Fluctuation bounds for subharmonic functions

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Abstract

We obtain bounds for the angular fluctuations of a subharmonic function in terms of the distribution of the angular mean. As an application, we establish positive Lyapunov exponents for matrix cocycles $A_{E,\lambda}(x) = \begin{pmatrix} E - \lambda g(x) & -1 \\ 1 & 0 \end{pmatrix}$ over a measure m preserving dynamical system $T : X \mapsto X$, where $g \in L^\infty(X)$ satisfies $C = \int_X \log|g(x)| dm(x) > -\infty$. For every $\delta > 0$, the set of parameters $\lambda > 0$ for which the cocycle $A_{E,\lambda}$ has Lyapunov exponent $\mu(A_{E,\lambda}) \geq \log(\lambda) + C - \delta$ approaches full Lebesgue density at $\lambda = \infty$. Especially, there are no realanalytic potentials g on $\mathbf{T}^d$ for which for fixed E and fixed measure preserving transformation T, the transfer cocycle $A_{E,\lambda}$ of the random Jacobi operator $\tau + \tau^* + \lambda g$ has a bounded Lyapunov exponent for $\lambda \to \infty$.

1 The homogenization theorem

Let f be a subharmonic function in the complex plane $\mathbf{C}$. By the Riesz decomposition theorem it is the sum of a potential $\int_{\mathbf{C}} \log|z - w| \; dk(w)$ and a harmonic function h. The maximal function $M_f(z) = \sup_{|w|=|z|} f(w)$ is a radial subharmonic function by Hadamard's three circle theorem. Also the mean function $C_f(z) = (2\pi)^{-1} \int_0^{2\pi} f(e^{i\theta} z) \; d\theta$ is a radial subharmonic function. Define $m_f(z) = \inf_{|w|=|z|} f(w)$. Define the measure $\rho_f(r) = (2\pi)^{-1} \int_0^{2\pi} dk(re^{i\theta}) \; d\theta$ on $\mathbf{R}^+$. By the Poisson-Jensen's formula, $C_f(r) = f(0) + \int_0^r \log(r/t) \; d\rho_f(t)$. (We use a notation, where $d\rho_f(t) = \rho(t) \; dt$ if ρ is absolutely continuous with respect to Lebesgue measure).
While estimates for the **upper angular fluctuation** $B_f(z) := M_f(z) - C_f(z) \in \mathbf{R}^+$ can be achieved usually quite well due to its boundedness, finding bounds for the **lower angular fluctuation** $A_f(z) := C_f(z) - m_f(z) \in \mathbf{R}^+ \cup \{\infty\}$ can be difficult. In general, there are no finite upper bounds on $A_f(z)$ as for example when the Riesz measure $dk = (2\pi)^{-1} \Delta f$ of f has atoms. The Homogenization Theorem below gives an upper bound of A_f in terms of the radial behavior of the mean function C_f.
Define $\mathcal{K}_l(\rho_f)(r) = 2^{-l} \sum_{k=l}^\infty \int_0^\infty \exp(-|\log(r/t)|2^k) \; d\rho_f(t)$, where we understand $\mathcal{K}_l(\rho_f) = \infty$ at points, where the sum does not have a finite limit e.g. when the measure ρ_f has an atom at r.
We say f is a **potential** if the harmonic part of f in the Riesz decomposition theorem is a constant.

Theorem 1.1 (Homogenization theorem) *If f is a potential, then $A_f(z) \leq \mathcal{K}(\rho_f)(|z|)$. If additionally $f(z) = f(-z)$ for all $z \in \mathbf{C}$, then $A_f(z) \leq \mathcal{K}_1(\rho_f)(|z|)$.*

Remark. For a general subharmonic function, we additionally could estimate the fluctuations of the harmonic function $h(z)$ on $|z| = r$ using Harnack's inequality.

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2 Homogenization

Let $\xi_k = \exp(2\pi i/2^k)$ denote the 2^k'th primitive root of unity.

Definition. For a subharmonic function f, define **the k'th homogenized subharmonic function** $f^{(k)}(z) = \sum_{j=1}^{2^k} f(\xi_k^j z)$.

Lemma 2.1 (Homogenization) *For a subharmonic function f one has $A_f(z) \leq \sum_{k=0}^{\infty} B_{f^{(k)}}(z)$. If $f(z) = f(\xi_l z)$ for all $z \in \mathbf{C}$, then $A_f(z) \leq \frac{1}{2^l} \sum_{k=l}^{\infty} B_{f^{(k)}}(z)$.*

Proof. Define $\xi_{k,m} = \xi_k \xi_{k-1} \cdots \xi_m$ for $m \leq k$. We apply recursively the identity $f^{(k)}(z) = f^{(k-1)}(z) + f^{(k-1)}(\xi_k z)$, to get

$$
\begin{aligned}
f^{(k)}(z) &= f^{(k-1)}(z) + f^{(k-1)}(\xi_k z) \\
&= f^{(k-1)}(z) + f^{(k-2)}(\xi_k z) + f^{(k-2)}(\xi_{k,k-1} z) \\
&= f^{(k-1)}(z) + f^{(k-2)}(\xi_k z) + f^{(k-3)}(\xi_{k,k-1} z) + f^{(k-3)}(\xi_{k,k-2} z) \\
&\cdots \\
&= f^{(k-1)}(z) + f^{(k-2)}(\xi_k z) + \cdots + f^{(1)}(\xi_{k,3} z) + f^{(0)}(\xi_{k,2} z) + f^{(0)}(\xi_{k,1} z)
\end{aligned}
$$

so that

$$
f^{(0)}(\xi_{k,1} z) = f^{(k)}(z) - \sum_{j=0}^{k-1} f^{(j)}(\xi_{k,j+1} z) \; .
$$

