Geometric Modular Action, Wedge Duality and Lorentz Covariance are Equivalent for Generalized Free Fields

Johanna Gaier[1] and Jakob Yngvason[2]

1. Abteilung für Finanz- und Versicherungsmathematik, TU Wien, Wiedener Hauptstraße 8–10, A-1040 Vienna, Austria

2. Institut für Theoretische Physik, Universität Wien Boltzmanngasse 5, A-1090 Vienna, Austria

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Abstract

The Tomita-Takesaki modular groups and conjugations for the observable algebras of space-like wedges and the vacuum state are computed for translationally covariant, but possibly not Lorentz covariant, generalized free quantum fields in arbitrary space-time dimension d. It is shown that for $d \geq 4$ the condition of geometric modular action (CGMA) of Buchholz, Dreyer, Florig and Summers [BDFS], Lorentz covariance and wedge duality are all equivalent in these models. The same holds for $d = 3$ if there is a mass gap. For massless fields in $d = 3$, and for $d = 2$ and arbitrary mass, CGMA does not imply Lorentz covariance of the field itself, but only of the maximal local net generated by the field.

1 Introduction

The importance of Tomita-Takesaki modular theory for both structural analysis and constructive aspects of quantum field theory has been amply manifested by important publications in recent years. We refer to [B1],[Sch] and [BFS] for extensive lists of references on this subject. The Bisognano-Wichmann theorem ([BW1], [BW2]), proved already in 1975, is the basic insight on which these developments are founded. It provides a geometrical interpretation of the modular objects associated with algebras generated by Poincaré covariant Wightman field operators localized in space-like wedges.

In 1992 Borchers [B2] discovered an important partial converse to this theorem. He showed that in two space-time dimensions the modular objects associated with a translationally covariant local net of von Neumann algebras and a vacuum state lead to a representation of the Poincaré group, even if no Lorentz covariance of the net is required at the outset. (See also [F] for a simplified proof of Borchers' theorem.) Such a geometrical interpretation of the modular objects is not always possible in higher dimensions, however, as can be seen from examples given in [Y].

By *postulating* a certain form of geometric action of the modular conjugations associated with space-like wedges and a given state ("Condition of geometric modular action") Buchholz et al. [BDFS] were able to construct a representation of the Poincaré group in space-time dimension 4 without even assuming translational invariance. The essence of the CGMA is the requirement that the modular conjugation of every wedge leaves the family of all wedge algebras invariant. As shown in [BFS] the spectrum condition for the translations follows from the additional requirement that the group generated by the conjugations contains the modular groups of the wedge algebras ("Modular stability condition"). Such a purely algebraic characterization of vacuum states has the potential for generalizations to a stability condition for quantum fields on curved space-times. Other important results relying on geometric actions of modular groups have been obtained, e.g., in [BGL], [G], [GL].

As a contribution to the understanding of the possible modular actions in quantum field theory when the Bisognano-Wichmann theorem does not apply we compute in this note the modular groups and the modular conjugations associated with the wedge algebras generated by translationally covariant generalized free quantum fields in arbitrary space-time dimension d. Such a computation was carried out in [Y] for two dimensional fields depending only on one light coordinate, and certain special cases in higher dimensions. Here we treat the general case (for single component, hermitian fields).

We investigate the geometrical significance of the modular objects, and in particular we answer the question when the adjoint action of the modular conjugation associated with a wedge algebra leaves the set of all wedge algebras invariant. We show that in $d \geq 4$ this is the case if and only if the two point function defining the field is Lorentz invariant. In fact, Lorentz invariance follows already from wedge duality for the field, i.e., if the algebra of a wedge is the commutant of the algebra of the opposite wedge, which is a consequence of CGMA cf. Prop. 4.3.1 in [BDFS]. Besides the explicit formulas for the modular objects this result is based on a lemma concerning the zeros of polynomials restricted to a mass shell (Lemma 5.2). The same conclusion can be drawn from the requirement that the modular groups act locally, i.e., transform observables localized in a bounded region into observables localized in another bounded region.

In view of the general result of [BDFS] the Lorentz covariance of fields satisfying CGMA is not a surprise, but it is important to note that this is not a consequence of [BDFS] alone. The point is that the same wedge algebras could *a priori* be generated by different fields and not all of them might be Lorentz covariant. In fact, the wedge algebras for a massless free field in $d = 3$ can be generated by certain derivatives of the field that do break Lorentz invariance. In $d = 3$, however, this massless case is the only exception: if there is a mass gap, then CGMA implies Lorentz covariance of the field. In $d = 2$ also massive fields without Lorentz covariance can fulfill CGMA. In the cases where Lorentz covariance of the field is broken but CGMA holds the minimal local net generated by the field operators is not Lorentz covariant, in contrast to the maximal net defined by intersections of wedge algebras which is strictly larger in these cases.

The bottom line is that in $d \geq 4$ the following conditions are all equivalent for the models considered: a) CGMA, b) wedge duality, c) local action of the modular groups d) Lorentz covariance of the field. This equivalence holds also for $d = 3$, provided there is a mass gap.

