ and has a (greatest) lower bound $c \in \mathbb{R}$, such that $\mathfrak{t}_f(\psi) \geq c \, \|\psi\|^2$.

Proof. This is the so-called KLMN theorem. For the proof, see [RS75, Theorem X.17]. ☐

4.4. Theorem. If the form $\mathfrak{t}_f$ is closed and has the greatest lower bound c as in the preceding theorem, then it belongs to a unique self-adjoint operator T_f which is characterized in terms of the square-root $\sqrt{T_f - c}$ by the domain $\mathcal{D}(\sqrt{T_f - c}) = \mathcal{Q}(\mathfrak{t}_f)$ and the equality

$$(\sqrt{T_f - c}\,\phi, \sqrt{T_f - c}\,\psi) + c(\phi, \psi) = \mathfrak{t}_f(\phi, \psi) \quad \text{for all} \quad \phi, \psi \in \mathcal{Q}(\mathfrak{t}_f) . \tag{24}$$

Proof. Again, we refer to the literature [RS80, Theorem VIII.15] or [Wei80, Theorem 5.36], where this result is known as the Friedrichs representation theorem. □

4.5. Remark. We do not adopt the usual term Friedrichs extension for T_f, because here it does not arise from the quadratic form of a given semibounded operator, and besides that, T_f is uniquely determined due to the closedness of the form. In the context of weighted Bergman spaces we call T_f a self-adjoint **Berezin-Toeplitz operator** and the function f its **symbol**.

5 Relation to Schrödinger operators

This section shows how a Berezin-Toeplitz operator can be extended to a family of Schrödinger operators. A major benefit is that the knowledge about Schrödinger operators can be used to find sufficient conditions for the semiboundedness of $\mathfrak{t}_f$ in terms of g and f. These ensure self-adjointness of the corresponding Berezin-Toeplitz operator.

5.1 Embedding the weighted Bergman space

Consider the unitary mapping

$$\begin{aligned} U : L^2(g\lambda) &\longrightarrow L^2(\mathbb{D}) && (25)\\ \psi &\longmapsto \sqrt{g}\psi\,, && (26)\end{aligned}$$

which simply amounts to a redistribution of the weight function. This mapping identifies $L^2_a(g\lambda)$ with a closed subspace of $L^2(\mathbb{D})$.

Remark. Note that the unitary equivalent $\widetilde{\mathcal{K}} = U\mathcal{K}U^\dagger$ of $\mathcal{K}$ is an orthogonal projection operator which maps $L^2(\mathbb{D})$ onto $U(L^2_a(g\lambda))$ and has the integral kernel $\widetilde{\kappa}(z,z') = \kappa(z,z')\sqrt{g(z)g(z')}$. However, unlike the case for $L^2_a(g\lambda)$, this kernel is continuous if and only if g is continuous. Following the previous notation we define $\widetilde{\kappa}_{z'} := \sqrt{g(z')}\,U\kappa_{z'}$.

5.2 A quadratic form and its null space

The purpose of this subsection is to construct a Schrödinger operator which is non-negative and has $U(L^2_a(g\lambda))$ as its null eigenspace.

5.1. Definition. We define the quadratic form

$$\begin{aligned} \mathfrak{r} : C_c^\infty(\mathbb{D}) &\longrightarrow \mathbb{R} && (27)\\ \psi &\longmapsto \|(i\partial_1 - \partial_2 + a)\psi\|^2 && (28)\end{aligned}$$

with the function

$$a(z) := (-i\partial_1 + \partial_2)\ln\sqrt{g(z)} \qquad (29)$$

where g is such that the local integrability condition $|a|^2 \in L^1_{loc}(\mathbb{D})$ holds. This form is closeable and the resulting form-closure domain $\mathcal{Q}(\mathfrak{r})$ is contained in the set $L^2(\mathbb{D})$. We say that the corresponding self-adjoint operator R obeys **Dirichlet boundary conditions**.