Subtracting the angular means on each side gives

$$
\begin{aligned}
f(\xi_{k,1} z) - C_f(z) &= f^{(k)}(z) - 2^k C_f(z) - \sum_{j=0}^{k-1} (f^{(j)}(\xi_{k,j+1} z) - 2^j C_f(z)) \\
&\geq (f^{(k)}(z) - 2^k C_f(z)) - \sum_{j=0}^{k-1} (M_{f^{(j)}}(z) - C_{f^{(j)}}(z)) \; .
\end{aligned}
$$

We apply this to the subharmonic functions $f_n(z) = 2^{n-1} \int_{-2^{-n}}^{2^{-n}} f(z e^{i 2\pi x}) \, dx$ for which $M_{f_n^{(j)}}(z) \leq M_{f^{(j)}}(z), C_{f_n}^{(j)} = C_f^{(j)}$ and so $B_{f_n^{(j)}} \leq B_{f^{(j)}}$. This gives

$$
f_n(\xi_{k,1} z) - C_f(z) \geq (f_n^{(k)}(z) - 2^k C_f(z)) - \sum_{j=0}^{k-1} B_{f^{(j)}}(z) \; .
$$

Taking the limit $k \to \infty$ gives, using $f_n^{(k)}(z) - 2^k C_f(z) = 0$ for $k \geq n$ and $\xi_{k,1} \to 1$ for $k \to \infty$

$$
(f_n(z) - C_{f_n}(z)) \geq - \sum_{j=0}^{\infty} B_{f^{(j)}}(z) \; .
$$

Since the right hand side is n-independent, we obtain

$$
\inf_n (f_n(z) - C_f(z)) \geq - \sum_{j=0}^{\infty} B_{f^{(j)}}(z) \; .
$$

Because subharmonic functions are upper semicontinuous, we have for given $\alpha > 0$ and $z \in \mathbf{C}$ that $f_n(z) < f(z) + \alpha$ for large n and therefore $\inf_n f_n(z) \leq f(z)$ and obtain

$$(f(z) - C_f(z)) \geq -\sum_{j=0}^{\infty} B_{f^{(j)}}(z) .$$

which is equivalent to the claim.

The case with symmetry $f(z) = f(\xi_l z)$ goes similar but starting with

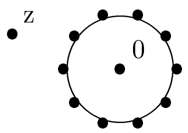
$$2^l f(\xi_{k,1} z) = f^{(k)}(z) - \sum_{j=l}^{k-1} f^{(j)}(\xi_{k,j+1} z) .$$

□

Remark. The case with more symmetry applies when $f(z) = f(z^{2^l})$ for all $z \in \mathbf{C}$. The case $f(z) = f(-z)$ is especially relevant in applications.

3 Proof of the homogenization theorem

We estimate first the upper fluctuation bounds $B_{f^{(k)}}(z)$, when f is the potential $f(z) = \log|z - a|$. In this case, $f^{(k)}(z)$ has a Riesz measure representing a circular equidistributed chain of charges with total charge 2^k.



The crucial fact is that for m charges located on the m'th roots of unity of the unit circle, the upper fluctuation $B_f(z)$ of the potential $f(z) = \log|z^m - 1|$ stays bounded above for $m \to \infty$ and decays exponentially $B_f(z) \leq \exp(-|\log(|z|)|m)$ for $|z| \neq 1$.

Lemma 3.1 (Potential of a circular chain of charges) *If* $f(z) = \log|z - a|$ *with* $|a| = r$, *there are sharp estimates*

$$B_{f^{(k)}}(z) \leq \begin{cases} \log|1 + (\frac{|z|}{r})^{2^k}| , & \text{for } |z| \leq r , \\ \log|1 + (\frac{r}{|z|})^{2^k}| , & \text{for } |z| \geq r \end{cases} .$$

Proof. Factoristion gives for every $m \in \mathbf{N}$

$$(z^m - 1) = \prod_{j=1}^{m} (z - \xi^j) ,$$

where ξ is the primitive complex m-th root of unity.

Therefore, $f^{(m)}(z) = \log|\prod_{j=1}^{m}(z - \xi^j)| = \log|z^m - 1|$ for $f(z) = \log|z - 1|$.

Assume now that $f(z) = \log|z - a|$ with $|a| = r$.