2 Definition of the Models

We consider a Hermitian Wightman field Φ that transforms covariantly under space-time translations, but not necessarily under Lorentz transformations. The general structure of the 2-point-function $\mathcal{W}_2(x-y) = \langle \Omega, \Phi(x)\Phi(y)\Omega \rangle$, where Ω denotes the vacuum state, follows from the Jost-Lehmann-Dyson representation (cf. e.g. [B3]); its Fourier transform can be written

$$\widetilde{\mathcal{W}}_2(p) = \int_0^\infty M(p, m^2)\Theta(p_0)\delta(p^2 - m^2)\,\mathrm{d}\rho(m^2) \tag{1}$$

where the Lehmann weight $\mathrm{d}\rho$ is a positive, tempered measure on $\mathbb{R}_+$ and $M(p, m^2)$ is for fixed m an even polynomial in $p = (p_0, \ldots, p_{d-1}) \in \mathbb{R}^d$, i.e.,

$$M(p, m^2) = M(-p, m^2), \tag{2}$$

with

$$M(p, m^2) \geq 0 \quad \text{for} \quad p \in H_m^+ := \{p \in \mathbb{R}^d : p^2 - m^2 = 0, p_0 \geq 0\} \tag{3}$$

and $\mathrm{d}\rho$-almost all m^2.

The Hilbert space of the field is the symmetric Fock space over the "one-field-space" $\mathcal{H}^{(1)}$, which is the L^2 space corresponding to the positive measure $\widetilde{\mathcal{W}}_2(p)\mathrm{d}^d p$ on the forward light cone $\mathrm{V}^+ = \{p \in \mathbb{R}^d : p^2 \geq 0, p_0 \geq 0\}$. We shall make use of the decomposition of $\mathcal{H}^{(1)}$ as a direct integral

$$\mathcal{H}^{(1)} = \int^\oplus \mathcal{H}_m^{(1)}\,\mathrm{d}\rho(m^2), \tag{4}$$

where $\mathcal{H}_m^{(1)} = L^2(\mathbb{R}^d, M(p, m^2)\Theta(p_0)\delta(p^2 - m^2)d^d p)$.

The smeared field operators $\Phi(f) = \int \Phi(x) f(x) dx$ are defined not only for test functions $f \in \mathcal{S}(\mathbb{R}^d)$, but also for distributions $f \in \mathcal{S}(\mathbb{R}^d)'$ such that the Fourier transform $\tilde{f}$ belongs to the L^2 space with respect to the measure $(\mathcal{W}_2(p) + \mathcal{W}_2(-p))\mathrm{d}p$ on $\mathrm{V} = \mathrm{V}^+ \cup -\mathrm{V}^+$. For real f, the field operator $\Phi(f)$ is uniquely determined by the restriction $\tilde{f}|_{\mathrm{V}+}$. It is a self adjoint operator on a natural domain in the Fock space and we may consider the unitary Weyl operators $W(f) = \exp \mathrm{i}\Phi(f)$. They satisfy the relation

$$W(f)W(g) = e^{-K(f,g)/2} W(f+g) \tag{5}$$

with

$$K(f,g) = \int \left(\widetilde{\mathcal{W}}_2(p) - \widetilde{\mathcal{W}}_2(-p) \right) \tilde{f}(-p) \tilde{g}(p) \mathrm{d}^d p. \tag{6}$$

Moreover,

$$\langle \Omega, W(f)\Omega \rangle = \exp \left(- \textstyle\int \widetilde{\mathcal{W}}_2(p) \tilde{f}(-p) \tilde{f}(p) dp \right). \tag{7}$$

If $\mathcal{O}$ is a subset of Minkowski space $\mathbb{R}^d$ we can define the following subspace of $\mathcal{H}_m^{(1)}$

$$\mathcal{H}_m^{(1)}(\mathcal{O}) := \text{closure of } \left\{ \tilde{g}|_{H_m^+} : \ g \in \mathcal{S}(\mathbb{R}^d), \ \operatorname{supp} g \subset \mathcal{O} \right\}. \tag{8}$$

We define the local algebra $\mathcal{M}(\mathcal{O})$ as the von Neumann algebra generated by the Weyl operators $W(f)$ (with real $f \in \mathcal{S}^1(\mathbb{R}^d)'$) such that

$$\tilde{f}|_{\mathrm{V}+} \in \int^{\oplus} \mathcal{H}_m^{(1)}(\mathcal{O}) \, \mathrm{d}\rho(m^2). \tag{9}$$

We remark that if the Lehmann weight does not decrease rapidly at infinity then $\mathcal{M}(\mathcal{O})$ can be larger than the algebra generated by the Weyl operators $W(f)$ with $\operatorname{supp} f \subset \mathcal{O}$, cf. [L]. This possibility, however, is independent of the issues of interest here. Our definition of $\mathcal{M}(\mathcal{O})$ simplifies things because it allows a complete reduction to the case of fixed mass.

