Solution of the quantum inverse problem

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Abstract

We derive a formula that expresses the local spin and field operators of fundamental graded models in terms of the elements of the monodromy matrix. This formula is a quantum analogue of the classical inverse scattering transform. It applies to fundamental spin chains, such as the XYZ chain, and to a number of important exactly solvable models of strongly correlated electrons, such as the supersymmetric t-J model or the the EKS model.

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Introduction

The quantum inverse scattering method was initiated some twenty years ago by E. K. Sklyanin and L. D. Faddeev [1, 2] and then mostly developed by the group of the Steklov mathematical institute at Leningrad (cf, for instance, [3, 4, 5, 6]). A pedagogical account of its most important aspects can be found in [7].

The quantum inverse scattering method got its name, since it arose as an attempt to develop a quantum version of the (classical) inverse scattering method [8, 9], which was successful in solving non-linear classical evolution equations, such as the Korteweg-de-Vries equation [10], the non-linear Schrödinger equation [11] or the sine-Gordon equation [12].

The classical inverse scattering method provides a mapping from a set of field variables satisfying non-linear evolution equations to a set of scattering data of an associated auxiliary problem. While the fields obey non-linear evolution equations, the scattering data evolve linearly in time. The solution of the initial value problem for the original non-linear evolution equations of the fields is achieved by first mapping the initial data to the scattering data at time $t = 0$, then using the linear time evolution of the scattering data, and finally applying the *inverse transformation* [13, 14] from scattering data to fields at a time $t > 0$.

In this paper we solve the 'inverse scattering problem' for quantum lattice models. The solution is remarkably simple.

Nowadays the term 'quantum inverse scattering method' usually refers to a method formulated for systems of finite length. The relation to the classical case is the following. The elements of the monodromy matrix, which appears in the formulation of the classical problem for systems of finite length, have simple Poisson brackets [4]. In the quantum case the Poisson brackets are replaced by commutators of quantum operators. These commutators remain simple after quantization. The quantum operators can be grouped into a matrix, which, by analogy to the classical case, is called (quantum) monodromy matrix. The elements of the quantum monodromy matrix obey a set of quadratic relations. They generate the so-called Yang-Baxter algebra. The structure of this algebra is determined by numerical functions of a complex spectral parameter, which again can be arranged in a matrix. This matrix is called the R-matrix. It satisfies the famous Yang-Baxter equation (see (40) below). The R-matrix and its associated Yang-Baxter algebra are the key concepts of the quantum inverse scattering method. These concepts are *algebraic*.

The Yang-Baxter algebra has two primary applications. First of all, it contains, in general, a rich commutative subalgebra generated by the trace of the monodromy matrix. The elements of this subalgebra have a natural interpretation as a set of commuting quantities belonging to a physical system. One of these quantities is interpreted as the Hamiltonian. The existence of a large set of commuting operators can not be directly utilized to diagonalize the Hamiltonian. In many cases, however, the Yang-Baxter algebra can be employed for this task. It can be used to simultaneously diagonalize all of the commuting quantities by a procedure called the algebraic Bethe ansatz [3]. This is the most important application of the Yang-Baxter algebra.

Inspite of the conceptual differences between classical and quantum inverse scattering method, both methods have an important point in common. They essentially rely on a mapping from local field variables to a set of non-local variables, which are the elements of the monodromy matrix. In the quantum case the inverse transformation, expressing the local fields in terms of the elements of the monodromy matrix, was not known until recently. It first appeared in the examples of the inhomogeneous XXX and XXZ spin-$\frac{1}{2}$ Heisenberg chains in the article [15].

The article [15] is part of a series of articles [16, 15, 17, 18] by Izergin, Kitanine, Maillet, Sanches de Santos and Terras. In this series an interesting new device, the 'factorizing F-matrix' [16] was introduced into the algebraic Bethe ansatz and its features were explored. This led to simplified derivations of a number of important results for the XXX and XXZ spin-$\frac{1}{2}$ chains. Among the rederived results are the norm formulae [19, 20] and the Slavnov formula [21] for the scalar product of a Bethe ansatz eigenstate with a non-eigenstate. The articles [15, 17, 18] also provide simplified derivations of various results for form factors [22] and correlation functions [23, 24, 25] and their generalization to the case of non-zero magnetic field.

From our point of view, the most interesting new result in [16, 15, 17, 18] is the solution of the quantum inverse problem for the periodic,and inhomogeneous XXX and XXZ spin-$\frac{1}{2}$ Heisenberg chains (cf [15]). This result appeared to be the most important new tool used in the rederivation of the determinant formulae for form factors [15] and in the derivation of multiple integral representations of correlation functions at finite magnetic field [18].

In this article we shall focus on the quantum inverse problem. We shall obtain an explicit solution, which is valid (i) in the homogeneous case*, (ii) for models with R-matrices of arbitrary higher dimension, and (iii), most generally, for fundamental graded models [26]. Upon specification to the cases of the inhomogeneous XXX and XXZ spin-$\frac{1}{2}$ Heisenberg chains our result reduces to the formula obtained in [15].

Our result in its most general form is given by the formula (86) below. This formula is valid for homogeneous as well as for inhomogeneous models. Important special cases considered in this article are the solution of the quantum inverse problem for the translationally invariant XYZ spin chain (cf equations (17)-(19) in section 1) and for the inhomogeneous t-J model (equations (91)-(98) in section 6).

Formula (86) expresses the local operators as *products* of the entries of the monodromy matrix evaluated at the inhomogeneities. The structure of the solution of the quantum inverse problem for periodic lattice models is thus much different from the structure of the solution of the classical inverse scattering problem. In the quantum case we have an explicit multiplicative formula. In the classical case the solution is implicit and additive. It reduces to the Gelfand-Levitan-Marchenko integral equations [13, 14].

The article is organized as follows. In section 1 we present the solution of the quantum inverse problem for the homogeneous XYZ spin-$\frac{1}{2}$ chain. Later this solution will appear as a special case of our general solution (86). We treat the case of the XYZ chain separately, since the proof greatly simplifies in the homogeneous case. In section 2 we remind to the reader the definitions of graded vector spaces and graded associative algebras. We introduce the notion of graded local projection operators which were recently defined in [26]. Section 3 reviews the construction of the Yang-Baxter algebra for fundamental graded models [26] and some important results about the graded version of the quantum inverse scattering method [5, 27, 26]. In section 4 we introduce canonical Fermi operators into the formalism. In section 5 we illustrate the abstract formalism developed in the preceding section through two examples which are important in physical applications. We consider the small polaron model [28] and the supersymmetric t-J model [29, 30, 31, 32, 33, 34, 35, 36]. In section 6 we present our main result (86) in its most general form, valid for inhomogeneous, fundamental graded models associated with R-matrices of arbitrary dimension. We specify our formula for the examples considered in

*The proof of [15] does not work in the homogeneous case, since the F-matrices are not invertible for homogeneous lattices.

section 5 and work out its homogeneous limit. In section 7 we give a general definition of the fermionic R-operator associated with a fundamental graded representation of the Yang-Baxter algebra. A fermionic R-operator was recently introduced in [37, 38] for a number of models important in physical applications. The fermionic R-operator is one of the tools we shall need for the proof of the solution (86) of the quantum inverse problem. Section 8 is devoted to this proof. We construct the shift operator for inhomogeneous, fundamental graded models and work out its properties. Our proof of the quantum invers problem solely relies on the properties of the shift operator. In particular, we do not use factorizing F-matrices (which are so far known only for the XXX and XXZ spin-$\frac{1}{2}$ chains) as in the article [15]. This makes our approach more general and powerful. Our article is concluded with a brief summary and a discussion of our new formulae.

1 Solution of the quantum inverse problem for the XYZ chain

In this section we shall start solving the quantum inverse problem for fundamental models by considering an important example. The result of this section will later appear as a special case of our main result, equation (86). We think, however, that the structure of our main result and of its proof is best understood by considering an example first. We shall assume that the reader is familiar with the basic ideas of the quantum inverse scattering method. Readers not familiar with those ideas are referred to section 3, where a brief review is provided.

The XYZ spin-$\frac{1}{2}$ chain is characterized by its R-matrix [39, 40, 3]

$$R(u) = \begin{pmatrix} a(u) & 0 & 0 & d(u) \\ 0 & b(u) & c(u) & 0 \\ 0 & c(u) & b(u) & 0 \\ d(u) & 0 & 0 & a(u) \end{pmatrix} \quad . \tag{1}$$

In a normalization, which assures the unitarity of the R-matrix (cf (85)), the

Boltzmann weights $a(u), \ldots, d(u)$ are given by

$$\begin{aligned} a(u) &= \frac{\operatorname{sn}(u+2\eta)}{\operatorname{sn}(u)+\operatorname{sn}(2\eta)} \quad , \quad b(u) = \frac{\operatorname{sn}(u)}{\operatorname{sn}(u)+\operatorname{sn}(2\eta)} \quad , \\ c(u) &= \frac{\operatorname{sn}(2\eta)}{\operatorname{sn}(u)+\operatorname{sn}(2\eta)} \quad , \quad d(u) = \frac{k\,\operatorname{sn}(u)\operatorname{sn}(2\eta)\operatorname{sn}(u+2\eta)}{\operatorname{sn}(u)+\operatorname{sn}(2\eta)} \quad . \end{aligned} \tag{2}$$

The R-matrix $R(u)$ is considered as acting on the tensor product $\mathbb{C}^2 \otimes \mathbb{C}^2$. $R(u,v) := R(u-v)$ is a solution of the Yang-Baxter equation (40) (see below). Hence, an exactly solvable spin chain can be associated with $R(u)$. Let us briefly recall the steps necessary for its construction.

Define $e_\alpha^\beta \in \operatorname{End}(\mathbb{C}^2)$, $\alpha, \beta = 1, 2$ by

$$e_1^1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad , \quad e_1^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , \quad e_2^1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad , \quad e_2^2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad . \tag{3}$$

The set $\{e_\alpha^\beta \in \operatorname{End}(\mathbb{C}^2) | \alpha, \beta = 1, 2\}$ is a basis of $\operatorname{End}(\mathbb{C}^2)$. The definition

$$e_j{}_\alpha^\beta = I_2^{\otimes(j-1)} \otimes e_\alpha^\beta \otimes I_2^{\otimes(L-j)} \quad , \tag{4}$$

for $j = 1, \ldots, L$, and I_2 being the 2×2 unit matrix, provides a basis of $\operatorname{End}(\mathbb{C}^2)^{\otimes L}$, which is the space of states of an L-site spin-$\frac{1}{2}$ quantum spin chain. The matrices $e_j{}_\alpha^\beta$ satisfy

$$[e_j{}_\alpha^\beta, e_k{}_\gamma^\delta] = 0 \quad \text{for } j \neq k \quad , \quad e_j{}_\alpha^\beta e_j{}_\gamma^\delta = \delta_\gamma^\beta e_j{}_\alpha^\delta \tag{5}$$

and have the meaning of local projection operators. Using the R-matrix (1) and the local projection operators we can define the L-matrix at site j as

$$L_j(u) = \sum_{\alpha,\beta,\gamma,\delta=1}^{2} R_{\gamma\delta}^{\alpha\beta}(u)\, e_\alpha^\gamma \otimes e_j{}_\beta^\delta = \begin{pmatrix} a(u)e_j{}_1^1 + b(u)e_j{}_2^2 & c(u)e_j{}_2^1 + d(u)e_j{}_1^2 \\ d(u)e_j{}_2^1 + c(u)e_j{}_1^2 & b(u)e_j{}_1^1 + a(u)e_j{}_2^2 \end{pmatrix} . \tag{6}$$

The L-matrix $L_j(u)$ is a 2×2 matrix in an auxiliary space. Its entries are operators acting on the space of states of an L-site spin-$\frac{1}{2}$ chain. The monodromy matrix of the corresponding homogeneous spin chain is the L-fold ordered product

$$T(u) = L_L(u) \dots L_1(u) \quad . \tag{7}$$

By construction the monodromy matrix gives a representation of the Yang-Baxter algebra with R-matrix $R(u)$ (see section 3 below). We have

$$PR(u-v)\big(T(u) \otimes T(v)\big) = \big(T(v) \otimes T(u)\big) PR(u-v) \quad , \tag{8}$$

where $P = \sum_{\alpha,\beta=1}^{2} e_\alpha^\beta \otimes e_\beta^\alpha$ is the permutation matrix on $\mathbb{C}^2 \otimes \mathbb{C}^2$. The Yang-Baxter algebra (8) is the basis for the solution of the XYZ chain by algebraic Bethe ansatz [3].

