Higher dimensional versions of the Folklore Theorem, II

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Abstract

We prove in this issue that a hyperbolic C^2 horseshoe with infinitely many branches satisfying certain bounded distortion condition supports an ergodic Sinai-Ruelle-Bowen measure if and only if its unstable Cantor sets have positive measure. This extends to generalized hyperbolic sets of saddle type similar results obtained for expanding Markov transformations in [11].

1 Introduction

We say that a Borel probability μ preserved by a C^2 diffeomorphism of a compact riemannian manifold is a Sinai-Ruelle-Bowen measure (SRB) if its conditional measures along leaves of the unstable lamination are absolutely continuous respect to the riemannian volume. One reason which makes interesting SRB measures is that they are *physically observable* since its ergodic basin has positive volume, in other words,

$$\mathrm{Vol}\left\{x \in M^m : \lim_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} \phi(f^k(x)) = \int \phi(z)d\mu(z) \ \ \forall\, \phi \in C^0(M)\right\} > 0$$

By this reason they are sometimes called *ergodic attractors*. Cf. [5] and [10].

As far as I know, hyperbolic dynamical systems are the unique where existence and properties of SRB measures are well understood. Major attempts to prove existence of SRB measures in non hyperbolic systems were made in the nineties at lower dimensional settings, for interval transformations as well as Hénon maps. See [3, Chapter V], [13] and references therein. Recently some new higher dimensional examples, including partially hyperbolic ones, exhibiting SRB measures with large unstable dimension were introduced. Cf. [1] and [14]. Nevertheless, to give effectively verified conditions for the existence of SRB mesures in systems which are hyperbolic in large parts of their phase space without being uniformly hyperbolic remains by large an open question.

The present note is aimed at proving an existence result of SRB measures for a class of topological Markov chains with infinitely many states, extending to saddle type sets previous results which generalizes several known versions of the

Folklore Theorem to higher dimensional Markov transformations with infinitely many expanding branches.

Recently, L.S. Young [14] proposed a general abstract model somewhat similar to that introduced here, where sufficient conditions are given for the existence of SRB measures in certain non hyperbolic systems. It is worth to mention that generalized horseshoes presented here appear naturally when inducing hyperbolicity in non-uniform hyperbolic systems. Namely, we can show that, given a C^2 diffeomorphism f of a riemannian manifold preserving a hyperbolic non-atomic measure μ, and given a point $x \in supp\ \mu$, there is a neighborhood $R = R(x)$ containing a horseshoe with infinitely many branches Ω obtained from a suitable sequence of returns defining countably many (in general infinite) hyperbolic branches $f^{n_i} : S_i \longrightarrow U_i$. Ω has the same topological picture and regular measure theoretical properties of generalized horseshoes introduced here. These results strongly suggests that the presence of horsershoes with infinitely many branches may be indeed a necessary and sufficient condition for the existence of SRB measures in certain non-uniformly hyperbolic systems. Pesin's theory techniques and ideas from [4] are been used to keep track this type of results. The following is our main result.

Folklore Theorem *Let* $\mathsf{R} = B^u \times B^s$ *be a rectangle in* $\mathbb{R}^m$ *and* $T : \Omega \circlearrowleft$ *be* C^2 *horseshoe constructed* R *defined by infinitely many* C^2 *hyperbolic branches* $T_i : S_i \longrightarrow U_i$ *such that* $\{S_i\}$ *covers* R *up to a subset of zero Lebesgue measure. Suppose in addition that the non-linear expansion along invariant manifolds is bounded from below by some* $\lambda > 1$ *and that the following* **bounded distortion condition** *holds:*

$$(\mathbf{BD}) \qquad \sup_{i>0} \frac{\sup_{z \in S_i} \|D^2_{kl} T_i(z)\|}{\inf_{z,w \in S_i} \|D_j T_i(z)\| \cdot \|D_j T_i(w)\|} \leq \Delta < +\infty \qquad (1)$$

for $j, k, l = 1, \cdots, m$ *and for every* $i > 0$. *Then* T *preserves a unique ergodic SRB measure* $\mu = \mu_\Omega$ *supported on* Ω. *Further, the ergodic basin of* μ *covers* R *up to a Lebesgue measure zero set.*

We refer to next Section for definitions and proofs. To the best of my knowledge these type of problems were considered for the first time by Newhouse and Jacobson. See [7] and references therein. Nevertheless, our approach not only provides higher dimensional generalizations but it also gives a new proof of [7, Theorem 1]. Here, as for Markov piecewise expanding endomorphisms, there is an essential difference between a finite and an infinite number of hyperbolic branches since derivatives grows with i and relations between the first and the second derivative become crucial. In this work we permit to the stable cylinder S_i to accumulate toward admissible manifolds (see below).

Condition (1) above is little bit stronger that bounded distortion condition (D1) in [7]. However, it allow us to go beyond dimension two and to consider generalized horseshoes with unstable Cantor set with positive measure as well, improving Jacobson-Newhouse's results, even in dimension two.

Roughly speaking what we do is to project the dynamics along the leaves of an stable lamination on local unstable manifolds $W = W^u_{loc}$. Then we introduce certain dynamically defined Carathéodory's structure (or shortly, a C-structure) [8, Chapter 1] on the unstable Cantor sets of Ω, proving that these projections are in fact measure-theoretical expanding transformations [6, Chapter III]. Then we use Ruelle's theorem to see prove that this projection leaves invariant a unique ergodic asymptotic measure, say μ_W, which is absolutely continuous wrt the C-structure. Let us stress that we don't require Jacobians to satisfy a Hölder condition, so we have to use a proof of Ruelle's theorem due to Walters [12, Theorem 8]. This put ourselves in a setting similar to previous [11]. Actually, we prove that μ_W is absolutely continuous wrt riemannian volume of W iff the Carathéodory dimensional characteristic defined by our C-structure attains its maximum on unstable Cantor sets and this occurs if and only if unstable Cantor sets has positive Lebesgue measure. Then we use that this dynamically defined C-structure is invariant by the holonomy of the stable lamination to "integrate" wrt the stable lamination getting an asymptotic SRB measure. Let us notice that this approach provides absolutely continuity of the stable lamination wrt Lebesgue measure without additional assumptions as condition (G4) used in [7].

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2 Proof of the Folklore Theorem

We recall first some definitions and terminology to state properly our main results. Let $\mathsf{R} = B^s \times B^u \subseteq \mathbb{R}^m$, where $B^s \subseteq \mathbb{R}^s$ and $B^u \subseteq \mathbb{R}^u$ are unit closed balls. We shall suppose that R is endowed with two cone fields $\mathsf{K}^u = \mathsf{K}^u(x)$ and $\mathsf{K}^s = \mathsf{K}^s(x)$ defined at every point in $x \in \mathsf{R}$. We also suppose that these cones families extend continuously to a slightly larger neighborhood $\hat{\mathsf{R}}$ containing R. We associate with these cone fields *admissible stable and unstable submanifolds* $\Gamma^s(\mathsf{R}) = \{\gamma^s\}$ and $\Gamma^u(\mathsf{R}) = \{\gamma^u\}$. Each $\gamma^s \in \Gamma^s(\mathsf{R})$ is the graph of a C^1 map $\phi : B^s \longrightarrow B^u$ such that $T_xW \subseteq \mathsf{K}^s(x)$ for every $x \in \gamma^s$. Likewise for admissible unstable submanifolds. Also *stable and unstable cylinders* can be defined. A compact connected subset subset $S \subseteq \mathsf{R}$ is an admissible stable cylinder if it admits a foliation by admissible stable submanifolds and if its unstable sections $S \cap \gamma^u$ are convex sets for every $\gamma^u \in \Gamma^u(\mathsf{R})$. Unstable cylinders are defined similarly.

A C^2 horseshoe Ω with infinitely many branches or a generalized horseshoe will be defined by a countable collection of maps $\{T_i : S_i \longrightarrow U_i\}$ where:

1. $\{S_i\}$ (resp.$\{U_i\}$) are countable collections of stable (resp. unstable) cylinders with *non overlapping interiors*, that is, $int\, S_i \cap int\, S_j = \emptyset$ whenever $i \neq j$;

2. each $T_i : S_i \longrightarrow U_i$ has a C^2 extension $\hat{T}_i : \hat{S}_i \longrightarrow \hat{U}_i$ to neighborhoods of S_i and U_i which are stable and unstable cylinders in $\hat{\mathsf{R}}$;

3. $\hat{T}_i$ maps hyperbolically $\hat{S}_i$ onto $\hat{U}_i$, that is, *preserving strictly the cone families* K^s *and* K^u: $\mathsf{K}^s(T_i(x)) \subseteq \operatorname{int} DT_i(x)\mathsf{K}^s(x)$ and $DT_i(x)\mathsf{K}^u(x) \subseteq \operatorname{int} \mathsf{K}^u(T_i(x))$.

