Inverse Scattering for the Nonlinear Schrödinger Equation. Reconstruction of the Potential and the Nonlinearity *

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Abstract

In this paper we consider the inverse scattering problem for the nonlinear Schrödinger equation on the line:

$$i\frac{\partial}{\partial t}u(t,x) = -\frac{d^2}{dx^2}u(t,x) + V_0(x)u(t,x) + \sum_{j=1}^{\infty} V_j(x)|u|^{2(j_0+j)}u(t,x).$$

We prove, under appropriate conditions, that the small-amplitude limit of the scattering operator determines uniquely, $V_j, j = 0, 1, \cdots$. Our proof gives also a method for the reconstruction of the $V_j, j = 0, 1, \cdots$.

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1 Introduction

We have recently studied the inverse scattering problem for the Schrödinger equation on the line [28]. We gave a method for the unique reconstruction of the potential and of a nonlinearity of the type $\lambda|u|^p u$. In this paper we extend the results of [28] to the case of a general class of inhomogeneous nonlinearities. The main new technical tool that allows us to consider this class of nonlinearities is the continuity of the Schrödinger wave operators on the Sobolev spaces $W_{k,p}$ [1], that we proved in [29].

Let us consider the following nonlinear Schrödinger equation with a potential (NLSP):

$$i\frac{\partial}{\partial t}u(t,x) = -\frac{d^2}{dx^2}u(t,x) + V_0(x)u(t,x) + F(x,u), u(0,x) = \phi(x), \tag{1.1}$$

where $t, x \in \mathbf{R}$, the potential, V_0, is a real-valued function and $F(x,u)$ is a complex-valued function.

The first problem that we consider is the construction of the scattering operator for the NLSP. For this purpose we first introduce some notations and definitions. As usual, we say that $F(x,u)$ is a C^k function of u in the real sense if for each $x \in \mathbf{R}$, $\Re F$ and $\Im F$ are C^k functions with respect to the real and imaginary parts of u. Below we assume that F is C^2 in the real sense and that $\left(\frac{\partial}{\partial x}F\right)(x,u)$ is C^1 in the real sense. If $F = F_1 + iF_2$ with F_1, F_2 real-valued, and $u = r + is$, $r, s \in \mathbf{R}$ we denote,

$$F^{(2)}(x,u) := \sum_{j=1}^{2}\left[\left|\frac{\partial^2}{\partial r^2}F_j(x,u)\right| + \left|\frac{\partial^2}{\partial r \partial s}F_j(x,u)\right| + \left|\frac{\partial^2}{\partial s^2}F_j(x,u)\right|\right], \tag{1.2}$$

$$\left(\frac{\partial}{\partial x}F\right)^{(1)}(x,u) := \sum_{j=1}^{2}\left[\left|\frac{\partial}{\partial r}\left(\frac{\partial}{\partial x}F_j\right)(x,u)\right| + \left|\frac{\partial}{\partial s}\left(\frac{\partial}{\partial x}F_j\right)(x,u)\right|\right]. \tag{1.3}$$

For any $\gamma \in \mathbf{R}$ let us denote by L^1_γ the Banach space of all complex-valued measurable functions, ϕ, defined on $\mathbf{R}$ and such that

$$\|\phi\|_{L^1_\gamma} := \int |\phi(x)|\,(1+|x|)^\gamma\, dx < \infty. \tag{1.4}$$