The monodromy matrix (7) is a 2×2 matrix in auxiliary space. Its entries are non-local operators acting on the space of states of the XYZ chain. We may write the monodromy matrix as

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \quad . \tag{9}$$

The quantum inverse problem is to express the local operators ${e_n}_\alpha^\beta$ in terms of the elements $A(u), \dots, D(u)$ of the monodromy matrix.

For the case at hand this problem is rather easily solved. Note that $a(0) = c(0) = 1$ and $b(0) = d(0) = 0$. It follows from (6) that

$$L_j(0) = \begin{pmatrix} {e_j}_1^1 & {e_j}_2^1 \\ {e_j}_1^2 & {e_j}_2^2 \end{pmatrix} = \sum_{\alpha,\beta=1}^{2} e_\alpha^\beta \otimes {e_j}_\beta^\alpha = P_{0j} \quad . \tag{10}$$

Here P_{0j} is the permutation operator that interchanges the auxiliary space with the j-th quantum space. Similarly, $P_{jk} = \sum_{\alpha,\beta=1}^{2} {e_j}_\alpha^\beta {e_k}_\beta^\alpha$ interchanges the j-th and k-th quantum spaces. Using equation (10) in equation (7) we obtain

$$T(0) = P_{0L} P_{0L-1} \dots P_{01} = P_{01} P_{1L} P_{1L-1} \dots P_{12} = P_{01} \hat{U} \quad . \tag{11}$$

Here $\hat{U} = P_{1L} P_{1L-1} \dots P_{12} = P_{12} P_{23} \dots P_{L-1L}$ is the cyclic shift operator in quantum space. In the second equation in (11) we have used the commutation relation $P_{jk}P_{jl} = P_{kl}P_{jk}$ for permutation operators. Let us write equation (11) in matrix form,

$$\begin{pmatrix} A(0) & B(0) \\ C(0) & D(0) \end{pmatrix} = \begin{pmatrix} e_{1}{}_1^1\,\hat{U} & e_{1}{}_2^1\,\hat{U} \\ e_{1}{}_1^2\,\hat{U} & e_{1}{}_2^2\,\hat{U} \end{pmatrix} \quad . \tag{12}$$

Comparing the matrix elements we find the relations

$$\hat{U} = A(0) + D(0) \quad , \tag{13}$$

$$\sigma_1^- = e_{1}{}_2^1 = B(0)\,\hat{U}^{-1} \quad , \tag{14}$$

$$\sigma_1^+ = e_{1}{}_1^2 = C(0)\,\hat{U}^{-1} \quad , \tag{15}$$

$$\sigma_1^z = e_{1}{}_1^1 - e_{1}{}_2^2 = \big(A(0) - D(0)\big)\,\hat{U}^{-1} \quad . \tag{16}$$

These relations constitute a solution of the quantum inverse problem for the local operators acting on the first lattice site. We may now simply use the shift operator to shift the site indices. Since $\hat{U}^{n-1}e_{1}{}_\alpha^\beta\hat{U}^{1-n} = e_{n}{}_\alpha^\beta$ and $\hat{U}^L = \mathrm{id}$, we obtain

$$\sigma_n^- = \hat{U}^{n-1}B(0)\,\hat{U}^{L-n} \quad , \tag{17}$$

$$\sigma_n^+ = \hat{U}^{n-1}C(0)\,\hat{U}^{L-n} \quad , \tag{18}$$

$$\sigma_n^z = \hat{U}^{n-1}\big(A(0) - D(0)\big)\,\hat{U}^{L-n} \quad . \tag{19}$$

Taking into account equation (13) we see that the right hand side of the equations (17)-(19) are entirely expressed in terms of the entries of the monodromy matrix. An alternative way of writing (17)-(19) is

$$e_{n}{}_\alpha^\beta = \hat{U}^{n-1}\,T_\alpha^\beta(0)\,\hat{U}^{L-n} = \big(A(0) + D(0)\big)^{n-1}\,T_\alpha^\beta(0)\,\big(A(0) + D(0)\big)^{L-n} \quad . \tag{20}$$

Equation (20) allows us to calculate expectation values of *local operators* by means of the Yang-Baxter algebra.

The remainder of this article will be devoted to the generalization of equation (20) to (i) arbitrary dimension of the R-matrix, (ii) the inhomogeneous case, and (iii) to fundamental graded models. It is important noting that our above solution of the quantum inverse problem does not depend on the specific features of the XYZ chain. Our calculation solely relied on the fact that the L-matrix evaluated at $u = 0$ turns into a permutation operator (cf equation (10)).

2 Graded vector spaces

In this section we shall recall the basic concepts of graded vector spaces and graded associative algebras. In the context of the quantum inverse scattering method these concepts were first used by Kulish and Sklyanin [5, 27]. We shall further recall the notions of 'graded local projection operators' and graded permutation operators. Graded local projection operators were introduced in the article [26]. They enable the definition of fundamental graded representations of the Yang-Baxter algebra, which will be given in the following section.

Graded vector spaces are vector spaces equipped with a notion of odd and even, that allows us to treat fermions within the formalism of the quantum inverse scattering method. Let us start with a finite dimensional local space of states V, on which we impose an additional structure, the parity, from the outset. Let $V = V_0 \oplus V_1$, $\dim V_0 = m$, $\dim V_1 = n$. We shall call $v_0 \in V_0$ even and $v_1 \in V_1$ odd. The subspaces V_0 and V_1 are called the homogeneous components of V. The parity p is a function $V_i \to \mathbb{Z}_2$ defined on the homogeneous components of V,

$$p(v_i) = i \quad , \quad i = 0, 1 \quad , \quad v_i \in V_i \quad . \tag{21}$$

The vector space V endowed with this structure is called a graded vector space or super space. Let us fix a basis $\{e_1, \dots, e_{m+n}\}$ of definite parity and let us define $p(\alpha) := p(e_\alpha)$.

The use of graded vector spaces within the quantum inverse scattering method requires the construction of an algebra of commuting and anticommuting *operators*. For this purpose we have to extend the concept of parity to operators in $\mathrm{End}(V)$ and to tensor products of these operators. Let $e_\alpha^\beta \in \mathrm{End}(V)$, $e_\alpha^\beta e_\gamma = \delta_\gamma^\beta e_\alpha$. The set $\{e_\alpha^\beta \in \mathrm{End}(V) | \alpha, \beta = 1, \dots m+n\}$ is a basis of $\mathrm{End}(V)$. Hence, the definition

$$p(e_\alpha^\beta) = p(\alpha) + p(\beta) \tag{22}$$

induces a grading on $\mathrm{End}(V)$ regarded as a vector space.

It is easy to see that an element $A = A_\beta^\alpha e_\alpha^\beta \in \mathrm{End}(V)$ is homogeneous with parity $p(A)$, if and only if

$$(-1)^{p(\alpha)+p(\beta)} A_\beta^\alpha = (-1)^{p(A)} A_\beta^\alpha \quad . \tag{23}$$

The latter equation implies for two homogeneous elements $A, B \in \mathrm{End}(V)$ that their product AB is homogeneous with parity

$$p(AB) = p(A) + p(B) \quad . \tag{24}$$

In other words, multiplication of matrices in $\mathrm{End}(V)$ preserves homogeneity, and, therefore, $\mathrm{End}(V)$ endowed with the grading (22) is a graded associative algebra [5].

Let us consider the L-fold tensorial power $(\mathrm{End}(V))^{\otimes L}$ of $\mathrm{End}(V)$. The definition (22) has a natural extension to $(\mathrm{End}(V))^{\otimes L}$, namely,

$$p(e_{\alpha_1}^{\beta_1} \otimes \cdots \otimes e_{\alpha_L}^{\beta_L}) = p(\alpha_1) + p(\beta_1) + \cdots + p(\alpha_L) + p(\beta_L) \quad . \tag{25}$$

From this formula it can be seen in a similar way as before, that homogeneous elements $A = A^{\alpha_1 \dots \alpha_L}_{\beta_1 \dots \beta_L} e_{\alpha_1}^{\beta_1} \otimes \cdots \otimes e_{\alpha_L}^{\beta_L}$ of $(\mathrm{End}(V))^{\otimes L}$ with parity $p(A)$ are characterized by the equation

$$(-1)^{\sum_{j=1}^L (p(\alpha_j) + p(\beta_j))} A^{\alpha_1 \dots \alpha_L}_{\beta_1 \dots \beta_L} = (-1)^{p(A)} A^{\alpha_1 \dots \alpha_L}_{\beta_1 \dots \beta_L} \quad , \tag{26}$$

which generalizes (23). Again the product AB is homogeneous with parity $p(AB) = p(A) + p(B)$, if A and B are homogeneous. Thus the definition (25) induces the structure of a graded associative algebra on $(\mathrm{End}(V))^{\otimes L}$.

Let us define the super-bracket

$$[X, Y]_\pm = XY - (-1)^{p(X)p(Y)} YX \tag{27}$$

for X, Y taken from the homogeneous components of $\mathrm{End}(V)$, and let us extend it linearly to $\mathrm{End}(V)$ in both of its arguments. Then, $\mathrm{End}(V)$ endowed with the super-bracket becomes the Lie-super algebra $\mathrm{gl}(m|n)$. Note that the above definition of a super-bracket makes sense in any graded algebra and is particularly valid in $(\mathrm{End}(V))^{\otimes L}$.

The following definition of 'graded local projection operators' [26] will be crucial for our definition of fundamental graded representations of the Yang-Baxter algebra in the next section. Define the matrices

$$e_{j\,\alpha}^{\ \beta} = (-1)^{(p(\alpha) + p(\beta)) \sum_{k=j+1}^L p(\gamma_k)} I_{m+n}^{\otimes (j-1)} \otimes e_\alpha^\beta \otimes e_{\gamma_{j+1}}^{\gamma_{j+1}} \otimes \cdots \otimes e_{\gamma_L}^{\gamma_L} \quad , \tag{28}$$

where I_{m+n} is the $(m+n) \times (m+n)$ unit matrix, and summation over double *tensor indices* (i.e. over $\gamma_{j+1}, \dots, \gamma_L$) is understood. We shall keep this sum convention throughout the remainder of this article. The index j on the left hand side of (28) will later refer to the j-th site of a physical lattice model and is called the site index. A simple consequence of the definition (28) for $j \neq k$ are the commutation relations

$$e_{j}{}_{\alpha}^{\beta} e_{k}{}_{\gamma}^{\delta} = (-1)^{(p(\alpha)+p(\beta))(p(\gamma)+p(\delta))} e_{k}{}_{\gamma}^{\delta} e_{j}{}_{\alpha}^{\beta} \quad . \tag{29}$$

It further follows from equation (28) that $e_{j}{}_{\alpha}^{\beta}$ is homogeneous with parity

$$p(e_{j}{}_{\alpha}^{\beta}) = p(\alpha) + p(\beta) \quad . \tag{30}$$

Hence, in agreement with intuition, equation (29) says that odd matrices with different site indices mutually anticommute, whereas even matrices commute with each other as well as with the odd matrices. For products of matrices $e_{j}{}_{\alpha}^{\beta}$ which are acting on the same site (28) implies the projection property

$$e_{j}{}_{\alpha}^{\beta} e_{j}{}_{\gamma}^{\delta} = \delta_{\gamma}^{\beta} e_{j}{}_{\alpha}^{\delta} \quad . \tag{31}$$

Equations (29) and (31) justify our terminology. The $e_{j}{}_{\alpha}^{\beta}$ are graded analogues of local projection operators. We call them graded local projection operators or projection operators, for short. Using the super-bracket (27), equations (29) and (31) can be combined into

$$[e_{j}{}_{\alpha}^{\beta}, e_{k}{}_{\gamma}^{\delta}]_{\pm} = \delta_{jk} \left(\delta_{\gamma}^{\beta} e_{j}{}_{\alpha}^{\delta} - (-1)^{(p(\alpha)+p(\beta))(p(\gamma)+p(\delta))} \delta_{\alpha}^{\delta} e_{j}{}_{\gamma}^{\beta} \right) \quad . \tag{32}$$

The right hand side of the latter equation with $j = k$ gives the structure constants of the Lie super algebra gl$(m|n)$ with respect to the basis $\{e_{j}{}_{\alpha}^{\beta}\}$.