We shall suppose that non linear expansion along unstable admissible manifolds is bounded from below, namely: $d_{\gamma^k}(T^k(x), T^k(y)) \geq \lambda^{n-k+1} d_{\gamma^0}(x,y)$ uniformly in x and y in $\gamma^0 = \gamma^u \cap P$, $k = 0, \cdots n$ and for every $\gamma^u \in \Gamma^u$, $P \in \wp_n$ and $n \geq 0$. Here $W^k = T^k(\gamma^0)$. We also suppose that a similar statement holds true for the inverse T^{-1}. Due to the bounded geometry of admissible submanifolds (see below) these conditions are equivalent to $\|T^k(x) - T^k(y)\| \geq C\lambda^{n-k+1}\|x-y\|$ for every x , $y \in \gamma^0 = \gamma^u \cap P$ and $k = 0, \cdots n$, $\gamma^u \in \Gamma^u$ and every stable cylinder of level n, $P \in \wp_n$. Admissible manifolds have the following useful properties which shall be used throughout:

i) let $W \in \Gamma^u$ and $W' \in \Gamma^s$ the W intersects transversally W' at a single point x and the angle $\angle(T_xW, T_xW')$ is uniformly bounded below;

ii) *the geometry of admissible manifolds is bounded*, that is, there is a constant $C = C(\Gamma) > 1$ only depending on γ and the diameter of R such that Vol (γ^u) and diam (γ^u) are bounded in $[C^{-1}, C]$. Further, $C^{-1} \leq \operatorname{dist}_{\gamma^u}(x,y) \,/\, \|x-y\| \leq C$ for every x, y in an unstable admissible submanifold $\gamma^u \in \Gamma^u$ where $\operatorname{dist}_{\gamma^u}$ and $\operatorname{diam}_{\gamma^u}$ are given by the intrinsic riemannian volume, as well as $\operatorname{Vol}_{\gamma^u}$. Likewise for for the admissible s-submanifolds;

iii) The set Γ^u (resp. Γ^s) is endowed with an structure of Banach space given by the identification with $C^1(B^u, B^s)$ equipped with the C^1 norm. In particular it is a complete metric space. Moreover, there is a well defined *graph transform* $\Gamma_T : \Gamma^u \longrightarrow \Gamma^u$ defined by $\Gamma_T(\gamma^u) = T(\gamma^u) \cap \mathsf{R}$. The following result, due to Aleeksev and Moser, will be used elsewhere without further comments: *The graph transform is a contraction, i.e. there exists* $0 < \theta < 1$ *such that* $dist_{C^1}(\Gamma_T(\gamma_1^u), \Gamma_T(\gamma_2^u)) \leq \theta \, dist_{C^1}(\gamma_1^u, \gamma_2^u)$. In addition, $(\gamma^u, \gamma^s) \longmapsto \gamma^u \cap \gamma^s$ is a Lipschitz map from $\Gamma^u \times \Gamma^s$ to R.

Given a family of hyperbolic branches $T_i : S_i \longrightarrow U_i$ as above we can define nested sequences of stable and unstable cylinders converging to two laminations of stable and unstable admissible manifolds $\mathcal{F}^s$ and $\mathcal{F}^u$, respectively. In fact, given a sequence $(i_0, \cdots, i_n, \cdots) \in \mathbb{N}^{\mathbb{N}}$ we set define

$$U_{i_0 \cdots i_{n-1}} = \bigcap_{k=0}^{n-1} T_{i_{n-1}} \circ \cdots \circ T_{i_0} U_{i_0} \qquad S_{i_0 \cdots i_{n-1}} = \bigcap_{k=0}^{n-1} T_{i_{n-1}}^{-1} \circ \cdots \circ T_{i_0}^{-1} S_{i_0}$$

where we by abuse of language we omit the domain of compositions. The sequence $\{S_{i_0 \cdots i_{n-1}}\}_{n \geq 0}$ is a nested sequence of stable cylinders. Similarly so $\{U_{i_0 \cdots i_{n-1}}\}_{n \geq 0}$. In fact, graph transform contraction properties implies that there is a unique admissible manifold $\gamma^s = \gamma^s(\mathbf{i})$ such that $d_{C^1}(S_{i_0 \cdots i_{n-1}}, \gamma^s)$

converges to zero as $n \longrightarrow +\infty$ and likewise for the unstable cylinders. Hence, we define $\mathcal{F}^s = \{\gamma^s(\mathbf{i}) : \mathbf{i} \in \mathbb{N}^{\mathbb{N}}\}$ (resp. $\mathcal{F}^u = \{\gamma^u(\mathbf{i}) : \mathbf{i} \in \mathbb{N}^{\mathbb{N}}\}$).

We set $\Omega = \bigcup \mathcal{F}^s \cap \bigcup \mathcal{F}^u$. Ω is a compact, perfect, subset of $\mathbb{R}^m$ contained in the cube R. Topologically it might be a Cantor set times an interval or a product of two Cantor sets or even it might fills R up to a measure zero set.

We can use the hyperbolic branches T_i to define a piecewise hyperbolic map $T : \bigcup_i int\, S_i \longrightarrow \bigcup_i int\, U_i$, by setting $T \mid int\, S_i = T_i$ and extends T to some well defined measurable transformation $\hat{T} : \bigcup_i S_i \longrightarrow \bigcup_i U_i$, which is C^2 smooth in a dense subset of its domain and preserves the admissible manifolds. Ω is the maximal invariant subset of $\hat{T}$ and $\mathcal{S} = \{x \in \bigcup_i \partial S_i : \quad x \in \partial S_i \cap \partial S_j\,, \quad \text{for some} \quad i, j > 0\}$ is its singular set. Extensions $\hat{T}$ are rather arbitrary and they are not unique. However, as long as their singular singular is negligible for all the purposes of the ergodic theory we can omit this arbitrariness, so we will continue denoting extensions $\hat{T}$ by T to avoid messy notations.

Given one such extension it is easy to see that $\mathcal{F}^s$ and $\mathcal{F}^u$ are T-invariant. Moreover, due to no-linear expansion properties along admissible manifolds, we conclude that $W^s(x, \mathsf{R}) = \gamma^s(\mathbf{i})$ and $W^u(x, \mathsf{R}) = \gamma^u(\mathbf{j})$ are the local stable manifold of x where $\{x\} = \gamma^s(\mathbf{i}) \cap \gamma^u(\mathbf{j})$. Indeed, $W^s(x, \mathsf{R}) = \{y \in \mathsf{R} : \|T^n(x) - T^n(y)\| \leq C\lambda^{-n} \quad \forall\, n \geq 0\}$ and similarly so for the unstable local manifold for backward orbits. Therefore Ω *is endowed with a hyperbolic product structure* in the sense that every point is the intersection of local invariant manifolds which intersects transversally at angle bounded from below.

Definition 2.1. A set Ω with a hyperbolic product structure and a dynamics $T : \Omega \circlearrowleft$ given by a collection of C^2 hyperbolic branches $T_i : S_i \longrightarrow U_i$ as defined above shall be called a *horseshoe with infinitely many branches or, shortly, a generalized horseshoe.*

We introduce for further use the following notation: $\wp_n = \{S_{i_0 \cdots i_{n-1}} : \mathbf{i} \in \mathbb{N}^{\mathbb{N}}\}$ for $n > 0$ and $\wp_0 = \{S_i\}_{i>0}$. Notice that $\wp_n = T^{-n}\wp_0$. $\wp_0$ is a Markov for Ω as it is easy to verify. Further, for every $x \in \Omega$ it holds $\{x\} = \bigcap_{n\in\mathbb{Z}} T^{-n}\wp_0(T^n(x))$. So $\bigvee_{n\in\mathbb{Z}} T^{-n}\wp_0$ generates the σ-algebra of Borelian sets of Ω. Also, $\bigvee_{n\geq 0} T^{-n}\wp_0$ generates the σ-algebra stable saturated of Borel subsets, that is those Borelians which are the reunion of stable leaves of $\mathcal{F}^s$. We shall denote $\wp$ the reunion of all cylinders in $\wp_n$ for $n \geq 0$.