If $\mathcal{O}$ is a fixed open subset of $\mathbb{R}^d$ such that its causal complement $\mathcal{O}'$ has a nonempty interior, then Ω is cyclic and separating for $\mathcal{M}(\mathcal{O})$ and we may consider the corresponding modular group $\Delta^{\mathrm{i}t}$ and modular conjugation J. Both are the second quantization of their restrictions to the one-field space $\mathcal{H}^{(1)}$ and we denote these restrictions by by $\delta^{\mathrm{i}t}$ and j respectively. Moreover, by our definition of $\mathcal{M}(\mathcal{O})$, we have a direct integral decomposition of these objects:

$$\delta^{\mathrm{i}t} = \int^{\oplus} \delta_m^{\mathrm{i}t} \, \mathrm{d}\rho(m^2), \qquad j = \int^{\oplus} j_m \, \mathrm{d}\rho(m^2). \tag{10}$$

Here $\delta_m^{\mathrm{i}t}$ and j_m the restrictions to the one-field space $\mathcal{H}_m^{(1)}$ of the modular objects for the field with 2-point-function

$$\widetilde{\mathcal{W}}_{2,m}(p) = M(p, m^2) \, \Theta(p_0) \, \delta(p^2 - m^2). \tag{11}$$

It is therefore sufficient to compute the modular objects for a fixed mass and we shall in the sequel drop the index m. We shall also write $M(p, m^2)$ simply as $M(p)$.

3 Computation of the modular objects for wedge algebras

We shall now compute δ^{it} and j for the field with two point function (11) and $\mathcal{O}$ a space like wedge W. Since the field is translationally covariant and general polynomials M are allowed in (11) it is sufficient to do this for some standard wedge. We choose for this purpose the “right wedge”

$$W_{\mathrm{R}} = \{x = (x_0, \ldots, x_{d-1}) \in \mathbb{R}^d : |x_0| < x_1\} \tag{12}$$

The modular objects to this wedge will be denoted δ_{R}^{it} and j_{R}. If Λ is a Lorentz transformation, then the modular objects for the wedge $W = \Lambda W_{\mathrm{R}}$ are the same as for W_{R}, but with the polynomial $M_\Lambda(p) := M(\Lambda^{-1}p)$ instead of M.

We introduce the light cone coordinates $p_\pm := p_0 \pm p_1$, and write the remaining components of p as $\hat{p} := (p_2, \ldots, p_{d-1})$. The two point function (11) can then be written as

$$\begin{aligned} \widetilde{\mathcal{W}}_2(p) &= M(p_+, p_-, \hat{p})\, \Theta(p_+)\, \delta(p_+ \cdot p_- - \hat{p}^2 - m^2) && (13)\\ &= p_+^{-1} M(p_+, p_+^{-1}(\hat{p}^2 + m^2), \hat{p})\, \Theta(p_+)\, \delta\left(p_- - p_+^{-1}(\hat{p}^2 + m^2)\right). && (14) \end{aligned}$$

Moreover, since M is a polynomial we can write

$$M(p_+, p_+^{-1}(\hat{p}^2 + m^2), \hat{p}) = p_+^{-2n} Q(p_+, \hat{p}) \tag{15}$$

with some $n \in \mathbb{N} \cup \{0\}$ and a polynomial $Q(p_+, \hat{p})$. The properties of M imply that Q satisfies

$$Q(p_+, \hat{p}) = Q(-p_+, -\hat{p}) \quad \text{and} \quad Q(p_+, \hat{p}) \geq 0. \tag{16}$$

We now consider Q as a polynomial in p_+, with coefficients that are polynomials in $\hat{p}$. Its zeros are algebraic functions of $\hat{p}$, and the properties (16) entail that every real zero $r_j(\hat{p})$ of $Q(\cdot, \hat{p})$ must be a double zero and every complex zero $z_k(\hat{p})$ comes together with its complex conjugate $z_k(\hat{p})^*$. Moreover, each real zero $r_j(\hat{p})$ is accompanied by a zero $-r_j(-\hat{p})$ and every complex zero $z_k(\hat{p})$ by $-z_k(-\hat{p})$.

All in all we can write

$$\widetilde{\mathcal{W}}_2(p) = \frac{1}{p_+} F(p_+, \hat{p})\, F(-p_+, -\hat{p})\, \Theta(p_+)\, \delta\left(p_- - \frac{\hat{p}^2 + m^2}{p_+}\right), \tag{17}$$

with

$$F(p_+, \hat{p}) = \frac{1}{(ip_+)^n} \cdot \prod_{j=1}^{J} (p_+ - r_j(\hat{p}))(p_+ + r_j(-\hat{p})) \prod_{k=1}^{K} (p_+ + z_k(\hat{p}))(p_+ - z_k(-\hat{p})^*), \tag{18}$$

where $r_j(\hat{p}) \in \mathbb{R}$ and $z_k(\hat{p}) \in \mathbb{C}$, $\mathrm{Im}\, z_k(\hat{p}) > 0$. Thus F has all the complex zeros of Q in the lower half plane and no zeros in the upper half plane, while

$$F(-p_+, -\hat{p}) = F(p_+, \hat{p})^* \tag{19}$$

has no zeros in the lower half plane. The real zeros of Q are evenly divided between $F(p_+, \hat{p})$ and $F(-p_+, -\hat{p})$.