Since any $m+n$-dimensional vector space over the complex numbers is isomorphic to $\mathbb{C}^{m+n}$, we may simply set $V = \mathbb{C}^{m+n}$. We may further assume that our homogeneous basis $\{e_\alpha \in \mathbb{C}^{m+n} \,|\, \alpha = 1, \dots, m+n\}$ is canonical, i.e. we may represent the vector e_α by a column vector having the only non-zero entry $+1$ in row α. Our basic matrices e_{α}^{β} are then $(m+n) \times (m+n)$-matrices with a single non-zero entry $+1$ in row α and column β.

Remark. The meaning of (28) becomes more evident by considering a simple example. Let $m = n = 1$ and $p(1) = 0$, $p(2) = 1$. Then, using (32), we obtain

$$[e_{j}{}_{1}^{2}, e_{k}{}_{1}^{2}]_{\pm} = \{e_{j}{}_{1}^{2}, e_{k}{}_{1}^{2}\} = 0 \quad , \tag{33}$$

$$[e_{j}{}_{2}^{1}, e_{k}{}_{2}^{1}]_{\pm} = \{e_{j}{}_{2}^{1}, e_{k}{}_{2}^{1}\} = 0 \quad , \tag{34}$$

$$[e_{j}{}_1^2, e_{k}{}_2^1]_\pm = \{e_{j}{}_2^1, e_{k}{}_1^2\} = \delta_{jk}(e_{j}{}_1^1 + e_{j}{}_2^2) = \delta_{jk} \tag{35}$$

for $j, k = 1, \dots, L$. The curly brackets in (33), (34) denote the anticommutator. The matrices $e_{j}{}_1^2$ and $e_{k}{}_2^1$ satisfy the canonical anticommutation relations for spinless Fermi operators. We can therefore identify $e_{j}{}_1^2 \to c_j$ and $e_{k}{}_2^1 \to c_k^+$. Introducing Pauli matrices $\sigma^+ = e_1^2$, $\sigma^- = e_2^1$ and $\sigma^z = e_1^1 - e_2^2$ we obtain, by carrying out the summation, the following explicit matrix representation from our basic definition (28),

$$c_j = I_2^{\otimes(j-1)} \otimes \sigma^+ \otimes (\sigma^z)^{\otimes(L-j)} \quad , \tag{36}$$

$$c_k^+ = I_2^{\otimes(k-1)} \otimes \sigma^- \otimes (\sigma^z)^{\otimes(L-k)} \quad . \tag{37}$$

This is the well-known Jordan-Wigner transformation [41] expressing Fermi operators for spinless fermions in terms of Pauli matrices. We may thus interpret equation (28) as a generalization of the Jordan-Wigner transformation. In general, equation (28) provides matrix representations not of Fermi operators but, more generally, of fermionic projection operators. Representations of Fermi operators can be obtained be taking appropriate linear combinations of matrices $e_{j}{}_\alpha^\beta$. This point will be elaborated in section 4 below. Note that an alternative generalization of the Jordan-Wigner transformation, which relies on representations of the Clifford algebra, was recently introduced in [42].

The permutation operator plays an important role in the construction of local integrable lattice models. It enters the expression for the shift operator on homogeneous lattices and the expression for the Hamiltonian of rational gl$(m|n)$ invariant models (see section 5 below). In the graded case the definition of the permutation operator requires the following modifications of signs,

$$P_{jk} = (-1)^{p(\beta)} e_{j}{}_\alpha^\beta e_{k}{}_\beta^\alpha \quad . \tag{38}$$

As indicated by its name, this operator induces the action of the symmetric group $\mathfrak{S}^L$ on the site indices of the matrices $e_{j}{}_\alpha^\beta$. The properties of P_{jk} (for $j \neq k$) are the same as in the non-graded case. They are easily derived from (29) and (31) and can be found, for instance, in [26]. Let $L = 2$. Then

$$P_{12} = (-1)^{p(\beta)} e_{1}{}_\alpha^\beta e_{2}{}_\beta^\alpha = (-1)^{p(\alpha)p(\beta)} e_\alpha^\beta \otimes e_\beta^\alpha = (-1)^{p(\alpha)p(\beta)} \delta_\delta^\alpha \delta_\gamma^\beta e_\alpha^\gamma \otimes e_\beta^\delta \quad . \tag{39}$$

From the right hand side of this equation we can read off the matrix elements of P_{12} with respect to the canonical basis of End$(V \otimes V)$.

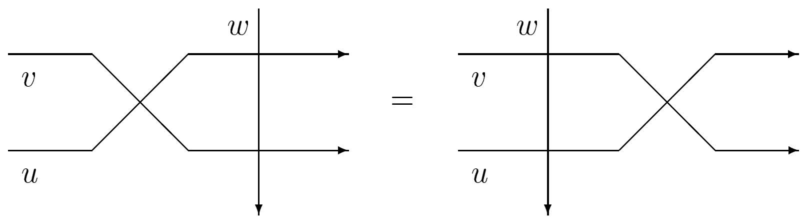

Figure 1: The Yang-Baxter equation is most easily memorized in graphical form

3 Fundamental graded models

In this section we shall recall the notion of *fundamental graded representations* of the Yang-Baxter algebra, which was recently introduced in [26]. For a given grading we shall associate a fundamental model with every solution of the Yang-Baxter equation that satisfies a certain compatibility condition (see (41) below).

For our present purpose it is most suitable to interpret the Yang-Baxter equation as a set of functional equations for the matrix elements of an $(m+n)^2 \times (m+n)^2$-matrix $R(u,v)$. We may represent the Yang-Baxter equation in graphical form as shown in Figure 1, where each vertex corresponds to a factor in the equation

$$R^{\alpha\beta}_{\alpha'\beta'}(u,v) R^{\alpha'\gamma}_{\alpha''\gamma'}(u,w) R^{\beta'\gamma'}_{\beta''\gamma''}(v,w) = R^{\beta\gamma}_{\beta'\gamma'}(v,w) R^{\alpha\gamma'}_{\alpha'\gamma''}(u,w) R^{\alpha'\beta'}_{\alpha''\beta''}(u,v) \quad . \tag{40}$$

Note that there is a direction assigned to every line in figure 1, which is indicated by the tips of the arrows. Therefore every vertex has an orientation, and vertices and R-matrices can be identified according to figure 2, where indices have been supplied to a vertex. Summation is over all inner lines in figure 1.

The construction of a graded Yang-Baxter algebra and its fundamental representation requires only minimal modifications compared to the non-graded case [7]. Let us assume we are given a solution of (40), which is compatible with the grading in the sense that [5]

$$R^{\alpha\beta}_{\gamma\delta}(u,v) = (-1)^{p(\alpha)+p(\beta)+p(\gamma)+p(\delta)} R^{\alpha\beta}_{\gamma\delta}(u,v) \quad . \tag{41}$$

$$R^{\alpha\beta}_{\gamma\delta}(u,v) \quad = $$

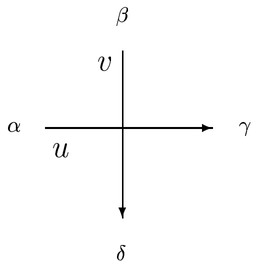

Figure 2: Identification of the R-matrix with a vertex

Define a graded L-matrix at site j as

$$\mathcal{L}_j{}^\alpha_\beta(u,v) = (-1)^{p(\alpha)p(\gamma)} R^{\alpha\gamma}_{\beta\delta}(u,v) e_j{}^\delta_\gamma \quad . \tag{42}$$

Equation (41) implies that the matrix elements of $\mathcal{L}_j(u,v)$ are of definite parity,

$$p(\mathcal{L}_j{}^\alpha_\beta(u,v)) = p(\alpha) + p(\beta) \quad . \tag{43}$$

Thus their commutation rules are given by

$$\mathcal{L}_j{}^\alpha_\beta(u,v)\mathcal{L}_k{}^\gamma_\delta(w,z) = (-1)^{(p(\alpha)+p(\beta))(p(\gamma)+p(\delta))}\mathcal{L}_k{}^\gamma_\delta(w,z)\mathcal{L}_j{}^\alpha_\beta(u,v) \quad . \tag{44}$$

It further follows from the Yang-Baxter equation (40) and from equation (41) that

$$\check{R}(u,v)\big(\mathcal{L}_j(u,w) \otimes_s \mathcal{L}_j(v,w)\big) = \big(\mathcal{L}_j(v,w) \otimes_s \mathcal{L}_j(u,w)\big)\check{R}(u,v) \quad . \tag{45}$$

As in the non-graded case the matrix $\check{R}(u,v)$ is defined by

$$\check{R}^{\alpha\beta}_{\gamma\delta}(u,v) = R^{\beta\alpha}_{\gamma\delta}(u,v) \quad . \tag{46}$$

The super tensor product [5] in equation (45) is to be understood as a super tensor product of matrices with non-commuting entries, $(A \otimes_s B)^{\alpha\gamma}_{\beta\delta} = (-1)^{(p(\alpha)+p(\beta))p(\gamma)} A^\alpha_\beta B^\gamma_\delta$. The super tensor product has the following important feature. Given matrices A, B, C, D with operator valued entries, which mutually commute according to the same rule as $\mathcal{L}_j$ and $\mathcal{L}_k$ in equation (44), we obtain for the product of two super tensor products

$$(A \otimes_s B)(C \otimes_s D) = AC \otimes_s BD \quad . \tag{47}$$

Equation (45) may be interpreted as defining a graded Yang-Baxter algebra with R-matrix R. We call $\mathcal{L}_j$ its fundamental graded representation.

Starting from (45) we can construct integrable lattice models as in the non-graded case [7]. Let us briefly recall the construction with emphasis on the modifications that appear due to the grading. Define a monodromy matrix $\mathcal{T}(u,v)$ as an L-fold ordered product of fundamental L-matrices,

$$\mathcal{T}(u,v) = \mathcal{L}_L(u,v)\ldots\mathcal{L}_1(u,v) \quad . \tag{48}$$

Due to equation (24) the matrix elements of $\mathcal{T}(u,v)$ are homogeneous with parity $p(\mathcal{T}^\alpha_\beta(u,v)) = p(\alpha)+p(\beta)$. Repeated application of (45) and (47) shows that this monodromy matrix is a representation of the graded Yang-Baxter algebra,

$$\check{R}(u,v)\big(\mathcal{T}(u,w)\otimes_s \mathcal{T}(v,w)\big) = \big(\mathcal{T}(v,w)\otimes_s \mathcal{T}(u,w)\big)\check{R}(u,v) \quad . \tag{49}$$

In the non-graded case ($n=0$) the super tensor product in (49) agrees with the usual tensor product. Let us now define the super trace as

$$\mathrm{str}(A) = (-1)^{p(\alpha)} A^\alpha_\alpha \quad . \tag{50}$$

It follows from (41) and (49) that

$$\big[\mathrm{str}(\mathcal{T}(u,w)), \mathrm{str}(\mathcal{T}(v,w))\big] = 0 \quad , \tag{51}$$

which is in complete analogy with the non-graded case.

Let us assume that $R(u,v)$ is a regular solution of the Yang-Baxter equation, $R^{\alpha\beta}_{\gamma\delta}(v,v) = \delta^\alpha_\delta \delta^\beta_\gamma$. Then (42) implies that

$$\mathcal{L}_j{}^\alpha_\beta(v,v) = (-1)^{p(\alpha)p(\beta)} e_j{}^\alpha_\beta \quad , \tag{52}$$

and we can easily see [26] that the super trace of the monodromy matrix evaluated at (v,v) generates a shift by one site,

$$\mathrm{str}(\mathcal{T}(v,v)) = P_{12}P_{23}\ldots P_{L-1L} =: \hat{U} \quad . \tag{53}$$

It follows that $\tau(u) := -\mathrm{i}\ln(\mathrm{str}(\mathcal{T}(u,v)))$ generates a sequence of local operators [43] which, as a consequence of (51), mutually commute,

$$\tau(u) = \hat{\Pi} + (u-v)\hat{H} + \mathcal{O}\left((u-v)^2\right) \quad . \tag{54}$$

$\hat{\Pi}$ in this expansion is the momentum operator. On a lattice, where the minimal possible shift is by one site, and thus $\hat{U}$ rather than $\hat{\Pi}$ is the fundamental geometrical operator, some care is required in the definition of $\hat{\Pi}$. As was shown in [44] a proper definition may be obtained by setting $\Pi := -\mathrm{i}\ln(\hat{U}) \mod 2\pi$ and expressing the function $f(x) = x \mod 2\pi$ by its Fourier sum. Then $\hat{\Pi}$ becomes a polynomial in $\hat{U}$.

$$\hat{\Pi} = \phi \sum_{m=1}^{L-1} \left(\frac{1}{2} + \frac{\hat{U}^m}{e^{-\mathrm{i}\phi m} - 1} \right) \quad , \tag{55}$$

where $\phi = 2\pi/L$. The first order term $\hat{H}$ in the expansion (54) may be interpreted as Hamiltonian. Using (53) it is obtained as

$$\hat{H} = \sum_{j=1}^{L} H_{jj+1} \quad , \tag{56}$$

where $H_{LL+1} = H_{L1}$ and

$$H_{jj+1} = -\mathrm{i}\,(-1)^{p(\gamma)(p(\alpha)+p(\gamma))}\, \partial_u\, \check{R}^{\alpha\beta}_{\gamma\delta}(u,v)\Big|_{u=v}\, e_j{}^\gamma_\alpha e_{j+1}{}^\delta_\beta \quad . \tag{57}$$

We would like to draw the readers' attention to the following points. (i) The R-matrix $\check{R}$ in equation (45) does *not* undergo a modification due to the grading. (ii) The only necessary compatibility condition which has to be satisfied in order to introduce a fundamental graded representation of the Yang-Baxter algebra associated with a solution of the Yang-Baxter equation is equation (41), which was introduced in [5]. Equation (41) is a rather weak constraint. The most important solutions of the Yang-Baxter equation, which appear in physical applications are compatible with a non-trivial grading (see the examples in section 5). Moreover, a given R-matrix may be compatible with different gradings, leading to different fundamental graded representations of the Yang-Baxter algebra [26].