If $V_0 \in L^1_1$ the differential expression $\tau := -\frac{d^2}{dx^2} + V_0(x)$ is essentially self-adjoint on the domain

$$D(\tau) := \left\{\phi \in L^2_C : \phi \,\text{and}\, \frac{d}{dx}\phi \,\text{are absolutely continuous and}\, \tau\phi \in L^2\right\}, \tag{1.5}$$

where L^2_C denotes the set of all functions on L^2 that have compact support. We denote by H the unique self-adjoint realization of τ. It is known (see [5], [31]) that H has a finite number of negative eigenvalues, that it has no positive or zero eigenvalues, that it has no singular-continuous spectrum and that the absolutely-continuous spectrum is $[0,\infty)$. By H_0 we denote the unique self-adjoint realization of $-\frac{d^2}{dx^2}$ with domain the Sobolev space $W_{2,2}$. The wave operators are given by:

$$W_{\pm} := s - \lim_{t\to\pm\infty} e^{itH}\, e^{-itH_0}. \tag{1.6}$$

It is proven in [20] that the limits in (1.6) exit in the strong topology in L^2 and that Range$W_{\pm} = \mathcal{H}_c$, the subspace of continuity of H. The adjoints of the $W_{\pm}$ are given by

$$W_{\pm}^{*} = s - \lim_{t\to\pm\infty} e^{itH_0}\, e^{-itH} P_c, \tag{1.7}$$

where P_c denotes the orthogonal projector onto $\mathcal{H}_c$. The scattering operator for the linear Schrödinger equation (equation (1.1) with $F=0$) is given by:

$$S_L := W_{+}^{*}\, W_{-}. \tag{1.8}$$

For any pair u, v of solutions to the stationary Schrödinger equation:

$$-\frac{d^2}{dx^2}u + V_0 u = k^2 u,\; k \in \mathrm{C}, \tag{1.9}$$

let $[u,v]$ denotes the Wronskian of u, v:

$$[u,v] := \left(\frac{d}{dx}u\right) v - u\frac{d}{dx}v. \tag{1.10}$$

Let $f_j(x,k), j = 1,2, \Im k \geq 0$, be the Jost solutions to (1.9) (see [6], [7], [5], [4] and [28]). A potential V_0 is said to be *generic* if $[f_1(x,0), f_2(x,0)] \neq 0$ and V_0 is said to be *exceptional* if $[f_1(x,0), f_2(x,0)] = 0$. If V_0 is *exceptional* there is a bounded solution to (1.9) with $k^2 = 0$ (a half-bound state or a zero-energy resonance). For these definitions and related issues see [14]. Note that the trivial potential, $V_0 = 0$, is *exceptional.* We denote: $V_0^{(l)} := \frac{d^l}{dx^l}V_0(x)$. Clearly, $V_0^{(0)} = V_0$. We designate,

$$M := \left\{ u \in C(\mathbf{R}, W_{1,p+1}) : \sup_{t\in\mathbf{R}} (1+|t|)^d \|u\|_{W_{1,p+1}} < \infty \right\}, \text{with norm} : \|u\|_M := \sup_{t\in\mathbf{R}} (1+|t|)^d \|u\|_{W_{1,p+1}}, \tag{1.11}$$

where $p \geq 1$, and $d := \frac{1}{2}\frac{p-1}{p+1}$. For functions $u(t,x)$ defined in $\mathbf{R}^2$ we denote $u(t)$, for $u(t,\cdot)$. The small-amplitude scattering operator is given in the following theorem.

THEOREM 1.1. *Suppose that* $V_0 \in L^1_\gamma$*, where in the* generic case $\gamma > 3/2$ *and in the* exceptional case $\gamma > 5/2$*, that* H *has no negative eigenvalues, and that*

$$N(V_0) := \sup_{x \in \mathbf{R}} \int_x^{x+1} |V_0(y)|^2 \, dy < \infty. \tag{1.12}$$

Furthermore, assume that F *is* C^2 *in the real sense, that* $F(x,0) = 0$*, and that for each fixed* $x \in \mathbf{R}$ *all the first order derivatives, in the real sense, of* F *vanish at zero. Moreover, suppose that* $\frac{\partial}{\partial x} F$ *is* $C^{(1)}$ *in the real sense and that* $\left(\frac{\partial}{\partial x} F\right)(x,0) = 0$ *for all* $x \in \mathbf{R}$*. We assume that the following estimates hold:*

