Strong-magnetic-field asymptotics of the integrated density of states for a random 3D Schrödinger operator

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Abstract. We consider the three-dimensional Schrödinger operator with constant magnetic field and bounded random electric potential. We investigate the asymptotic behaviour of the integrated density of states for this operator as the norm of the magnetic field tends to infinity.

1 Introduction

Let $\mathbf{b} := (0, 0, b)$, $b > 0$, $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$. Introduce the unperturbed selfadjoint Schrödinger operator $H_0(b) := \left(i\nabla + \frac{\mathbf{b}\wedge\mathbf{x}}{2}\right)^2$ defined originally on $C_0^\infty(\mathbb{R}^3)$ and then closed in $L^2(\mathbb{R}^3)$. It is well-known that for each $b > 0$ we have

$$\sigma(H_0(b)) = \sigma_{\text{ess}}(H_0(b)) = [b, +\infty) \tag{1.1}$$

where $\sigma(H_0(b))$ denotes the spectrum of the operator $H_0(b)$, and $\sigma_{\text{ess}}(H_0(b))$ - its essential spectrum (see e.g. [A.H.S]).

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $V_\omega(\mathbf{x})$, $\omega \in \Omega$, $\mathbf{x} \in \mathbb{R}^3$, be a real random field. We assume that V_ω is $\mathbb{G}^3$-ergodic with $\mathbb{G} = \mathbb{Z}$ or $\mathbb{G} = \mathbb{R}$ (see [K, Section 3.1] or [P.Fi, Section 1B]). In other words, there exists an ergodic group of automorphisms $\mathcal{T}_\mathbf{k} : \Omega \to \Omega$, $\mathbf{k} \in \mathbb{G}^3$, such that

$$V_\omega(\mathbf{x} + \mathbf{k}) = V_{\mathcal{T}_\mathbf{k}\omega}(\mathbf{x}), \ \mathbf{x} \in \mathbb{R}^3, \ \omega \in \Omega, \ \mathbf{k} \in \mathbb{G}^3. \tag{1.2}$$

We recall that ergodicity of a group G of measure preserving automorphisms of Ω means that the invariance of a given set $\mathcal{A} \in \mathcal{F}$ under the action of G (i.e. $g\mathcal{A} = \mathcal{A}$ for each $g \in G$) implies either $\mathbb{P}(\mathcal{A}) = 1$ or $\mathbb{P}(\mathcal{A}) = 0$.
Let $\mathbf{x} \in \mathbb{R}^3$. We shall write occasionally $\mathbf{x} = (X, z)$ with $X \in \mathbb{R}^2$ and $z \in \mathbb{R}$. We suppose that V_ω is $\mathbb{G}$-ergodic with $\mathbb{G} = \mathbb{Z}$ or $\mathbb{G} = \mathbb{R}$ in the direction of the magnetic field (or, in brief, in the z-direction), i.e. that the subgroup $\{\mathcal{T}_\mathbf{k} | \mathbf{k} = (0, 0, k), \ k \in \mathbb{G}\}$ is ergodic.
Further, we assume that the realizations of V_ω are almost surely uniformly bounded, i.e. we have

$$c_0 := \text{ess} - \sup_{\omega \in \Omega} \sup_{\mathbf{x} \in \mathbb{R}^3} |V_\omega(\mathbf{x})| < \infty. \tag{1.3}$$

Finally, for simplicity we suppose that the realizations of V_ω are almost surely continuous.

Example: Let $\alpha_\mathbf{j} : \Omega \to \mathbb{R}$, $\mathbf{j} \in \mathbb{Z}^3$, be independent identically distributed almost surely bounded random variables. Assume that $q : \mathbb{R}^3 \to \mathbb{R}$ is a continuous function satisfying

$$|q(\mathbf{x})| \leq c\langle \mathbf{x} \rangle^{-\beta}, \quad c > 0, \quad \beta > 3.$$

Set

$$V_\omega(\mathbf{x}) := \sum_{\mathbf{j} \in \mathbb{Z}^3} \alpha_\mathbf{j}(\omega) \ q(\mathbf{x} - \mathbf{j}), \ \omega \in \Omega, \ \mathbf{x} \in \mathbb{R}^3.$$

Then the random field V_ω is $\mathbb{Z}^3$-ergodic (see [K, Model II, Section 3.3] or [P.Fi, Example 1.15a, p.23]), $\mathbb{Z}$-ergodic in the direction of the magnetic field (as a matter of fact, in all directions; see [E.K.Sch.S, Example 2, p.615]). Moreover, it is obvious that almost surely the realizations of V_ω are uniformly bounded and continuous.