Before turning to our next subject let us introduce the inhomogeneous generalization

$$\mathcal{T}(u; \xi_1, \dots, \xi_L) = \mathcal{L}_L(u, \xi_L) \dots \mathcal{L}_1(u, \xi_1) \tag{58}$$

of the monodromy matrix (48). This monodromy matrix satisfies (49). For $\xi_1 = \dots = \xi_L = v$ it turns into $\mathcal{T}(u,v)$ defined in (48). We shall formulate our main result below for the inhomogeneous model generated by $\mathcal{T}(u; \xi_1, \dots, \xi_L)$.

4 Fermi operators

In the article [26] it was explained how the various graded objects introduced in the previous section can be expressed in terms of Fermi operators. The key observation is that, as far as the matrices ${e_j}^\beta_\alpha$ are concerned, all calculations of the previous section rely on the commutation relations (29) and on the projection property (31). Fermionic projection operators satisfy the same equations. We may thus say that the matrices ${e_j}^\beta_\alpha$ are matrix representations of fermionic projection operators. As we have seen in the previous section, the matrices ${e_j}^\beta_\alpha$ are suitable for formulating a graded version of the quantum inverse scattering method. For the physical interpretation of the models constructed from a given solution of the Yang-Baxter equation, however, it is convenient to introduce Fermi operators into the formalism.

A general construction of fermionic projection operators for fermions of arbitrary su(N) spin was presented in [26]. Rather than repeating those results let us illustrate them by example.

Let us consider spinless fermions on a ring of L lattice sites,

$$\{c_j, c_k\} = \{c_j^\dagger, c_k^\dagger\} = 0 \quad , \quad \{c_j, c_k^\dagger\} = \delta_{jk} \quad , \quad j, k = 1, \dots, L \quad . \tag{59}$$

It is easy to verify that the entries $(X_j)^\alpha_\beta$ of the matrix

$$X_j = \begin{pmatrix} 1 - n_j & c_j \\ c_j^\dagger & n_j \end{pmatrix} \tag{60}$$

are fermionic projection operators. Define ${X_j}^\beta_\alpha = (X_j)^\alpha_\beta$. Then

$${X_j}^\beta_\alpha {X_j}^\delta_\gamma = \delta^\beta_\gamma {X_j}^\delta_\alpha \quad . \tag{61}$$

The operators ${X_j}^\beta_\alpha$ carry parity, induced by the anti commutation rule (59) for the Fermi operators. For $j \neq k$ ${X_j}^\beta_\alpha$ and ${X_k}^\delta_\gamma$ anticommute, if both are build up of an odd number of Fermi operators, and otherwise commute. This fact can be expressed as follows. Let $p(1) = 0$, $p(2) = 1$ and $p({X_j}^\beta_\alpha) = p(\alpha) + p(\beta)$. Then ${X_j}^\beta_\alpha$ is odd (contains an odd number of Fermi operators), if $p({X_j}^\beta_\alpha) = 1$, and even, if $p({X_j}^\beta_\alpha) = 0$. The commutation rules for the projectors ${X_j}^\beta_\alpha$ are thus

$${X_j}^\beta_\alpha {X_k}^\delta_\gamma = (-1)^{(p(\alpha)+p(\beta))(p(\gamma)+p(\delta))} {X_k}^\delta_\gamma {X_j}^\beta_\alpha \quad . \tag{62}$$

Now (61) and (62) are of the same form as (31) and (29), respectively. Since the calculations in the previous section relied solely on (29) and (31), we may simply replace $e_j{}^\beta_\alpha \rightarrow X_j{}^\beta_\alpha$ in equations (42) and (57).

Fermionic representations compatible with arbitrary grading can be constructed by considering several species of fermions and graded products of projection operators. We shall explain this for the case of two species. This is the case most interesting for applications, since we may interpret the two species as up- and down-spin electrons. We have to attach a spin index to the Fermi operators, $c_j \rightarrow c_{j\sigma}$, $\sigma = \uparrow, \downarrow$, $\{c_{j\sigma}, c_{k\tau}^\dagger\} = \delta_{jk}\delta_{\sigma\tau}$. Accordingly, there are two species of projection operators, $X_j{}^\beta_\alpha \rightarrow X^{\sigma}_j{}^\beta_\alpha$.

Let us define projection operators for electrons by the tensor products

$$X_j{}^{\beta\delta}_{\alpha\gamma} = (-1)^{(p(\alpha)+p(\beta))p(\gamma)} X^{\downarrow}_j{}^\beta_\alpha X^{\uparrow}_j{}^\delta_\gamma = \left(X_j^\downarrow \otimes_s X_j^\uparrow\right)^{\alpha\gamma}_{\beta\delta} \quad . \tag{63}$$

Then

$$X_j{}^{\beta\delta}_{\alpha\gamma} X_j{}^{\beta'\delta'}_{\alpha'\gamma'} = \delta^\beta_{\alpha'}\delta^\delta_{\gamma'} X_j{}^{\beta'\delta'}_{\alpha\gamma} \quad . \tag{64}$$

$X_j{}^{\beta\delta}_{\alpha\gamma}$ inherits the parity from $X^{\downarrow}_j{}^\beta_\alpha$ and $X^{\uparrow}_j{}^\delta_\gamma$. The number of Fermi operators contained in $X_j{}^{\beta\delta}_{\alpha\gamma}$ is the sum of the number of Fermi operators in $X^{\downarrow}_j{}^\beta_\alpha$ and $X^{\uparrow}_j{}^\delta_\gamma$. Hence $p(X_j{}^{\beta\delta}_{\alpha\gamma}) = p(X^{\downarrow}_j{}^\beta_\alpha) + p(X^{\uparrow}_j{}^\delta_\gamma) = p(\alpha) + \cdots + p(\delta)$, and the analog of (62) holds for $X_j{}^{\beta\delta}_{\alpha\gamma}$, too. Again we present all projection operators in form of a matrix $(X_j)^{\alpha\gamma}_{\beta\delta} = X_j{}^{\beta\delta}_{\alpha\gamma}$,

$$\begin{aligned} X_j &= X_j^\downarrow \otimes_s X_j^\uparrow \\ &= \begin{pmatrix} (1-n_{j\downarrow})(1-n_{j\uparrow}) & (1-n_{j\downarrow})c_{j\uparrow} & c_{j\downarrow}(1-n_{j\uparrow}) & c_{j\downarrow}c_{j\uparrow} \\ (1-n_{j\downarrow})c_{j\uparrow}^\dagger & (1-n_{j\downarrow})n_{j\uparrow} & -c_{j\downarrow}c_{j\uparrow}^\dagger & -c_{j\downarrow}n_{j\uparrow} \\ c_{j\downarrow}^\dagger(1-n_{j\uparrow}) & c_{j\downarrow}^\dagger c_{j\uparrow} & n_{j\downarrow}(1-n_{j\uparrow}) & n_{j\downarrow}c_{j\uparrow} \\ -c_{j\downarrow}^\dagger c_{j\uparrow}^\dagger & -c_{j\downarrow}^\dagger n_{j\uparrow} & n_{j\downarrow}c_{j\uparrow}^\dagger & n_{j\downarrow}n_{j\uparrow} \end{pmatrix} . \end{aligned} \tag{65}$$

Here we used the standard ordering of matrix elements of tensor products, corresponding to a renumbering $(11) \rightarrow 1$, $(12) \rightarrow 2$, $(21) \rightarrow 3$, $(22) \rightarrow 4$. Within this convention $X_j{}^{\beta\delta}_{\alpha\gamma}$ is replaced by $X_j{}^\beta_\alpha$, $\alpha, \beta = 1, \ldots, 4$, which then satisfies (61) and (62) with grading $p(1) = p(4) = 0$, $p(2) = p(3) = 1$.

Note that Fermi operators can be obtained as linear combinations of projection operators. We have, for instance, $c_{j\uparrow} = X_{j}{}_1^2 + X_{j}{}_3^4$.

So far we have considered the case of spinless fermions with two-dimensional local space of states and grading $m = n = 1$, and the case of electrons with four-dimensional space of states and grading $m = n = 2$. There are four different possibilities to realize (29) and (31) in case of a three-dimensional local space of states, $m + n = 3$. They can be obtained by deleting the α's row and column of the matrix X_j in equation (65), $\alpha = 1, 2, 3, 4$. (61) and (62) remain valid, since the operators $X_{j}{}_\alpha^\beta$ are projectors.

An alternative way [45, 46, 42] of introducing Fermi operators into the quantum inverse scattering method is by applying the Jordan-Wigner transformation [41] to the non-graded L-matrix and then pulling out the non-local factors. This approach was of primary importance, for instance, for a fermionic formulation of the Yang-Baxter algebra of the Hubbard model [46] and led to the discovery of a SO(4)-invariant form of the monodromy matrix of the Hubbard model [46, 44, 47]. In general, however, we prefer the method presented above, since the approach of [45, 46, 42] so far did not lead to general formulae such as (42) or (57) and may have unpleasant side effects, such as a twist of boundary conditions or the appearance of numerous factors of 'i' in the equations.

5 Examples

Before turning to our main result let us present several examples in order to provide an idea to the reader which applications we have in mind.

We shall start with the XXZ spin chain. The R-matrix of the XXZ spin chain can, for instance, be written as

$$R(u,v) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(u,v) & c(u,v) & 0 \\ 0 & c(u,v) & b(u,v) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad , \tag{66}$$

where

$$b(u,v) = \frac{\mathrm{sh}(u-v)}{\mathrm{sh}(u-v+\mathrm{i}\kappa)} \quad , \quad c(u,v) = \frac{\mathrm{sh}(\mathrm{i}\kappa)}{\mathrm{sh}(u-v+\mathrm{i}\kappa)} \quad . \tag{67}$$

The R-matrix (66) is compatible with the grading $p(1) = 0$, $p(2) = 1$ (cf equation (41)). The corresponding L-matrix then follows from (42),

$$\mathcal{L}_j(u,v) = \begin{pmatrix} e_{j}{}_1^1 + b(u,v) e_{j}{}_2^2 & c(u,v) e_{j}{}_2^1 \\ c(u,v) e_{j}{}_1^2 & b(u,v) e_{j}{}_1^1 - e_{j}{}_2^2 \end{pmatrix} \quad . \tag{68}$$

Using (56), (57) we obtain the Hamiltonian

$$\hat{H} = \frac{-1}{\sin(\kappa)} \sum_{j=1}^{L} \Big\{ e_{j}{}_2^1 e_{j+1}{}_1^2 + e_{j+1}{}_2^1 e_{j}{}_1^2 - \cos(\kappa) \left(e_{j}{}_1^1 e_{j+1}{}_2^2 + e_{j+1}{}_1^1 e_{j}{}_2^2 \right) \Big\} \quad . \tag{69}$$

We may now replace the matrices $e_{j}{}_\alpha^\beta$ by the fermionic projectors $X_{j}{}_\alpha^\beta$, equation (60). Then

$$\mathcal{L}_j(u,v) = \begin{pmatrix} (1 - n_j) + b(u,v) n_j & c(u,v)\, c_j^+ \\ c(u,v)\, c_j & b(u,v)(1 - n_j) - n_j \end{pmatrix} \tag{70}$$

and

$$\hat{H} = \frac{-1}{\sin(\kappa)} \sum_{j=1}^{L} \big\{ c_j^+ c_{j+1} + c_{j+1}^+ c_j + 2\, \cos(\kappa)\, n_j n_{j+1} \big\} + 2 \mathrm{ctg}(\kappa) \hat{N} \quad , \tag{71}$$

where $\hat{N} = \sum_{j=1}^{L} n_j$ is the particle number operator. The Hamiltonian (71) defines the 'small polaron model' [28]. Note that the algebraic Bethe ansatz for the small polaron model gets slightly modified compared to the 'non-graded' spin chain case ($p(1) = p(2) = 0$), since due to the grading $p(1) = 0$, $p(2) = 1$ there appear certain minus signs in the Yang-Baxter algebra (49).