We shall now give explicit formulas for $\delta_{\mathrm{R}}^{\mathrm{it}}$ and j_{R}. Note that every $\varphi \in \mathcal{H}_m^{(1)}$ can be regarded as a function of $p_+ > 0$ and $\hat{p} \in \mathbb{R}^{d-2}$, since on the mass shell $p_- = p_+^{-1}(\hat{p}^2 + m^2)$.

3.1. THEOREM. *On the one particle space $\mathcal{H}^{(1)}$ the modular group associated with $\mathcal{M}(W_{\mathrm{R}})$ and Ω has the form*

$$(\delta_{\mathrm{R}}^{\mathrm{it}}\varphi)(p_+, \hat{p}) = \frac{F(e^{2\pi t}p_+, \hat{p})}{F(p_+, \hat{p})}\varphi(e^{2\pi t}p_+, \hat{p}) \tag{20}$$

where F is given by (18). The corresponding modular conjugation is

$$(j_{\mathrm{R}}\varphi)(p_+, \hat{p}) = \frac{F(-p_+, \hat{p})}{F(p_+, \hat{p})}\varphi(p_+, -\hat{p})^* \tag{21}$$

Proof. One can easily check that $\delta_{\mathrm{R}}^{\mathrm{it}}$ is unitary for all t and j_{R} is anti-unitary. The same holds then for the second quantized operators $\Delta_{\mathrm{R}}^{\mathrm{it}}$ and J_{R}. To show that $\Delta_{\mathrm{R}}^{\mathrm{it}}$ is indeed the modular group associated with the vacuum state on $\mathcal{M}(W_{\mathrm{R}})$ it is necessary to verify that $\sigma_t := \mathrm{ad}\Delta_{\mathrm{R}}^{\mathrm{it}}$ defines an automorphism group of $\mathcal{M}(W_{\mathrm{R}})$ and that the KMS condition

$$\langle \Omega, \sigma_t W(f)W(g)\Omega\rangle = \langle \Omega, W(f)\sigma_{t-\mathrm{i}}W(g)\Omega\rangle \tag{22}$$

holds for Weyl operators localized in W_{R}.

By Eq. (20), (5) and (7) the action of $\Delta_{\mathrm{R}}^{\mathrm{it}}$ on the Weyl operators is

$$\Delta_{\mathrm{R}}^{\mathrm{it}}W(f)\Delta_{\mathrm{R}}^{-it} = W(f_t) \tag{23}$$

with

$$\tilde{f}_t(p_+, p_-, \hat{p}) = \frac{F(e^{2\pi t}p_+, \hat{p})}{F(p_+, \hat{p})}\tilde{f}(e^{2\pi t}p_+, e^{-2\pi t}p_-, \hat{p}). \tag{24}$$

(Note that on the positive and negative mass shells $p_+p_- = \hat{p}^2 + m^2$.) Test functions f with support in W_{R} are characterized by analyticity and decay properties of the Fourier transform $\tilde{f}$: For fixed $\hat{p}$, $\tilde{f}$ is analytic in

$$\mathcal{T}_{\mathrm{R}} = \{(p_+, p_-) \in \mathbb{C}^2 : \mathrm{Im}\ p_+ > 0,\ \mathrm{Im}\ p_- < 0\} \tag{25}$$

and decays rapidly at infinity in this domain. The same conditions apply if f is a distribution w.r.t. the light cone variables $x_\pm$, but $\tilde{f}$ may increase like an inverse polynomial as p_+ or p_- approach the real axis. Since F has no zeros in p_+ in the open upper half plane, it is evident that f_t satisfies these conditions if f does. Hence the group $\mathrm{ad}\Delta_{\mathrm{R}}^{\mathrm{i}t}$ leaves $\mathcal{M}(W_{\mathrm{R}})$ invariant.

The KMS condition can be verified by essentially the same computation as the corresponding statement for fields on a light ray in [Y].

To show that (21) is the modular conjugation we note first that the set of state vectors $\varphi \in \mathcal{H}^{(1)}$, such that $\varphi = \tilde{f}|_{H_m^+}$ with $f \in \mathcal{S}(\mathbb{R}^d)$ and $\mathrm{supp}\, f \in W_{\mathrm{R}}$, is a core for the restriction s to $\varphi \in \mathcal{H}^{(1)}$ of the S operator corresponding to $\mathcal{M}(W_{\mathrm{R}})$ and Ω. The latter is defined by $SW(f)\Omega = W(f)^*\Omega$ for $\mathrm{supp}\, f \in W_{\mathrm{R}}$. Such φ are analytic in p_+ in the upper half plane, and

$$\delta_{\mathrm{R}}^{1/2}\varphi(p_+,\hat{p}) = \frac{F(-p_+,\hat{p})}{F(p_+,\hat{p})}\varphi(-p_+,\hat{p}) \tag{26}$$

by analytic continuation of (20) to $t = -\mathrm{i}\pi/2$. On the other hand,

$$s\varphi(p_+,\hat{p}) = \varphi(-p_+,-\hat{p})^* \tag{27}$$

Using (19) we see that j_{R} satisfies

$$s = j_{\mathrm{R}}\delta_{\mathrm{R}}^{1/2} \tag{28}$$

as required for the modular conjugation. □

4 Duality and modular action for a fixed wedge

As next topic we discuss duality and the geometrical significance of the modular objects for the right wedge. In particular we compare them with the corresponding objects for the left wedge

$$W_{\mathrm{L}} = \{x = (x_0,\ldots,x_{d-1}) \in \mathbb{R}^d : |x_0| < -x_1\}. \tag{29}$$