Now, let $T : \Omega \circlearrowleft$ be a C^2 generalized horseshoe with bounded distortion and expansion coefficient bounded from below. We introduce, as we did previously for Markov transformations, a dynamically defined Charathéodory's structure to study some regularity properties of unstable Cantor sets of a generalized horseshoe. For this we let $\gamma^u \in \Gamma^u$ and define a generating net of subsets on γ^u taking the intersections of γ^u with cylinders in $\wp$:

$$\wp(\gamma^u) = \{P \cap \gamma^u : P \in \wp_n \text{ is an stable n-cylinder for some } \ n \geq 0\}.$$

Then we define an outer measure in γ^u, using the Carathéodory construction, by taking coverings by $\wp(\gamma^u)$-sets and using a riemannian volume Vol^u_γ as set function. More precisely, given $a > 0$ we define an outer measure for these unstable Cantor sets as follows:

$$\mathcal{D}_{a,\wp,\gamma^u}(X) = \lim_{\delta \to 0^+} \inf_{\mathcal{U}} \sum_{i=1}^{+\infty} \mathrm{Vol}_{\gamma^u}(U_i)^a \qquad X \subseteq \gamma^u,$$

taking infimum over all the δ-covering by $\wp(\gamma^u)$-sets. A countable family $\mathcal{U} = \{U_i\}_{i>0}$ $\mathcal{U} \subseteq \wp$ with $\mathrm{diam}\,(U_i) < \delta$, will be called *$\delta$-coverings by $\wp(\gamma^u)$-sets.* (Throughout this work Vol_{γ^u} will always denote the volume on $\gamma^u \in \Gamma^u$ defined by its intrinsic riemannian metric and $\mathrm{Vol}_{\gamma^u}(X) = \mathrm{Vol}\,(X \cap \gamma^u)$, for every Borel subset $X \subseteq \mathbb{R}^m$.) This construction yields a well defined and non-trivial measure class and a Carathéodory's dimensional characteristic which shall be called the *dynamical measure and the dynamical dimension of Ω associated to the generating family $\wp$.* Compare [8, Chapter 1] and [8, Chapter 4]. We denote these by $\mathcal{D}_{\alpha,\gamma^u}$ and $\dim_{\gamma^u,\wp}$, respectively. Changing the riemannian volume gives equivalent measures. Also we can enlarge the covering family $\wp$ using the bounded distortion property. But these procedures at most changes the measure by bounded constants so this holds inside a well defined measure class. The following is our first main result

Theorem A *Let $T : \Omega \circlearrowleft$ be a C^2 generalized horseshoe with bounded distortion and expansion coefficient bounded from below. Then there exists a unique ergodic measure $\mu = \mu_\Omega$ such that for every local unstable manifold $W = W^u(x, \mathsf{R})$*

1. $\dim_{\mathcal{D},W}(\Omega) = h_\mu(T) \,/\, \int \log\, J^uT(x) d\mu(x)$;
2. *μ_W, the projection of μ_Ω along the stable lamination onto W, is equivalent to $\mathcal{D}_{\alpha,W}$ and bounded by a constant $C = C(\Delta, \lambda, \Gamma) > 1$. Further, $\mu_W(P) \asymp Vol_W(P)^\alpha$ for every $P \in \wp$;*
3. *the stable lamination $\mathcal{F}^s$ of a generalized horseshoe is absolutely continuous wrt the measure classes $\{\mathcal{D}_{\alpha,\gamma^u}\}_{\gamma^u \in \Gamma^u}$ and it has a continuous and bounded Jacobian:*

$$\frac{d\,H^*\mathcal{D}_{\alpha,\overline{\gamma}^u}}{d\,\mathcal{D}_{\alpha,\gamma^u}}(x) = \left[\prod_{i=1}^{+\infty} \frac{J(T \mid \gamma^u)(T^i(x))}{J(T \mid \overline{\gamma}^u)(T^i(H(x)))}\right]^\alpha . \tag{2}$$

The following is an useful regularity property of the stable lamination.

Here $H_{\gamma^u,\overline{\gamma}^u}$ is the holonomy map of the local stable lamination $\mathcal{F}^s$ defined by unstable admissible unstable sections $\gamma^u, \overline{\gamma}^u \in \Gamma^u(\mathsf{R})$: $H_{\gamma^u,\overline{\gamma}^u}(x) = W^s(x, R) \cap \overline{\gamma}^u$, for every $x \in \gamma^u \cap \bigcup_{x \in \Omega} W^s(x, \mathsf{R})$. $J(T \mid \gamma^u)$ will denote throughout this work the Jacobian of T restricted to γ^u with respect to the intrinsic riemannian volume. $J^uT(x) = J(T \mid W^u(x, \mathsf{R}))(x)$ is the unstable Jacobian of T wrt to the intrinsic riemannian volume of the local unstable manifold $W^u(x, \mathsf{R})$. (1) follows from (2) using Billingsley's method. See [11] and compare [8, Appendix III]. Being independent of $W \in \mathcal{F}^u$ we can introduce the following

Definition 2.2. $\dim_u(\Omega) = \dim_{\mathcal{D},W}(\Omega)$, for every local unstable manifold $W \in \mathcal{F}^u$ and is called the unstable dimension of Ω.

Theorem B *Let $T : \Omega \circlearrowleft$ be a C^2 generalized horseshoe with bounded distortion and expansion coefficient bounded from below. Then, the following statements are equivalent:*

1. $h_\mu(T) = \int \log\, J^u T(x) d\mu(x)$*;*
2. $Vol_W(\Omega \cap W) > 0$ *for some local unstable manifold* $W \in \mathcal{F}^u$*;*
3. *the volume* $Vol_W(\Omega)$ *of the unstable Cantor sets* $\Omega \cap W^u(x, \mathsf{R})$ *is uniformly bounded away from zero and* $\mathcal{F}^s$ *is absolutely continuous wrt Lebesgue with uniformly bounded Jacobians;*
4. μ_Ω *is absolutely continuous wrt the riemannian volume along the local unstable manifolds.*

In addition, if any of the above conditions hold the stable invariant lamination $\mathcal{F}^s$ *is absolutely continuous wrt Lebesgue and it has a bounded Jacobian, namely*

$$\frac{d\, H^* \mathcal{L}_{\overline{\gamma}^u}}{d\, \mathcal{L}_{\gamma^u}}(x) = \prod_{i=1}^{+\infty} \frac{J(T \mid \gamma^u)(T^i(x))}{J(T \mid \overline{\gamma}^u)(T^i(H(x)))} \tag{3}$$

for every pair of admissible unstable manifolds $\gamma^u, \overline{\gamma}^u \in \Gamma^u$*. Here* $\mathcal{L}_{\gamma^u}$ *is the Lebesgue measure class of* γ^u*. Compare [10]. Therefore, the ergodic basin of the asymptotic measure contains a set of positive volume, so* μ_Ω *is physically observable.*

Proof. We notice that $\mathcal{D}_{1,\gamma^u}$ represents the Lebesgue measure class of γ^u. This is a straightforward consequence of the definitions and the fact that $\wp$ generates the stable lamination. Compare [11, Theorem C]. In particular, $\mathrm{Vol}_{\gamma^u}(X) > 0$ implies $\dim_{\mathcal{D},W}(X) = 1$. Thus, $\dim_{\mathcal{D},W}(\Omega) = 1$ implies that μ_W, the projection of μ along $\mathcal{F}^s$ onto the local unstable manifolds $W = W^u(x, \mathsf{R})$, is equivalent to the restriction of the Lebesgue measure class $\mathcal{L}_W$ to $W \cap \Omega$, bounded by constants not depending on $x \in \Omega$ neither on $W \in \mathcal{F}^u$. In particular, the volume $\mathrm{Vol}_W(\Omega)$ of the unstable Cantor sets $\Omega \cap W^u(x, \mathsf{R})$ is uniformly bounded away from zero, by the absolute continuity of the lamination and since its Jacobians are uniformly bounded. As $\mathrm{Vol}_W(\Omega) > 0$ for some local unstable manifold $W = W^u(x, \mathsf{R})$ implies $\dim_{\mathcal{D},W}(\Omega) = 1$, we see that all these conditions are equivalent. To finish, we notice that the ergodic basin of $\mu = \mu_\Omega$, defined as

$$W^s(\mu) = \{x \in \mathsf{R} : \lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi(T^k(x)) = \int \phi(z) d\mu(z) \;\; \forall\, \phi \in C^0(M)\},$$

contains the stable set of Ω, $W^s(\Omega) = \bigcup_{x \in \Omega} W^s(x, \mathsf{R})$. Now, $W^s(\Omega)$ has positive volume, in view of the absolute continuity of the stable lamination wrt the Lebesgue measure, using a Fubini's theorem argument, hence $\mathrm{Vol}\,(W^s(\mu)) > 0$ and we are done. $\square$

Next Theorem can be deduced from the above results, using Moran's covers arguments as we did before for hyperbolic repellors. Cf. [8, Chapter 7] and [11, Theorem D].

