Rigidity for periodic magnetic fields

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Abstract. We study the motion of a charge on a conformally flat Riemannian torus in the presence of magnetic field. We prove that for any non-zero magnetic field there always exist orbits of this motion which have conjugate points. We conjecture that the restriction of conformal flatness of the metric is not essential for this result. This would provide a "twisted" version of the recent generalisation of Hopf's rigidity result obtained by Burago and Ivanov.

1 Introduction

It was proved by E. Hopf [9], that for Riemannian 2-torus with non-zero curvature there always exist geodesics with conjugate points. In [5] this result was finaly generalised to higher dimensions. We refer the reader to [13], [7], [6] and [10] for a non-complete list of previous important contributions. It was discovered in [1], [2] (see also [4]) that the nature of E. Hopf rigidity is not entirely Riemannian and can be established for other variational problems. In each case this requires some new integral geometric tool adapted for the system under consideration.

In this paper we give a proof of such a rigidity result for the motion of a charged particle on a torus in the presence of the magnetic field, provided the Riemannian metric is conformally flat. This result gives the twisted version of a theorem by A. Knauf and C .Croke-A. Fathi ([6] and [13]). We formulate the conjecture that the restriction of conformal flatness is not essential for the result. This, being true, would provide a twisted version of Burago-Ivanov

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theorem.

In the situation we consider, it would be reasonable to think that there always exist a periodic orbit with conjugate points. This fact is, however, much more difficult to approach. It is not proved to the best of our knowledge even in the usual untwisted case. Moreover, it seems that the fact of existence of periodic orbit is still not completely answered for arbitrary magnetic field on the torus. We refer the reader to a survey paper by V. Ginzburg [12] for known results and techniques.

2 Main Results

Magnetic field on a torus $\mathbb{T}^n$ is a closed 2-form which we will denote by β. Consider the cotangent bundle $T^*\mathbb{T}^n$ and define the symplectic structure twisted by β.

$$\omega = \omega_0 + \pi^*\beta$$

where ω_0 is a standard structure and π is the canonical projection.

For a Riemannian metric g on $\mathbb{T}^n$ consider the Hamiltonian flow g^t of the function H

$$H = \frac{1}{2} < p, p >_g$$

computed with the help of the symplectic structure ω. Let me remind that no conjugate points condition for the orbit ϑ of g^t means that for any two points $x, y \in \vartheta$

$$g_*^t V(x) \pitchfork V(y)$$

where t is a time difference between y and x and $V(x)$, $V(y)$ are vertical subspaces at x, y.

Theorem 1 *Suppose that the Riemannian metric g is conformally flat. Then there always exist orbits of g^t on the level of $\{H = 1/2\}$ with conjugate points unless the 2-form β vanishes identically.*

Remark 1 *In other words, no conjugate points condition implies $\beta \equiv 0$ and then it follows from [6] and [13] that the metric is flat too. Actually, we will see it once more by our computations later.*

The proof which is suggested in this paper follows the original scheme by E. Hopf. However, it might need serious modifications if one tries to generalize the result for arbitrary Riemannian metric.

The first ingredient of the proof is to construct measurable field of Lagrangian subspaces $l(x) \subseteq T_x(T^*\mathbb{T}^n)$ for any $x \in \{H = 1/2\}$. This field can be constructed by the following limit procedure:

$$l(x) = lim_{t\to+\infty} g_*^t V\left(g^{-t}(x)\right)$$

It was first used by E. Hopf [9] and L. Green [11] for Riemannian case. We refer the reader to recent paper [8] for the proof in a general optical case.

It follows from the very construction of l that the field l is invariant under g^t and is transversal to the vertical field V everywhere.

With the construction of l Theorem 1 is a corollary of the following

Theorem 2 *Let l be a measurable field of Lagragian subspaces invariant under the flow g^t. If l is transversal to V everywhere then β vanishes identically, and the metric g is flat.*

This last theorem has the following dynamical interpretation.