On $D(H_0(b))$ define the perturbed Schrödinger operator

$$H(b, \omega) := H_0(b) + V_\omega, \ b > 0, \ \omega \in \Omega.$$

It follows from (1.1) and (1.3) that almost surely we have

$$\sigma(H(b, \omega)) \subseteq [b - c_0, +\infty). \tag{1.4}$$

The aim of this paper is to study the asymptotic behaviour as $b \to \infty$ of the integrated density of states (IDOS) for the operator $H(b, \omega)$. In order to recall the definition of

the IDOS we need several auxiliary concepts.
Let φ_r, $r \in \mathbb{R}_+$, and φ be non-decreasing functions defined on a common domain $I \subseteq \mathbb{R}$. We shall write

$$v - \lim_{r\to\infty} \varphi_r = \varphi$$

if we have $\lim_{r\to\infty} \varphi_r(t) = \varphi(t)$ at all continuity points t of the function φ. In this case the function φ is called the vague limit as $r \to \infty$ of the family φ_r (cf [K, p.313]).

Further, let $T = T^*$ be a selfadjoint operator in a Hilbert space. Denote by $P_{\mathcal{I}}(T)$ its spectral projection corresponding to the interval $\mathcal{I} \subset \mathbb{R}$. Set

$$N(\lambda; T) := \operatorname{rank} P_{(-\infty,\lambda)}(T), \ \lambda \in \mathbb{R},$$

$$n_\pm(s; T) := \operatorname{rank} P_{(s,+\infty)}(\pm T), s > 0.$$

On the Sobolev space $\mathrm{H}^2\left(\left(-\frac{R}{2}, \frac{R}{2}\right)^3\right)$, $R > 0$, with Dirichlet boundary conditions define the operator $H_{0,R}(b) := \left(i\nabla + \frac{\mathbf{b}\wedge\mathbf{x}}{2}\right)^2$. Put

$$\mathcal{D}_b(.) := v - \lim_{R\to\infty} R^{-3} N(\,.\,; H_{0,R}(b) + V_\omega). \tag{1.5}$$

The function $\mathcal{D}_b(\mu)$, $\mu \in \mathbb{R}$, is called the IDOS for the operator $H(b,\omega)$. It is well-known that almost surely the vague limit (1.5) exists and the quantity $\mathcal{D}_b(\mu)$ is non-random (see e.g. [Bro.H.L], [Ma], [U], and the references cited there).
Note that (1.4) implies that almost surely $\inf \sigma(H_{0,R}(b) + V_\omega) \geq b - c_0$ for all $R > 0$. Therefore,

$$\mathcal{D}_b(\mu) = 0, \quad \mu < b - c_0. \tag{1.6}$$

For $\mu \in \mathbb{R}$ set

$$D_b(\mu) := \frac{b}{2\pi^2} \sum_{k=1}^{\infty} (\mu - (2k-1)b)_+^{1/2}.$$

By [CdV, Theorem 3.1] the estimates

$$(R - R_0)^3 D_b(\mu - CR_0^{-2} - c_0) \leq N(\mu; H_{0,R}(b) + V_\omega) \leq$$

$$R^3 D_b(\mu + c_0), \ \mu \in \mathbb{R}, \ R > 0, \ R_0 \in (0, R),$$

hold with C independent of μ, R, and R_0. Then (1.5) easily implies

$$D_b(\mu - c_0) \leq \mathcal{D}_b(\mu) \leq D_b(\mu + c_0), \ \mu \in \mathbb{R}. \tag{1.7}$$

In this paper we study the asymptotic behaviour as $b \to \infty$ of $\mathcal{D}_b(\lambda + b)$, the parameter $\lambda \in \mathbb{R}$ being fixed.

2 Statement of the main result

On $\mathrm{H}^2\left(\left(-\frac{R}{2}, \frac{R}{2}\right)\right)$ with Dirichlet boundary conditions define the operator $\chi_{0,R} := -\frac{d^2}{dz^2}$.

Proposition 2.1 *Let $\mathbb{G} = \mathbb{Z}$ or $\mathbb{G} = \mathbb{R}$. Let $f_\omega(z)$, $\omega \in \Omega$, $z \in \mathbb{R}$, be a real $\mathbb{G}$-ergodic random field whose realizations are almost surely uniformly bounded and continuous. Then for each $\lambda \in \mathbb{R}$ the limit*

$$\varrho(\lambda; f) := \lim_{R\to\infty} R^{-1}\, N(\lambda; \chi_{0,R} + f_\omega)$$

exists almost surely. Moreover, the function $\varrho(\lambda; f)$ is non-random, and continuous with respect to $\lambda \in \mathbb{R}$.

The proof of the existence and the non-randomness of $\varrho(\lambda; f)$ for much more general ergodic fields f_ω can be found in [K, Chapter 7]. The continuity of $\varrho(\lambda; f)$ which is guaranteed by the fact that $\chi_{0,R} + f_\omega$ is an *ordinary* differential operator, is discussed in [P.Fi, Chapter III].