5.2. Remarks. Despite the construction of R via (28), in general the null-space $\mathcal{N}(R) := R^{-1}(\{0\})$ is only a closed subspace of $U(L^2_a(g\lambda))$. On the other hand, if instead of $C_c^\infty(\mathbb{D})$ we start with the maximal form domain, which means the set of all $\psi \in L^2(\mathbb{D})$ for which the expression on the right-hand side of (28) makes sense and is finite, then the resulting self-adjoint operator is said to obey Neumann boundary conditions and possesses the null-space $U(L^2_a(g\lambda))$. For this reason it seems like an artificial complication to introduce Dirichlet boundary conditions here. Nevertheless, the preceding definition is needed as a preparation for the probabilistic expression in Section 6.

If a is absolutely continuous we can formally write R as a Schrödinger-type operator

$$R = (i\partial_1 + a_1)^2 + (i\partial_2 + a_2)^2 + v \tag{30}$$

with $v := \partial_2 a_1 - \partial_1 a_2$. However, in order to consider R as the usual form sum [BHL00, Remark 2.7] we need additional regularity assumptions on a and v; see also the following subsection.

Next we establish conditions which guarantee the inclusion of $U(L^2_a(g\lambda))$ in the domain of R.

5.3. Proposition. If either $\mathbb{D} = \mathbb{C}$ or the operator T_f for $f(z) = \inf_{y \in \partial\mathbb{D}} |z-y|^{-2}$ is bounded on $L^2_a(g\lambda)$, then the image $U(L^2_a(g\lambda))$ is contained in the form-closure domain $\mathcal{Q}(\mathfrak{r})$ and consequently $\mathcal{N}(R) = U(L^2_a(g\lambda))$.

Proof. In case $\mathbb{D} = \mathbb{C}$ the argument follows a standard procedure, compare with [Sim79b] or [CFKS87, Theorem 1.13]. The more general case treated here demands an adaptation.

The proof proceeds in two steps: First we will show that for any function $\psi \in U(L^2_a(g\lambda))$, the space of compactly supported, in $\mathbb{D}$ essentially bounded functions $L_c^\infty(\mathbb{D})$, provides a sequence which converges to ψ in the sense of the form norm. The second step is a standard mollifier argument, which shows that $C_c^\infty(\mathbb{D})$ is dense in $L_c^\infty(\mathbb{D}) \cap \mathcal{Q}(\mathfrak{r})$.

Step 1: Consider a function $\psi \in U(L^2_a(g\lambda))$. We perform two alterations: a mollified cutoff near $\partial\mathbb{D}$ (if $\mathbb{D}^c$ is nonempty) and a smooth cutoff towards infinity.

The details are as follows: Let $D_n := \{z \in \mathbb{D} : |z-y| \geq 1/n \text{ for all } y \in \partial\mathbb{D}\}$. Obviously the sequence of characteristic functions χ_{D_n} converges pointwise to one on $\mathbb{D}$ as $n \to \infty$. In addition, consider a real-valued function $\eta \in C_0^\infty(\mathbb{C})$ which is non-negative and satisfies $\eta(0) = \max_{z\in\mathbb{C}} \eta(z) = 1$ as well as the gradient bound $|\nabla\eta| \leq 1$. We define a sequence $\eta_n(z) = \eta(z/n)$ which also tends to one from below. Furthermore, let the non-negative function $\delta_1 \in C_c^\infty(\mathbb{C})$ be an approximate δ-function, which means $\int_C \delta_1(z) d\lambda = 1$, and $\delta = 0$ for all

$|z| > 1$. We define a sequence of approximations $\delta_n(z) := n^2\delta_1(nz)$ to smear out χ_{D_n} by convolution, $(\chi_{D_n} * \delta_n)(z) := \int_{\mathbb{D}} d\lambda(y)\, \chi_{D_n}(y)\delta_n(x-y)$.