Theorem C *Let $T : \Omega \circlearrowleft$ be a C^2 generalized conformal horseshoe with bounded distortion and expansion coefficient bounded from below. Then there exists a unique ergodic measure $\mu = \mu_\Omega$ such that*

1. $\dim_{\mathcal{H}}(\Omega \cap W^u(x, \mathsf{R}))$ *is constant for every local unstable manifold* $W = W^u(x, \mathsf{R})$*;*

2. *the stable lamination $\mathcal{F}^s$ is Lipschitz;*

3. *μ_W, the projection of μ_Ω along the stable lamination onto W, is equivalent to the Hausdorff measure bounded by a constant which only depends on the bounded geometry of admissible manifolds, the distortion and the non linear expansion coefficient; further, the following strong Frostman's distribution property holds:*

$$C^{-1} \leq \frac{\mu_W(B(x,r) \cap W)}{r^{\dim_{\mathcal{H}}(\Omega \cap W)}} \leq C \qquad \forall\, x \in \Omega \quad W = W^u(x, \mathsf{R})).$$

Proof. By bounded distortion diam $(\wp_n(x) \cap W^u(x, \mathsf{R})) \asymp \|J^u T^n(x)\|^{-1}$ are uniformly bounded for every $x \in \Omega$ and $n \geq 0$. We use this to introduce Moran's covers $\wp_r$ for every $0 < r < 1$ associated to $\wp$. For these Moran's covers there exists a universal constant $M > 1$ such that $B(x, r) \cap \gamma^u$ intersects at most M atoms in the family $\wp_r(\gamma^u)$, for every $x \in \Lambda$, $0 < r < 1$ and admissible unstable manifold γ^u. Moreover, of Moran's covers follows because

$$B(x, C^{-1}\, r) \cap \gamma^u \subseteq \wp_r(x) \cap \gamma^u \subseteq B(x, C\, r) \cap \gamma^u \tag{4}$$

for every $0 < r < 1$ and $x \in \Omega$ and some universal constant $C > 1$ only depending on the distortion and the bounded geometry of admissible manifolds. Cf. [8, Chapter 7].

By Theorem A there exists an ergodic T-invariant probability $\mu = \mu_\Omega$ such that $\mu_W(\wp_r(x)) \asymp \mathrm{Vol}\,(\wp_r(x))^{\dim_u(\Omega)}$, for every $x \in \Omega$ and $0 < r < 1$. Therefore, $\mu_W(B(x,r) \cap W) \asymp r^{u \dim_u(\Omega)}$ for every $x \in \Omega$, $0 < r < 1$ and a suitable constant $C > 1$ independent of x and r. So, $\dim_{\mathcal{H}}(\Omega \cap W) = u \dim_u(\Omega)$ and the asymptotic measure μ_Ω, the dynamical measure and the Hausdorff measure shall be equivalent. Finally, by Frostman's uniform distribution property and the existence of a bounded Jacobian with respect to the Hausdorff measure for the holonomies of the stable lamination $\mathcal{F}^s$ we see that

$$\|x - y\|^\alpha \asymp \mathcal{H}_{\alpha, \gamma^u}([x, y]) \asymp \mathcal{H}_{\alpha, \overline{\gamma}^u}([H(x), H(y)]) \asymp \|H(x) - H(y)\|^\alpha,$$

where $\alpha = \dim_{\mathcal{H}}(\Omega \cap W)$, $\mathcal{H}_{\alpha,\gamma^u}$ is the restriction of the Hausdorff measure to the admissible unstable manifold γ^u and $H = H_{\gamma^u, \overline{\gamma}^u}$ the holonomy map defined by γ^u and $\overline{\gamma}^u$. Constants are uniform, by previous remarks and the bounded geometry of admissible manifolds. Hence,

$$\frac{\|H(x)-H(y)\|}{\|x-y\|} \asymp 1 \qquad \forall\, x,y \in \gamma^u \cap \bigcup_{z\in\Omega} W^s(z,\mathsf{R}),$$

meaning that $C^{-1}\|x-y\| \leq \|H(x)-H(y)\| \leq C\|x-y\|$, $\mathcal{F}^s$ is Lipschitz, and we are done. $\square$

3 Bounded distortion and bounded geometry estimates

Along this Section $T : \Omega \circlearrowleft$ will be a C^2 hyperbolic horseshoe defined by countably (possibly infinite) many hyperbolic branches $T_i : S_i \longrightarrow U_i$. We suppose in addition that the collection T_i have non-linear expansion bounded from below and bounded distortion **(BD)**.

Lemma 3.1. *There is constant $C > 1$, only depending on the distortion, the expansion coefficient and the bounded geometry of the admissible manifolds such that, for every $\gamma^u \in \Gamma^u$ and $n \geq 0$ it holds*

$$\frac{J(T^n \mid \gamma^u)(x)}{J(T^n \mid \gamma^u)(y)} \leq \exp(C d_u(T^n(x), T^n(y))), \tag{5}$$

whenever $\wp_n(x) = \wp_n(y)$ and $x, y \in \gamma^u$.

Proof. Using bounded distortion condition **(BD)** we prove that for every $\gamma^u \in \Gamma^u$ and $i > 0$ the following estimate holds:

$$\frac{\sup_{z\in\gamma^u\cap S_i} \|\nabla \log J(T_i \mid \gamma^u)(z)\|}{\inf_{w\in\gamma^u\cap S_i} \|D(T_i \mid \gamma^u)(w)\|} \leq \Delta < +\infty. \tag{6}$$

This is a straightforward computation. Now, we use (6) and an standard argument to get (5). Let $\gamma^u \in \Gamma^u$ an unstable admissible submanifold and denote $\gamma^0 = \gamma^u \cap \wp_n(x)$, $\gamma^i = T^i(\gamma^0)$:

$$\begin{aligned}
\log \frac{J(T^n \mid \gamma^u)(x)}{J(T^n \mid \gamma^u)(y)} &\leq \sum_{i=0}^{n-1} |\log J(T \mid \gamma^i)(T^i(x)) - \log J(T \mid \gamma^i)(T^i(y))| \\
&\leq \sum_{i=0}^{n-1} \sup_{z\in\gamma^i} \|\nabla \log J(T \mid \gamma^i)(z)\| \cdot \|T^i(x) - T^i(y)\| \\
&\leq \sum_{i=0}^{n-1} \frac{\sup_{z\in\gamma^i} \|\nabla \log J(T \mid \gamma^i)(z)\|}{\inf_{w\in\gamma^i} \|DT^{n-i}(w)\|} \cdot \|T^n(x) - T^n(y)\| \\
&\leq \sum_{i=0}^{n-1} \frac{\sup_{z\in\gamma^i} \|\nabla \log J(T \mid \gamma^i)(z)\|}{\inf_{w\in\gamma^{i+1}} \|DT^{n-i-1}(w)\| \inf_{w\in\gamma^i} \|DT(w)\|} \cdot \|T^n(x) - T^n(y)\|
\end{aligned}$$

$$\leq C(\Gamma)\Delta \sum_{i=0}^{n-1} \lambda^{-(n-i-1)} \cdot \|T^n(x) - T^n(y)\|$$

$$\leq C(\Gamma)\Delta \sum_{n=0}^{+\infty} \lambda^{-n} \cdot \|T^n(x) - T^n(y)\| \leq C(\Gamma)^2 \Delta (1-\lambda^{-1}) d_u(T^n(x), T^n(y)),$$

using the bounded geometry of admissible manifolds and condition (6) $\square$

Corollary 3.1. *There is a constant $C = C(\Delta, \lambda, \Gamma) > 1$ such that, for every admissible unstable manifold $\gamma^u \in \Gamma^u$ and every $n > 0$ it holds*

$$C^{-1} \inf_{\gamma^u \in \Gamma^u} Vol(\gamma^u) \leq J(T^n \mid \gamma^u)\, Vol_{\gamma^u}(\wp_n(x)) \leq C \sup_{\gamma^u \in \Gamma^u} Vol(\gamma^u)$$

Lemma 3.2. *Let $\gamma^u = \gamma^u(x)$ and $\overline{\gamma}^u = \gamma^u(y)$ be two admissible unstable manifolds passing by x and y in Ω, respectively. Suppose that $y \in W^s(x, \mathsf{R})$ and denote*

$$h(x,y) = \left[\prod_{i=1}^{+\infty} \frac{J(T \mid \gamma^u)(T^i(x))}{J(T \mid \overline{\gamma}^u)(T^i(y))} \right]^{\alpha}.$$

Then $h(x,y) \leq \exp(C d_s(x,y))$ for a constant $C = C(\Delta, \lambda, \Gamma) > 1$ where $d_s(x,y) = d_{W^s(x,\mathsf{R})}(x,y)$. We have also

$$\frac{h(x,y)}{h(x',y')} \leq \max\{e^{C d_u(x,x')}, e^{C d_u(y,y')}\},$$

for every $x' \in \gamma^u(x)$ and $y' \in \gamma^u(y)$ with $y' \in W^s(x', \mathsf{R})$, where $d_u(x,x') = d_{\gamma^u(x)}(x,x')$, likewise $d_u(y,y')$.