For $|z| < r = |a|$ and $f^{(k)}(z) = 2^k \log(r) + \log|(z/r)^{2^k} - 1|$, one gets

$$B_{f^{(k)}}(z) \leq \log|1 + (\frac{|z|}{r})^{2^k}| \; .$$

For $|z| > r = |a|$ and with $f^{(k)}(z) = 2^k \log|z| + \log|1 - (r/z)^{2^k}|$

$$B_{f^{(k)}}(z) \leq \log|1 + (\frac{r}{|z|})^{2^k}| \; .$$

□

The proof of Theorem (1.1) is obtained along the following steps:

(i) If $f(z) = \log|z - a|$ then $B_{f^{(k)}}(z) \leq \exp(-|\log(r/|z|)|2^k)$. Proof: Because $\log(1 + x) \leq x$ for all $x \geq 0$, we obtain $\log|1 + (|z|/r)^{2^k}| \leq (|z|/r)^{2^k} = \exp(-|\log(r/|z|)|2^k)$ for $|z| > r$ and $\log|1 + r/|z|)^{2^k}| \leq (r/|z|)^{2^k} = \exp(-|\log(r/|z|)|2^k)$ for $|z| < r$. Apply Lemma (3.1).

(ii) "Averaging subharmonic functions does not increase fluctuations". If $\sum_i \alpha_i = 1, \alpha_i > 0$ and $f = \sum_i \alpha_i f_i$, where f_i are subharmonic functions which are potentials, then $M_f(z) \leq \sum_i \alpha_i M_{f_i}(z)$ and $C_f(z) = \sum_i \alpha_i C_{f_i}(z)$ so that $B_f(z) \leq \sum_i \alpha_i B_{f_i}(z)$. More generally, if $x \mapsto f_x(z)$ is a random variable over a probability space $(X, \mathcal{A}, \mu)$ for all z and $z \mapsto f_x(z)$ is subharmonic, then $f(z) = \int_X f_x(z) \, d\mu(x)$ is subharmonic and $B_f(z) \leq \int_X B_{f_x}(z) \, d\mu(x)$.

(iii) Especially, if $f(z) = \int_{\mathbf{C}} f_w \, dk(w)$ where dk is a finite, positive Borel measure in $\mathbf{C}$ and f_w are subharmonic potentials for which B_{f_w} is bounded above, then f is a subharmonic potential and $B_f(z) \leq \int_{\mathbf{C}} B_{f_w} \, dk(w)$.

(iv) If $\text{supp}(dk) \subset \{|z| = r\}$, then $B_{f^{(k)}}(z) \leq \exp(-|\log(r/|z|)|2^k)$.
Proof. From (i), the result follows for measures on $\{|z| = r\}$ which are finite point measures. The general case follows from (iii) by integration over the angles.

(v) For a general finite Borel measure ρ on $\mathbf{R}^+$, we have

$$B_{f^{(k)}}(z) \leq \int_0^\infty \exp(-|\log(\frac{t}{|z|})|2^k) \, d\rho_f(t) \; .$$

Proof. By (ii) and (iv) by integration over the radius.

(vi) The claim of the theorem: from Lemma (2.1) and (v)

$$A_f(z) \leq \sum_{k=0}^\infty B_{f^{(k)}}(z) \leq \sum_{k=0}^\infty \int_0^\infty \exp(-|\log(\frac{t}{|z|})|2^k) \, d\rho(t) = \mathcal{K}(\rho)(z)$$

allowing the right hand side to be $+\infty$.

4 The homogenization transformation

The **homogenization transformations**

$$\mathcal{K}_l : \rho \mapsto \frac{1}{2^l} \sum_{k=l}^\infty \int_{\mathbf{R}^+} \exp(-|\log(\frac{r}{t})|2^k) \, d\rho(t)$$

are defined on the linear space of all finite Borel measures ρ on $\mathbf{R}^+ = [0,\infty)$. The image of $\mathcal{K}_l(\rho)$ is a $\mathbf{R}^+ \cup \{+\infty\}$-valued function on $\mathbf{R}^+$.

Proposition 4.1 ($\mathcal{K}_1$ for $\rho \in L^\infty(\mathbf{R}^+)$.) *If $\rho \in L^\infty(\mathbf{R}^+)$ then $\mathcal{K}_1(\rho)(z) \leq 2||\rho||_\infty |z|$. Especially, $\mathcal{K}_1(\rho) \in L^\infty_{loc}(\mathbf{R}^+ \setminus \{0\})$.*

Proof. Using $\int_0^r (t/r)^m \, dt = r/(1+m)$ and $\int_r^\infty (r/t)^m \, dt = r/(m-1)$, we estimate

$$\begin{aligned} 2\mathcal{K}_1(\rho)(r) &= \sum_{k=1}^\infty \int_0^r (\frac{t}{r})^{2^k} \rho(t)dt + \sum_{k=1}^\infty \int_r^\infty (\frac{r}{t})^{2^k} \rho(t)dt \\ &\leq ||\rho||_\infty (\sum_{k=1}^\infty \int_0^r (\frac{t}{r})^{2^k} dt + \sum_{k=1}^\infty \int_r^\infty (\frac{r}{t})^{2^k} dt) \\ &= 2||\rho||_\infty r \sum_{k=1}^\infty \frac{2^k}{2^{2k}-1} \leq 2||\rho||_\infty r \sum_{k=1}^\infty \frac{1}{(2^k - 2^{-k})} \leq 4||\rho||_\infty r \ . \end{aligned}$$

□

The next proposition can be applied if C_f is Lipschitz on some interval $[r/a, ra]$ in which case ρ_f is bounded on that interval.