By an analogous computation as for the right wedge these are given by

$$(\delta_{\mathrm{L}}^{\mathrm{i}t}\varphi)(p_+,\hat{p}) = \frac{F(-e^{-2\pi t}p_+,-\hat{p})}{F(-p_+,-\hat{p})}\varphi(e^{-2\pi t}p_+,\hat{p}) \tag{30}$$

and

$$(j_{\mathrm{L}}\varphi)(p_+,\hat{p}) = \frac{F(p_+,-\hat{p})}{F(-p_+,-\hat{p})}\varphi(p_+,-\hat{p})^*. \tag{31}$$

Wedge duality for the left and right wedge, i.e., $\mathcal{M}(W_{\mathrm{R}})' = \mathcal{M}(W_{\mathrm{L}})$, holds if and only if the modular conjugations coincide, i.e., $j_{\mathrm{R}} = j_{\mathrm{L}}$. By (21) and (31) the condition for this is

$$\frac{F(-p_+,\hat{p})}{F(p_+,\hat{p})} = \frac{F(p_+,-\hat{p})}{F(-p_+,-\hat{p})}, \tag{32}$$

which by (19) can be written

$$F(p_+,-\hat{p})F(p_+,\hat{p}) = F(p_+,-\hat{p})^*F(p_+,\hat{p})^*. \tag{33}$$

Since $F(p_+,\pm\hat{p})$, regarded as a function of p_+, has all its complex zeros in the lower half plane, we see that this holds if and only if F has no complex zeros at all.

Let us now consider the geometric action of the modular conjugation. If M has only real zeros in p_+, then duality holds and hence $J_{\rm R}\mathcal{M}(W_{\rm R})J_{\rm R} = \mathcal{M}(W_{\rm L})$. A complex zero, on the other hand, implies that the pre factor $F(-p_+,\hat{p})/F(p_+,\hat{p})$ in the definition of $j_{\rm R}$ is not analytic in the lower half plane. Hence in general $j_{\rm R}\varphi$ is not analytic in p_+ in the lower half plane for $\varphi \in \mathcal{H}^{(1)}(W_{\rm R})$. This implies that $j_{\rm R}\varphi$ is in general not contained in $\mathcal{H}^{(1)}(W)$ for any wedge of the form of $W_{\rm L}+a$, $a \in \mathbb{R}^d$, and hence $J_{\rm R}\mathcal{M}(W_{\rm R})J_{\rm R}$ is not contained in any translate of $\mathcal{M}(W_{\rm L})$. A localization of $J_{\rm R}\mathcal{M}(W_{\rm R})J_{\rm R}$ in any other wedge algebra is excluded since for general $\varphi \in \mathcal{H}^{(1)}(W_{\rm R})$, $\varphi(p_+,-\hat{p})^*$ has no further analyticity properties beyond the analytic continuation in p_+ to the lower half plane which follows from the localization of φ in $W_{\rm R}$.

We summarize these findings as follows.

4.1. PROPOSITION. *The following are equivalent*

(i) $\mathcal{M}(W_{\rm R})' = \mathcal{M}(W_{\rm L})$

(ii) $J_{\rm R}\mathcal{M}(W_{\rm R})J_{\rm R}$ *is contained in* $\mathcal{M}(W)$ *for some wedge* W.

(iii) The rational function

$$p_+ \mapsto M(p_+, p_+^{-1}(\hat{p}^2+m^2),\hat{p}) \tag{34}$$

has only real zeros, for all $\hat{p} \in \mathbb{R}^{d-2}$.

Our last concern in this section is the local action of the modular group. The general theorem of Borchers [B2] implies that translates of $\mathcal{M}(W_{\rm R})$ are mapped onto algebras of the same type:

$$\Delta_{\rm R}^{\rm it}\mathcal{M}(W_{\rm R}+a)\Delta_{\rm R}^{-\rm it} = \mathcal{M}(W_{\rm R}+\Lambda(t)a) \tag{35}$$