Now consider $\phi_n := \psi\eta_n(\chi_{D_n} * \delta_n)$. Clearly $\phi_n \to \psi$ in $L^2(\mathbb{D})$. To show that the convergence is also with respect to the form-norm, we use the triangle inequality,

$$\begin{aligned} \|(i\partial_1 - \partial_2 + a)\phi_n\| &= \|\psi(i\partial_1 - \partial_2)\eta_n(\chi_{D_n} * \delta_n)\| \\ &\leq \|\psi(\chi_{D_n} * \delta_n)(i\partial_1 - \partial_2)\eta_n\| + \|\psi\eta_n(i\partial_1 - \partial_2)(\chi_{D_n} * \delta_n)\| \,. \end{aligned} \tag{31}$$

The first term on the right-hand side vanishes in the limit $n \to \infty$ by dominated convergence; the second term needs a closer look. Note that the neigborhood of the boundary contains the support $\operatorname{supp}(i\partial_1 - \partial_2)(\delta_n * \chi_{D_n}) \subset \{z \in \mathbb{D} : |z - y| > 2/n \,\forall y \in \partial\mathbb{D}\}$. By Hölder's inequality we can estimate $|\nabla\delta_n * \chi_{D_n}| \leq n\lambda(\operatorname{supp}\delta_1)\max|\nabla\delta_1|$. In consequence there is some constant $c \geq 0$ such that $|(i\partial_1 - \partial_2)(\chi_{D_n} * \delta_n)| \leq c\sqrt{f}$ with the function f given in the statement of the theorem. Hence by the assumption on the boundedness of T_f we can apply dominated convergence and the second term in (31) also vanishes as $n \to \infty$.

Step 2: Let $\psi \in L_c^\infty(\mathbb{D}) \cap \mathcal{Q}(\mathfrak{t})$ be given. We borrow an argument from [BHL00, Lemma B.3, Assertion 3)]. Consider the previously defined δ_n for only suitably large n such that $\psi_n := \psi * \delta_n \in C_c^\infty(\mathbb{D})$.

We have $\lim_{n\to\infty} \|\phi * \delta_n - \phi\| = 0$ for all $\phi \in L^2(\mathbb{D})$. Therefore it remains to show that after the estimate

$$\|(i\partial_1 - \partial_2 + a)(\psi_n - \psi)\| \leq \|(i\partial_1 - \partial_2)(\psi_n - \psi)\| + \|a(\psi_n - \psi)\| \tag{32}$$

each term on the right-hand side converges to zero.

The first term is taken care of by the identity

$$\|(i\partial_1 - \partial_2)(\psi_n - \psi)\| = \|((i\partial_1 - \partial_2)\psi) * \delta_n - (i\partial_1 - \partial_2)\psi\| \tag{33}$$

because the assumption $|a|^2 \in K_{loc}$ implies $a \in L^2_{loc}(\mathbb{D})$ and

$$\|(i\partial_1 - \partial_2)\psi\| \leq \|(i\partial_1 - \partial_2 + a)\psi\| + \|a\psi\| < \infty \,. \tag{34}$$

To ensure that the second term in (32) vanishes we pass to a subsequence of ψ_n which is almost everywhere convergent and select a compact set C which contains the support of all but finitely many ψ_n. The bound $|\psi_n(x)| \leq \chi_C \|\psi\|_\infty$ for $x \in \mathbb{D}$ together with $a \in L^2_{loc}(\mathbb{D})$ allows dominated convergence which completes the proof. □

5.4. Remark. When $\mathbb{D}$ is a proper subset of the complex numbers, the condition on the functions in $U(L^2_a(g\lambda))$ amounts to controlling their decay towards the boundary $\partial\mathbb{D}$, which relates to the intuitive understanding of Dirichlet boundary conditions.

5.3 Extending Berezin-Toeplitz operators to Schrödinger operators

Henceforth, we say f and g are **admissible** if they are such that $|a|^2 \in K_{loc}$ and $\partial_2 a_1 - \partial_1 a_2, f \in K_\pm$.