Proposition 4.2 ($\mathcal{K}$ for $\rho \in L^\infty_{loc}(\mathbf{R}^+)$) *Assume $\rho(s) \leq M$ for almost all $s \in [r/a, ar]$, with $a > 1$ and let $c = \rho(\mathbf{C})$ be the total mass of ρ. Then $\mathcal{K}(\rho)(z) \leq 2Mr(1-1/a) + (c/2)/(1-1/a)$. Especially, $\mathcal{K}(\rho) \in L^\infty_{loc}(\mathbf{R}^+)$. Taking $a = 2$, we get $\mathcal{K}_1(\rho)(r) \leq Mr + c$ for $r \in [r/2, 2r]$.*

Proof. We get from $\int_{r/a}^r (t/r)^m \, dt = (1-1/a^{m+1})r/(1+m) \leq (1-1/a)r/(1+m)$, $\int_r^{r+a} (r/t)^m \, dt = (1-1/a^{m-1})r/(m-1) \leq (1-1/a)r/(m-1)$ the first term. The second term comes by assuming the worst case, where we have a Dirac measure of maximal mass c at r/a. □

Proposition 4.3 ($\mathcal{K}_1$ for $\rho \in L^p(\mathbf{R}^+)$) *Given $1 < p < \infty$ define q by $1/p + 1/q = 1$. For $\rho \in L^p(\mathbf{R}^+)$,*

$$2\mathcal{K}_1(\rho)(z) \leq ||\rho||_p |z|^{1/q} (2/q)^{1/q} 1/(1-2^{-1/q}) \ .$$

Especially, $\mathcal{K}_1(\rho) \in L^q_{loc}(\mathbf{R}^+ \setminus \{0\})$.

Proof. Write $r = |z|$. With Hölder's inequality one has

$$\int_0^r (\frac{t}{r})^m \rho(t)dt \leq ||(\frac{t}{r})^m||_{L^q(0,r)} ||\rho||_p = (\frac{r}{qm+1})^{1/q} ||\rho||_p$$

and

$$\int_r^\infty (\frac{r}{t})^m \rho(t)dt \leq ||(\frac{r}{t})^m||_{L^q(r,\infty)} ||\rho||_p = (\frac{r}{qm-1})^{1/q} ||\rho||_p \ ,$$

so that

$$\int_0^r (\frac{t}{r})^m \rho(t)dt + \int_r^\infty (\frac{r}{t})^m \rho(t)dt \leq r^{1/q} (\frac{2qm}{q^2 m^2 - 1})^{1/q} ||\rho||_p \leq r^{1/q} (2/qm)^{1/q} ||\rho||_p \ .$$

Summing up the right hand side for $m = 2^k$ gives

$$2\mathcal{K}_1(\rho)(z) \leq ||\rho||_p |z|^{1/q} (2/q)^{1/q} \sum_{k=1}^{\infty} (2^{1/q})^{-k} = ||\rho||_p |z|^{1/q} (2/q)^{1/q} 1/(1-2^{-1/q}) \; .$$

□

Examples. For $p = 2$, we get $\mathcal{K}_1(\rho)(z) \leq (2+\sqrt{2})||\rho||_2 \sqrt{|z|}$. The limiting case $p \to \infty$ becomes the L^∞ case treated above. For $p = 1$, the estimate breaks down.

Definition. Given a subharmonic function f, the potential $f^*(r) = f_{\rho_f}(r) = \int_0^\infty \log|r-s| \; d\rho_f(s)$ is called the $*$**-function** of f. Define $C_\rho(r) = \frac{1}{2\pi} \int_0^{2\pi} f^*(r\exp(i\theta)) \; d\theta$. The function f_ρ is obtained from f by sweeping all masses to a positive (or negative) real axes. The $*$ function appears in results on growth estimates for subharmonic functions.

Proposition 4.4 (Bounds by potential of $*$-function)

$$C_\rho(r) - f_\rho(r) \leq \mathcal{K}(\rho)(r) \leq 2(C_\rho(r) - f_\rho(r)) \; .$$

If $f(z) = f(\xi_l z)$ for all $z \in \mathbf{C}$, then $\mathcal{K}_l(\rho)(r) \leq 2^{-(l-1)}(C_\rho(r) - f_\rho(r))$.

Proof. We can assume that ρ has no atom at r because in that case, all three terms are $+\infty$.

(i)

$$\begin{aligned}
\int_0^r \log|r-s| \; d\rho(s) &= \int_0^r \log|1-\frac{s}{r}| \; d\rho(s) + \int_0^r \log|r| \; d\rho(s) \\
\int_r^\infty \log|r-s| \; d\rho(s) &= \int_r^\infty \log|1-\frac{r}{s}| \; d\rho(s) + \int_r^\infty \log|s| \; d\rho(s) \; .
\end{aligned}$$

(ii)

$$\begin{aligned}
\int_0^r \log|1-\frac{s}{r}| \; d\rho(s) &= -\sum_{k=1}^\infty \int_0^r (\frac{s}{r})^k/k \; d\rho(s) \\
\int_r^\infty \log|1-\frac{r}{s}| \; d\rho(s) &= -\sum_{k=1}^\infty \int_r^\infty (\frac{r}{s})^k/k \; d\rho(s) \; .
\end{aligned}$$

(iii) For $0 < a < 1$,

$$\sum_{k=0}^\infty a^{2^k} \leq 2 \sum_{k=1}^\infty \frac{a^k}{k} \; .$$

Proof. Match each term on the left with a sum on the right hand side: $a \leq 2a/1, a^2 \leq 2a^2/2, a^4 \leq 2(a^3/3 + a^4/4), a^8 \leq 2(a^5/5 + a^6/6 + a^7/7 + a^8/8)$, etc.