for all $a \in \mathbb{R}^d$, with $\Lambda(t)$ a Lorentz boost. Observables localized in bounded domains, however, are in general not localized in a bounded domain after transformation by ${\rm ad}\Delta_{\rm R}^{\rm it}$. In fact, if $\mathcal{O}$ is bounded, then $\varphi \in \mathcal{H}^{(1)}(\mathcal{O})$ is the restriction to the mass shell of an entire analytic function. This analyticity is in general destroyed by the pre factor $F(e^{2\pi t}p_+,\hat{p})/F(p_+,\hat{p})$, unless $F(p_+,\hat{p})$ and $F(e^{2\pi t}p_+,\hat{p})$ have the same set of zeros. This holds only if $M(p_+,p_+^{-1}(\hat{p}^2+m^2),\hat{p})$ has the form $p_+^{2n}C(\hat{p})$ for some $n \in \mathbb{Z}$. If M is independent of $\hat{p}$, then there is at least no dislocalization in the directions along the edge of the wedge, but the example $M(p) = p_0^2$ mentioned in [Y] (this corresponds to the time derivative of the free field) has $F(p_+,\hat{p}) = (2{\rm i}p_+)^{-2}(p_+ + {\rm i}(\hat{p}^2+m^2)^{1/2})^2$ and here $F(e^{2\pi t}p_+,\hat{p})/F(p_+,\hat{p})$ also dislocalizes in the $\hat{x}$ variables if there are such variables at all, i.e., if $d \geq 3$.

5 Duality and modular action for all wedges

In the last section we dealt with a fixed wedge and saw in particular that duality for W_{R} and W_{L} holds if and only if M has only real zeros in p_+ on the mass shell. For $d = 2$ this is the complete answer to the question when wedge duality holds and this does not necessarily imply Lorentz covariance of the field.

We shall now see how the picture changes in dimensions $d \geq 3$. We start with the local action of the modular groups.

5.1. PROPOSITION. *Suppose $d \geq 3$ and the modular group for every wedge acts locally on the net generated by the field. Then M is constant on the mass shell.*

Proof. By the discussion in the last section local action of the modular group $\Delta_{\mathrm{R}}^{\mathrm{it}}$ requires that $M(p_+, p_+^{-1}(\hat{p}^2 + m^2), \hat{p})$ has the form $p_+^{2n} C(\hat{p})$ for some $n \in \mathbb{Z}$ and a function (polynomial) C depending only on $\hat{p}$. If M_Λ has the same form for all Λ then in particular we have for the Lorentz boosts $\Lambda_W(t)$ corresponding to an arbitrary wedge W and boost parameter t

$$M(\Lambda_W(t)^{-1} p) = D(\Lambda_W(t)) M(p) \tag{36}$$

with $D(\Lambda_W(t)) = (\exp(2\pi t))^{2n_W}$ for some $n_W \in \mathbb{Z}$. Moreover, since this holds for all p on the mass shell, we conclude that $D(\Lambda_{W_1}(t)\Lambda_{W_2}(s)) = D(\Lambda_{W_1}(t))D(\Lambda_{W_2}(s))$ for any two boosts in arbitary directions. Since any Lorentz transformation can be written as a product of boosts, we obtain in this way a one dimensional representation of the Lorentz group. If $d \geq 3$ this implies that D is constant, and hence, since the Lorentz group acts transitively on the mass shell, that M is constant on the mass shell. □

The requirement that wedge duality holds for *all* wedges also restricts the possible structure of M drastically in higher dimensions than 2. This is due to the following

5.2. LEMMA. *Let $M(p_+, p_-, \hat{p})$ be an even polynomial on $\mathbb{R}^d$ with $d \geq 4$. If the rational function*

$$p_+ \mapsto M_\Lambda(p_+, p_+^{-1}(\hat{p}^2 + m^2), \hat{p}) \tag{37}$$

has only real zeros for every Lorentz transformation Λ and every $\hat{p} \in \mathbb{R}^{d-2}$, then M is constant on the mass shell H_m^+. The same holds for $d = 3$ if $m > 0$.

Proof. We denote the rational function $M(p_+, p_+^{-1}(\hat{p}^2 + m^2), \hat{p})$ by $R(p_+, \hat{p})$ for short. If Λ is a Lorentz transformation, then the passage from M to M_Λ replaces $R(p_+, \hat{p})$ by $R_\Lambda(p_+, \hat{p}) = R((\Lambda^{-1}p)_+, (\Lambda^{-1}p)\hat{})$. Suppose now that R is *not* constant. Since Λ is invertible it is clear that R_Λ is not constant either for any Λ.

We shall show that there exists a Lorentz transformation Λ and a $\hat{p} \in \mathbb{R}^{d-1}$ such that $p_+ \mapsto R_\Lambda(p_+, \hat{p})$ has a complex (i.e. not real) zero.

The function R has the form

$$R(p_+, \hat{p}) = \sum_{n\in\mathbb{Z}} p_+^n a_n(\hat{p}) \tag{38}$$

where the a_n are polynomials in $\hat{p}$, and $a_n \equiv 0$ except for finitely many n. Likewise,

$$R_\Lambda(p_+, \hat{p}) = \sum_{n\in\mathbb{Z}} (\Lambda^{-1}p)_+^n a_n(\Lambda^{-1}\hat{p}) = \sum_{n\in\mathbb{Z}} p_+^n a_n^\Lambda(\hat{p}) \tag{39}$$

with different coefficients $a_n^\Lambda(\hat{p})$. The first remark is that there is at least one $n \neq 0$ such that a_n^Λ is not identically zero for some Λ. In fact, suppose R is independent of p_+, i.e., $R(p_+, \hat{p}) = a_0(\hat{p})$. Since the polynomial a_0 is not constant by assumption, it depends nontrivially on p_i for at least one i, $2 \leq i \leq d-1$, i.e., it contains a term $p_i^\nu b_\nu(p_2, \ldots, p_{i-1}, p_{i+1}, \ldots p_{d-1})$ with $\nu \neq 0$. If Λ is a rotation by $\pi/2$ in the $1i$ plane, then $(\Lambda^{-1}p)_i = p_1$. For p on the mass shell

$$p_1 = \tfrac{1}{2}(p_+ - p_-) = \tfrac{1}{2}(p_+ - (\hat{p}^2 + m^2)p_+^{-1}) \tag{40}$$

and inserting this for $(\Lambda^{-1}p)_i$ we see that R_Λ is not independent of p_+. To simplify notation we denote this R_Λ again by R.