$$F^{(2)}(x,u) = O\left(|u|^{p-2}\right), \; \left(\frac{\partial}{\partial x} F\right)^{(1)} (x,u) = O\left(|u|^{p-1}\right), \; u \to 0, \text{ uniformly for } x \in \mathbf{R}, \tag{1.13}$$

for some $\rho < p < \infty$*, and where* ρ *is the positive root of* $\frac{1}{2}\frac{\rho-1}{\rho+1} = \frac{1}{\rho}$*. Then, there is a* $\delta > 0$ *such that for all* $\phi_- \in W_{2,2} \cap W_{1,1+\frac{1}{p}}$ *with* $\|\phi_-\|_{W_{2,2}} + \|\phi_-\|_{W_{1,1+\frac{1}{p}}} \leq \delta$ *there is a unique solution,* u*, to (1.1) such that* $u \in C(\mathbf{R}, W_{1,2}) \cap M$ *and,*

$$\lim_{t \to -\infty} \|u(t) - e^{-itH}\phi_-\|_{W_{1,2}} = 0. \tag{1.14}$$

Moreover, there is a unique $\phi_+ \in W_{1,2}$ *such that*

$$\lim_{t \to \infty} \|u(t) - e^{-itH}\phi_+\|_{W_{1,2}} = 0. \tag{1.15}$$

Furthermore, $e^{-itH}\phi_\pm \in M$ *and*

$$\left\| u - e^{-itH}\phi_\pm \right\|_M \leq C \left\| e^{-itH}\phi_\pm \right\|_M^p, \tag{1.16}$$

$$\|\phi_+ - \phi_-\|_{W_{1,2}} \leq C \left[\|\phi_-\|_{W_{2,2}} + \|\phi_-\|_{W_{1,1+\frac{1}{p}}} \right]^p. \tag{1.17}$$

The scattering operator, $S_V : \phi_- \hookrightarrow \phi_+$ *is injective on* $W_{1,1+\frac{1}{p}} \cap W_{2,2}$*.*

In Theorem 1.1 we do not need to restrict F in such a way that energy is conserved. Moreover, $\rho \approx 3.56$. We prove Theorem 1.1 in Section 2 adapting to this case the proof, of a similar result, given in [30] in the case of the nonlinear Klein-Gordon equation. There many results on scattering for the nonlinear Schrödinger equation in the case where $V_0 = 0$. See [23], [24], [25], [13], [11], [16], [3], [8], [2] and the references quoted there. In [10] the direct scattering for (1.1) with $F = F(u)$ was studied for $n \geq 3$. The corresponding inverse problem was considered in [27].

To reconstruct the potential, V_0, we introduce below the scattering operator that relates asymptotic states that are solutions to the linear Schrödinger equation with potential zero:

$$S := W_+^* \, S_V \, W_-. \tag{1.18}$$

We now reconstruct S_L from S.

THEOREM 1.2. *Suppose that the assumptions of Theorem 1.1 are satisfied. Then for every* $\phi_- \in W_{2,2} \cap W_{1,1+\frac{1}{p}}$

$$\frac{d}{d\epsilon} S(\epsilon\phi)\Big|_{\epsilon=0} = S_L \phi, \tag{1.19}$$

where the derivative in the left-hand side of (1.19) exists in the strong convergence in $W_{1,2}$.

COROLLARY 1.3. *Under the conditions of Theorem 1.1 the scattering operator, S, determines uniquely the potential V_0.*

Proof: By Theorem 1.2 S determines uniquely S_L. From S_L we get the reflection coefficients for linear Schrödinger scattering on the line (see Section 9.7 of [15] and [28]). As H has no bound states we uniquely reconstruct V_0 from one of the reflection coefficients by using any method for inverse scattering on the line(see for example [6], [7], [5], [12], [4], [9]).