Proof. We can find a C^2 foliation of R by admissible unstable leaves, say $\gamma^u = \gamma^u(z)$ passing by $\gamma^u(x)$ and $\gamma^u(y)$. By the bounded distortion condition **(BD)** we get again

$$\frac{\sup_{w \in \gamma^s(z)} \|\nabla \log J(T_i \mid \gamma^u(w))(w)\|}{\inf_{w \in \gamma^s(z)} \|DT_i(w)\|} \leq \Delta < +\infty. \tag{7}$$

for every leave $\gamma^s(z)$ contained in S_i, every $z \in \mathsf{R}$ and $i > 0$. Let us denote $J_i(z) = J(T_i \mid \gamma^u(z))(z)$. Arguing similarly as we did before we get

$$\log \frac{J_i(T^j(x))}{J_i(T^j(y))} \leq \sup_{w \in W^s(T^j(x),\mathsf{R})} \|\nabla \log J_i(w)\| \cdot \|T^j(x) - T^j(y)\|$$

$$\leq \frac{\sup_{w \in W^s(T^j(x),\mathsf{R})} \|\nabla \log J_i(w)\|}{\inf_{w \in W^s(T^j(x),\mathsf{R})} \|DT_i(w)\|} \|T^{j+1}(x) - T^{j+1}(y)\|.$$

Thus $\log(J_i(T^j(x)) \,/\, J_i(T^j(y))) \leq \Delta\, C(\Gamma) \lambda^{-(j+1)} d_s(x,y)$ and then

$$\log h(x,y) \leq \sum_{j=1}^{+\infty} \Delta\, C(\Gamma) \lambda^{-(j+1)} d_s(x,y) = \Delta\, C(1-\lambda^{-1})\lambda^{-2} d_s(x,y).$$

Further, $h(x,y)\,/\,h(x',y') \le \exp\left(C(d_s(x,y) - d_s(x',y')\right)$. Therefore,

$$\frac{h(x,y)}{h(x',y')} \le \max\left\{\exp(Cd_u(x,x')), \exp(Cd_u(y,y')\right)\,,$$

for some $C = C(\Delta,\lambda,\Gamma) > 1$ as claimed. □

Corollary 3.2. *There is a constant $C = C(\Delta,\lambda,\Gamma) > 1$ such that, for any two admissible manifolds γ^u and $\overline{\gamma}^u$ in Γ^u it holds $Vol_{\gamma^u}(P) \asymp Vol_{\overline{\gamma}^u}(P)$ for every $P \in \wp$.*

In particular, the volume of the cylinder $\mathrm{Vol}\,(P)$ is comparable with the volume of its unstable sections $\gamma^u \cap P$, i.e. $\mathrm{Vol}_{\gamma^u}(P) \asymp \mathrm{Vol}\,(P)$, bounded by uniform constants which do not depend on p neither $\gamma^u \in \Gamma^u$.

Proof. It follows from Corollary 3.1 and Lemma 3.2 that for every $n > 0$

$$\frac{\mathrm{Vol}_{\gamma^u}(\wp_n(x))}{\mathrm{Vol}_{\overline{\gamma}^u}(\wp_n(x))} \le C\,\frac{\sup_{\gamma^u\in\Gamma^u}\ \mathrm{Vol}\,(\gamma^u)}{\inf_{\gamma^u\in\Gamma^u}\ \mathrm{Vol}\,(\gamma^u)},$$

and similarly so the lower bound. □

Previous Corollary has two important consequences:

- the holonomy of the stable lamination is absolutely continuous wrt the dynamical class $\{\mathcal{D}_{\alpha,\gamma^u}\}_{\gamma^u\in\Gamma^u}$ and it has a bounded Jacobian;
- the measure of the unstable Cantor sets cut by $W = W^u(x,\mathsf{R})$ are uniformly bounded, i.e. $C^{-1} \le \mathcal{D}_{\alpha,W}(\Omega) \le C$ for every $W \in \mathcal{F}^u$.

Both conclusions follows from the (obvious) estimate $\mathcal{D}_{\alpha,\gamma^u}(B) \asymp \mathcal{D}_{\alpha,\overline{\gamma}^u}(H(B))$ bounded by constants $C = C(\Delta,\lambda,\Gamma) > 1$, for every γ^u and $\overline{\gamma}^u$, where $H = H_{\gamma^u,\overline{\gamma}^u}$ is the holonomy transformation defined by these unstable admissible manifolds.

We shall prove later that $h = h_{\gamma^u,\overline{\gamma}^u}$ below is the Jacobian of H wrt the dynamical measure class:

$$h(x) = \left[\prod_{i=1}^{+\infty} \frac{J(T \mid \gamma^u)(T^i(x))}{J(T \mid \overline{\gamma}^u)(T^i(H(x)))}\right]^{\alpha}.$$

The following result is an straightforward consequence of Lemma 3.2.

Lemma 3.3. *For every γ^u and $\overline{\gamma}^u$ in Γ^u it holds*

$$h(x) \le \exp(Cd_s(x,H(x))) \tag{8}$$

for a constant $C = C(\Delta,\lambda,\Gamma) > 1$. Here $d_s(x,y) = d_{W^s(x,\mathsf{R})}(x,y)$ for $y \in W^s(x,\mathsf{R})$. Also, for every x and y in γ^u

$$\frac{h(x)}{h(y)} \le \max\left\{e^{Cd_u(x,y))}\,,\, e^{Cd_u(H(x),H(y)))}\right\}, \tag{9}$$

where $d_u(x,y) = d_{W^u(x,\mathsf{R})}(x,y)$, for $y \in W^u(x,\mathsf{R})$.

4 Asymptotic unstable measures in generalized horseshoes

Lemma 4.1. *Let Ω be a C^2 hyperbolic horseshoe defined by infinitely many hyperbolic branches $T_i : S_i \longrightarrow U_i$ satisfying bounded distortion condition* **(BD)**. *Suppose in addition that the dynamical measure class in dimension $\alpha = \dim_u(\Omega)$ is nontrivial, i.e. $0 < \mathcal{D}_{\alpha,\gamma^u}(\Omega) < +\infty$. Then, for every local unstable manifold $W = W^u(x, \mathsf{R})$ there is a unique Borel probability μ_W equivalent to the dynamical measure $\mathcal{D}_{\alpha,W}$ and a constant $C = C(\Delta, \gamma, \Gamma) > 1$ only depending in the distortion and the bounded geometry of the admissible manifolds such that*

$$C^{-1} \leq \frac{\mu_W(P)}{(Vol\ P)^{\dim_u(\Omega)}} \leq C, \tag{10}$$

for every geometrical cylinder $P \in \mathcal{P}$.

We will prove this result through several lemmas below. For this we begin recalling the definitions of a measurable transformation of a measure space. Let (X, d) be a complete metric space. $T : (X, \mathcal{A}, \mu) \circlearrowleft$ is a *measure-theoretical expanding transformation* if there exists a sequence of partitions $\wp_n$ such that:

1. $\overline{\bigcup_{P\in\wp_n} P} = X$, $\inf_{P\in\wp_n} \mu(T(P)) > 0$;
2. for each $n \geq 0$ and $P \in \wp_{n+1}$, $T(P)$ is a union of atoms of $\wp_n \mod (0)$ and $T \mid P$ is one-to-one;
3. there are $0 < \lambda < 1$, $K > 0$ such that

   $$d(x, y) \leq K\lambda^n\, d(T^n(x), T^n(x))$$

   for each $n \geq 0,\ x \in X,\ y \in \wp_n(x)$;
4. for each pair P , $Q \in \wp_n$ there is a $n > 0$ such that $\mu(T^{-n}(P) \cap Q) > 0$;
5. there are $J : X \longrightarrow \mathbb{R}^+$, $0 < \beta < 1$ and $C > 0$ such that:

   (a) for each $n \geq 0$ and a Borelian B contained in a atom of $\wp$:

   $$\mu(T(B)) = \int_B J(x) d\mu(x);$$

   (b) for each x , y *in* $\wp_n$:

   $$\left| \frac{J(x)}{J(y)} - 1 \right| \leq C\, d(T(x), T(y))^\beta.$$

Cf. [Chapter III][6]. It is proved in [6] that $T : (X, \mathcal{A}, \mu) \circlearrowleft$ leaves invariant a unique ergodic Borel probability absolutely continuous wrt μ and its density is bounded away from zero and infinite. Actually, this is a version of a remarkable and well known result due to D. Ruelle.