For such f and g we define a family $\{\mathfrak{s}_f^{(\nu)}\}_{\nu>0}$ of quadratic forms

$$\mathfrak{s}_f^{(\nu)} : \mathcal{Q}(\mathfrak{s}_f^{(\nu)}) \longrightarrow \mathbb{R} \tag{35}$$

$$\psi \longmapsto \nu\, \mathfrak{r}(\psi) + \int_{\mathbb{D}} d\lambda f\, |\psi|^2 \, . \tag{36}$$

The domain is independent of ν given by $\mathcal{Q}(\mathfrak{s}_f^{(\nu)}) := \mathcal{Q}(\mathfrak{r}) \cap \{\psi : \left\|\sqrt{f^+}\psi\right\| < \infty\}$, because the negative part of the second term in (36) is, due to the assumption, infinitesimally form-bounded with respect to $\mathfrak{r}$.

With the above regularity assumptions on a, $v = \partial_2 a_1 - \partial_1 a_2$ and f the corresponding self-adjoint, semibounded operator $S_f^{(\nu)}$ is a Schrödinger operator defined in the usual form sense [BHL00], which is implicitly understood in the expression

$$S_f^{(\nu)} = \nu\left[(i\partial_1 + a_1)^2 + (i\partial_2 + a_2)^2 + v\right] + f \, . \tag{37}$$

5.5. Theorem. If f and g obey the conditions $|a|^2 \in K_{loc}$ and $\partial_2 a_1 - \partial_1 a_2, f \in K_\pm$, and the conclusion $\mathcal{N}(R) = U(L_a^2(g\lambda))$ of Proposition 5.3 holds, then the form $\mathfrak{t}_f$ is semibounded and closed on $\mathcal{Q}(\mathfrak{t}_{f^+})$. Therefore, f defines a self-adjoint semibounded operator T_f on the closure $\overline{\mathcal{Q}(\mathfrak{t}_f)} \subset L_a^2(g\lambda)$.

Proof. First we note $\mathfrak{s}_f^{(\nu)}(U\psi) = \mathfrak{t}_f(\psi)$ for any $\nu > 0$ and $\psi \in \mathcal{Q}(\mathfrak{t}_f)$. Thus, we only need to show that the restriction of $\mathfrak{s}_f^{(\nu)}$ to the closed subspace $\mathcal{N}(\mathfrak{r}) = U(L_a^2(g\lambda))$ is again a closed and semibounded form.

To show closedness, assume a sequence $(\psi_n)_{n\in\mathbb{N}}$ in $\mathcal{N}(\mathfrak{r})$ which is Cauchy with respect to the form-norm. Then by the closedness of $\mathfrak{s}_f^{(\nu)}$ the sequence has a limit $\psi \in \mathcal{Q}(\mathfrak{s}_f^{(\nu)})$. However, this limit must also be contained in $\mathcal{N}(\mathfrak{r})$, because $\mathcal{N}(\mathfrak{r})$ is a closed subspace and the sequence $(\psi_n)_{n\in\mathbb{N}}$ converges with respect to the usual norm on $L^2(\mathbb{D})$.

Semiboundedness follows from the inequality

$$\inf\{\mathfrak{s}_f^{(\nu)}(\psi) : \|\psi\| = 1\} \le \inf\{\mathfrak{s}_f^{(\nu)}(\psi) : \psi \in \mathcal{N}(\mathfrak{r}) \text{ and } \|\psi\| = 1\} \, . \tag{38}$$

$\square$

5.6. Remarks. As stated, the above theorem does not imply that $\mathfrak{t}_f$ is densely defined. Therefore, T_f might be self-adjoint only on a Hilbert-subspace of $L_a^2(g\lambda)$.

In analogy with Theorem 4.3, it is sufficient for the closedness and semiboundedness of $\mathfrak{t}_f$ when for some $\nu > 0$ the negative part f^- can be incorporated as a form-bounded perturbation of $\mathfrak{s}_{f^+}^{(\nu)}$ with relative form bound strictly less than one. However, this condition is not as easy to characterize in terms of f as the stronger assumption in the preceding theorem.

6 Probabilistic representation of Berezin-Toeplitz semigroups

In this section we derive a probabilistic expression for the sesqui-analytic integral kernel of Berezin-Toeplitz semigroups. The major steps are the reconstruction of T_f via the monotone convergence of $\mathfrak{s}_f^{(\nu)}$ for $\nu \to \infty$ and the application of the Feynman-Kac-Itô formula.