■

In the case where $F(x,u) = \sum_{j=1}^{\infty} V_j(x) |u|^{2(j_0+j)} u$ we can also reconstruct the $V_j, j = 1, 2, \cdots$. We first introduce some notation. For $\lambda > 0$ and $\acute{x} \in \mathbf{R}$ we denote by H_λ the following self-adjoint operator in L^2:

$$H_\lambda := H_0 + V_\lambda(x), \text{ where } V_\lambda(x) = \frac{1}{\lambda^2} V_0(\frac{x}{\lambda} + \acute{x}). \tag{1.20}$$

Since H has no eigenvalues, we have that H_λ has no eigenvalues, i.e., $H_\lambda > 0$. Let Φ be any continuous and bounded function on $\mathbf{R}$ that has a bounded derivative and such that $\Phi(k) = 0$, for $|k| \leq k_1$, and $\Phi(k) = 1$, for $|k| \geq k_2$; for some $0 < k_1 < k_2$. We define $\Phi(H_\lambda)$ by functional calculus. It follows from Corollary 1.2 of [29] that $\Phi(H_\lambda)$ is a bounded operator on $W_{1,p}$, $1 < p < \infty$. Also, $\Phi(H_\lambda)$ is bounded on $W_{2,2}$(see equation (2.26) in Section 2).

LEMMA 1.4. *Suppose that the conditions of Theorem 1.1 are satisfied, and moreover, that* $F(x,u) = \sum_{j=1}^{\infty} V_j(x)|u|^{2(j_0+j)}u$, $j_0 \geq (p-3)/2$, *for* $|u| \leq \eta$, *for some* $\eta > 0$, *and where* $V_j \in W_{1,\infty}$ *with* $\|V_j\|_{W_{1,\infty}} \leq C^j, j = 1,2,\cdots$, *for some constant* C. *Then, for any* $\phi \in W_{2,2} \cap W_{1,1+\frac{1}{p}}$ *there is an* $\epsilon_0 > 0$ *such that for all* $0 < \epsilon < \epsilon_0$*:*

$$((S_V - I)(\epsilon\phi),\, \phi)_{L^2} = \sum_{j=1}^{\infty} \epsilon^{2(j_0+j)+1} \left[\int \int \, dt\, dx\, V_j(x) \left| e^{-itH}\phi \right|^{2(j_0+j+1)} + Q_j \right], \tag{1.21}$$

where $Q_1 = 0$ *and* $Q_j, j > 1$, *depends only on* ϕ *and on* V_k *with* $k < j$. *Moreover, for any* $\acute{x} \in \mathbf{R}$, *and any* $\lambda > 0$, *we denote,* $\phi_\lambda(x) := [\Phi(H_\lambda)\phi]\,(\lambda(x - \acute{x}))$. *Then, if* $\Phi(H_0)\,\phi \neq 0$*:*

$$V_j(\acute{x}) = \frac{\lim_{\lambda\to\infty} \lambda^3 \int\int \, dt\, dx\, V_j(x) \left| e^{-itH}\phi_\lambda \right|^{2(j_0+j+1)}}{\int\int \, dt\, dx\, |e^{-itH_0}\Phi(H_0)\phi|^{2(j_0+j+1)}}. \tag{1.22}$$

COROLLARY 1.5. *Under the conditions of Lemma 1.4 the scattering operator, S, determines uniquely the potentials* $V_j, j = 0,1,\cdots$.

Proof: By Corollary 1.3, S determines uniquely V_0. Then the wave operators, $W_\pm$, are uniquely determined, and by (1.18), S determines uniquely S_V. Finally by (1.21) and (1.22) S_V determines uniquely $V_j, j = 1,2,\cdots$.

The method to reconstruct the potentials $V_j, j = 0,1,\cdots$, is as follows. First we obtain S_L from S using (1.19). By any standard method for inverse scatering for the linear Schrödinger equation on the line we reconstruct V_0. We then reconstruct S_V from S using (1.18). Finally (1.21) and (1.22) give us, recursively, $V_j, j = 1,2,\cdots$.