Now, we projects the dynamics in Ω along the stable lamination to get an expanding transformation. Namely, we let $\tau(x) = H_{W(T(x)),W} \circ T(x)$, where $W = W^u(x, \mathsf{R})$ and $W(T(x)) = W^u(T(x), \mathsf{R})$. We shall denote below $\Lambda = \Omega \cap W^u(x, \mathsf{R})$ the unstable Cantor set cut by $W = W^u(x, \mathsf{R})$ and $\tau : \Lambda \longrightarrow \Lambda$ the projection of $T \mid W^u(x, \mathsf{R})$ along the stable leaves. Now put the metric $d^*(x, y) = \operatorname{dist}_{\mathcal{H}} d(W^s(x, R), W^s(y, R))$ on Λ, where $\operatorname{dist}_{\mathcal{H}}$ is the Hausdorff distance of sets. (Λ, d^*) is a compact metric space. The Markov partitions $\wp(n, W)$ and non-linear expansion can be used to verify the above axioms proving that $\tau : (\Lambda, \mathcal{D}_{\alpha,W}) \circlearrowleft$ satisfies the above conditions except possibly by the Hölder property of the Jacobian. Actually, by conformality of the dynamical measure class and the absolute continuity of the lamination implies that τ has a Jacobian wrt $\mathcal{D}_{\alpha,W}$. In fact,

$$J_\alpha \tau(x) = \frac{d\, H^* \mathcal{D}_{\alpha,W}}{d\, \mathcal{D}_{\alpha,W(T(x))}}(T(x))[J^u T(x)]^\alpha,$$

for every local unstable manifold $W = W^u(x, \mathsf{R})$, where by conformality we refer to the following

Lemma 4.2. *Let $h : W \longrightarrow \mathbb{R}^m$ be a $C^{1+\epsilon}$ embedding, then $\mathcal{D}_{\alpha,W}(h(B)) = \int_B (Jh(x))^\alpha \, d\mathcal{D}_{\alpha,W}(x)$, for every Borel subset $B \subseteq \Omega \cap W$. Jh is the Jacobian of h wrt intrinsic riemannian volume in W.*

This was proved in [11] for hyperbolic repellors. The proof for horseshoes of saddle type is similar and we omit it. Notice too for further use that the several metrics that we are using are indeed comparable, bounded by constants which only depend on the bounded geometry of the admissible manifolds, i.e.

$$d_u(x, y) \asymp \|x - y\| \asymp d^*(x, y) \quad \text{for every} \;\; y \in W^u(x, \mathsf{R}). \tag{11}$$

One can keep track the proof of Ruelle's theorem using bounded distortion estimates in previous Section, in spite that Jacobians might not be Hölder . Compare [6, Chapter III] and [12, Theorem 8].

Actually, let $\psi = -\log J_\alpha \tau$ and $\mathcal{L}_\psi h(x) = \sum_{\{y:\tau(y)=x\}} e^{\psi(y)} h(y)$ be the Ruelle operator. Then, $\mathcal{L}_\psi^* \mathcal{D}_{\alpha,W} = \mathcal{D}_{\alpha,W}$, where $\mathcal{L}_\psi^*$ is the adjoint of the Ruelle operator acting over the Borel measures. We seek for a τ-invariant measure μ_W absolutely continuous wrt $\mathcal{D}_{\alpha,W}$ which is obtained by redistributing it mass through the dynamics, that is

$$\mu_W(B) = \lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} \frac{\mathcal{D}_{\alpha,W}(\tau^{-k} B)}{\mathcal{D}_{\alpha,W}(\Lambda)},$$

for every Borel subset $B \subseteq \Lambda$. Indeed, let $\mu_n = 1\,/\,n \sum_{k=0}^{n-1} (\tau^k)^* \mathcal{D}_{\alpha,W} \,/\, \mathcal{D}_{\alpha,W}(\Lambda)$. μ_n is absolutely continuous wrt $\mathcal{D}_{\alpha,W}$ and it has a Jacobian, namely $h_n(x) = 1\,/\,n \sum_{k=0}^{n-1} \sum_{y \in \tau^{-n} x} e^{S_n \psi(y)}$, where $S_n \psi(x) = \sum_{i=0}^{n-1} \psi(\tau^i(x))$.

The following conditions are sufficient to prove the existence of an asymptotic measure.

$$\sum_{y \in \tau^{-n} x} e^{S_n \psi(y)} \le K \quad \forall \; x \; \in \Lambda, \tag{12}$$

for some constant $K > 0$ and that

$$C_\psi(x,y) = \sup_{n>0} \sup_{y\in\tau^{-n}x} \sum_{i=0}^{n-1} \left[\psi(\tau^i(x)) - \psi(\tau^i(y))\right] \tag{13}$$

exists, is bounded by a constant C_ψ, and $C_\psi(x,y) \longrightarrow 0$ as $d(x,x') \longrightarrow 0$. Cf. [12, Theorem 8]. Condition (12) and (13) imply that the sequence $\{h_n\}$ is uniformly bounded and equicontinuous. h will be a limit point of this sequence which exists by Arzelá-Ascoli theorem. Standard arguments show that h is the unique positive eigenfunction with $\int h(x)d\,\mathcal{D}_{\alpha,W}(x) = \mathcal{D}_{\alpha,W}(\Lambda)$. $d\mu_W(x) = h(x)d\mathcal{D}_{\alpha,W}(x)\,/\,\mathcal{D}_{\alpha,W}(\Lambda)$ is the asymptotic measure we are looking for.

Inequalities (12) and (13) above will be consequence of the following result and arguments in [6, Chapter III, Lemma 1.4].

Lemma 4.3. *There exists a constant* $C = C(\Delta,\lambda,\Gamma) > 1$ *such that*

$$J_\alpha\tau^n(x)\,/\,J_\alpha\tau^n(y) \le \exp(C\,d^*(\tau^n(x),\tau^n(y)))$$

and

$$|J_\alpha\tau^n(x)\,/\,J_\alpha\tau^n(y) - 1| \le C\,d^*(\tau^n(x),\tau^n(y)))$$

whenever $\wp(n,W)(x) = \wp(n,W)(y)$.

Proof. We shall denote below $C = C(\Gamma)$ all the various constants which only depends on the bounded geometry of the admissible submanifolds. Now, denote $J^u(x) = J^uT(x)$ and $h(z) = \prod_{i=1}^{+\infty} J^u(T^iz)\,/\,J^u(T^iH(z))$. Then,

$$\frac{J_\alpha\tau^n(x)}{J_\alpha\tau^n(y)} = \left[\prod_{k=0}^{n-1} \frac{h(T(\tau^k(x)))}{h(T(\tau^k(y)))} \cdot \frac{J^u(\tau^k(x))}{J^u(\tau^k(y))}\right]^\alpha .$$

Now $|\log\, J^u(\tau^k(x)) - \log\, J^u(\tau^k(y))| \le \sup_z \|\nabla\, J^u(z)\| \cdot \|\tau^k(x) - \tau^k(y)\|$. Hence

$$|\log\, J^u(\tau^k(x)) - \log\, J^u(\tau^k(y))| \le C \sup_z \|\nabla\, J^u(z)\| \cdot \|T^k(x) - T^k(y)\|,$$

since $\|\tau^k(x) - \tau^k(y)\| \asymp d^*(\tau^k(x),\tau^k(y)) = d^*(T^k(x),T^k(y)) \asymp \|T^k(x) - T^k(y)\|$, by the bounded geometry of the admissible manifolds. Therefore, using the bounded distortion condition as we did in the above Section we conclude that for every $n > 0$,

$$\begin{aligned}\sum_{k=0}^{n-1} |\log\, J^u(\tau^k(x)) - \log\, J^u(\tau^k(y))| &\le C\sum_{k=0}^{n-1} \lambda^{-k} d^*(\tau^n(x),\tau^n(y)) \\ &\le C(1-\lambda^{-1})d^*(\tau^n(x),\tau^n(y)).\end{aligned}$$

Therefore,

$$\log\left[\prod_{k=0}^{n-1} \frac{J^u(\tau^k(x))}{J^u(\tau^k(y))}\right]^\alpha \le 2C(1-\lambda^{-1})d^*(\tau^n(x),\tau^n(y)).$$

Also, log $h(T(\tau^k(x)))\,/\,h(T(\tau^k(y)))$ is bounded from above by

$$C\max\{d_u(T(\tau^k(x)),T(\tau^k(y)))\,,\,d_u(\tau^{k+1}(x),\tau^{k+1}(y))\},$$

and this, in turn, is bounded by $2Cd^*(\tau^{k+1}(x),\tau^{k+1}(y))$. Thus, for every $n>0$ we have

$$\begin{aligned}\log\left[\prod_{k=0}^{n-1}\frac{h(T(\tau^k(x)))}{h(T(\tau^k(y)))}\right] &\le 2C\sum_{k=0}^{n-1}\lambda^{-n-k-1}d^*(\tau^n(x),\tau^n(y))\\ &\le C(1-\lambda^{-1})d^*(\tau^n(x),\tau^n(y)),\end{aligned}$$

absorbing the various constants into some $C=C(\Gamma)$. That finish the proof. $\square$

The above Lemma and arguments in [12] or [6] proves the existence of μ_W. We use somewhere that $\mathcal{D}_{\alpha,W}(\Omega)\asymp 1$. In fact,

$$\int_\Lambda h_n(x)d\mathcal{D}_{\alpha,W}(x)=\mathcal{D}_{\alpha,W}(\Omega),$$

hence, by Lemma 4.3 $e^{-\delta}\mathcal{D}_{\alpha,W}(\Omega)\le|h_n(x)|\le e^{\delta}\mathcal{D}_{\alpha,W}(\Omega)$, for some $\delta>0$ only depending on the distortion, expanding coefficient bounds and the bounded geometry of admissible manifolds, proving (12). Lemma 4.3 and (12) permit to prove property (13).