Since we may now assume that R depends nontrivially on p_+, we can write

$$R(p_+, \hat{p}) = p_+^{-2n} A(\hat{p}) B(p_+, \hat{p}) \tag{41}$$

where $A(\hat{p})$ is a polynomial and

$$B(p_+, \hat{p}) = p_+^{2\ell} + \hat{B}(p_+, \hat{p}) \tag{42}$$

with $\ell \geq 1$ and $\hat{B}(p_+, \hat{p})$ a polynomial in p_+ of degree lower than 2ℓ. The coefficients of this polynomial are real analytic functions of $\hat{p}$ on some open set in $\mathbb{R}^{d-1}$. We write $\hat{p} = (p_2, \tilde{p})$ with $\tilde{p} \in \mathbb{R}^{d-3}$ (if $d = 3$ there is no $\tilde{p}$) and fix $\tilde{p}$. Then B can be regarded as a polynomial in p_+ with coefficients that are real analytic in p_2 on some open interval. The coefficient to the highest power of p_+ is independent of p_2.

If B has a complex zero in p_+ for some p_2 there is nothing more to be proved. On the other hand, if all zeros of B are real we may apply a theorem of Rellich [R] (see also [AKLM]), from which it follows that there is a real analytic function $r(\cdot)$, so that $p_+ = r(p_2)$ is a zero of B, and hence of R, for all p_2 in some open interval. Since M and hence R is even, we may assume that $r(p_2) > 0$. (The case $r(p_2) \equiv 0$ would mean that M on the mass shell has the form $p_+^{2n} C(\hat{p})$. As shown in Proposition 5.1 this can not hold in all Lorentz systems unless M is constant on the mass shell.)

It is convenient to replace the variables $(p_+, p_2, \tilde{p})$ on the mass shell by the variables $(p_1, p_2, \tilde{p})$:

$$p_1 = \tfrac{1}{2}(p_+ - p_-) = \tfrac{1}{2}(p_+ - (p_2^2 + \tilde{p}^2 + m^2)p_+^{-1}). \tag{43}$$

The inverse transformation is

$$p_+ = p_0 + p_1 = (p_1^2 + p_2^2 + \tilde{p}^2 + m^2)^{1/2} + p_1. \tag{44}$$

Inserting $p_+ = r(p_2)$ in (43) we obtain a real analytic curve

$$p_1 = s(p_2) \tag{45}$$

of zeros of B, and hence of R, in the 12-plane.

The function R_Λ has a corresponding curve of zeros at fixed $\tilde{p}$ for any Lorentz transformation Λ that affects only the variables p_0, p_1 and p_2. This curve is given by $(\Lambda p)_+ = r((\Lambda p)_2)$, or equivalently in the variables p_1, p_2, by $(\Lambda p)_1 = s((\Lambda p)_2)$. The point $p \in \mathbb{R}^d$ is here always on the mass shell.

Returning to the original curve $p_1 = s(p_2)$ there are two possibilities:

- The curve is a straight line segment.
- There is a point $\bar{p}_2$, such that the second derivative $s''(\bar{p}_2) \neq 0$.

We deal with the second case first.

By Taylor expansion we have

$$s(p_2) = s(\bar{p}_2) + s'(\bar{p}_2)(p_2 - \bar{p}_2) + \tfrac{1}{2}s''(\bar{p}_2)(p_2 - \bar{p}_2)^2(1 + g(p_2 - \bar{p}_2)) \tag{46}$$

whith some real analytic function g satisfying $g(t) \to 0$ for $|t| \to 0$. Let Λ be a rotation in the 12 plane by an angle φ, determined by $\cot\varphi = s'(\bar{p}_2)$. This transformation rotates the curve so that the tangent which previously had the slope $s'(\bar{p}_2)$ becomes parallel to the 1-axis. Moreover, the point $(s(\bar{p}_2), \bar{p}_2)$ is rotated into another point, $(a, b) = (\cos\varphi\, \bar{p}_2 + \sin\varphi\, s(\bar{p}_2), -\sin\varphi\, \bar{p}_2 + \cos\varphi\, s(\bar{p}_2))$, while the curvature, $\frac{1}{2}s''(\bar{p}_2) =: c \neq 0$ remains unchanged. Hence the equation of the the rotated curve, i.e., $(\Lambda p)_1 = s((\Lambda p)_2)$, has the form

$$p_2 = b + c(p_1 - a)^2(1 + h(p_1 - a)), \tag{47}$$

with $a, b, c \in \mathbb{R}$, $c \neq 0$ and where h is real analytic with $h(t) \to 0$ for $t \to 0$.