(iv) From (iii), we get

$$\mathcal{K}(\rho)(r) = \sum_{k=0}^\infty \int_{\mathbf{R}^+} \exp(-|\log(r/t)|2^k) \; d\rho(t)$$

$$
\begin{aligned}
&\le 2\sum_{k=0}^{\infty}\int_{\mathbf{R}^+}\exp(-|\log(r/t)|k)/k\; d\rho(t)\\
&= -2\left(\int_0^r \log|1-\frac{s}{r}|\; d\rho(s) + \int_r^{\infty}\log|1-\frac{r}{s}|\; d\rho(s)\right)\\
&= -2\left(\int_0^{\infty}\log|r-s|\; d\rho(s) + \int_0^r \log|r|\; d\rho(s) + \int_r^{\infty}\log(s)\; d\rho(s)\right)
\end{aligned}
$$

so that

$$
\mathcal{K}(\rho)(r) \le 2\left(-f_\rho(r) + \int_0^r \log(r)\; d\rho(s) + \int_r^{\infty}\log(s)\; d\rho(s)\right) .
$$

(v) Subtracting and adding $\int_0^r \log(s)\; d\rho(s)$ in the second and third term respectively gives, using $\rho(\{r\}) = 0$

$$
\mathcal{K}(\rho)(r) \le 2\left(-f_\rho(r) + \int_0^r \log(\frac{r}{s})\; d\rho(s) + f_\rho(0)\right) .
$$

(vi) With the Poisson-Jensen formula $C_\rho(r) = \int_0^r \log(\frac{r}{s})\; d\rho(s) + f_\rho(0)$, the estimate in (v) can be rewritten as

$$
\mathcal{K}(\rho)(r) \le 2(C_\rho(r) - f_\rho(r)) .
$$

(vii) The inequality $-(f_\rho(r) - C_\rho(r)) \le \mathcal{K}(\rho)(r)$ is a consequence of the Homogenization theorem applied to the case, when dk has support in the positive real line.

(viii) If $f(z) = f(\xi_l z)$, things are similar. □

Remarks.
1) The inequalities can be rewritten as $A_{f^*}(r) \le \mathcal{K}(\rho_f)(r) \le 2A_{f^*}(r)$. If $f(z) = f(-z)$, then $\mathcal{K}_1(\rho_f)(r) \le A_{f^*}(r)$. While the lower bound $A_{f^*}(r) \le \mathcal{K}(\rho_f)(r)$ is sharp as the example shows, when $f = f^*$, we don't expect the upper bound to be sharp. Is a general estimate $\mathcal{K}(\rho_f)(r) \le aA_{f^*}(r)$ with $a < 2$ possible?
2) It follows that $\mathcal{K}(\rho)(r)$ is finite if ρ is absolutely continuous with respect to the $\alpha > 0$-dimensional Hausdorff measure in an open interval containing r.
3) The set $S_\rho = \{t \in \mathbf{R}^+, \mathcal{K}(\rho)(t) = \infty\}$ is a G_δ polar set. Especially, it has Lebesgue measure zero.
4) The potential $f(x) = \int_{\mathbf{R}^+}\log|x-w|\; d\rho(w)$ is as a function on the real line equal to the Hilbert transform of the function $h(x) = \int_0^x d\rho(y)$. The Hilbert transform maps α-Hölder spaces or L^p spaces into itself. We see especially that if ρ is bounded then h is Lipschitz and f α-Hölder for all $\alpha \in (0,1)$. Because also C_f is Lipschitz if ρ is bounded, Proposition (4.4) implies that if ρ is bounded, then $\mathcal{K}(\rho)$ is sandwiched between two Hölder continuous maps.

Corollary 4.1 *There exists a real sequence* $r_n \to 0$ *such that* $\limsup_{n\to\infty}\mathcal{K}(\rho)(r_n) = 0$.

Proof. It is a consequence of the Milloux-Schmidt inequality [1] that for any subharmonic function f with $f(0) > -\infty$ one has $\limsup_{r\to 0}\inf_\theta f(r\exp(i\theta)) = f(0)$. Therefore $\limsup_{r\to 0} f_\rho(r) = f_\rho(0)$. Together with $\lim_{r\to 0} C_\rho(r) = C_\rho(0)$ which follows from the upper-semicontinuity and the submean inequality for subharmonic functions, we get $\limsup_{r\to 0} 2(f_\rho(r) - C_\rho(r)) = 0$. The claim follows from Proposition (4.4). □

5 Radially mollified subharmonic functions

The homogenization theorem is especially useful if $d\rho_g$ is bounded. It can be applied to cases, where ρ is smeared out radially in such a way that the mollified ρ comes from a radially mollified subharmonic function.

Definition. Let ϕ be smooth positive function on $\mathbf{R}^+$ of compact support $K_\phi \subset \mathbf{R}^+$ satisfying $\phi(x) = \phi(1/x)$. If f is a subharmonic function, define a **radially mollified function** as $f_\phi(z) = \int_{\mathbf{R}} f(\beta z)\, d\phi(\beta)$.