Theorem 3 *Suppose that the energy shell $\{H = 1/2\}$ is smoothly foliated by Lagrangian tori homologous to the zero section of $T^*\mathbb{T}^n$. Then the 2-form β vanishes identically and the metric is flat.*

Remark 2 *In the paper [13], such a situation is called total integrability and simple examples of totally integrable magnetic geodesic flows are given. Let me remark that the Lagrangian torii in all these examples cannot be homologous to the zero section as it follows from Theorem 3. It is an interesting, completely open problem to characterise totally integrable magnetic geodesic flows.*

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3 Proofs

Let me explain first how Theorem 3 follows from Theorem 2. Since the torii are Lagrangian and homologous to the zero section then the 2-form β must be exact. Denote by α the primitive 1-form. Then one can easily see that the flow g^t is equivalent to the Hamiltonian flow $\tilde{g}^t$ of the function $\tilde{H} = 1/2 < p - \alpha, p - \alpha >_g$ with respect to the standard structure ω_0. This equivalence is given by the diffeomorphism $(q, p) \to (q, p + \alpha)$ which is fiber preserving.

Note that the function $\tilde{H}$ is strictly convex with respect to p. It follows from generalised Birkhoff theorem (see [3] for its most general form and for the survey and discussions) that all the Lagrangian torii are the sections of the cotangent bundle. But then the distribution of their tangent spaces meets the conditions of Theorem 2.

Proof of Theorem 2

We shall work in standard coordinates (q, p) on $T^*\mathbb{T}^n$, such that the Riemannian metric g is given by

$$ds^2 = \frac{1}{2\lambda}\left(dq_1^2 + \cdots + dq_n^2\right).$$

Then

$$H(p, q) = \frac{\lambda}{2}\left(p_1^2 + \cdots + p_n^2\right) \tag{3.1}$$

Write $\beta = d\alpha + \gamma$, where α is a 1-form and γ is a 2-form, having constant coefficients in the coordinates $(q_1 \cdots q_n)$, $\gamma = \Sigma_{i<j}\, \gamma_{ij}\, dq_i \;\wedge\; dq_j$. By the change of coordinates

$$(q, p) \to (q, p + \alpha)$$

we have an equivalent Hamiltonian flow $\tilde{g}^t$ of

$$\tilde{H}(p, q) = H(p - \alpha, q) = \frac{1}{2}\lambda\left((p_1 - \alpha_1)^2 + \cdots + (p_n - \alpha_n)^2\right)$$

with respect to the symplectic form

$$\tilde{\omega} = \omega_0 + \gamma = \Sigma_{i=1}^n dp_i \wedge dq_i + \Sigma_{i<\gamma}\gamma_{ij}\, dq_i \;\wedge\; dq_j.$$

Denote by $\tilde{l}$ the invariant distribution of Lagrangian subspaces with respect to $\tilde{\omega}$. At any point $x = (q, p) \in \{\tilde{H} = 1/2\}$ it is given by a matrix $A(x)$:

$$dp = A(p, q)dq$$

The condition to be Lagrangian is equivalent to

$$A^T - A = \Gamma$$

where Γ is the skew symmetric matrix of γ. Then A is a measurable matrix function and satisfies the following Ricatti equation along the flow $\tilde{g}^t$.

$$\dot{A} + (A + \Gamma)\tilde{H}_{pp}A + (A + \Gamma)\tilde{H}_{pq} + \tilde{H}_{qp}A + \tilde{H}_{qq} = 0$$

where $\tilde{H}_{pp}$, $\tilde{H}_{pq}$, $\tilde{H}_{qq}$ are the matrices of second derivatives of $\tilde{H}$. Then it can be written in the form

$$\begin{aligned}\dot{A} \;&+\; \left((A + \Gamma)\tilde{H}_{pp}^{1/2} + \tilde{H}_{qp}\tilde{H}_{pp}^{-1/2}\right)\left(\tilde{H}_{pp}^{1/2}A + \tilde{H}_{pp}^{-1/2}\tilde{H}_{pq}\right) \\ &+\; \left(\tilde{H}_{qq} - \tilde{H}_{qp}\tilde{H}_{pp}^{-1}\tilde{H}_{pq}\right) = 0\end{aligned}$$