By analytic continuation, R_Λ vanishes also for complex points p_1 satisfying this equation. It is clear that if $(p_2 - b)/c$ is negative and sufficiently small, then there is a solution for p_1 with a nonvanishing imaginary part. By Eq. (44) this corresponds to a p_+ with nonvanishing imaginary part. (Note that p_2 and $\tilde{p}$ are still real.) Hence R_Λ has a complex zero in p_+ for some $(p_2, \tilde{p}) \in \mathbb{R}^{d-1}$.

If the curve (45) is a straight line, we can by a rotation transform it to a line parallel to the p_1 axis,

$$p_2 = k \tag{48}$$

with a constant k. A Lorentz boost in the 2-direction with parameter α transforms (48) into

$$p_2 = (\cosh\alpha)k + (\sinh\alpha)(k^2 + p_1^2 + \tilde{p}^2 + m^2)^{1/2}. \tag{49}$$

If $k^2 + \tilde{p}^1 + m^2 > 0$ we are back to the case considered before. This can always be achieved by choosing $\tilde{p} \neq 0$ if $d \geq 4$, and it holds also for $d = 3$ if $m > 0$. Thus we have again found a Λ, this time a composition of a rotation and a Lorentz boost, such that R_Λ has a complex zero in p_+. $\square$

The following examples show that wedge duality and Lorentz invariance are not necessarily equivalent in lower dimensions than 4.

Examples

1. Consider a massless field in $d = 3$ with $M(p) = (a \cdot p)^{2n}$ where $a = (a_0, a_1, a_2)$ is a space-like, or light like vector in $\mathbb{R}^3$. (The exponent $2n$ guarantees the required positivity and symmetry.) It is clear that M_Λ has the same form for all Lorentz transformations Λ. Vanishing of M is the same as vanishing of $a \cdot p$, and on the mass shell

$$\begin{aligned} a \cdot p &= \tfrac{1}{2}a_0(p_+ + p_2^2 p_+^{-1}) - \tfrac{1}{2}a_1((p_+ - p_2^2 p_+^{-1}) - a_2 p_2 \\ &= p_+^{-1}\left[\tfrac{1}{2}(a_0 - a_1)p_+^2 - (a_2 p_2)p_+ + \tfrac{1}{2}(a_0 + a_1)p_2^2\right]. \end{aligned} \tag{50}$$

The discriminant of the quadratic equation for p_+ is

$$(a_2 p_2)^2 - (a_0 - a_1)(a_0 + a_1)p_2^2 = -(a \cdot a)p_2^2 \geq 0 \tag{51}$$

for all real p_2, if a is space-like or light-like. Thus there are only real zeros. But M is not constant on the mass shell, unless $a = 0$.

2. In $d = 2$ we may also consider mass $m > 0$ (fields without a mass gap, depending only on one light cone coordinate, are discussed in [Y]): With M as above we have on the mass shell

$$a \cdot p = \tfrac{1}{2}a_0(p_+ + m^2 p_+^{-1}) - \tfrac{1}{2}a_1((p_+ - m^2 p_+^{-1}) \tag{52}$$

$$= \tfrac{1}{2}p_+^{-1}\left[(a_0 - a_1)p_+^2 + (a_0 + a_1)m^2\right]. \tag{53}$$

Again, if a is space-like or light like there are only real zeros.

Remark. In both these examples the minimal net of von Neumann algebras, $\mathcal{M}_{\min}(\mathcal{O}) = \{W(f) : \operatorname{supp} f \subset \mathcal{O}\}''$ generated by the field is different from the maximal net $\mathcal{M}_{\max}(\mathcal{O}) = \mathcal{M}_{\min}(\mathcal{O}')'$, if $\mathcal{O}$ is bounded. However, for every wedge W we have $\mathcal{M}_{\min}(W) = \mathcal{M}_{\max}(W)$, and $\mathcal{M}_{\max}(\mathcal{O}) = \cap_{W \supset \mathcal{O}}\mathcal{M}_{\min}(W)$. (This is a well known consequence of wedge duality, cf. e.g., Lemma 4.1 in [BY]). Moreover, $\mathcal{M}_{\max}(\cdot)$ is Lorentz covariant in both examples. In fact, it is straightforward to verify (cf. Section 3 in [Y]) that for a space-like or light like, the Lorentz covariant field Φ_0 and the non-Lorentz covariant derivatives $a\cdot\partial\Phi_0$ generate the same wedge algebras. In particular, CGMA also holds in these examples, because the wedge algebras are generated by a Lorentz covariant field.

Putting everything together we finally obtain the main conclusion of this note:

5.3. THEOREM. *If $d \geq 4$ the following are equivalent for the generalized free field models considered*

(i) CGMA

(ii) Wedge duality for all wedges

(iii) Local action of the modular groups of all wedges

(iv) Lorentz covariance of the field

For models with a mass gap this holds also for $d = 3$.

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