Lemma 5.1 *If f is a subharmonic potential, then $f_\phi(z)$ is a subharmonic potential, ρ_{f_ϕ} has the same smoothness as ϕ and $|\rho_{f_\phi}(r)| \leq ||\phi_\infty||\rho_f(K_\phi r)$.*

Proof. A positive average of subharmonic functions $f_\beta(z) = f(\beta z)$ is subharmonic.

Taking the Laplacian in the distributional sense on $f_\phi(z) = \int_{\mathbf{R}} f(\beta z)\ d\phi(\beta)$ shows $dk_{f_\phi} = \int_0^\infty dk(\beta z)\ d\phi(\beta)$ so that f_ϕ is a potential.

The function $\rho_{f_\phi}(r) = \int_0^\infty \rho(\beta r)\ d\phi(\beta) = \int_0^\infty \rho(\beta^{-1} r)\ d\phi(\beta) = \int_{-\infty}^\infty \tilde{\rho}(r' - \beta')\tilde{\phi}(\beta')\ d\beta' = (\tilde{\phi} \star \tilde{\rho})(r')$ is in logarithmic coordinates $r' = \log(r)$ a convolution with $\tilde{\phi}(r') = \phi(\exp(r'))$ and has therefore the same smoothness as ϕ.

The estimate $|\rho_{f_\phi}(r)| \leq ||\phi_\infty||\rho(K_\phi r)$ is obtained by assuming ρ to have an atom of mass $\rho(K_\phi r)$ at r in which case we have equality. $\square$

Remark. While we could also smooth f using the standard smoothing theorem in the theory of subharmonic functions, the radial smoothing will be more convenient for the applications later on.

Notation. Denote by $|Y|$ the Lebesgue measure of a measurable subset Y of $\mathbf{R}$. For $Y \subset \mathbf{R}$ and $x \in \mathbf{R}$ call $\limsup_{\epsilon \to 0} |Y \cap [x - \epsilon, x + \epsilon]|/(2\epsilon)$ the **Lebesgue density** of Y at x. (It is equal to $M1_Y(x)$, where Mf denotes the Hardy-Littlewood maximal function of f).

We have already used the consequence of the Milloux-Schmidt inequality that for any subharmonic function f $\limsup_{\lambda \to 0} \inf_\theta f(\lambda \exp(i\theta)) = f(0)$. The following Corollary is stronger and provides also a new proof to this fact:

Theorem 5.1 *Let f be an arbitrary subharmonic function and let $z \in \mathbf{C}$ given for which $f(z) > -\infty$. For every $\delta > 0$ and every line $z + \lambda e^{i\theta}$ through z, the Lebesgue density of $Y = \{\lambda \in \mathbf{R} \mid f(z + \lambda e^{i\theta}) > f(z) - \delta\ \}$ at $\lambda = 0$ is equal to 1.*

Proof. Because the claim is obviously true for harmonic functions, it is enough to prove the statement for potentials $f(z) = \int_{\mathbf{C}} \log|z - w|\ dk(w)$. After a translation, we can assume $z = 0$. If K is a compact subset of $\mathbf{C}$, then $dk(K) < \infty$, otherwise, f would be constant $-\infty$. We have $B_f(z) = M_f(z) - C_f(z) \to 0$ for $z \to 0$. Let I be an open ball around 0 such that $B_f(z) \leq \delta/3$ for $z \in I$ and therefore $M_f(z) - f(0) \leq \delta/3$ for $z \in I$. Define $g(z) = (f(z) + f(-z))/2$. We will show that the Lebesgue density of $Y = \{\lambda \mid g(\lambda e^{i\theta}) \geq g(0) - \delta/3\}$ is 1 at $\lambda = 0$. This implies the claim because in $Y \cap I$, $g(\lambda e^{i\theta}) > g(0) - \delta/3 = f(0) - \delta/3$, implies together with $M_f(-\lambda e^{i\theta})) - f(0) \leq \delta/3$ that $f(\lambda e^{i\theta}) = 2g(\lambda e^{i\theta}) - f(-\lambda e^{i\theta}) \geq 2f(0) - 2\delta/3 - (f(0) - \delta/3) = f(0) - \delta$.

Let ϕ be a radial mollifier function with support in $[1/2, 2]$ and let g_ϕ be the corresponding radially mollified subharmonic potential. Applying Lemma (5.1) and the Homogenization Theorem, we get, writing $\rho_\phi = \rho_{g_\phi}$

$$A_{g_\phi}(z) \leq \mathcal{K}_1(\rho_\phi)(r) \leq 2||\rho_\phi||\rho([r/2, 2r])r$$

and the right hand side is $\leq \delta/3$ for small enough r.