6.1. Definition. For a given $z, z' \in \mathbb{C}$ and $t, \nu > 0$ we define the integral with respect to the **pinned Wiener measure**

$$\int (\bullet)\, d\mu^{(\nu)}_{z,0;z',t} := \frac{1}{4\pi t\nu}\, \mathrm{e}^{-\,|z-z'|^2/4t\nu}\, \mathbb{E}[\bullet] \tag{39}$$

via the expectation $\mathbb{E}[\bullet]$ with respect to the two-dimensional Brownian bridge measure with diffusion constant ν. Both measures are concentrated on the set of continuous paths $\{b : [0,t] \to \mathbb{C}, b(0) = z \text{ and } b(t) = z'\}$ which are pinned at the start and endpoint. As Gaussian stochastic process the Brownian bridge is uniquely determined by its mean

$$\mathbb{E}[b(s)] = z + (z'-z)\frac{s}{t}\,, \qquad s \in [0,t] \tag{40}$$

and covariance

$$\mathbb{E}\Big[\overline{b(r)}b(s)\Big] - \mathbb{E}\Big[\overline{b(r)}\Big]\mathbb{E}[b(s)] = 4\nu\left(\min\{r,s\} - \frac{rs}{t}\right) \tag{41}$$

$$\mathbb{E}[b(r)b(s)] - \mathbb{E}[b(r)]\mathbb{E}[b(s)] = 0 \qquad r,s \in [0,t]\,. \tag{42}$$

6.2. Definition. Given $\mathbb{D}$, the random variable $\mathrm{T}_{\mathbb{D}} := \inf\{s > 0 : b(s) \in \mathbb{D}^c\}$ is called the **first exit time** of the process. By convention, we define $\mathrm{T}_{\mathbb{D}}$ to be infinite on the set for which b never leaves $\mathbb{D}$.

6.3. Proposition. If f and g are admissible and the conclusion $\mathcal{N}(R) = U(L^2_a(g\lambda))$ of Proposition 5.3 holds, then for $\nu \to \infty$ the semigroup generated by $S_f^{(\nu)}$ converges strongly,

$$\lim_{\nu\to\infty} \mathrm{e}^{-tS_f^{(\nu)}}\psi = \mathrm{e}^{-tS_f^{(\infty)}}\widetilde{E}\psi = U\mathrm{e}^{-tT_f}EU^\dagger\psi\,, \tag{43}$$

where $\psi \in L^2(\mathbb{D})$, $E = E^\dagger E$ projects onto the closure $\overline{\mathcal{Q}(\mathfrak{t}_f)}$, and $\widetilde{E} := UEU^\dagger$.

Proof. The limit $\nu \to \infty$ of $\mathfrak{s}_f^{(\nu)}$ yields a non-densely defined form

$$\mathfrak{s}_f^{(\infty)} : \psi \mapsto \lim_{\nu\to\infty} \mathfrak{s}_f^{(\nu)}(\psi) \tag{44}$$

with the domain $\mathcal{Q}(\mathfrak{s}_f^{(\infty)}) = U(\mathcal{Q}(\mathfrak{t}_f))$. This last equality follows from Proposition 5.3 and the definition of the embedding.

The monotone convergence implies that $s_f^{(\infty)}$ is closed [Sim78] and semiboundedness follows from that of $\mathfrak{s}_f^{(\nu)}$ for any $\nu > 0$. All these properties hold then for $\mathfrak{t}_f$ as well, via the identity $\mathfrak{s}_f^{(\infty)}(U\psi) = \mathfrak{t}_f(\psi)$ valid for all $\psi \in \mathcal{Q}(\mathfrak{t}_f)$, and thus give rise to a semibounded self-adjoint operator $T_f = U^\dagger S_f^{(\infty)} U$ on the closure $\overline{\mathcal{Q}(\mathfrak{t}_f)} \subset L^2_a(g\lambda)$.