Lemma 2.1 *Assume that the hypotheses of Proposition 2.1 hold. Let $\mathcal{T} : \Omega \to \Omega$ be a measure preserving automorphism. Then we have*

$$\lim_{R\to\infty} R^{-1}\, N(\lambda; \chi_{0,R} + f_{\mathcal{T}\omega}) = \lim_{R\to\infty} R^{-1}\, N(\lambda; \chi_{0,R} + f_\omega). \tag{2.1}$$

Proof. By [K, Chapter 6, Theorem 7] we have

$$\lim_{R\to\infty} R^{-1}\, N(\lambda; \chi_{0,R} + f_\omega) = \sup_{R>0} R^{-1} \mathbb{E}\left(N(\lambda; \chi_{0,R} + f_\omega)\right) \equiv$$

$$\sup_{R>0} R^{-1} \int_\Omega N(\lambda; \chi_{0,R} + f_\omega)\, d\mathbb{P}(\omega)$$

where $\mathbb{E}$ is used as the symbol of the mathematical expectation. Analogously,

$$\lim_{R\to\infty} R^{-1}\, N(\lambda; \chi_{0,R} + f_{\mathcal{T}\omega}) = \sup_{R>0} R^{-1} \int_\Omega N(\lambda; \chi_{0,R} + f_{\mathcal{T}\omega})\, d\mathbb{P}(\omega).$$

Since $\mathcal{T}$ is a measure preserving automorphism, we get

$$\sup_{R>0} R^{-1} \int_\Omega N(\lambda; \chi_{0,R} + f_{\mathcal{T}\omega})\, d\mathbb{P}(\omega) = \sup_{R>0} R^{-1} \int_\Omega N(\lambda; \chi_{0,R} + f_\omega)\, d\mathbb{P}(\omega)$$

which entails (2.1). $\diamondsuit$

Our assumptions concerning V_ω imply that the random field $f_\omega = V_\omega(X, .)$ depending on the parameter $X \in \mathbb{R}^2$, satisfies the hypotheses of Proposition 2.1. Moreover, if $\mathbb{G} = \mathbb{Z}$, then the function $\varrho(\lambda; V(X, .))$ is $\mathbb{Z}^2$-periodic with respect to $X \in \mathbb{R}^2$. In order to see this, one may apply Lemma 2.1 with $\mathcal{T} = \mathcal{T}_{\mathbf{k}_0}$ (see (1.2)) with $\mathbf{k}_0 = (K, 0)$, $K \in \mathbb{Z}^2$, and conclude that

$$\varrho(\lambda; V(X+K, .)) = \varrho(\lambda; V(X, .)), \ \lambda \in \mathbb{R}, \ X \in \mathbb{R}^2.$$

Note that the continuity of $V_\omega(\mathbf{x})$ with respect to $\mathbf{x} \in \mathbb{R}^3$, and the continuity of $\varrho(\lambda; f)$ with respect to $\lambda \in \mathbb{R}$, imply the continuity of $\varrho(\lambda; V(X, .))$ with respect to $X \in \mathbb{R}^2$. Taking into account also its periodicity, we find that $\varrho(\lambda; V(X, .))$, $X \in \mathbb{R}^2$, is uniquely determined by its values for $X \in \left(-\frac{1}{2}, \frac{1}{2}\right)^2$.
Similarly, if $\mathbb{G} = \mathbb{R}$, the quantity $\varrho(\lambda; V(X, .))$ is independent of $X \in \mathbb{R}^2$. In order to see this, one may apply Lemma 2.1 with $\mathcal{T} = \mathcal{T}_{\mathbf{x}_0}$ (see (1.2)) with $\mathbf{x}_0 = (X, 0)$, $X \in \mathbb{R}^2$, and conclude that

$$\varrho(\lambda; V(X, .)) = \varrho(\lambda; V(0, .)), \ \lambda \in \mathbb{R}, \ X \in \mathbb{R}^2.$$

Finally, the elementary estimate

$$N(\lambda; \chi_{0,R} + V_\omega(X, .)) \leq \frac{R}{\pi} (\lambda + c_0)_+^{1/2}, \ X \in \left(-\frac{1}{2}, \frac{1}{2}\right)^2, \ R > 0,$$

implies

$$\varrho(\lambda; v(X, .)) \leq \frac{1}{\pi} (\lambda + c_0)_+^{1/2}, \ X \in \left(-\frac{1}{2}, \frac{1}{2}\right)^2.$$

For $\lambda \in \mathbb{R}$ set

$$\delta(\lambda) := \begin{cases} \int_{\left(-\frac{1}{2}, \frac{1}{2}\right)^2} \varrho(\lambda, V(X, .))\, dX & \text{if} \quad \mathbb{G} = \mathbb{Z}, \\ \varrho(\lambda, V(0, .)) \quad \text{if} \quad \mathbb{G} = \mathbb{R}. \end{cases}$$

Obviously, $\delta(\lambda)$ is continuous with respect to λ.