Our next example is the well known family [27] of graded rational R-matrices

$$R^{\alpha\beta}_{\gamma\delta}(u,v) = a(u,v)(-1)^{p(\alpha)p(\beta)}\, \delta^\alpha_\gamma \delta^\beta_\delta + d(u,v)\, \delta^\alpha_\delta \delta^\beta_\gamma \quad , \tag{72}$$

where

$$a(u,v) = \frac{u - v}{u - v + \mathrm{i}} \quad , \quad d(u,v) = \frac{\mathrm{i}}{u - v + \mathrm{i}} \quad . \tag{73}$$

$R(u,v)$ solves the Yang-Baxter equation (40) for arbitrary matrix dimension $N^2 \times N^2$ and arbitrary grading $p : \{1, \dots, N\} \to \mathbb{Z}_2$.

Remark. Note the following subtlety. The grading introduced in (72) is *independent* of the grading that enters the definition (28) of the matrices ${e_j}_\alpha^\beta$. Let $q : \{1, \dots, N\} \to \mathbb{Z}_2$ arbitrary. Then, because of the Kronecker deltas in (72),

$$R^{\alpha\beta}_{\gamma\delta}(u,v) = (-1)^{q(\alpha)+q(\beta)+q(\gamma)+q(\delta)}\, R^{\alpha\beta}_{\gamma\delta}(u,v) \quad , \tag{74}$$

i.e. the compatibility condition (41) is satisfied for arbitrary p and q. For example, let $N = 2$, $p(1) = p(2) = 0$. Then $R(u,v)$ is the R-matrix of the XXX spin-$\frac{1}{2}$ Heisenberg chain, which is compatible with the grading $q(1) = 0$, $q(2) = 1$ leading to a special case of the small polaron Hamiltonian introduced above.

Let us now elaborate on the case $p = q$. Since $a(v,v) = 0$ and $d(v,v) = 1$, the R-matrix defined in equation (72) is regular. Furthermore,

$$\partial_u\, \check{R}^{\alpha\beta}_{\gamma\delta}(u,v)\Big|_{u=v} = \mathrm{i}\left[\delta^\alpha_\gamma \delta^\beta_\delta - (-1)^{p(\alpha)p(\beta)}\, \delta^\alpha_\delta \delta^\beta_\gamma\right] \quad . \tag{75}$$

Thus, using $p = q$ in equations (56) and (57) we find the Hamiltonian

$$\hat{H} = -\sum_{j=1}^{L} (P_{jj+1} - 1) \quad , \tag{76}$$

where P_{jj+1} is the graded permutation operator defined in (38). Clearly $\hat{H}$ commutes with the generators

$$E^\beta_\alpha = \sum_{j=1}^{L} {e_j}^\beta_\alpha \tag{77}$$

of the graded Lie algebra $\mathrm{gl}(m|n)$.

The family of Hamiltonians (75) based on graded permutations includes a number of models that are interesting for applications in physics. In the 'non-graded' case ($p(\alpha) = 0$, $\alpha = 1, \dots, N$) we have the XXX spin-$\frac{1}{2}$ chain and its su(N) generalizations. The case $m = 1$, $n = 2$ leads us to the supersymmetric t-J model [35]. In order to see this we shall employ the fermionization scheme of the previous section. We start with the set of projectors obtained from the matrix X_j in (65) by deleting its fourth row and column. We obtain the reduced matrix

$$X_j = \begin{pmatrix} (1-n_{j\downarrow})(1-n_{j\uparrow}) & (1-n_{j\downarrow})c_{j\uparrow} & c_{j\downarrow}(1-n_{j\uparrow}) \\ (1-n_{j\downarrow})c_{j\uparrow}^\dagger & (1-n_{j\downarrow})n_{j\uparrow} & -c_{j\downarrow}c_{j\uparrow}^\dagger \\ c_{j\downarrow}^\dagger(1-n_{j\uparrow}) & c_{j\downarrow}^\dagger c_{j\uparrow} & n_{j\downarrow}(1-n_{j\uparrow}) \end{pmatrix} \quad . \tag{78}$$

The entries $(X_j)^\alpha_\beta$ of this matrix form a complete set of projection operators on the local space of states spanned by the basis states $|0\rangle$, $c_{j\uparrow}^+|0\rangle$, $c_{j\downarrow}^+|0\rangle$. Double occupancy is excluded on this space of states. Let $X_j{}^\beta_\alpha = (X_j)^\alpha_\beta$. The operator

$$X_j{}^\alpha_\alpha = 1 - n_{j\uparrow}n_{j\downarrow} \tag{79}$$

projects the local Hilbert space of electrons onto the space with no double occupancy. The global projection operator for a chain of L sites is given by the product

$$\Delta = \prod_{j=1}^{L}(1 - n_{j\uparrow}n_{j\downarrow}) \quad . \tag{80}$$

The permutation operator P_{jk} is given by equation (38) with $X_j{}^\beta_\alpha$ replacing $e_j{}^\beta_\alpha$. The summation in (38) is now over three values, $\alpha, \beta = 1, 2, 3$, and the grading is $p(1) = 0$, $p(2) = p(3) = 1$. An elegant way of taking into account the simplifications arising from the restriction to the Hilbert space with with no double occupancy is to consider $P_{jk}\Delta$ instead of P_{jk}. Since $n_{j\uparrow}n_{j\downarrow}\Delta = 0$, we obtain

$$(P_{jk} - 1)\Delta = \Delta(c_{j\sigma}^\dagger c_{k\sigma} + c_{k\sigma}^\dagger c_{j\sigma})\Delta - 2(S_j^a S_k^a - \tfrac{1}{4}n_j n_k)\Delta - (n_j + n_k)\Delta \quad . \tag{81}$$

Here we have introduced the electron density $n_j = n_{j\uparrow} + n_{j\downarrow}$ and the spin densities

$$S_j^a = \tfrac{1}{2}\sigma_{\alpha\beta}^a c_{j\alpha}^\dagger c_{j\beta} \quad . \tag{82}$$

The σ^a, $a = x, y, z$, are the Pauli matrices, and we identify 1 with $\uparrow$ an 2 with $\downarrow$ in the summation over α and β. Inserting (81) into the expression (75) for the Hamiltonian we obtain the familiar Hamiltonian of the supersymmetric t-J model [29, 30, 32, 33, 34, 35, 36].

Let us also write down the corresponding L-matrix, which follows from equation (42),

$$\mathcal{L}_j(u,v) = a(u,v) + d(u,v) \begin{pmatrix} X_j{}^1_1 & X_j{}^1_2 & X_j{}^1_3 \\ X_j{}^2_1 & -X_j{}^2_2 & -X_j{}^2_3 \\ X_j{}^3_1 & -X_j{}^3_2 & -X_j{}^3_3 \end{pmatrix} \quad . \tag{83}$$

This form of the L-matrix suggests a similar form for the monodromy matrix of the corresponding inhomogeneous model,

$$\mathcal{T}(u;\xi_1,\dots,\xi_L) = \begin{pmatrix} A(u) & B_1(u) & B_2(u) \\ C_1(u) & D^1_1(u) & D^1_2(u) \\ C_2(u) & D^2_1(u) & D^2_2(u) \end{pmatrix} \quad . \tag{84}$$

6 Solution of the quantum inverse problem

We are now ready to formulate our main result, which is a formula that expresses the local projection matrices $e_j{}^\beta_\alpha$ for fundamental graded models in terms of the elements of the monodromy matrix. We shall assume we are given a solution of the Yang-Baxter equation (40) which is regular and unitary. Unitarity means that $R(u,v)$ satisfies the equation

$$R^{\alpha\beta}_{\gamma\delta}(u,v) R^{\delta\gamma}_{\alpha'\beta'}(v,u) = \delta^\alpha_{\beta'} \delta^\beta_{\alpha'} \quad . \tag{85}$$

Let p be a grading that is compatible with the R-matrix in the sense of equation (41), and let $\mathcal{T}(u) = \mathcal{T}(u;\xi_1,\dots,\xi_L)$ be the corresponding inhomogeneous monodromy matrix (58). Then we have the following formula,

$$e_n{}^\beta_\alpha = (-1)^{p(\alpha)p(\beta)} \prod_{j=1}^{n-1} \mathrm{str}(\mathcal{T}(\xi_j)) \cdot \mathcal{T}^\beta_\alpha(\xi_n) \cdot \prod_{j=n+1}^{L} \mathrm{str}(\mathcal{T}(\xi_j)) \quad . \tag{86}$$

Equation (86) is our main result. It constitutes a solution of the quantum inverse problem for fundamental graded models. We shall prove it in the remaining sections of this article. For $m = 2$, $n = 0$ ($p(1) = p(2) = 0$) equation (86) reduces to a result recently obtained by Kitanine et al. [15]. Note that because of (51) no ordering is required for the products on the right hand side of (86).

Before proceeding with the proof of (86) let us illustrate the equation through the examples of the previous section. Note that the functions $b(u,v)$, $c(u,v)$ in (67) and $a(u,v)$, $d(u,v)$ in (73) have been chosen in such a way that the corresponding R-matrices (66), (72) satisfy the unitarity condition (85).

For the small polaron model the monodromy matrix is of the form

$$\mathcal{T}(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \quad . \tag{87}$$

Using the fermionization (60) we obtain from (86),

$$c_n^+ = \prod_{j=1}^{n-1} \bigl(A(\xi_j) - D(\xi_j)\bigr) \cdot B(\xi_n) \cdot \prod_{j=n+1}^{L} \bigl(A(\xi_j) - D(\xi_j)\bigr) \quad , \tag{88}$$

$$c_n = \prod_{j=1}^{n-1} \bigl(A(\xi_j) - D(\xi_j)\bigr) \cdot C(\xi_n) \cdot \prod_{j=n+1}^{L} \bigl(A(\xi_j) - D(\xi_j)\bigr) \quad , \tag{89}$$

$$n_n = -\prod_{j=1}^{n-1} \bigl(A(\xi_j) - D(\xi_j)\bigr) \cdot D(\xi_n) \cdot \prod_{j=n+1}^{L} \bigl(A(\xi_j) - D(\xi_j)\bigr) \quad . \tag{90}$$