Our formula (1.22) is an extension to our case of the reconstruction algorithm of [22]. In [22] Strauss proved that in the case $V_0 = 0$ and $F(x,u) = V(x)|u|^{p-1}u$, $x \in \mathbf{R}^n$, $p > 4$ if $n = 1$, $p > 3$ if $n = 2$, $p \geq 3$ if $n \geq 3$, and $V(x)$ a real-valued potential whose derivatives up to order l are bounded, with $l > 3n/4$, then, the scattering operator uniquely determines V.

2 The Proofs

Proof of Theorem 1.1 : The basic input for the proof of Theorem 1.1 is the $L^p - L^{\acute{p}}$ estimate that we proved in [28], namely,

$$\left\| e^{-itH} \right\|_{\mathcal{B}\left(L^p, L^{\acute{p}}\right)} \leq \frac{C}{t^{\left(\frac{1}{p}-\frac{1}{2}\right)}},\, t > 0, \tag{2.1}$$

for $1 \leq p \leq 2$ and where $\frac{1}{p} + \frac{1}{\acute{p}}$. For any pair of Banach spaces X, Y, we denote by $\mathcal{B}(X, Y)$ the Banach space of all bounded operators from X into Y. By $W_{k,p}, k = 0, 1, \cdots, 1 \leq p \leq \infty$, we denote the Sobolev space [1] of all functions on L^p such that all its derivatives of order up to k are functions in L^p. By $\| \cdot \|_{k,p}$ we denote the norm in $W_{k,p}$. Clearly, $W_{0,p} = L^p$. In Theorem 1.1 of [29] we proved that $W_{\pm}$ and $W_{\pm}^*$ are bounded operators on $W_{1,p}, 1 < p < \infty$. By Theorem 3 in page 135 of [21],

$$\|F^{-1}(1+q^2)^{k/2}(Ff)(q)\|_{L^p}, \tag{2.2}$$

is a norm that is equivalent to the norm of $W_{k,p}, 1 < p < \infty$. Then, by Corollary 1.2 of [29]

$$\|(I+H)^{k/2} f\|_{L^p}, \tag{2.3}$$

defines a norm that is equivalent to the norm of $W_{k,p}, k = 0, 1, 1 < p < \infty$. Below we use this equivalence without further comments. In particular, it follows from (2.3) that estimate (2.1) holds in the norm on $\mathcal{B}(W_{1,p}, W_{1,\acute{p}})$.

By Sobolev's imbedding theorem [1] L^∞ is continuously imbedded in $W_{1,2}$. It follows that F is locally Lipschitz continuous on $W_{1,2}$. Then, by standard arguments, $u \in C(\mathbf{R}, W_{1,2}) \cap M$ is a solution to (1.1) with $\lim_{t \to -\infty} \|u(t) - e^{-itH}\phi\|_{W_{1,2}} = 0$, for some $\phi \in W_{1,2}$, if and only if u is a solution to the following integral equation:

$$u = e^{-itH}\phi + \int_{-\infty}^{t} e^{-i(t-\tau)H} F(x, u(\tau))\, d\tau. \tag{2.4}$$

As we prove below the integral in the right-hand side of (2.4) converges absolutely in $W_{1,2}$ and in M. For $u \in M$ we denote

$$\mathcal{Q}u(t) := \int_{-\infty}^{t} e^{-i(t-\tau)H} F(x, u(\tau))\, d\tau. \tag{2.5}$$

It follows from (2.1), from the generalized Young inequality (see [18]) and since L^∞ is continuously imbedded in $W_{1,p+1}$ that

$$\|\mathcal{Q}u(t) - \mathcal{Q}v(t)\|_{W_{1,p+1}} \leq C\,(1+|t|)^{-d}(\|u\|_M + \|v\|_M)^{p-1}\|u-v\|_M, \tag{2.6}$$

where we used that $p\,d > 1$. The constants C in (2.6) can be taken uniform in closed balls in M. By (2.6) with $v(t) = 0$:

$$\|\mathcal{Q}u(t)\|_{W_{1,2}}^2 \leq C\Re \int_{-\infty}^{t} d\tau\, \left(\sqrt{I+H}F(x,u(\tau)), \sqrt{I+H}\mathcal{Q}u(\tau)\right)_{L^2} \leq C\int_{-\infty}^{t} d\tau\, \|F(x,u)(\tau)\|_{W_{1,1+1/p}} \times$$

$$(1+|\tau|)^{-d}\,\|u\|_M^p \leq C\int_{-\infty}^{t} d\tau\,\|u\|^p_{W_{1,p+1}}\;(1+|\tau|)^{-d}\|u\|^p_M \leq C\int_{-\infty}^{t} d\tau\,(1+|\tau|)^{-d(p+1)}\,\|u\|_M^{2p}$$
$$\leq C(1+\max[0,-t])^{-(d+dp-1)}\,\|u\|_M^{2p}. \tag{2.7}$$

By (2.6) with $v(t)=0$, the integral in the right-hand side of (2.4) converges in M and by (2.7) the converge holds also in $W_{1,2}$. Let us first prove the uniqueness. For u, v any pair of solutions to (1.1) that satisfy (1.14) we have that,

$$u(t)-v(t)=\mathcal{Q}u(t)-\mathcal{Q}v(t). \tag{2.8}$$

Let us denote $u_T := \chi_{(-\infty,T)}(t)\,u(t)$, where $\chi_{(-\infty,T)}(t)$ is the characteristic function of $(-\infty,T), T\in\mathbf{R}$. v_T is similarly defined. It follows from (2.8) that

$$\|u_T(t)-v_T(t)\|_M < 1/2\|u_T(t)-v_T(t)\|_M, \text{ for some } T \text{ negative enough.} \tag{2.9}$$

It follows that $u(t)=v(t)$ for $t\leq T$. Hence, as F is locally Lipschitz in $W_{1,2}$, the standard uniqueness result implies that $u=v$. The uniqueness of ϕ_+ is obvious by the unitarity of e^{-itH} in $W_{1,2}$.

Equation (1.12) and Theorem 2.7.1 in page 35 of [20] imply that, $D(H)=D(H_0)=W_{2,2}$, and that the following norm is equivalent to the norm of $W_{2,2}$:

$$\|(H+I)\phi\|_{L^2}\,. \tag{2.10}$$

Then, by Sobolev's imbedding theorem,

$$\|e^{-itH}\phi_-\|_{W_{1,p+1}} \leq C\|e^{-itH}\phi_-\|_{W_{2,2}} \leq C\left\|(H+I)e^{-itH}\phi_-\right\|_{L^2} = C\,\|(H+I)\phi_-\|_{L^2} \leq C\,\|\phi_-\|_{W_{2,2}}\,. \tag{2.11}$$

By (2.1) and (2.11):

$$\left\|e^{-itH}\phi_-\right\|_M \leq C\left[\|\phi_-\|_{W_{2,2}} + \|\phi_-\|_{W_{1,1+\frac{1}{p}}}\right]. \tag{2.12}$$

For $R>0$ let us denote: $M_R := \{u\in M : \|u\|_M\leq R\}$. Let us take R so small that $C(2R)^{p-1}\leq 1/2$, with C as in (2.6), and $\delta>0$ so small that $C\delta\leq R/4$, with C as in (2.12). It follows from (2.6) and (2.12)that the map $u\hookrightarrow e^{-itH}\phi_- + \mathcal{Q}F(x,u)$ is a contraction from M_R into M_R for all $\phi_-\in W_{2,2}\cap W_{1,1+\frac{1}{p}}$ with $\|\phi_-\|_{W_{2,2}} + \|\phi_-\|_{W_{1,1+\frac{1}{p}}} \leq \delta$. The contraction mapping theorem (see Theorem V.18 in page 151 of

[17]) implies that this map has a unique fixed point. This fixed point is a solution to (2.4) in M_R, and furthermore,

$$\|u\|_M \leq C \left\| e^{-itH}\phi_- \right\|_M + \frac{1}{2}\|u\|_M . \tag{2.13}$$

Then,

$$\|u\|_M \leq C \left\| e^{-itH}\phi_- \right\|_M . \tag{2.14}$$

Equation(1.16) for ϕ_- follows from (2.4), (2.6) with $v=0$ and (2.14). By (2.4) and (2.7) $u \in C(\mathbf{R}, W_{1,2})$ and (1.14) holds.