Because $\sup_{\beta \in [1/2,2]} B_g(\beta r) \to 0$ for $r \to 0$ and $A_g(\beta r) \leq \delta/3$ for a set of $\beta \in [1/2, 2]$ which reaches Lebesgue measure $3/2$ for $r \to 0$, the normalized Lebesgue measure of parameters $s \in [r/2, 2r]$ with $A_g(s) \leq \delta/3$ approaches 1 for $r \to 0$. $\square$

This result shows that for "most" points w near z, $f(w)$ is near $f(z)$. A subharmonic function f behaves in a probabilistic sense like a continuous function: if a sequence $z_n \to z$ is chosen at 'random' in the product probability space of a nested sequence of open intervals $I_n \subset \mathbf{R}$ around 0 (all equipped with the normalized Lebesgue measure and $\bigcap_n I_n = \{0\}$) , then with probability 1, a point $x = \{x_n\}$ in the product probability space satisfies $f(z + e^{i\theta} x_n) \to f(z)$. For example, if f is the at 0 discontinuous subharmonic function $f = \sum_{n=1}^\infty 2^{-n} \log|z - 2^{-n}|$ and x_n is an independent, identically distributed sequence of random variables with uniform distribution in $[-1, 1]$ then $f(x_n/n) \to f(0)$ with probability 1.

6 Lyapunov exponents of matrix cocycles

Let $(X, \mathcal{A}, m)$ be a probability space and let $T : X \mapsto X$ be a measure preserving invertible transformation. Given $g \in L^\infty(X)$ for which $\int_X \log|g(x)| \, dm(x)$ is finite. Given $E, \lambda \in \mathbf{R}$, the matrix-valued map $A_\lambda(x) = \begin{pmatrix} E - \lambda g(x) & -1 \\ 1 & 0 \end{pmatrix}$ defines a cocycle $(n, x) \mapsto A_\lambda^n(x) = A_\lambda(T^{n-1}x) \cdots A_\lambda(x)$. Denote by $\mu(A_\lambda) = \lim_{n\to\infty} n^{-1} \int_X \log ||A_\lambda^n(x)|| \, dm(x)$ the Lyapunov exponent of A_λ. Define $f(\epsilon) = \mu(B_\epsilon)$, where $\epsilon = 1/\lambda$ and $B_\epsilon(x) := \begin{pmatrix} \epsilon E + g(x) & -\epsilon \\ \epsilon & 0 \end{pmatrix}$. We have $\mu(A_\lambda) = f(\epsilon) - \log|\epsilon| = f(\epsilon) + \log|\lambda|$. The aim is to estimate the subharmonic function $\epsilon \mapsto f(\epsilon)$ from below, using $f(0) = \int_X \log|g(x)| \, dx > -\infty$. The later will be a standing assumption in this section.

Lemma 6.1 *f is a subharmonic function which is a potential: there exists a constant C such that $f(\epsilon) = C + \int_{\mathbf{C}} \log|\epsilon - w| \, dk(w)$. The total mass of the Riesz measure dk is 1. If there exists a measure preserving involution S on X such that $g(x) = -g(S(x))$ for almost all $x \in X$, then $f(\epsilon) = f(-\epsilon)$.*

Proof. It is an observation of Herman that the Lyapunov exponent of an analytically parameterized cocycle A is subharmonic as the lim inf of subharmonic functions $f_n(\epsilon) = 2^{-n} \int_X \log ||A_\epsilon^{2^n}(x)|| \, dm(x)$ satisfying $f_n \leq f_m$ for $n \leq m$.

By the Riesz decomposition theorem, the Lyapunov exponent can be written as $f(\epsilon) = \int_{\mathbf{C}} \log|\epsilon - w| \, dk(w) + h(\epsilon)$, where $\epsilon \mapsto h(\epsilon)$ is harmonic.

With the constant $C = \mu(B_0)$, we have for $\epsilon \to \infty$ $f(\epsilon) = C + \log|\epsilon| + O(1)$. This shows that the order of f (defined as $\limsup_{r\to\infty} \log|M_f(r)|/\log|r|$) is equal to 1 implying that the harmonic function h in the Riesz decomposition theorem must be constant.

If $g(x) = -g(S(x))$, the pointwise Lyapunov exponent $\mu(\lambda, x) = \lim_{n\to\infty} n^{-1} \log ||A_\lambda^n(x)||$ (which exists according to Kingman's subadditive ergodic theorem m-almost everywhere), satisfies $\mu(-\lambda, S(x)) = \mu(\lambda, x)$ so that the average over X satisfies $\mu(-\lambda) = \mu(\lambda)$ implying $f(\epsilon) = f(-\epsilon)$. □

Theorem 6.1 *For a fixed dynamical system, there exists a sequence $\lambda_m \to \infty$ such that*

$$\limsup_{m\to\infty} \mu(A_{\lambda_m}) - \log|\lambda_m| - \int_X \log|g(x)|\, dx \geq 0\ .$$

Proof. This is consequence of Corollary (4.1) and the homogenization theorem. □

Example. For $g(x) = 2x + 2\cos(x)$ we obtain $\limsup_{m\to\infty} \mu(A_{\lambda_m}) - \log(|\lambda_m|/2) \geq 0$.

Theorem 6.2 *Let ϕ be a radial mollifier function. Then*

$$\int_0^\infty \mu(A_{\lambda\beta})\, d\phi(\beta) \geq \log|\lambda| + \int_X \log|g(x)|\, dx - 2||\phi||_\infty/\lambda\ .$$

Let ϕ be a radial mollifier function such that $\rho_\phi(s) \leq M_\phi$ for almost all $s \in [r/2, 2r]$. Then

$$\int_0^\infty \mu(A_{\lambda\beta})\, d\phi(\beta) \geq \log(|\lambda|) + \int_X \log|g(x)|\, dx - 2M_\phi/\lambda - 2c\ ,$$

where $c = \rho(\mathbf{C})$.