A similar set of equations holds for the local operators of the supersymmetric t-J model. The monodromy matrix was presented in equation (84). Since the grading is $p(1) = 0$, $p(2) = p(3) = 1$, we have $\mathrm{str}(\mathcal{T}(u)) = A(u) - \mathrm{tr}(D(u))$, and

$$(1 - n_{n\downarrow}) c_{n\uparrow}^+ = \prod_{j=1}^{n-1} \mathrm{str}(\mathcal{T}(\xi_j)) \cdot B_1(\xi_n) \cdot \prod_{j=n+1}^{L} \mathrm{str}(\mathcal{T}(\xi_j)) \quad , \tag{91}$$

$$c_{n\downarrow}^+ (1 - n_{n\uparrow}) = \prod_{j=1}^{n-1} \mathrm{str}(\mathcal{T}(\xi_j)) \cdot B_2(\xi_n) \cdot \prod_{j=n+1}^{L} \mathrm{str}(\mathcal{T}(\xi_j)) \quad , \tag{92}$$

$$(1 - n_{n\downarrow}) c_{n\uparrow} = \prod_{j=1}^{n-1} \mathrm{str}(\mathcal{T}(\xi_j)) \cdot C_1(\xi_n) \cdot \prod_{j=n+1}^{L} \mathrm{str}(\mathcal{T}(\xi_j)) \quad , \tag{93}$$

$$c_{n\downarrow} (1 - n_{n\uparrow}) = \prod_{j=1}^{n-1} \mathrm{str}(\mathcal{T}(\xi_j)) \cdot C_2(\xi_n) \cdot \prod_{j=n+1}^{L} \mathrm{str}(\mathcal{T}(\xi_j)) \quad , \tag{94}$$

$$S_n^+ = -\prod_{j=1}^{n-1} \mathrm{str}(\mathcal{T}(\xi_j)) \cdot D_1^2(\xi_n) \cdot \prod_{j=n+1}^{L} \mathrm{str}(\mathcal{T}(\xi_j)) \quad , \tag{95}$$

$$S_n^- = -\prod_{j=1}^{n-1} \mathrm{str}(\mathcal{T}(\xi_j)) \cdot D_2^1(\xi_n) \cdot \prod_{j=n+1}^{L} \mathrm{str}(\mathcal{T}(\xi_j)) \quad , \tag{96}$$

$$S_n^z = -\frac{1}{2}\prod_{j=1}^{n-1} \mathrm{str}(\mathcal{T}(\xi_j)) \cdot \big(D_1^1(\xi_n) - D_2^2(\xi_n)\big) \cdot \prod_{j=n+1}^{L} \mathrm{str}(\mathcal{T}(\xi_j)) \quad , \tag{97}$$

$$(1-n_{n\downarrow})(1-n_{n\uparrow}) = \prod_{j=1}^{n-1} \mathrm{str}(\mathcal{T}(\xi_j)) \cdot A(\xi_n) \cdot \prod_{j=n+1}^{L} \mathrm{str}(\mathcal{T}(\xi_j)) \quad . \tag{98}$$

On the restricted Hilbert space of the supersymmetric t-J model, where double occupancy of lattice sites is excluded, the operators on the left hand side of equations (91)-(94) are appropriate creation and annihilation operators. The operator $(1-n_{n\downarrow})c_{n\uparrow}^+$, for instance, creates an up-spin electron on site n, provided this site is not occupied by a down-spin electron. The local spin operators S_n^a $(a = x, y, z)$ were introduced in equation (82). S_n^+ and S_n^- in (95) and (96) are defined as $S_n^+ = S_n^x + \mathrm{i}S_n^y = c_{n\uparrow}^+ c_{n\downarrow}$ and $S_n^- = S_n^x - \mathrm{i}S_n^y = c_{n\downarrow}^+ c_{n\uparrow}$. These operators induce a local spin flip. The operator $(1-n_{n\downarrow})(1-n_{n\uparrow})$ on the left hand side of (98) acts as $1-(n_{n\uparrow}+n_{n\downarrow})$ on the restricted Hilbert space of the supersymmetric t-J model and thus essentially gives the local particle number operator.

Note that equations (91)-(98) after appropriate replacement of the monodromy matrix also apply to the 't-0 model' (the infinite coupling limit of the Hubbard model below half-filling) which was solved by algebraic Bethe ansatz in [26].

We would like to stress, that our main result, equation (86), also holds for homogeneous, translationally invariant models, for which $\xi_j = v = 0$, for $j = 1, \ldots, L$. In this case (86) takes a particularly simple form, since $\mathrm{str}(\mathcal{T}(0)) = \hat{U}$, where $\hat{U}$ is the homogeneous shift operator (53). Using this fact equation (86) turns into

$$e_{n\alpha}^{\ \beta} = (-1)^{p(\alpha)p(\beta)}\, \hat{U}^{n-1} \mathcal{T}_\alpha^\beta(0)\, \hat{U}^{L-n} \quad . \tag{99}$$

Performing a similarity transformation with $\hat{U}^{1-n}$ we obtain the amazingly simple result

$$e_{1\alpha}^{\ \beta} = (-1)^{p(\alpha)p(\beta)}\, \mathcal{T}_\alpha^\beta(0)\, \hat{U}^{-1} \quad , \tag{100}$$

which in section 1 was obtained for the special case of the XYZ chain. To give another example, equations (88)-(90), for instance, are in the homogeneous case equivalent to

$$c_1^+ = B(0)\, \hat{U}^{-1} \quad , \quad c_1 = C(0)\, \hat{U}^{-1} \quad , \quad n_1 = -D(0)\, \hat{U}^{-1} \quad . \tag{101}$$

7 The fermionic R-operator

The role of the matrix $\check{R}(u,v)$ in the graded Yang-Baxter algebra (45) is to switch the order of the two auxiliary spaces. The definition of an operator that similarly switches the order of quantum spaces in a product of two L-matrices requires appropriate use of the grading. Recently, such an operator was introduced for several important models by Umeno, Shiroishi and Wadati [37, 38] and was called fermionic R-operator. Here we give a general definition of the fermionic R-operator associated with a solution $R(u,v)$ of the Yang-Baxter equation (40). For a given grading and a solution $R(u,v)$ of the Yang-Baxter equation (40) that is compatible with this grading (cf (41)) we define

$$\mathcal{R}_{jk}^f(u,v) = (-1)^{p(\gamma)+p(\alpha)(p(\beta)+p(\gamma))}\, R_{\gamma\delta}^{\alpha\beta}(u,v) e_{j\alpha}^{\ \gamma} e_{k\beta}^{\ \delta} \quad . \tag{102}$$

The fermionic R-operator will be an important tool in the proof of our main result (86). Let us summarize its properties in the following lemma.

Lemma 1. *Properties of the fermionic R-operator.*

(i) Evenness. The fermionic R-operator is even,

$$p(\mathcal{R}_{jk}^f(u,v)) = 0 \quad . \tag{103}$$

(ii) Bilinear relation. The fermionic R-operator satisfies

$$\mathcal{R}_{jk}^f(\xi_j,\xi_k)\mathcal{L}_k(u,\xi_k)\mathcal{L}_j(u,\xi_j) = \mathcal{L}_j(u,\xi_j)\mathcal{L}_k(u,\xi_k)\mathcal{R}_{jk}^f(\xi_j,\xi_k) \quad . \tag{104}$$

(iii) Yang-Baxter equation. The fermionic R-operator satisfies the following form of the Yang-Baxter equation,

$$\mathcal{R}_{12}^f(u,v)\mathcal{R}_{13}^f(u,w)\mathcal{R}_{23}^f(v,w) = \mathcal{R}_{23}^f(v,w)\mathcal{R}_{13}^f(u,w)\mathcal{R}_{12}^f(u,v) \quad . \tag{105}$$

(iv) Regularity. If $R(u,v)$ is regular, say $R^{\alpha\beta}_{\gamma\delta}(v,v) = \delta^\alpha_\delta \delta^\beta_\gamma$, then

$$\mathcal{R}^f_{jk}(v,v) = P_{jk} \quad , \tag{106}$$

where P_{jk} is the graded permutation operator (38).

(v) Unitarity. If $R(u,v)$ is unitary (cf equation (85)), then $\mathcal{R}^f_{jk}(u,v)$ is unitary in the sense that

$$\mathcal{R}^f_{jk}(u,v)\mathcal{R}^f_{kj}(v,u) = \mathrm{id} \quad . \tag{107}$$

Proof. (i) The evenness of the fermionic R-operator is a direct consequence of the compatibility condition (41).

(ii) Using the commutation relations (29) and the projection property (31) for the matrices ${e_j}^\beta_\alpha$ as well as the compatibility condition (41) and the Yang-Baxter equation (40), the matrix elements on both sides of (104) can be reduced to

$$\begin{aligned}&(-1)^{\{p(\alpha)p(\beta)+p(\beta)p(\gamma)+p(\gamma)p(\alpha)+p(\beta'')(p(\gamma)+p(\gamma''))\}}\\ &\qquad R^{\beta\gamma}_{\beta'\gamma'}(\xi_1,\xi_2) R^{\alpha\gamma'}_{\alpha'\gamma''}(u,\xi_2) R^{\alpha'\beta'}_{\alpha''\beta''}(u,\xi_1) {e_1}^{\beta''}_\beta {e_2}^{\gamma''}_\gamma \quad .\end{aligned}$$

(iii) The proof is similar to the proof of (ii). Using (29), (31) and (41), (40) both sides of equation (105) reduce to

$$\begin{aligned}&(-1)^{\{p(\alpha)p(\beta)+p(\beta)p(\gamma)+p(\gamma)p(\alpha)+p(\alpha'')(p(\alpha)+p(\alpha''))+p(\beta'')(p(\gamma)+p(\gamma''))\}}\\ &\qquad R^{\beta\gamma}_{\beta'\gamma'}(\xi_1,\xi_2) R^{\alpha\gamma'}_{\alpha'\gamma''}(u,\xi_2) R^{\alpha'\beta'}_{\alpha''\beta''}(u,\xi_1) {e_1}^{\alpha''}_\alpha {e_2}^{\beta''}_\beta {e_3}^{\gamma''}_\gamma \quad .\end{aligned}$$

(iv)

$$(-1)^{p(\gamma)+p(\alpha)(p(\beta)+p(\gamma))} \delta^\alpha_\delta \delta^\beta_\gamma {e_j}^\gamma_\alpha {e_k}^\delta_\beta = (-1)^{p(\beta)} {e_j}^\beta_\alpha {e_k}^\alpha_\beta = P_{jk} \quad .$$

(v) Using (29), (31) and (41) we obtain

$$\mathcal{R}^f_{jk}(u,v)\mathcal{R}^f_{kj}(v,u) = (-1)^{p(\beta)(p(\alpha)+p(\beta'))} R^{\alpha\beta}_{\gamma\delta}(u,v) R^{\delta\gamma}_{\alpha'\beta'}(v,u) {e_j}^{\beta'}_\alpha {e_k}^{\alpha'}_\beta \quad ,$$

and the assertion follows from (85).

□

8 The shift operator

In this section we shall use the fermionic R-operator introduced above in order to define the shift operator for inhomogeneous graded models. We shall explore the properties of the shift operator and shall eventually use these properties to prove our main result (86).

Let us start with a slightly more general concept. The inhomogeneous monodromy matrix defined in (58) is an *ordered* product of L-matrices. In the following we shall indicate the order of the factors by supplying subscripts to the monodromy matrix

$$\mathcal{T}_{1\dots L}(u;\xi_1,\dots,\xi_L) = \mathcal{T}(u;\xi_1,\dots,\xi_L) = \mathcal{L}_L(u,\xi_L)\dots\mathcal{L}_1(u,\xi_1) \quad . \tag{108}$$

As can be seen from (103) and (104) the fermionic R-operator $\mathcal{R}^f_{jj+1}(\xi_j,\xi_{j+1})$ interchanges the two neighbouring factors $\mathcal{L}_{j+1}(u,\xi_{j+1})$ and $\mathcal{L}_j(u,\xi_j)$ in the monodromy matrix. Since the symmetric group $\mathfrak{S}^L$ is generated by the transpositions of nearest neighbours, the L-matrices on the right hand side of (108) can be arbitrarily reordered by application of an appropriate product of fermionic R-operators. This means that for every $\tau\in\mathfrak{S}^L$ there exists an operator $\mathcal{R}^\tau_{1\dots L}(\xi_1,\dots,\xi_L)$, which is a product of fermionic R-operators and induces the action of the permutation $\tau\in\mathfrak{S}^L$ on the inhomogeneous monodromy matrix,

$$\begin{aligned}\mathcal{R}^\tau_{1\dots L}(\xi_1,\dots,\xi_L)\mathcal{T}_{1\dots L}(u;\xi_1,\dots,\xi_L) = \\ \mathcal{T}_{\tau(1)\dots\tau(L)}(u;\xi_{\tau(1)},\dots,\xi_{\tau(L)})\mathcal{R}^\tau_{1\dots L}(\xi_1,\dots,\xi_L) \quad .\end{aligned} \tag{109}$$

The non-graded analogue of this operator was introduced in [16].