Theorem 2.1 *Let $\mathbb{G} = \mathbb{Z}$ or $\mathbb{G} = \mathbb{R}$. Let V_ω be a real $\mathbb{G}^3$-ergodic random field whose realizations are almost surely uniformly bounded and continuous. Assume in addition that V_ω is $\mathbb{G}$-ergodic in the direction of the magnetic field. Then we have*

$$\lim_{b \to \infty} b^{-1} \mathcal{D}_b(\lambda + b) = \frac{1}{2\pi} \delta(\lambda), \ \lambda \in \mathbb{R}. \tag{2.2}$$

Remark: For definiteness, we shall prove Theorem 2.1 in the case $\mathbb{G} = \mathbb{Z}$. The proof in the case $\mathbb{G} = \mathbb{R}$ is quite similar and only simpler.

The asymptotics as $b \to \infty$ of the IDOS for the two-dimensional Schrödinger operator with constant magnetic field has been extensively investigated during the last two decades (see e.g. [Br.G.I], [Bro.H.L], [M.Pu], [Pu.Sc], [W]). As far as the authors are informed, no results concerning the strong-magnetic-field asymptotics of the IDOS for

the three-dimensional Schrödinger operator considered in this paper, are known. Beside the quantity $\mathcal{D}_b(\lambda+b)$ whose main asymptotic term is obtained in (2.2), we could consider more general quantities $\mathcal{D}_b(\lambda_2+\epsilon b)-\mathcal{D}_b(\lambda_1+\epsilon b)$ with $\lambda_j \in \mathbb{R}$, $j=1,2$, $\lambda_1<\lambda_2$, and $\epsilon \in \mathbb{R}$. Recall that the numbers $\{(2k-1)b)\}_{k\geq 1}$ are called Landau levels. For this reason we shall refer to the asymptotics as $b\to\infty$ of $\mathcal{D}_b(\lambda_2+(2k-1)b)-\mathcal{D}_b(\lambda_1+(2k-1)b)$, $k\in\mathbb{Z}$, $k\geq 1$, as the asymptotics of the IDOS near the k^{th} Landau level. Analogously, if $\epsilon>1$ is not an odd integer, we shall refer to the asymptotics as $b\to\infty$ of $\mathcal{D}_b(\lambda_2+\epsilon b)-\mathcal{D}_b(\lambda_1+\epsilon b)$ as the asymptotics of the IDOS far from the Landau levels. Note that the case $\epsilon<1$ is trivial since (1.6) implies $\mathcal{D}_b(\lambda+\epsilon b)=0$, $\lambda\in\mathbb{R}$, $\epsilon<1$, provided that b is big enough.
Since (2.2) entails

$$\lim_{b\to\infty} b^{-1}\left(\mathcal{D}_b(\lambda_2+b)-\mathcal{D}_b(\lambda_1+b)\right)=\frac{1}{2\pi}\left(\delta(\lambda_2)-\delta(\lambda_1)\right),$$

we can say that Theorem 2.1 contains the asymptotics of the IDOS near the first Landau Level. The problems of obtaining the first asymptotic term of the IDOS near the higher Landau levels and far from the Landau levels remain open as far as the methods used in this paper are not directly applicable to them. We hope to solve these problems in a future work. Here we would like to note that we have

$$\lim_{b\to\infty} b^{-1}\left(D_b(\lambda_2+\epsilon b)-D_b(\lambda_1+\epsilon b)\right)=\frac{1}{2\pi^2}\left((\lambda_2)_+^{1/2}-(\lambda_1)_+^{1/2}\right), \tag{2.3}$$

if $\epsilon>1$ is an odd integer, and

$$\lim_{b\to\infty} b^{-1/2}\left(D_b(\lambda_2+\epsilon b)-D_b(\lambda_1+\epsilon b)\right)=\frac{1}{4\pi^2}(\lambda_2-\lambda_1)\sum_{1\leq k<(\epsilon+1)/2}(\epsilon-(2k-1))^{-1/2}, \tag{2.4}$$

if $\epsilon>1$ is not an odd integer. Combining (2.3) (respectively, (2.4)) with (1.7) we obtain generically the correct asymptotic order of the IDOS near the higher Landau levels (respectively, far from the Landau levels).