Introduce the function $a(p,q) = trA(p,q)$. Then it follows

$$\dot{a} + tr\left(\tilde{H}_{qq} - \tilde{H}_{qp}\tilde{H}_{pp}^{-1}\tilde{H}_{pq}\right) \leq 0$$

with equality possible only when $A = -\tilde{H}_{pp}^{-1}\tilde{H}_{pq}$. Integrate now this inequality with respect to the invariant measure $\tilde{\mu}$ on the energy level $\tilde{H} = 1/2$. We get the following

$$\sigma(\tilde{H}) = \int tr\left(\tilde{H}_{qq} - \tilde{H}_{qp}\tilde{H}_{pp}^{-1}\tilde{H}_{pq}\right) d\tilde{\mu} \leq 0 \tag{3.2}$$

with equality possible only for $A = -\tilde{H}_{pp}^{-1}\tilde{H}_{pq}$ everywhere on the level $\{\tilde{H} = 1/2\}$. On the other hand, we shall compute $\sigma(\tilde{H})$ by the following:

Lemma 1 *For any Hamiltonian function $H(p,q)$ convex and symmetric in p $H(p,q) = H(-p,q)$ it follows that $\sigma(\tilde{H}) = \sigma(H)$ where $\tilde{H}(p,q) = H(p-\alpha(q),q)$ for any 1-form α. Here $\sigma(H)$ is computed by the integral (3.2) with respect to the hamiltonian H and the invariant measure μ of g^t.*

Lemma 2 *If H is given by (3.1) then $\sigma(H) \geq 0$ and equality is achieved only when $\lambda = const$.*

Remark 3 *The last Lemma is actually the inequality proved in [13], but here it appears in the Hamiltonian version.*

Let me complete the proof of the theorem, postponing ones of the Lemmas.

Comparing (3.2) with Lemmas 1 and 2, one concludes that the metric is Euclidean and

$$A = -\tilde{H}_{pp}^{-1}\tilde{H}_{pq}, \quad \text{where} \quad \tilde{H} = \frac{1}{2}\Sigma_{i=1}^{n}\left(p_i - \alpha_i\right)^2$$

Then

$$A = (A_{ij}) = -\left(\tilde{H}_{p_i q_j}\right) = \left(\frac{\partial \alpha_i}{\partial q_j}\right) \tag{3.3}$$

Compute and plug the second derivatives in the Ricatti equation. We have

$$(tr\dot{A}) = -tr\left(\tilde{H}_{qq} - \tilde{H}_{qp}\tilde{H}_{pp}^{-1}\tilde{H}_{pq}\right) = \Sigma_{ij=1}^{n}\left(p_j - \alpha_j\right)\frac{\partial^2 \alpha_j}{\partial q_i^2}$$

Differentiating trA explicitly along the flow $\tilde{g}^t$, one gets another expression

$$(tr\dot{A}) = \Sigma_{j=1}^{n}\frac{\partial(trA)}{\partial q_j}\tilde{H}_{p_j} = \Sigma_{ij=1}^{n}(p_j - \alpha_j)\frac{\partial^2 \alpha_i}{\partial q_i\,\partial q_j}$$

Comparing these two expression we obtain

$$\frac{\partial(trA)}{\partial q_j} = \Delta\alpha_j \quad \text{for} \quad j = 1, \dots n.$$

Then the compatibility condition gives

$$\Delta\left(\frac{\partial\alpha_k}{\partial q_j} - \frac{\partial\alpha_j}{\partial q_k}\right) = 0 \quad \text{for any} \quad j, k = 1, \dots n.$$

This implies

$$\frac{\partial\alpha_k}{\partial q_j} - \frac{\partial\alpha_j}{\partial q_k} \equiv const = 0 \quad \text{and thus} \quad d\alpha = 0.$$

Also by (3.3) we have $A^T - A = \Gamma = 0$. So $\beta = d\alpha + \gamma = 0$. This completes the proof of the theorem. □