By the monotone convergence of forms the self-adjoint operators associated with $\mathfrak{s}_f^{(\nu)}$ converge in the strong resolvent sense [Sim78], which in turn implies strong convergence of the semigroups they generate [RS80, Theorem S.14]. $\square$

6.4. Theorem. Provided f and g are such that $|a|^2 \in K_{loc}$ and $v, f \in K_\pm$, and the conclusion of Proposition 5.3 holds, then for $t > 0$ the continuous integral kernel of the semigroup generated by $S_f^{(\nu)}$ converges in the limit $\nu \to \infty$ pointwise to the kernel of the semigroup associated with the generator $S_f^{(\infty)}$ on $\widetilde{E}(L^2(\mathbb{D}))$,

$$\lim_{\nu\to\infty} \mathrm{e}^{-tS_f^{(\nu)}}(z, z') = \left(\mathrm{e}^{-tS_f^{(\infty)}}\widetilde{E}\right)(z, z'). \tag{45}$$

Proof. The proof heavily borrows from the strategy of [BLW99a] which is accommodated here to the case of unbounded f. The key to the present generalization is the use of monotone convergence. Fundamental to the proof is the well-known Feynman-Kac-Itô formula [BHL00] which expresses the continuous integral kernel of the semigroup $e^{-tS_f^{(\nu)}}$ as

$$\begin{aligned}\mathrm{e}^{-tS_f^{(\nu)}}(z, z') = \int_{\{\mathrm{T}_\mathbb{D} > t\}} \exp\Big\{\frac{1}{2}\int_0^t \Big[a(b(s))\overline{db(s)} - \overline{a(b(s))}db(s)\Big] \\ - \int_0^t ds\,(\nu v + f)(b(s))\Big\}\, d\mu^{(\nu)}_{z,0;z',t}\end{aligned} \tag{46}$$

where in this case it does not matter whether the stochastic integral in the exponent is interpreted in the Itô sense or according to Fisk and Stratonovich [RW87], because $\partial_1 a_1 + \partial_2 a_2 = 0$.

For notational convenience we fix a reference diffusion constant $\nu_0 > 0$ and abbreviate for $\alpha \geq 0, w \in \mathbb{C}$

$$\eta_w^{(\alpha)}(z) := \mathrm{e}^{-tS_{\alpha f}^{(\nu_0)}}(z, w). \tag{47}$$

As preparation for the main part of the proof we state some properties of $\eta_w^{(\alpha)}$. Note that each $\eta_w^{(\alpha)}$ is a bounded and continuous function [BHL00, Theorem 4.1] and lies in $L^2(\mathbb{D})$, which follows from the inequality

$$\|\eta_w^{(\alpha)}\| \leq \sup_{w\in\mathbb{D}} \|\eta_w^{(\alpha)}\| = \|\mathrm{e}^{-tS_{\alpha f}^{(\nu)}}\|_{2,\infty} \tag{48}$$

where the last term is the finite operator norm of $e^{-tS^{(\nu)}_{\alpha f}}$ considered as mapping from $L^2(\mathbb{D})$ to $L^\infty(\mathbb{D})$ [BHL00, Estimate (2.39)].

Also note that $w \mapsto \eta^{(\alpha)}_w$ is a strongly continuous mapping, because of the identity $(\eta^{(\alpha)}_w, \eta^{(\alpha)}_{w'}) = \exp\bigl(-2tS^{(\nu)}_{\alpha f}\bigr)(w, w')$ and due to the continuity of the kernel [BHL00, Theorem 6.1].

In addition, the mapping $\alpha \mapsto \eta^{(\alpha)}_w$ is also strongly continuous because the integrand in (46) can be estimated by $\exp\bigl(\int_0^t ds\,(-\nu v + \alpha_0 f^-)\bigr)$ with some large α_0 and then dominated convergence applies.

As the main part of the proof we show that

$$\lim_{\nu\to\infty} \mathrm{e}^{-tS^{(\nu)}_f}(z, z') = (\widetilde{\kappa}_z, \mathrm{e}^{-tS^{(\infty)}_f}\,\widetilde{E}\,\widetilde{\kappa}_{z'}) \tag{49}$$

which by an analogue of Corollary 3.6 in connection with Subsection 5.A constitutes the continuous integral kernel for $\exp\bigl(-tS^{(\infty)}_f\bigr)\widetilde{E}$ on $L^2(\mathbb{D})$.