μ_W is the unique ergodic τ invariant measure absolutely continuous wrt $\mathcal{D}_{\alpha,W}$. Furthermore, as τ is topologically mixing, it holds

$$\mu_W(B)=\lim_{n\to+\infty}\frac{\mathcal{D}_{\alpha,W}(\tau^{-n}B)}{\mathcal{D}_{\alpha,W}(\Lambda)}.$$

The asymptotic measure μ_W satisfies

$$e^{-2\delta}\le\mu_W(\wp_n(x))J_\mu\tau^n(x)\le e^{2\delta}\tag{14}$$

for every $n\ge 0$, μ-a.e. $x\in\Lambda$. Here $J_\mu\tau$ is the Jacobian of τ wrt the asymptotic measure μ_W. We get these bounds using an standard argument. In fact, $J_\mu\tau^n=J_\alpha\tau^n\cdot h\circ\tau^n\,/\,h$, hence:

$$\frac{J_\mu\tau^n(x)}{J_\mu\tau^n(y)}\in[\frac{\inf_{x\in\Lambda}h(x)]}{\sup_{x\in\Lambda}h(x)},\frac{\sup_{x\in\Lambda}h(x)]}{\inf_{x\in\Lambda}h(x)}]\in[e^{-2\delta},e^{2\delta}],$$

for every $x\,,\,y\in W\,\cap\,P$ and $P\in\wp_n$. As $\tau^n(P)=\Omega\cap W$, then $\mu_W(\tau^n(P)=1$, for every $P\in\wp(n,W)$ the Markov partition of n-cylinders of Λ. Since $\mathcal{D}_{\alpha,W}(\Omega)\asymp 1$ we get a constant $C=C(\Delta,\lambda,\Gamma)>1$ such that, for every $n\ge 0$

$$C^{-1}\mathcal{D}_{\alpha,W}(\Omega)\le\mathcal{D}_{\alpha,W}(\wp_n(x))J_\alpha\tau^n(x)\le C\,\mathcal{D}_{\alpha,W}(\Omega),\tag{15}$$

for every admissible unstable manifold $W\in\Gamma^u$ and for every stable n-cylinder in $\wp_n$. The following shows that μ_W is equivalent to $\mathcal{D}_{\alpha,W}$ bounded by constants which do not depend on $W\in\mathcal{F}^u$.

Lemma 4.4. *The asymptotic measure μ_W is equivalent to the dynamical measure $\mathcal{D}_{\alpha,W}$. Further, there is a constant $C = C(\Delta, \lambda, \Gamma) > 1$ such that the ratios $\mu_W(B) / \mathcal{D}_{\alpha,W}(B)$ are bounded in $\mathcal{D}_{\alpha,W}(\Omega)[C^{-1}, C]$ for every Borel subset $B \subseteq \Lambda$ of positive measure.*

Proof. Observe that $J_\mu \tau^n(x) / J_\alpha \tau^n(x) \in [e^{-2\delta}, e^{2\delta}]$. Therefore, combining (15) and (14) yields

$$\left(\frac{e^{2\delta}C}{\mathcal{D}_{\alpha,W}(\Omega)}\right)^{-1} \leq \mu_W(\wp_n(x)) / \mathcal{D}_{\alpha,W}(\wp_n(x)) \leq \frac{e^{2\delta}C}{\mathcal{D}_{\alpha,W}(\Omega)}, \tag{16}$$

So, for every Borel subset $\mu_W(B) \asymp \mathcal{D}_{\alpha,W}(B)$, since $\wp(W)$ is a generator. □

Lemma 4.5. *Let $\alpha = \dim_u(\Omega)$. Then there is a constant $C = C(\Delta, \lambda, \Gamma) > 1$ such that for every $n > 0$ and $P \in \wp_n$ the following holds*

$$C^{-1}\mathcal{D}_{\alpha,W}(\Omega) \leq \frac{\mathcal{D}_{\alpha,W}(P \cap \Lambda)}{Vol_W(P)^\alpha} \leq C\,\mathcal{D}_{\alpha,W}(\Omega).$$

Proof.

$$\frac{\mathcal{D}_{\alpha,W}(\Omega)}{\sup_{\gamma^u \in \Gamma^u} \mathrm{Vol}\,(\gamma^u)^\alpha} \leq \frac{\mathcal{D}_{\alpha,W}(T^n(P \cap \Lambda))}{\mathrm{Vol}_W(T^n(P))^\alpha} \leq C\,\frac{\mathcal{D}_{\alpha,W}(P \cap \Lambda)}{\mathrm{Vol}_W(P)^\alpha}.$$

Likewise we prove

$$\frac{\mathcal{D}_{\alpha,W}(P \cap \Lambda)}{\mathrm{Vol}_W(P)^\alpha} \leq C\,\frac{\mathcal{D}_{\alpha,W}(\Omega)}{\inf_{\gamma^u \in \Gamma^u} \mathrm{Vol}\,(\gamma^u)^\alpha},$$

where C only depends on the distortion Δ, the expansion coefficient λ and the bounded geometry of the admissible manifolds. We finish the proof, using previous lemma and the bounded geometry of the admissible manifolds. □

Lemma 4.4 and Lemma 4.5 show that the ratios $\mathcal{D}_{\alpha,W}(P) / \mathrm{Vol}_W(P)^\alpha$ and $\mu_W(P) / \mathcal{D}_{\alpha,W}(P)$ are both comparable with $\mathcal{D}_{\alpha,W}(\Omega)$, bounded by constants $C = C(\Delta, \lambda, \gamma, \mathsf{R}) > 1$, for every local unstable manifold $W = W^u(x, \mathsf{R})$ and every cylinder $P \in \wp$ of the generating net. Therefore, $\mu_W(P) \asymp \mathrm{Vol}_W(P)^\alpha$ for every cylinder $P \in \wp$, bounded by constants only depending on the distortion, the expansion coefficient and the bounded geometry of Ω. That concludes the proof of Lemma 4.1.

5 Proof of Theorem A

In view of arguments in preceding Section, to prove Theorem A, we only have to show that the unstable Cantor sets of a C^2 horseshoe with infinitely many branches have non-trivial dynamical measure. For this we will show that Ω is the limit of a sequence of horseshoes with finitely many branches Ω_n and show that the asymptotic measure μ_Ω is a limit of certain Gibbs measures supported on Ω_n.

Lemma 5.1. *Let $\Omega \subseteq \mathbb{R}^m$ be a C^2 hyperbolic horseshoe with infinitely many branches and bounded distortion. Then there exists a sequence $\Omega_n \subseteq \Omega$ of hyperbolic horseshoes with finitely many branches and ergodic measures μ_n such that*

1. $\Omega = \overline{\bigcup_{n>0} \Omega_n}$;
2. $\mu_{n,W}(P) \asymp Vol_W(P)^{\dim_u(\Omega_n)}$ *bounded by some constant $C > 1$ for every local unstable manifold $W = W^u(x, \mathsf{R})$ and for every P in a family of stable cylinders generating the stable lamination of Ω_n where $\mu_{n,W}$ is the projection of μ_n onto W along the stable lamination $\mathcal{F}^s$.*

Suppose proved Lemma 5.1 and let μ^* some limit measure of the sequence μ_n, then: $\mu^*_W(P) \geq C^{-1}\mathrm{Vol}_W(P)^\alpha$ and $\mu^*_W(int\, P) \leq C\,\mathrm{Vol}_W(P)^\alpha$, for every P in the generating net $\wp$. This implies that $0 < \mathcal{D}_{\alpha,W}(\Omega) < +\infty$, so $\dim_u(\Omega) = \lim_{n\to+\infty} \dim_u(\Omega_n)$ and there is an asymptotic measure μ_W for every local unstable manifold $W = W^u(x, \mathsf{R})$. Actually $\mu^*_W = \mu_W$ since they are τ invariant, ergodic and μ_W is absolutely continuous wrt μ^*_W. In fact, $\mu^*_W(P) \geq C^{-1}\mathrm{Vol}_W(P)^\alpha \geq C\mu_W(P)$ for every $P \in \wp$ and a suitable constant $C = C(\Delta, \lambda, \Gamma) > 1$ and this proves our claim. So, $\mu^* = \mu_\Omega$ is the unique ergodic T invariant asymptotic measure that we are looking for and $\{\mu_W\}_{W\in\mathcal{F}^u}$ is a family of conditional measures of μ_Ω respect to the measurable decomposition defined by $\xi(x) = \Omega \cap W^u(x, \mathsf{R})$. That completes the proof of Theorem A.