Let us construct $\mathcal{R}^\tau_{1\dots L}(\xi_1,\dots,\xi_L)$ explicitly. We shall use the shorthand notation $\mathcal{R}^\tau_{1\dots L}=\mathcal{R}^\tau_{1\dots L}(\xi_1,\dots,\xi_L)$, $\mathcal{T}_{1\dots L}(u)=\mathcal{T}_{1\dots L}(u;\xi_1,\dots,\xi_L)$ and $\mathcal{R}^f_{jk}=\mathcal{R}^f_{jk}(\xi_j,\xi_k)$ whenever the order of the inhomogeneities $\xi_1,\dots,\xi_L$ is the same as the order of the corresponding lattice sites. For $j=1,\dots,L-1$ define $\sigma_j\in\mathfrak{S}^L$ by

$$\sigma_j(k) = \begin{cases} j+1 & \text{if } k=j,\\ j & \text{if } k=j+1,\\ k & \text{else.}\end{cases} \tag{110}$$

The $\sigma_j \in \mathfrak{S}^L$ are transpositions of nearest neighbours. It follows from (103), (104) that

$$\mathcal{R}^f_{jj+1}\, \mathcal{T}_{1\dots L}(u) = \mathcal{T}_{\sigma_j(1)\dots\sigma_j(L)}(u)\, \mathcal{R}^f_{jj+1} \quad . \tag{111}$$

This means that $\mathcal{R}^{\sigma_j}_{1\dots L} = \mathcal{R}^f_{jj+1}$. Choose $\tau \in \mathfrak{S}^L$ arbitrarily. Then

$$\mathcal{R}^f_{\tau(j),\tau(j+1)}\, \mathcal{T}_{\tau(1)\dots\tau(L)}(u) = \mathcal{T}_{\tau\sigma_j(1)\dots\tau\sigma_j(L)}(u)\, \mathcal{R}^f_{\tau(j),\tau(j+1)} \quad . \tag{112}$$

Since the transpositions of nearest neighbours σ_j, $j = 1, \dots, L-1$, generate the symmetric group $\mathfrak{S}^L$, there is a finite sequence $(j_p)_{p=1}^n$, such that $\tau = \sigma_{j_1} \dots \sigma_{j_n}$. Let $\tau_p = \sigma_{j_1} \dots \sigma_{j_p}$, $p = 1, \dots, n$ and $\tau_0 = \mathrm{id}$. Then $\tau = \tau_n$, and, using (112), we conclude that

$$\mathcal{R}^f_{\tau_{p-1}(j_p),\tau_{p-1}(j_p+1)}\, \mathcal{T}_{\tau_{p-1}(1)\dots\tau_{p-1}(L)}(u) = \mathcal{T}_{\tau_p(1)\dots\tau_p(L)}(u)\, \mathcal{R}^f_{\tau_{p-1}(j_p),\tau_{p-1}(j_p+1)} \quad , \tag{113}$$

for $p = 1, \dots, n$. By iteration of the latter equation we obtain

$$\mathcal{R}^f_{\tau_{n-1}(j_n),\tau_{n-1}(j_n+1)} \dots \mathcal{R}^f_{\tau_1(j_2),\tau_1(j_2+1)} \mathcal{R}^f_{j_1,j_1+1}\, \mathcal{T}_{1\dots L}(u) = \\ \mathcal{T}_{\tau(1)\dots\tau(L)}(u)\, \mathcal{R}^f_{\tau_{n-1}(j_n),\tau_{n-1}(j_n+1)} \dots \mathcal{R}^f_{\tau_1(j_2),\tau_1(j_2+1)} \mathcal{R}^f_{j_1,j_1+1} \quad . \tag{114}$$

Thus we have constructed an explicit expression for the operator $\mathcal{R}^\tau_{1\dots L}$,

$$\mathcal{R}^\tau_{1\dots L} = \mathcal{R}^f_{\tau_{n-1}(j_n),\tau_{n-1}(j_n+1)} \dots \mathcal{R}^f_{\tau_1(j_2),\tau_1(j_2+1)} \mathcal{R}^f_{j_1,j_1+1} \quad . \tag{115}$$

Let us now specify to the case, when τ is equal to the cyclic permutation $\gamma = \sigma_1 \dots \sigma_{L-1}$. Then $j_p = p$, $p = 1, \dots, L-1$, in our above construction, and $\gamma_p = \sigma_1 \dots \sigma_p$. Thus $\gamma_{p-1}(j_p) = \gamma_{p-1}(p) = 1$, and $\gamma_{p-1}(j_p+1) = \gamma_{p-1}(p+1) = p+1$. Using (115) we obtain

$$\mathcal{R}^\gamma_{1\dots L} = \mathcal{R}^f_{1L} \mathcal{R}^f_{1L-1} \dots \mathcal{R}^f_{12} \quad . \tag{116}$$

The operator $\mathcal{R}^\gamma_{1\dots L}$ induces a shift by one site on the inhomogeneous monodromy matrix. Now (112) implies that

$$\mathcal{R}^\gamma_{\gamma(1)\dots\gamma(L)} \mathcal{T}_{\gamma(1)\dots\gamma(L)}(u) = \mathcal{T}_{\gamma^2(1)\dots\gamma^2(L)}(u) \mathcal{R}^\gamma_{\gamma(1)\dots\gamma(L)} \quad . \tag{117}$$

It follows by multiplication by $\mathcal{R}^\gamma_{1\dots L}$ from the right, that

$$\mathcal{R}^{\gamma^2}_{1\dots L} = \mathcal{R}^\gamma_{\gamma(1)\dots\gamma(L)} \mathcal{R}^\gamma_{1\dots L} \quad . \tag{118}$$

Iterating the above steps we arrive at the following lemma.

Lemma 2. *The operator*

$$\mathcal{R}^{\gamma^n}_{1\ldots L} = \mathcal{R}^{\gamma}_{\gamma^{n-1}(1)\ldots\gamma^{n-1}(L)}\mathcal{R}^{\gamma}_{\gamma^{n-2}(1)\ldots\gamma^{n-2}(L)}\cdots\mathcal{R}^{\gamma}_{1\ldots L} \quad , \tag{119}$$

where

$$\mathcal{R}^{\gamma}_{\gamma^{p-1}(1)\ldots\gamma^{p-1}(L)} = \mathcal{R}^{f}_{pp-1}\cdots\mathcal{R}^{f}_{p1}\mathcal{R}^{f}_{pL}\cdots\mathcal{R}^{f}_{pp+1} \quad , \tag{120}$$

generates a shift by n sites on the inhomogeneous lattice, i.e.

$$\mathcal{R}^{\gamma^n}_{1\ldots L}\,\mathcal{T}_{1\ldots L}(u) = \mathcal{T}_{n+1\ldots L1\ldots n}(u)\,\mathcal{R}^{\gamma^n}_{1\ldots L} \quad . \tag{121}$$

Since $\gamma^L = \mathrm{id}$, we conclude from (121) that

$$\mathcal{R}^{\gamma^L}_{1\ldots L}\,\mathcal{T}_{1\ldots L}(u) = \mathcal{T}_{1\ldots L}(u)\,\mathcal{R}^{\gamma^L}_{1\ldots L} \quad . \tag{122}$$

If $\mathcal{R}^f_{jk}$ is unitary, we have the following stronger result.

Lemma 3. *Let $\mathcal{R}^f_{jk}$ be unitary (cf equation 107). Then*

$$\mathcal{R}^{\gamma^L}_{1\ldots L} = \mathrm{id} \quad . \tag{123}$$

Proof. Let us first prove the case $L = 2$. Then $\mathcal{R}^{\gamma}_{12} = \mathcal{R}^f_{12}$ and $\mathcal{R}^{\gamma^2}_{12} = \mathcal{R}^{\gamma}_{\gamma1\gamma2}\mathcal{R}^{\gamma}_{12} = \mathcal{R}^{\gamma}_{21}\mathcal{R}^{\gamma}_{12} = \mathcal{R}^f_{21}\mathcal{R}^f_{12} = \mathrm{id}$. The last equation holds, since by hypothesis, $\mathcal{R}^f_{12}$ is unitary.

For the case $L > 2$ we start from the Yang-Baxter equation (105),

$$\mathcal{R}^f_{L,L-n}\mathcal{R}^f_{L,j}\mathcal{R}^f_{L-n,j} = \mathcal{R}^f_{L-n,j}\mathcal{R}^f_{L,j}\mathcal{R}^f_{L,L-n} \quad . \tag{124}$$

By iterated use of (124) we obtain

$$\begin{aligned}\mathcal{R}^f_{L,L-n}(\mathcal{R}^f_{L,L-n-1}\cdots\mathcal{R}^f_{L,1})(\mathcal{R}^f_{L-n,L-n-1}\cdots\mathcal{R}^f_{L-n,1}) = \\ (\mathcal{R}^f_{L-n,L-n-1}\cdots\mathcal{R}^f_{L-n,1})(\mathcal{R}^f_{L,L-n-1}\cdots\mathcal{R}^f_{L,1})\mathcal{R}^f_{L,L-n} \quad , \end{aligned} \tag{125}$$

for $n = 1, \ldots, L-2$.

Let us introduce the truncated cyclic permutations $\gamma_p = \sigma_1 \ldots \sigma_{p-1}$, $p = 2, \ldots, L$, as above. γ_p induces a cyclic shift on the p-tuple $(1, \ldots, p)$ and leaves the $(L-p)$-tuple $(p+1,\dots,L)$ invariant. Using (125), it follows that

$$\begin{aligned}
&\mathcal{R}^f_{L,L-n}\dots\mathcal{R}^f_{L,1}\,\mathcal{R}^\gamma_{\gamma^{L-n-1}(1)\dots\gamma^{L-n-1}(L)}\\
&\qquad = \mathcal{R}^f_{L,L-n}\,(\mathcal{R}^f_{L,L-n-1}\dots\mathcal{R}^f_{L,1})(\mathcal{R}^f_{L-n,L-n-1}\dots\mathcal{R}^f_{L-n,1})\,\cdot\\
&\qquad\qquad\qquad\qquad\cdot\,(\mathcal{R}^f_{L-n,L}\dots\mathcal{R}^f_{L-n,L-n+1})\\
&\qquad = (\mathcal{R}^f_{L-n,L-n-1}\dots\mathcal{R}^f_{L-n,1})(\mathcal{R}^f_{L,L-n-1}\dots\mathcal{R}^f_{L,1})\,\cdot\\
&\qquad\qquad\qquad\qquad\cdot\,\underbrace{\mathcal{R}^f_{L,L-n}\mathcal{R}^f_{L-n,L}}_{=\ \mathrm{id}}(\mathcal{R}^f_{L-n,L-1}\dots\mathcal{R}^f_{L-n,L-n+1})\\
&\qquad = (\mathcal{R}^f_{L-n,L-n-1}\dots\mathcal{R}^f_{L-n,1})(\mathcal{R}^f_{L-n,L-1}\dots\mathcal{R}^f_{L-n,L-n+1})\,\cdot\\
&\qquad\qquad\qquad\qquad\cdot\,(\mathcal{R}^f_{L,L-n-1}\dots\mathcal{R}^f_{L,1})\\
&\qquad = \mathcal{R}^{\gamma_{L-1}}_{\gamma_{L-1}^{L-n-1}(1),\dots,\gamma_{L-1}^{L-n-1}(L-1),\,L}\,\mathcal{R}^f_{L,L-n-1}\dots\mathcal{R}^f_{L,1}\quad .
\end{aligned}\tag{126}$$

Hence,

$$\begin{aligned}
\mathcal{R}^{\gamma^L}_{1\dots L} &= \mathcal{R}^\gamma_{\gamma^{L-1}(1)\dots\gamma^{L-1}(L)}\,\mathcal{R}^\gamma_{\gamma^{L-2}(1)\dots\gamma^{L-2}(L)}\dots\mathcal{R}^\gamma_{1\dots L}\\
&= \mathcal{R}^f_{L,L-1}\dots\mathcal{R}^f_{L,1}\,\mathcal{R}^\gamma_{\gamma^{L-2}(1)\dots\gamma^{L-2}(L)}\,\mathcal{R}^\gamma_{\gamma^{L-3}(1)\dots\gamma^{L-3}(L)}\dots\mathcal{R}^\gamma_{1\dots L}\\
&= \mathcal{R}^{\gamma_{L-1}}_{\gamma_{L-1}^{L-2}(1),\dots,\gamma_{L-1}^{L-2}(L-1),\,L}\,\mathcal{R}^f_{L,L-2}\dots\mathcal{R}^f_{L,1}\,\mathcal{R}^\gamma_{\gamma^{L-3}(1)\dots\gamma^{L-3}(L)}\dots\mathcal{R}^\gamma_{1\dots L}\\
&= \mathcal{R}^{\gamma_{L-1}^{L-1}}_{1\dots L} = \mathcal{R}^{\gamma_{L-2}^{L-2}}_{1\dots L} = \dots = \mathcal{R}^{\gamma_2^2}_{1\dots L}\quad .
\end{aligned}\tag{127}$$

Since $\gamma_2 = \sigma_1$ and $\mathcal{R}^{\sigma_1}_{1\dots L} = \mathcal{R}^f_{12}$, the latter equation reduces the proof of Lemma 3 for $L > 2$ to the case $L = 2$, which was proved above. $\square$