Finally, note that (1.7) implies immediately

$$\lim_{b\to\infty} b^{-3/2}\mathcal{D}_b(\lambda+\epsilon b)=D_1(\epsilon),\ \lambda\in\mathbb{R},\ \epsilon>1,$$

which however yields only the rough estimate

$$\mathcal{D}_b(\lambda_2+\epsilon b)-\mathcal{D}_b(\lambda_1+\epsilon b)=o(b^{3/2}),\quad b\to\infty.$$

The methods we apply are relatively simple. First of all, we give an equivalent representation of $\mathcal{D}_b(\lambda+b)$ which is more convenient for our purposes. Namely, on $D(H_0(b))$ we define the operator

$$\tilde{H}(b,\omega,\lambda,R):=H_0(b)-b+(V_\omega-\lambda-1)\mathbf{1}_{\left(-\frac{R}{2},\frac{R}{2}\right)^3},\ b>0,\ \lambda\in\mathbb{R},\ \omega\in\Omega, \tag{2.5}$$

which has purely discrete negative spectrum, and show that we probability one we have

$$\mathcal{D}_b(\lambda + b) = \lim_{R\to\infty} R^{-3} N(-1; \tilde{H}(b, \omega, \lambda, R)) \tag{2.6}$$

provided that $\lambda + b$ is a continuity point of $\mathcal{D}_b$ (see Proposition 4.1 below).
Moreover, we apply the Birman-Schwinger principle (see [B]), and similarly to [R 1] – [R 3] we employ the Kac-Murdock-Szegö theorem in order to reduce the study of the asymptotics as $b \to \infty$ of $\mathcal{D}_b(\lambda + b)$ to the asymptotic analysis as $R \to \infty$ and $b \to \infty$ of the traces of the powers of certain trace-class operators $t_{b,R,\omega}$ (see (3.3) below). The Birkhoff-Khintchine ergodic theorem plays a crucial rôle in this analysis.
The paper is organized as follows. In Section 3 we investigate the asymptotics of $R \to \infty$ and $b \to \infty$ of Tr $t^l_{b,R,\omega}$, $l \geq 1$, and some related traces. Section 4 contains auxiliary results. In particular, we demonstrate (2.6) as well as an analogous formula concerning $\varrho(\lambda, V(X, .))$, $\lambda \in \mathbb{R}$, $X \in \left(-\frac{1}{2}, \frac{1}{2}\right)^2$. Finally, the proof of Theorem 2.1 can be found in Section 5.

3 Trace asymptotics

Define the operator $r := \left(-\frac{\partial^2}{\partial z^2} + 1\right)^{-1/2}$, bounded and selfadjoint in $L^2(\mathbb{R}^3)$. Evidently,

$$(r^2 u)(x, y, z) = \frac{1}{2} \int_{\mathbb{R}} e^{-|z-z'|} u(x, y, z')\, dz', \quad u \in L^2(\mathbb{R}^3).$$

Introduce the orthogonal projection $p_b : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ by

$$(p_b u)(x, y, z) = \int_{\mathbb{R}^2} \mathcal{P}_b(x, y; x', y') u(x', y', z)\, dx' dy'$$

where

$$\mathcal{P}_b(x, y; x', y') := \frac{b}{2\pi} \exp\left\{-\frac{b}{4}\left[(x - x')^2 + (y - y')^2 + 2i(xy' - yx')\right]\right\}. \tag{3.1}$$

Obviously, p_b commutes with $H_0(b)$ and $\frac{\partial}{\partial z}$ (and, hence, with r). Moreover, we have

$$H_0(b) p_b u = \left(-\frac{\partial^2}{\partial z^2} + b\right) p_b u, \quad u \in D(H_0(b)).$$

Fix $\lambda \in \mathbb{R}$ and for brevity set

$$Q_\omega(\mathbf{x}) \equiv Q_\omega(\mathbf{x}; \lambda) = V_\omega(\mathbf{x}) - \lambda - 1, \quad \mathbf{x} \in \mathbb{R}^3, \quad \omega \in \Omega.$$

Evidently, almost surely we have

$$|Q_\omega(\mathbf{x})| \leq c_1, \ \mathbf{x} \in \mathbb{R}^3, \tag{3.2}$$

with $c_1 := c_0 + |\lambda + 1|$. Put

$$\tilde{Q}_{\omega,R}(\mathbf{x}) := Q_\omega(\mathbf{x})\mathbf{1}_{(-\frac{R}{2},\frac{R}{2})^3}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3.$$

Define the operator

$$t_{b,R}(Q_\omega) := p_b r \tilde{Q}_{\omega,R} r p_b, \tag{3.3}$$

compact and selfadjoint in $L^2(\mathbb{R}^3)$. It is easy to check that we have

$$\|p_b r |\tilde{Q}_{\omega,R}|^{1/2}\|_2^2 = \frac{b}{4\pi}\int_{(-\frac{R}{2},\frac{R}{2})^3} |Q_\omega(\mathbf{x})|\, d\mathbf{x} \leq \frac{bR^3}{4\pi} c_1 \tag{3.4}$$

where $\|.\|_2$ denotes the Hilbert-Schmidt norm. Therefore, the operator $t_{b,R}(Q_\omega)$ is in the trace class. Set

$$M_1(b) := \frac{b}{4\pi}\mathbb{E}\left(\int_{(-\frac{1}{2},\frac{1}{2})^3} Q_\omega(X,z)\, dX dz\right).$$