Proof of Lemma 5.1: let us take Ω_n the horseshoe generated by the first n branches $T_i : S_i \longrightarrow U_i$, $i = 1, \cdots, n$. Clearly, Ω is the topological limit of this sequence.

Let Ω be a C^2 hyperbolic horseshoe with finitely many branches $T : \bigcup_i S_i \longrightarrow \bigcup_i U_i$ in a cube $\mathsf{R} \subseteq \mathbb{R}^m$. Then the following hold:

1. $\alpha = \dim_u(\Omega)$ is the solution to the Bowen equation
$$P(T, -\alpha \log J^uT) = 0$$
for every local unstable manifold $W = W^u(x, \mathsf{R})$;
2. $0 < \mathcal{D}_{\alpha,W}(\Omega) < +\infty$ for every $W = W^u(x, \mathsf{R})$;
3. let μ_α be equilibrium state for the potential $-\alpha \log J^uT$. Then, the projection of μ_α onto $W = W^u(x, \mathsf{R})$ along the stable leaves is the asymptotic measure μ_W.

To prove this we recall that the unstable fiber bundle of a horseshoe with finitely many branches satisfies a Hölder condition. In particular the unstable Jacobian J^uT is Hölder continuous. So we apply Ruelle's theorem to the potential $\phi = -\alpha \log J^uT$ to get a Gibbs measure μ_α. That is, there is a constant $C > 1$ such that

$$C^{-1} \leq \frac{\mu(\wp_n(x))}{\exp(-Pn + S_n\phi(x))} \leq C \tag{17}$$

for every $x \in \Omega$, $n \geq 0$, where $P = P(T, \phi)$ is the topological pressure and $S_n\phi(x) = \sum_{k=0}^{n-1} \phi(T^k(x))$. μ_ϕ is an equilibrium state for ϕ in the following sense $P(T, \phi) = h_\mu(T) + \int \phi(x) d\mu(x)$. Cf. [2] for definitions and proofs. Properties (1) and (2) are entirely similar to those proved for hyperbolic repellors in [11] and we omit the proof. Now, let μ_α be the Gibbs measure for the potential $\phi = -\alpha \log J^u T$. As $\alpha = \dim_u(\Omega)$ is the solution the Bowen equation one shows that $\mu_\alpha(\wp_n(x)) \asymp \mathrm{Vol}\,(\wp_n(x))^\alpha$, using the Gibbs property and volume lemma (15). Therefore, the projection of μ_α onto $W = W^u(x, \mathsf{R})$ and μ_W are equivalent. Being ergodic and τ invariant we conclude that are equal.

Now, let μ_n be the Gibbs measure of $\phi = -\dim_u(\Omega_n) \log J^u T$. The projection of μ_n onto any local unstable manifold $W = W^u(x, \mathsf{R})$ is equal to the asymptotic measure μ_W associated to Ω_n. So $\mu_{n,W}(P) \asymp \mathrm{Vol}_W(P)^{\dim_u(\Omega_n)}$ bounded by constants $C = C(\Delta, \lambda, \Gamma) > 1$ only depending on the distortion properties and the bounded geometry of the admissible manifolds of Ω, so we are done.

6 Computing the Jacobian of the stable lamination

In this Section Ω always denote a C^2 generalized horseshoe constructed in a cube R. Γ^u (resp. Γ^s) are the admissible unstable (resp. stable) manifolds. $\mathcal{F}^s = \{W^s(x, \mathsf{R}) : x \in \Omega\}$ is the stable lamination and $W^s(\Omega) = \bigcup_{x \in \Omega} W^u(x, \mathsf{R})$ is the *stable set of* Ω. As before $\mathcal{D}_{\alpha,\gamma^u}$ is the dynamical unstable measure and $\alpha = \dim_u(\Omega)$. J_α is the Jacobian wrt the dynamical measure class $\mathcal{D}_{\alpha,\gamma^u}$.

Lemma 6.1. *Let $\Lambda = W^s(\Omega) \cap \gamma^u$ and let $h : \Lambda \longrightarrow \mathbb{R}^m$ be a topological embedding. Suppose $\{g_n : W \longrightarrow \mathbb{R}^m\}$ is a sequence of C^2 embeddings such that*

$$g_n \mid \Lambda \rightrightarrows h \quad \text{and} \quad J_\alpha(g_n) \mid \Lambda \rightrightarrows J,$$

uniformly. Then, h is absolutely continuous respect to $\mathcal{D}_{\alpha,\gamma^u}$ and $J_\alpha(h) = J$.

Proof. It is an straightforward adaptation of [9, Lemma 3.2]. □

Let $\gamma^u, \overline{\gamma}^u \in \Gamma^u(\mathsf{R})$. They are cross-sections of the stable lamination so they define a holonomy transformation $h = H_{\gamma^u,\overline{\gamma}^u}$. Now we compute its Jacobian. Fix $p \in W^s(\Omega) \cap \gamma^u$ and $q = h(p) \in W^s(\Omega) \cap \overline{\gamma}^u$.

First we take a C^2 foliation $\mathcal{G}$ of R by admissible stable manifolds. Now, define a sequence of foliations $\mathcal{G}^n$ as the families of discs obtained from iteration by T^{-n}, namely:

$$p \longmapsto \mathcal{G}^n(p) := \text{connected component of } T^{-n}\mathcal{G}(T^n(p)) \cap \mathsf{R} \text{ containing p.}$$

Note that $\mathcal{G}^n$ is a C^2 foliation of the (disjoint) reunion of stable geometrical cylinder of generation n, $\bigcup_{P \in \wp_n} P$. By construction we have holonomy maps $g_n = G^n_{\gamma^u,\overline{\gamma}^u} : D_n \longrightarrow \bar{D}_n$, where $D_n = \gamma^u \cap \bigcup_{P \in \wp_n} P$ and $\bar{D}_n = \overline{\gamma}^u \cap \bigcup_{P \in \wp_n} P$.

Lemma 6.2.
$$\mathcal{G}^n \longrightarrow \mathcal{F}^s \qquad \textit{uniformly}$$

This follows from Graph Transform theory. Therefore:

- $g_n \longrightarrow h$ uniformly and $Q_n = g_n(p) \longrightarrow q$;
- each g_n admits a Jacobian $J_\alpha(g_n)$ wrt $\mathcal{D}_{\alpha,W}$ since the dynamical measure class is conformal.

By a expansion along admissible unstable manifolds or, if you prefer, by λ-lemma arguments we can find two compact discs $\mathbb{D}_n \subseteq D_n$ (resp. $\bar{\mathbb{D}}_n \subseteq \bar{D}_n$) containing $p \in W^s(\Omega) \cap \gamma^u$ (resp. $q \in W^s(\Omega) \cap \overline{\gamma}^u$ such that

$$T^n(\mathbb{D}_n) = \text{ connected component of } T^n(\gamma^u) \text{ containing } p_n = T^n(p),$$

respectively for the other point $q_n = T^n(Q_n)$ and $T^n(\overline{\gamma}^u)$. Indeed, $\mathbb{D}_n(p)$ (resp. $\bar{\mathbb{D}}_n$) is the connected component of D_n containing p (resp. Q_n). Further,

$$\operatorname{dist}_{C^1}(T^n(\mathbb{D}_n), T^n(\bar{\mathbb{D}}_n)) \longrightarrow 0^+. \tag{18}$$

Now, factorize g_n as follows

$$g_n = T^{-n} \circ G^0_{p_n,q_n} \circ (T^n \mid \mathbb{D}_n)$$

where $G^0_{p_n,q_n} = G^0_{T^n(\mathbb{D}_n),\,T^n(\bar{\mathbb{D}}_n)}$. Using that the dynamical measure is conformal we conclude

$$J_\alpha(g_n)(y) = (J(T^{-n} \mid T^n(\bar{\mathbb{D}}_n))^\alpha(T^n(g_n(y)))\, J(G^0_{p_n,q_n})^\alpha(T^n(y))\, J(T^n \mid \mathbb{D}_n)^\alpha(y)$$

for every $y \in \mathbb{D}_n(p)$. By (18) $G^0_{p_n,q_n} \longrightarrow id$ uniformly. Thus $J(G^0_{p_n,q_n}) \longrightarrow 1$ uniformly, too. Now,

$$J(T^{-n} \mid T^n(\bar{\mathbb{D}}_n))(T^n(g_n(y)) = [J(T^n \mid \bar{\mathbb{D}}_n)(g_n(y))]^{-1}.$$

From here and Lemma 6.1 we conclude

$$J_\alpha(h)(p) = \lim_{n\to+\infty} \left[\frac{J(T^n \mid \gamma^u)(p)}{J(T^n \mid \overline{\gamma}^u)(h(p))}\right]^\alpha .$$

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