Proof of Lemma 1

We have to show that σ is the same for H and $\tilde{H} = H(p-\alpha(q), q)$. The direct computation gives the following expressions for the matrices of second derivatives.

$$\begin{aligned}
\tilde{H}_{pp} &= H_{pp} \\
\tilde{H}_{pq} &= -H_{pp}D\alpha + H_{pq}, \quad \tilde{H}_{qp} = -(D\alpha)^T H_{pp} + H_{qp} \\
\tilde{H}_{qq} &= (D\alpha)^T H_{pp}(D\alpha) - H_{qp}D\alpha - (D\alpha)^T H_{pq} + H_{qq} + M \\
\text{where} \quad M &= (M_{ij}), \quad M_{ij} = \Sigma_{k=1}^n H_{p_k} \frac{\partial^2\alpha_k}{\partial q_i\, \partial q_j}
\end{aligned}$$

Thus we have after performing change of variables $(q, p) \to (q, p - \alpha)$

$$\sigma(\tilde{H}) = \sigma(H) + \int_{\left\{H=\frac{1}{2}\right\}} \Sigma_{k,i=1}^n H_{p_k} \frac{\partial^2\alpha_k}{\partial q_i^2} d\mu$$

By the symmetry assumption this additional integral vanishes because the integrand is odd function and the level is even. This completes the proof of the lemma. □

Proof of Lemma 2

The second derivative for the Hamiltonian $H = \lambda/2\,(p_1^2 + \cdots + p_n^2)$ are given by the formulas

$$H_{p_ip_j} = \lambda\delta_{ij}, \quad H_{q_iq_j} = \frac{1}{2}(\lambda)_{q_iq_j}\left(p_1^2 + \cdots + p_n^2\right), \quad H_{p_iq_j} = \lambda_{q_j}p_i$$

Thus for the points of the energy level $\left\{H = \frac{1}{2}\right\}$

$$tr\left(H_{qq} - H_{qp}H_{pp}^{-1}H_{pq}\right) = \frac{1}{2\lambda}\Delta\lambda - \frac{1}{2\lambda^2}(grad\lambda)^2$$

The invariant measure μ on $\left\{H = \frac{1}{2}\right\}$ is computed from the condition

$$d\mu \wedge dH = \omega_0^n$$

and then can be easily computed and equals

$$d\mu = \left(\frac{1}{\sqrt{\lambda}}\right)^n d\sigma dq$$

where $d\sigma$ is a standard measure on unit sphere in p space. Then we have

$$\sigma(H) = Vol(S^{n-1})\int\left(\frac{1}{2\lambda}\Delta\lambda - \frac{1}{2\lambda^2}(grad\lambda)^2\right)\left(\frac{1}{\sqrt{\lambda}}\right)^n dq$$

Integrating by parts we obtain

$$\sigma(H) = Vol(S^{n-1})\left(\frac{n-2}{4}\right)\int\lambda^{-2-\frac{n}{2}}(grad\lambda)^2 dq$$

From this expression the assertion follows. □

4 Some Open Problems

Let me discuss here some natural open problems.

In the proof of Theorem 3 we noticed first that the form β must be exact and then showed that it vanishes. It may happen that the condition of existence of invariant distribution of Lagrangian non-vertical subspaces also implies that the form β is exact. We were not able, however, to find a direct proof of this fact.

It is an important problem to study those magnetic fields having integrable twisted geodesic flow. The simplest possible topology of the phase portrait is one for the totally integrable flows. It is not clear how to characterise those flows even for dimension 2. One natural class of examples is for S^1 symmetric magnetic field $\beta = \beta(q_1)dq_1 \wedge dq_2$. We don't know if there are other examples. Note that the Lagrangian torii of the foliation can not be homologous to the zero section (theorem 3) .

One would like to generalise our results to the case of any Riemannian metric on $\mathbb{T}^n$. However, it may happen that twisted version of Burago-Ivanov proof should be found. It should be mentioned that our approach works each time when it is related to a good choice of coordinates in the phase space. It is not clear to me however, if the coordinates of [5] are good for our approach.

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