Let $l \geq 2$. Put

$$M_l(b) :=$$

$$\frac{b}{2\pi}\mathbb{E}\left(\int_{(-\frac{1}{2},\frac{1}{2})^3} Q_\omega(X_1,z_1)\int_{\mathbb{R}^{3(l-1)}} \Pi_{s=2}^l Q_\omega(X_1 + b^{-1/2}X_s, z_1 + z_s)\right.$$
$$\left.\psi_l(z_2,\dots,z_l)\Psi_l(X_2,\dots,X_l) dX_2\dots dX_l\, dz_2\dots dz_l\; dX_1 dz_1\right),$$

where

$$\psi_l(z_2,\dots,z_l) := \frac{1}{2^l}\exp\left\{-|z_2| - |z_l| - \sum_{s=2}^{l-1}|z_{s+1} - z_s|\right\},$$

$$\Psi_l(X_2,\dots,X_l) \equiv \Psi_l(x_2,y_2,\dots,x_l,y_l) := \frac{1}{(2\pi)^{l-1}}\exp\{-\Phi_l(x_2,y_2,\dots,x_l,y_l)\},$$

and

$$\Phi_l(x_2,y_2,\dots,x_l,y_l) :=$$

$$\frac{1}{4}\left\{{x_2}^2 + {y_2}^2 + {x_l}^2 + {y_l}^2 + \sum_{s=2}^{l-1}\left((x_{s+1}-x_s)^2 + (y_{s+1}-y_s)^2 + 2i(x_{s+1}y_s - y_{s+1}x_s)\right)\right\}.$$

Note that $\psi_l \in L^1(\mathbb{R}^{l-1})$, $\Psi_l \in L^1(\mathbb{R}^{2(l-1)})$. Hence, (3.2) implies that the integral defining $M_l(b)$ is absolutely convergent.

Proposition 3.1 *Almost surely we have*

$$\lim_{R\to\infty} R^{-3}\mathrm{Tr}\; t_{b,R}(Q_\omega)^l = M_l(b), \quad l \geq 1, \tag{3.5}$$

where the operator $t_{b,R}(Q_\omega)$ is defined in (3.3).

Proof. We shall demonstrate (3.5) in the generic case $l \geq 2$.
It is easy to verify that Tr $t_{b,R}(Q_\omega)^l = \mathrm{Tr}\left(p_b r^2 \tilde{Q}_{\omega,R}\right)^l$, and that Tr $\left(p_b r^2 \tilde{Q}_{\omega,R}\right)^l$ can be written in a standard way as an integral over $\mathbb{R}^{3l}$ of the diagonal value of the integral kernel of the operator $\left(p_b r^2 Q_{\omega,R}\right)^l$. Hence, we have

$$\mathrm{Tr}\; t_{b,R}(Q_\omega)^l =$$

$$\int_{\mathbb{R}^{3l}} {\Pi_{s=1}^l}' \left(\frac{1}{2}\mathcal{P}_b(X_{s+1}, X_s)e^{-|z_{s+1}-z_s|}\right) \Pi_{s=1}^l \tilde{Q}_{\omega,R}(X_s, z_s) dX_2 \dots dX_l \, dz_2 \dots dz_l$$

where the notation ${\Pi_{s=1}^l}'$ means that in the product of l factors the variables X_{l+1} and z_{l+1}, should be set equal respectively to X_1, and z_1. Changing the variables

$$X_1 = X_1', \; X_s = X_1' + b^{-1/2}X_s', \; s = 2, \dots, l,$$

$$z_1 = z_1', \; z_s = z_1' + z_s', \; s = 2, \dots, l,$$

we get

$$\mathrm{Tr}\; t_{b,R}(Q_\omega)^l =$$

$$\frac{b}{2\pi}\int_{\left(-\frac{R}{2},\frac{R}{2}\right)^3} Q_\omega(X_1', z_1') \int_{\mathbb{R}^{3(l-1)}} \Pi_{s=2}^l \tilde{Q}_{\omega,R}(X_1' + b^{-1/2}X_s', z_1' + z_s')$$
$$\psi_l(z_2', \dots, z_l')\Psi_l(X_2', \dots, X_l') dX_2' \dots dX_l' \, dz_2' \dots dz_l' \; dX_1' dz_1'. \tag{3.6}$$