Our next lemma can be used to establishes a connection between the inhomogeneous monodromy matrix (108) and the shift operator (119)

Lemma 4. *Let* $X = X^\alpha_\beta e^\beta_\alpha \in \mathrm{End}(\mathbb{C}^{m+n})$ *and let* $R(u,v)$ *be regular, say,* $R^{\alpha\beta}_{\gamma\delta}(v,v) = \delta^\alpha_\delta\delta^\beta_\gamma$. *Then*

$$\mathrm{str}(X\mathcal{T}_{n\dots L1\dots n-1}(\xi_n)) = (-1)^{p(\alpha)+p(\alpha)p(\beta)}X^\alpha_\beta {e_n}_\alpha^{\ \beta}\,\mathcal{R}^\gamma_{\gamma^{n-1}1\dots\gamma^{n-1}L}\quad . \tag{128}$$

Proof.

$$
\begin{aligned}
&\mathrm{str}(X\mathcal{T}_{n\ldots L1\ldots n-1}(\xi_n))\\
&= (-1)^{p(\alpha)} X^{\alpha}_{\beta}\, \mathcal{L}_{n-1}{}^{\beta}_{\beta_{n-1}}(\xi_n,\xi_{n-1})\ldots \mathcal{L}_{1}{}^{\beta_2}_{\beta_1}(\xi_n,\xi_1)\mathcal{L}_{L}{}^{\beta_1}_{\beta_L}(\xi_n,\xi_L)\,\cdot\\
&\qquad\qquad \ldots \mathcal{L}_{n+1}{}^{\beta_{n+2}}_{\beta_{n+1}}(\xi_n,\xi_{n+1})\,(-1)^{p(\alpha)p(\beta_{n+1})}\, e_{n}{}^{\beta_{n+1}}_{\alpha}\\
&= (-1)^{\left\{p(\alpha)+p(\alpha)p(\beta)+\sum_{\substack{j=1\\ j\neq n}}^{L}(p(\beta_j)+p(\alpha_j)p(\beta_j))\right\}}\\
&\qquad \cdot X^{\alpha}_{\beta}\,\delta^{\beta}_{\alpha_{n-1}}\delta^{\beta_{n-1}}_{\alpha_{n-2}}\ldots\delta^{\beta_2}_{\alpha_1}\delta^{\beta_1}_{\alpha_L}\delta^{\beta_L}_{\alpha_{L-1}}\ldots\delta^{\beta_{n+2}}_{\alpha_{n+1}} e_{n}{}^{\beta_{n+1}}_{\alpha}\\
&\qquad \cdot \mathcal{L}_{n+1}{}^{\alpha_{n+1}}_{\beta_{n+1}}(\xi_n,\xi_{n+1})\ldots\mathcal{L}_{L}{}^{\alpha_L}_{\beta_L}(\xi_n,\xi_L)\mathcal{L}_{1}{}^{\alpha_1}_{\beta_1}(\xi_n,\xi_1)\\
&\qquad\qquad \ldots\mathcal{L}_{n-1}{}^{\alpha_{n-1}}_{\beta_{n-1}}(\xi_n,\xi_{n-1})\\
&= (-1)^{\left\{p(\alpha)+p(\alpha)p(\beta)+\sum_{\substack{j=1\\ j\neq n,n+1}}^{L}(p(\beta_j)+p(\alpha_j)p(\beta_j))\right\}}\\
&\qquad \cdot X^{\alpha}_{\beta}\, e_{n}{}^{\beta}_{\alpha} e_{n}{}^{\beta_{n-1}}_{\alpha_{n-1}} e_{n}{}^{\beta_{n-2}}_{\alpha_{n-2}}\ldots e_{n}{}^{\beta_1}_{\alpha_1} e_{n}{}^{\beta_L}_{\alpha_L}\ldots e_{n}{}^{\beta_{n+2}}_{\alpha_{n+2}}\\
&\qquad \cdot (-1)^{p(\beta_{n+1})+p(\alpha_{n+1})p(\beta_{n+1})} e_{n}{}^{\beta_{n+1}}_{\alpha_{n+1}}\mathcal{L}_{n+1}{}^{\alpha_{n+1}}_{\beta_{n+1}}(\xi_n,\xi_{n+1})\\
&\qquad \cdot \mathcal{L}_{n+2}{}^{\alpha_{n+2}}_{\beta_{n+2}}(\xi_n,\xi_{n+2})\ldots\mathcal{L}_{L}{}^{\alpha_L}_{\beta_L}(\xi_n,\xi_L)\mathcal{L}_{1}{}^{\alpha_1}_{\beta_1}(\xi_n,\xi_1)\\
&\qquad\qquad \ldots\mathcal{L}_{n-1}{}^{\alpha_{n-1}}_{\beta_{n-1}}(\xi_n,\xi_{n-1})\\
&= (-1)^{\left\{p(\alpha)+p(\alpha)p(\beta)+\sum_{\substack{j=1\\ j\neq n,n+1}}^{L}(p(\beta_j)+p(\alpha_j)p(\beta_j))\right\}}\\
&\qquad \cdot X^{\alpha}_{\beta}\, e_{n}{}^{\beta}_{\alpha} e_{n}{}^{\beta_{n-1}}_{\alpha_{n-1}} e_{n}{}^{\beta_{n-2}}_{\alpha_{n-2}}\ldots e_{n}{}^{\beta_1}_{\alpha_1} e_{n}{}^{\beta_L}_{\alpha_L}\ldots e_{n}{}^{\beta_{n+2}}_{\alpha_{n+2}}\\
&\qquad \cdot \mathcal{L}_{n+2}{}^{\alpha_{n+2}}_{\beta_{n+2}}(\xi_n,\xi_{n+2})\ldots\mathcal{L}_{L}{}^{\alpha_L}_{\beta_L}(\xi_n,\xi_L)\mathcal{L}_{1}{}^{\alpha_1}_{\beta_1}(\xi_n,\xi_1)\\
&\qquad\qquad \ldots\mathcal{L}_{n-1}{}^{\alpha_{n-1}}_{\beta_{n-1}}(\xi_n,\xi_{n-1})\,\mathcal{R}^{f}_{n,n+1} \qquad (129)\\
&= (-1)^{p(\alpha)+p(\alpha)p(\beta)} X^{\alpha}_{\beta} e_{n}{}^{\beta}_{\alpha}\,\mathcal{R}^{f}_{n,n-1}\ldots\mathcal{R}^{f}_{n,1}\mathcal{R}^{f}_{n,L}\ldots\mathcal{R}^{f}_{n,n+1}\\
&= (-1)^{p(\alpha)+p(\alpha)p(\beta)} X^{\alpha}_{\beta}\, e_{n}{}^{\beta}_{\alpha}\mathcal{R}^{\gamma}_{\gamma^{n-1}1\ldots\gamma^{n-1}L}\quad . \qquad (130)
\end{aligned}
$$

Here we used the regularity in the first equation. In the second equation we reversed the order of factors and introduced a product of Kronecker deltas.

In the third equation we used the identity

$$\delta^{\beta}_{\alpha_{n-1}} \delta^{\beta_{n-1}}_{\alpha_{n-2}} \dots \delta^{\beta_2}_{\alpha_1} \delta^{\beta_1}_{\alpha_L} \delta^{\beta_L}_{\alpha_{L-1}} \dots \delta^{\beta_{n+2}}_{\alpha_{n+1}} e_n{}^{\beta_{n+1}}_{\alpha}$$
$$= e_n{}^{\beta}_{\alpha} e_n{}^{\beta_{n-1}}_{\alpha_{n-1}} e_n{}^{\beta_{n-2}}_{\alpha_{n-2}} \dots e_n{}^{\beta_1}_{\alpha_1} e_n{}^{\beta_L}_{\alpha_L} \dots e_n{}^{\beta_{n+1}}_{\alpha_{n+1}} \quad , \tag{131}$$

which follows from (31). In the fourth equation we used that

$$\mathcal{R}^f_{jk} = (-1)^{p(\beta)+p(\alpha)p(\beta)} e_j{}^{\beta}_{\alpha} \mathcal{L}_k{}^{\alpha}_{\beta}(\xi_j, \xi_k) \tag{132}$$

and the fact that $\mathcal{R}^f_{jk}$ is even. In the fifth equation we iterated the two previous steps of our calculation. Finally in the sixth equation the formula (120) entered. □

Setting $X^{\alpha}_{\beta} = \delta^{\alpha}_{\beta}$ in (128) and using the cyclic invariance of the super trace we obtain the following corollary to Lemma 4.

Corollary.

$$\mathcal{R}^{\gamma}_{\gamma^{n-1}1 \dots \gamma^{n-1}L} = \mathrm{str}(\mathcal{T}_{1 \dots L}(\xi_n)) \quad . \tag{133}$$

Equation (133) is the inhomogeneous analogue of equation (53).

Lemma 5. *We have the following expression for the shift operator in terms of the elements of the monodromy matrix,*

$$\mathcal{R}^{\gamma^n}_{1 \dots L} = \prod_{j=1}^{n} \mathrm{str}(\mathcal{T}_{1 \dots L}(\xi_j)) \quad . \tag{134}$$

If $R(u,v)$ is unitary (cf equation (85)), then $\mathcal{R}^{\gamma^n}_{1 \dots L}$ is invertible with inverse

$$\left(\mathcal{R}^{\gamma^n}_{1 \dots L}\right)^{-1} = \prod_{j=n+1}^{L} \mathrm{str}(\mathcal{T}_{1 \dots L}(\xi_j)) \quad . \tag{135}$$

Proof. The lemma follows from Lemma 2, Lemma 3 and Corrolary 1 to Lemma 4. □

We are now prepared to prove our main result, equation (86).

Proof of equation (86). Using Lemma 2, Lemma 4, the corollary to Lemma 4 and Lemma 5 we obtain

$$\begin{aligned} &\mathrm{str}(X\mathcal{T}_{n\dots L1\dots n-1}(\xi_n)) \\ &\quad= \mathcal{R}_{1\dots L}^{\gamma^{n-1}} \mathrm{str}(X\mathcal{T}_{1\dots L}(\xi_n)) \left(\mathcal{R}_{1\dots L}^{\gamma^{n-1}}\right)^{-1} \\ &\quad= \prod_{j=1}^{n-1} \mathrm{str}(\mathcal{T}_{1\dots L}(\xi_j)) \cdot \mathrm{str}(X\mathcal{T}_{1\dots L}(\xi_n)) \cdot \prod_{j=n}^{L} \mathrm{str}(\mathcal{T}_{1\dots L}(\xi_j)) \\ &\quad= (-1)^{p(\alpha)+p(\alpha)p(\beta)} X^{\alpha}_{\beta} {e_n}^{\beta}_{\alpha} \mathrm{str}(\mathcal{T}_{1\dots L}(\xi_n)) \quad . \end{aligned} \tag{136}$$

It follows that

$$\begin{aligned} &(-1)^{p(\alpha')+p(\alpha')p(\beta')} X^{\alpha'}_{\beta'} {e_n}^{\beta'}_{\alpha'} = \\ &\quad \prod_{j=1}^{n-1} \mathrm{str}(\mathcal{T}_{1\dots L}(\xi_j)) \cdot \mathrm{str}(X\mathcal{T}_{1\dots L}(\xi_n)) \cdot \prod_{j=n+1}^{L} \mathrm{str}(\mathcal{T}_{1\dots L}(\xi_j)) \quad . \end{aligned} \tag{137}$$

Finally, by specifying $X^{\alpha'}_{\beta'} = (-1)^{p(\alpha')+p(\alpha')p(\beta')} \delta^{\alpha'}_{\alpha} \delta^{\beta}_{\beta'}$, we arrive at equation (86). □

Summary

In this article we obtained an explicit solution of the quantum inverse problem for fundamental graded models. Our main result is the general formula (86). This formula expresses the local projection operators ${e_n}^{\beta}_{\alpha}$, which represent local spins and local fields, in terms of the elements of the monodromy matrix. The formula and its proof essentially simplify for translationally invariant models (all inhomogeneities coincide, $\xi_j = \xi$). In the translationally invariant case the proof is based on the representation of the shift operator as a product of permutation matrices.

We presented explicit formulae for the solution of the quantum inverse problem for the XYZ quantum spin chain (17)-(19) and for the supersymmetric t-J model of strongly correlated electrons (91)-(98). For the XYZ chain the local projection operators coincide with local Pauli matrices (quantum spin operators).

We are planing to use our results in forthcoming publications in order to obtain multiple integral representations for correlation functions of fundamental graded models.

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