Let us define:

$$\phi_+ = \phi_- + \int_{-\infty}^{\infty} e^{i\tau H} F(x, u(\tau))\, d\tau. \tag{2.15}$$

Estimating as in (2.7) we prove that $\phi_+ \in W_{1,2}$ and that

$$\|\phi_+ - \phi_-\|_{W_{1,2}} \leq C \|u\|_M^p . \tag{2.16}$$

Equation (1.17) follows from (2.12), (2.14) and (2.16). By (2.15) and (2.4):

$$u(t) = e^{-itH}\phi_+ - \int_t^{\infty} e^{-i(t-\tau)H} F(x, u(\tau))\, d\tau, \tag{2.17}$$

and we prove (1.15) estimating as in (2.7). Furthermore, arguing again as in (2.7) we prove that

$$\left\| \int_t^{\infty} e^{-i(t-\tau)H} F(x, u(\tau))\, d\tau \right\|_M \leq C \|u\|_M^p . \tag{2.18}$$

By (2.17) and (2.18), $e^{-itH}\phi_+ \in M$. Using once more (2.17) and arguing as in the proof of (2.14) we prove that,

$$\|u\|_M \leq C \left\| e^{-itH}\phi_+ \right\|_M . \tag{2.19}$$

Equation (1.16) for ϕ_+ follows from (2.17), (2.18) and (2.19).

If $\phi_+ = S\phi_- = 0$, it follows from (2.17) that

$$u(t) = -\int_t^{\infty} e^{-i(t-\tau)H} F(x, u(\tau))\, d\tau. \tag{2.20}$$

Let us denote: $u_T(t) := \chi_{(T,\infty)}(t) u(t)$. It follows from (2.20) (estimating as in (2.9)) that

$$\|u_T(t)\|_M \leq \frac{1}{2}\|u_T(t)\|_M , \text{ for some } T \text{ positive enough.} \tag{2.21}$$

By (2.21), $u(t) = 0$ for $t \geq T$. Then, by standard uniqueness in $C(\mathbf{R}, W_{1,2})$ (recall that F is locally Lipschitz continuous on $W_{1,2}$), $u(t) = 0$. Finally, by (1.15) $\phi_- = 0$, proving that S_V is injective.

Proof of Theorem 1.2 : Since, $S(0) = 0$, and $W_\pm$ are bounded on $W_{2,2} \cap W_{1,1+\frac{1}{p}}$ [29], it is enough to to prove that

$$s - \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \left(S_V(\epsilon\phi) - \epsilon\phi \right) = 0. \tag{2.22}$$

By (2.12) and (2.14) with ϕ_- replaced by $\epsilon\phi$:

$$\|u\|_M \leq C\epsilon \left[\|\phi_-\|_{W_{2,2}} + \|\phi_-\|_{W_{1,1+\frac{1}{p}}} \right]. \tag{2.23}$$

We prove (2.22) estimating the integral in the right-hand side of (2.15) as in (2.7) and using (2.23).