Our next step is to show that if we replace in (3.6) all the functions $\tilde{Q}_{\omega,R}$ by Q_ω, the error will be of order $o(R^3)$ as $R \to \infty$. More precisely, we set

$$\tilde{M}_{l,R}(b) :=$$

$$\frac{b}{2\pi}\int_{\left(-\frac{R}{2},\frac{R}{2}\right)^3} Q_\omega(X_1, z_1) \int_{\mathbb{R}^{3(l-1)}} \Pi_{s=2}^l Q_\omega(X_1 + b^{-1/2}X_s, z_1 + z_s)$$
$$\psi_l(z_2, \dots, z_l)\Psi_l(X_2, \dots, X_l) dX_2 \dots dX_l \, dz_2 \dots dz_l \; dX_1 dz_1, \quad l \geq 2,$$

write

$$\mathrm{Tr}\left(t_{b,R}(Q_\omega)\right)^l = \tilde{M}_{l,R}(b) + \mathcal{E}_l(R, b, \omega), \; l \geq 2, \tag{3.7}$$

and shall demonstrate that almost surely we have

$$\lim_{R\to\infty} R^{-3}\mathcal{E}_l(R, b, \omega) = 0. \tag{3.8}$$

Evidently, $\mathcal{E}_l(R, b, \omega)$ admits the estimate

$$|\mathcal{E}_l(R, b, \omega)| \leq$$

$$\frac{b}{2\pi}c_1^l R^3 \int_{\left(-\frac{1}{2},\frac{1}{2}\right)^3} \int_{\mathbb{R}^{3(l-1)}} \Pi_{s=2}^l E_{R,b}(X_1, \dots, X_l, z_1, \dots, z_l)$$

$$\psi_l(z_2,\ldots,z_l)|\Psi_l(X_2,\ldots,X_l)|dX_2\ldots dX_l\, dz_2\ldots dz_l\; dX_1 dz_1$$

where

$$E_{R,b}(X_1,\ldots,X_l,z_1,\ldots,z_l) :=$$

$$\left(1-\Pi_{s=2}^l \mathbf{1}_{\left(-\frac{R}{2},\frac{R}{2}\right)^3}(RX_1+b^{-1/2}X_s, Rz_1+z_s)\right).$$

Since $\psi_l\Psi_l \in L^1(\mathbb{R}^{3(l-1)})$, $\|E_{R,b}\|_{L^\infty\left(\left(-\frac{1}{2},\frac{1}{2}\right)^3\times\mathbb{R}^{3(l-1)}\right)} = 1$ for every $b>0$ and $R>0$, and

$$\lim_{R\to\infty} E_{R,b}(X_1,\ldots,X_l,z_1,\ldots,z_l) = 0$$

for almost every $(X_1,\ldots,X_l,z_1,\ldots,z_l) \in \left(-\frac{1}{2},\frac{1}{2}\right)^3 \times \mathbb{R}^{3(l-1)}$, the dominated convergence theorem yields (3.8).
Set $L = L(R) = \mathrm{ent}\left(\frac{R}{2}\right)$ where $\mathrm{ent}(x)$ denotes the integer part of $x\in\mathbb{R}$. Obviously,

$$R^{-3}\tilde{M}_{l,R}(b) = (2L+1)^{-3}\tilde{M}_{l,2L+1}(b) + o(1),\; R\to\infty. \tag{3.9}$$

Introduce the random variables

$$\gamma_{\mathbf{g}}(\omega,b) := \frac{b}{2\pi}\int_{\left(-\frac{1}{2},\frac{1}{2}\right)^3} Q_\omega(X_1+\mathbf{g},z_1)\int_{\mathbb{R}^{3(l-1)}} \Pi_{s=2}^l Q_\omega(X_1+\mathbf{g}+b^{-1/2}X_s, z_1+z_s)$$

$$\psi_l(z_2,\ldots,z_l)\Psi_l(X_2,\ldots,X_l)dX_2\ldots dX_l\, dz_2\ldots dz_l\; dX_1 dz_1, \quad l\geq 2, \quad \mathbf{g}\in\mathbb{Z}^3.$$

Set

$$\Gamma_L = \left\{\mathbf{g} = (g_1,g_2,g_3)\in\mathbb{Z}^3 \,|\, |g_j|\leq L,\; j=1,2,3\right\},\; L\in\mathbb{Z}, L\geq 1.$$

It is easy to verify that

$$\tilde{M}_{l,2L+1} = \sum_{\mathbf{g}\in\Gamma_L}\gamma_{\mathbf{g}}(\omega,b). \tag{3.10}$$

On the other hand, it is obvious that for each $\mathbf{k}\in\mathbb{Z}^3$ we have

$$\gamma_{\mathbf{g}+\mathbf{k}}(\omega,b) = \gamma_{\mathbf{g}}(\mathcal{T}_{\mathbf{k}}\omega,b),\; \mathbf{g}\in\mathbb{Z}^3,\; \omega\in\Omega,$$