To see (49), we use the semigroup property and rewrite the integral kernel as scalar product

$$\mathrm{e}^{-tS^{(\nu)}_f}(z, z') = \Bigl(\eta^{(\nu_0/\nu)}_z, \exp\bigl(-tS^{(\nu-2\nu_0)}_{(\nu-2\nu_0)f/\nu}\bigr)\eta^{(\nu_0/\nu)}_{z'}\Bigr) \tag{50}$$

which converges in the limit $\nu \to \infty$ to

$$\lim_{\nu\to\infty} \mathrm{e}^{-tS^{(\nu)}_f}(z, z') = (\eta^{(0)}_z, \mathrm{e}^{-tS^{(\infty)}_f}\,\widetilde{E}\,\eta^{(0)}_{z'})\,. \tag{51}$$

This can be deduced from the strong continuity of $\eta^{(\alpha)}_z$ in α, the strong convergence stated in Theorem 6.3 together with the uniform boundedness (according to the Banach-Steinhaus theorem) of the operators $\exp\bigl(-tS^{(\nu-2\nu_0)}_{(\nu-2\nu_0)f/\nu}\bigr)$.

To finish the proof, we observe that the right-hand side of (51) is an integral kernel for $\exp(-tS^{(\infty)}_f)$ on $\widetilde{E}(L^2(\mathbb{D}))$ which is, in addition, continuous in z and z' and therefore coincides with the right-hand side of (49). The continuity of (51) is clear, and with $\widetilde{E}\exp(-tS^{(\nu_0)}_0) = \widetilde{E}$ it can be checked that it indeed constitutes an integral kernel. $\square$

6.5. Corollary. Combined with the Feynman-Kac-Itô formula (46) and the identity

$$\mathrm{e}^{-tT_f}(z, z') = \bigl(U^\dagger \mathrm{e}^{-tS^{(\infty)}_f}\,\widetilde{E}\,U\bigr)(z, z')\,, \tag{52}$$

the result (45) provides a probabilistic expression for the continuous integral kernel of the semigroup generated by T_f on $\overline{\mathcal{Q}(\mathfrak{t}_f)} \subset L^2_a(g\lambda)$. This is the **generalized Daubechies-Klauder formula**, which states that for admissible f

and g and for $t > 0$,

$$\mathrm{e}^{-tT_f}(z, z') = \lim_{\nu \to \infty} \sqrt{\frac{g(z')}{g(z)}} \int_{\{\mathrm{T}_{\mathbb{D}} > t\}} \exp\Big\{ \frac{1}{2} \int_0^t \Big[a(b(s)) \overline{db(s)} - \overline{a(b(s))} db(s) \Big] - \int_0^t ds \, (\nu v + f)(b(s)) \Big\} \, d\mu^{(\nu)}_{z,0;z',t} \,. \quad (53)$$

In particular, the choice $f = 0$ yields the reproducing kernel of the weighted Bergman space which g characterizes.

6.6. Remarks. According to [BLW99b, BLW99a], due to the significance of ν the expression for the integral kernel of the Berezin-Toeplitz semigroup in (53) is called an **ultra-diffusive limit**. By rescaling the Brownian bridge as in [BLW99a, Equation (11)], one may alternatively restate (53) as a long-time limit. Another version of the result pointed out there implies that the integrand in (53) can be re-expressed in terms of a complex Itô-stochastic integral. All these versions will be omitted here.

Reading from the right to the left, (53) can be interpreted as a quantization formula, which constructs from the functions f and g the semigroup generated by T_f and thereby specifically selects the relevant Hilbert-space $\overline{\mathcal{Q}(\mathrm{t}_f)}$ on which T_f is properly defined as a self-adjoint operator. In the context of quantization, $\ln g$ is interpreted as a Kähler potential on $\mathbb{D}$. To our knowledge, without additional symmetry requirements the setting considered here still lacks the proof of a correspondence principle [BMS94, Eng95], for which probabilistic techniques might provide helpful tools.