Proof of Lemma 1.4 By the contraction mapping theorem, [17]

$$u(t) = e^{-itH}\epsilon\phi + \sum_{n=1}^{\infty} \mathcal{Q}^n e^{-itH}\epsilon\phi. \tag{2.24}$$

Equation (1.21) follows from (2.15) and (2.24). By Sobolev's imbedding theorem [1], $W_{2,2} \subset L^q$, $2 \leq q \leq \infty$. Then, estimating as in (2.11) we prove that, $\left\| e^{-itH}\phi \right\|_{L^q} \leq C_q \left\| e^{-itH}\phi \right\|_{W_{2,2}} \leq C_q \|\phi\|_{W_{2,2}}, 2 \leq q \leq \infty$, and it follows from (2.1) that:

$$\int\int dt\, dx \left| e^{-itH}\phi_\lambda \right|^{2(j_0+j+1)} < \infty,\ j = 1, 2, \cdots. \tag{2.25}$$

It follows from (1.12) and from Theorem 2.7.1 in page 35 of [20] that

$$C_1 \|\phi\|_{W_{2,2}} \leq \|(H_\lambda + I)\phi\|_{L^2} \leq C_2 \|\phi\|_{W_{2,2}}, \tag{2.26}$$

for some constants C_1, C_2. Moreover, since $N(V_\lambda) \leq \frac{1}{\lambda} N(V), \lambda \geq 1$, it follows from the proof of Theorem 2.7.1 in page 35 of [20] that we can take fixed C_1 and C_2 for all $\lambda \geq 1$. By Corollary 1.2 of [29] $\Phi(H_\lambda)$ is bounded on $W_{1,1+\frac{1}{p}}$ and by (2.26) it is bounded on $W_{2,2}$. To prove (1.22) we denote: $\tilde{t} := \lambda^2 t$ and $\tilde{x} := \lambda(x - \acute{x})$. Then, we observe that,

$$\left(e^{-i\tilde{t}H_\lambda}\Phi(H_\lambda)\phi \right)(\tilde{x}) = \left(e^{-itH}\phi_\lambda \right)(x). \tag{2.27}$$

It follows from (2.27) that,

$$I_j := \lambda^3 \int\int dt\, dx\, V_j(x) \left| e^{-itH}\phi_\lambda \right|^{2(j_0+j+1)} = \int\int d\tilde{t}\, d\tilde{x}\, V_j(\frac{\tilde{x}}{\lambda} + \acute{x}) \left| e^{-i\tilde{t}H_\lambda}\Phi(H_\lambda)\phi \right|^{2(j_0+j+1)} (\tilde{x}). \tag{2.28}$$

By Theorem VIII.20 in page 286 of [17] and (2.26)

$$s - \lim_{\lambda\to\infty} e^{-i\tilde{t}H_\lambda}\Phi(H_\lambda)\phi = e^{-i\tilde{t}H_0}\Phi(H_0)\phi, \tag{2.29}$$

where the limit exists in the strong topology on $W_{2,2}$. By Sobolev's imbedding theorem, the limit in (2.29) also exists in the strong topology on L^q, $2 \leq q \leq \infty$, and moreover,

$$\left\| e^{-i\tilde{t}H_\lambda}\Phi(H_\lambda)\phi \right\|_{L^q} \leq C_q \left\|\phi\right\|_{W_{2,2}}, 2 \leq q \leq \infty,\ \lambda \geq 1. \tag{2.30}$$

By Theorem 2.4 of [28] and since $e^{-itH}\Phi(H_\lambda)$ is bounded in L^2 with bound bounded by $\|\Phi\|_{L^\infty}$ it follows by interpolation that (see [21]):

$$\left\| e^{-i\tilde{t}H_\lambda}\Phi(H_\lambda)\phi \right\|_{L^{p+1}} \leq C \frac{1}{|t|^d} \left\|\phi\right\|_{L^{1+\frac{1}{p}}}, \tag{2.31}$$

with $d := \frac{1}{2}\frac{p-1}{p+1}$. Since $\|V_\lambda\|_{L^1_1} \leq C\|V_0\|_{L^1_1}$, $\lambda \geq 1$, with C independent of λ, we can take a fixed C in (2.31) for all $\lambda \geq 1$. Equation (1.22) follows from (2.29), (2.30), (2.31) and the dominated convergence theorem, observing that $2(j_0 + j + 1) \geq p + 1$, that $d(p + 1) > 1$ and that V_j is continuous.

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