Proof. The statements follow from the Homogenization theorem, Lemma (5.1) and the estimates in Proposition (4.1) rsp. Proposition (4.2) which can be applied using Lemma (6.1). □

Example. For $g(x) = 2\cos(x)$, where $S(x) = x + \pi$ produces the symmetry $f(\epsilon) = f(-\epsilon)$, this gives $\int_{1/2}^2 \mu(A_{\lambda\beta})\, d\phi(\beta) \geq \log(|\lambda|/2) - 2|\phi|_\infty/|\lambda|$.

The set of parameters for which the cocycle has positive Lyapunov exponent reaches full Lebesgue density at $\lambda \to \infty$. This follows from Theorem (5.1). We state this here in a more quantitative way:

Corollary 6.1 *There exists a constant C such that for all λ_0 the Lebesgue measure of parameters $\lambda \in [\lambda_0/2, 2\lambda_0]$ for which $\mu(A_\lambda) = 0$ is $\leq C(3\lambda_0/2)/\log|\lambda_0|$.*

Proof. For $\lambda \in [\lambda_0/2, 2\lambda_0]$ we have $\mu(A_\lambda) \leq c_1 + \log(|\lambda_0|) + C_1/|\lambda_0|$ and $\int_{1/2}^2 \mu(A_{\lambda \cos})\, d\phi(\beta) \geq c_2 + \log(|\lambda_0|) - C_2/|\lambda_0|$. Therefore, for each λ_0 the normalized Lebesgue measure x of the subset $\{\lambda \in [\lambda_0/2, 2\lambda_0] \mid \mu(A_\lambda) = 0\ \}$ satisfies $(1-x) \geq (c_1 + \log(|\lambda_0|) - C_2/|\lambda_0|)/(c_2 + \log(|\lambda_0|) + C_1/|\lambda_0|)$ which implies that there exists a constant C such that $x \leq C/\log(\lambda_0)$ for some C. To get the Lebesgue measure, we have to multiply this by $|[\lambda_0/2, 2\lambda_0]| = 3\lambda/2$. □

The homogenization result can be applied to realanalytic maps $T : \mathbf{T}^2 \to \mathbf{T}^2$ which extend to entire maps in $\mathbf{C}^2$. An example is the **Chirikov-Standard map family**

$$T_\lambda(x, y) = (2x - y + \lambda\sin(x), x)$$

which is a family of Lebesgue measure-preserving maps on $\mathbf{T}^2$ and which extends to an entire map on $\mathbf{C}^2$. The Jacobean $dT_\lambda^n(x,y)$ is an analytically parameterized cocycle: the matrix entries $\lambda \mapsto dT_\lambda^n(x,y)$ are complex polynomials in λ. Therefore, for fixed n, the map

$$\lambda \mapsto \mu_n(\lambda) := \frac{1}{n} \int_{\mathbf{T}^2} \log ||dT_\lambda^n(x,y)||\; dxdy$$

is subharmonic as an integral of subharmonic functions. (Note that $T^n(x,y)$ is now complex in general for $(x,y) \in \mathbf{T}^2$.) Applying Corollary (4.1) and Corollary (4.1) gives information about the **finite time Lyapunov exponents** which are of physical interest.

Corollary 6.2 *Let T_λ be the Chirikov-Standard family. For fixed n, there exists a real sequence $\lambda_m \to \infty$ such that the finite time Lyapunov exponent μ_n satisfies $\mu_n(\lambda_m) - \log(\lambda_m/2) \to 0$ for $m \to \infty$. Moreover, for every $\delta > 0$ and every $n \in \mathbf{N}$, the set of parameters λ satisfying $\mu_k(\lambda) \geq \log(\lambda/2) - \delta$ simultaneously for all $k = 1, \dots, n$ has full Lebesgue density at $\lambda = \infty$.*

Our argument does not allow us here to pass to the limit $n \to \infty$. Nevertheless, this Corollary supports optimism that there should be a positive answer to the open question, whether for the Chirikov-Standard map $\mu_n(\lambda) \geq \log(|\lambda|/2)$ for all $\lambda \in \mathbf{R}$ and all $n \geq 1$.

Theorem (5.1) has an application in spectral theory:

Corollary 6.3 *Assume the transfer cocycle A_E of a discrete ergodic selfadjoint Jacobi matrix has a positive Lyapunov exponent at some energy $E \in \mathbf{R}$ for which the spectrum $\sigma(L)$ has positive Lebesgue density* $\limsup_{\epsilon \to 0} |\sigma(L) \cap \{[E-\epsilon, E+\epsilon]\}|/(2\epsilon) > 0$. *Then L has some spectrum which is not absolutely continuous.*

Proof. We apply Theorem (5.1) to the complex parameterization $E \mapsto A_E$. Let $S = \{E \in \mathbf{R} \mid \mu(A_E) > 0\}$. If $\mu(A_E) > 0$, then $\limsup_{\epsilon \to 0} |S \cap \{[E-\epsilon, E+\epsilon]\}|/(2\epsilon) = 1$. Therefore $S \cap \sigma(L)$ must have positive Lebesgue measure. Kotani theory tells that on the set $S \cap \sigma(L)$ the spectrum is not absolutely continuous. $\square$

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