6.7. Examples. Now we revisit the examples from Section 3 again and list the functions a and v belonging to each weight function g considered there.

No.	$\mathbb{D}$	$g(z)$	$a(z)$	$v(z)$
1.	$\mathbb{C}$	$\frac{1}{\pi} e^{-\lvert z \rvert^2}$	iz	-2
2.	$\mathbb{C}$	$\frac{2j+1}{\pi}(1 + \lvert z \rvert^2)^{-2j-2}$	$\frac{2(j+1)iz}{1+\lvert z \rvert^2}$	$-\frac{4(j+1)}{(1+\lvert z \rvert^2)^2}$
3.a)	$\{\lvert z \rvert^2 < 1\}$	$\frac{2k-1}{\pi}(1 - \lvert z \rvert^2)^{2k-2}$	$\frac{2(k-1)iz}{1-\lvert z \rvert^2}$	$-\frac{4(k-1)}{(1-\lvert z \rvert^2)^2}$
3.b)	$\mathbb{C}$	$\frac{2\lvert z \rvert^{2k-1}}{\pi\Gamma(2k)} K_{2k-1}(2\lvert z \rvert)$	$\frac{Z_{2k-2}\, iz}{Z_{2k-1}}$	$-\frac{2}{Z_{2k-1}^2}\big(Z_{2k-2}Z_{2k-1} + \lvert z \rvert^2 (Z_{2k-2}^2 - Z_{2k-1}Z_{2k-3})\big)$

In the last row we used the abbreviation $Z_\sigma := \lvert z \rvert^\sigma K_\sigma(2\lvert z \rvert)$.

The following remarks examine the validity of equation (53) for each case.

6.8. Remarks. The primary condition for the validity of (53) in a specific situation is that the weight function leads to admissible a and v. If this is satisfied, any Kato-decomposable symbol f allows a probabilistic expression for the semigroup generated by T_f.

By inspection we decide whether each example can be used in the formula (53). For the examples 1 and 2 the function a is continuous and v is even bounded, hence they admit the probabilistic representation according to (53).

Unfortunately, for the example 3.a) either $k = 1$ and Proposition 5.3 is not satisfied or the scalar potential v is not Kato decomposable because of a strong negative singularity towards the boundary of the disc. Therefore the probabilistic representation is not valid in either case. However, the desired monotone form convergence can be recovered with the modification of the domain suggested in Remark 5.2 which is associated with Neumann boundary conditions.

On the other hand, the vector and scalar potentials a and v emerging from the Barut-Girardello representation 3.b) are admissible for $k > 1$, which can be read off from the asymptotics: $\lim_{r\to 0} r^{|\sigma|} K_\sigma(2r) = \frac{1}{2}\int_0^\infty t^{-\sigma-1} e^{-1/t}\, dt$ for $\sigma \neq 0$, and $\lim_{r\to 0} K_0(2r)/\ln 2r = -1$.

Within the setting of example 1 the identity (53) is in essence a result by [DK85]. The identity corresponding to example 2 has already been worked out in [BLW99a], where it is also compared to a similar formula derived in [DK85]. These last two alternatives are among the few known ways to obtain a mathematically well-founded path integral for spin. The weighted Bergman space of example 3.b) was not previously known to admit a formula of type (53), which can now serve as an alternative to path-integral formulas that do not contain genuine path measures [IKG92, FF97].

7 Conclusion

In this paper we have tried to indicate the key principles behind the construction of Daubechies and Klauder and have thus derived a natural generalization. As a major spin-off we developed criteria for self-adjointness of Berezin-Toeplitz operators. Once the relation to Schrödinger operators is established, this is an immediate consequence.

As to further ramifications, we point out that with a suitable analyticity argument one could obtain from (53) the probabilistic expression for the unitary group e^{-itT_f} which was a primary motivation for [DK85, DK86]. It might also be interesting to investigate random Berezin-Toeplitz operators, for which the probabilistic representation seems to offer an appropriate analytic framework. Finally, it deserves mentioning that the concept of path transformations is also applicable in the context of (53) in order to relate the resolvents of certain Berezin-Toeplitz operators [Bod].

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