(see (1.2)). Hence, the sequence $\{\gamma_{\mathbf{g}}\}_{\mathbf{g}\in\mathbb{Z}^3}$ is a $\mathbb{Z}^3$-ergodic random field. Therefore, we can apply the Birkhoff-Khintchine theorem (see e.g. [K, Theorem 2, Section 3.1] or [P.Fi, Proposition 1.7]). As a result we get almost surely

$$\lim_{L\to\infty}(2L+1)^{-3}\sum_{\mathbf{g}\in\Gamma_L}\gamma_{\mathbf{g}}(\omega,b) = \mathbb{E}(\gamma_{\mathbf{0}}) \equiv M_l(b), \quad l\geq 2. \tag{3.11}$$

Now the combination of (3.7)-(3.11) yields (3.5) with $l\geq 2$.
The proof in the case $l=1$ is similar but much simpler. $\diamondsuit$

Fix $X \in \left(-\frac{1}{2}, \frac{1}{2}\right)^2$. Introduce the operator

$$\tau_R(Q_\omega(X)) := \left(-\frac{d^2}{dz^2}+1\right)^{-1/2} Q_\omega(X,z)\mathbf{1}_{\left(-\frac{R}{2},\frac{R}{2}\right)}(z) \left(-\frac{d^2}{dz^2}+1\right)^{-1/2} \tag{3.12}$$

which is compact and selfadjoint in $L^2(\mathbb{R}_z)$, and depends on the parameters $X \in \left(-\frac{1}{2}, \frac{1}{2}\right)^2$ and $\omega \in \Omega$. It is easy to check that the operator $\tau_R(Q_\omega(X))$ is in the trace class.
For $X \in \left(-\frac{1}{2}, \frac{1}{2}\right)^2$ set

$$\mu_1(X) := \frac{1}{2}\mathbb{E}\left(\int_{\left(-\frac{1}{2},\frac{1}{2}\right)} Q_\omega(X,z)\,dz\right),$$

$$\mu_l(X) :=$$

$$\mathbb{E}\left(\int_{\left(-\frac{1}{2},\frac{1}{2}\right)} Q_\omega(X,z_1)\int_{\mathbb{R}^{l-1}} \Pi_{s=2}^{l} Q_\omega(X,z_1+z_s)\psi_l(z_2,\ldots,z_l)dz_2\ldots z_l\right), \quad l \geq 2.$$

By analogy with Proposition 3.1 we can demonstrate the following

Proposition 3.2 *Almost surely we have*

$$\lim_{R\to\infty} R^{-1}\mathrm{Tr}\,\tau_R(Q_\omega(X))^l = \mu_l(X),\ l \geq 1,\ X \in \left(-\frac{1}{2}, \frac{1}{2}\right)^2.$$

Set $m_l := \int_{\left(-\frac{1}{2},\frac{1}{2}\right)^2} \mu_l(X)dX,\ l \geq 1.$

Proposition 3.3 *We have*

$$\lim_{b\to\infty} b^{-1}M_l(b) = \frac{1}{2\pi}m_l,\ l \geq 1. \tag{3.13}$$

Proof. Obviously $M_1(b) = \frac{b}{2\pi}m_1$ which entails immediately (3.13) with $l = 1$.
Let $l \geq 2$. Using the fact that p_1 is an orthogonal projection, and taking into account the explicit form of its kernel $\mathcal{P}_1$ (see (3.1)), we get

$$\int_{\mathbb{R}^{2(l-1)}} \Psi_l(X_2,\ldots,X_l)dX_2\ldots dX_l =$$

$$2\pi\int_{\mathbb{R}^{2(l-1)}} \mathcal{P}_1(0;X_2)\mathcal{P}_1(X_2;X_3)\ldots\mathcal{P}_1(X_{l-1};X_l)\mathcal{P}_1(X_l;0)dX_2\ldots dX_l =$$

$$2\pi\mathcal{P}_1(0;0) = 1.$$

(cf. [R 2, pp.16-17]). Hence, we have

$$m_l = \frac{1}{2\pi}\mathbb{E}\left(\int_{\left(-\frac{1}{2},\frac{1}{2}\right)^3} Q_\omega(X_1,z_1)\int_{\mathbb{R}^{3(l-1)}} \psi_l(z_2,\dots,z_l)\Psi_l(X_2,\dots,X_l)\right.$$

$$\left.\Pi_{s=2}^l Q_\omega(X_1,z_1+z_s)dX_2\dots dX_l\,dz_2\dots dz_l\,dX_1dz_1\right).$$

Then, obviously,

$$b^{-1}M_l(b) - m_l = \frac{1}{2\pi}\mathbb{E}\left(\int_{\left(-\frac{1}{2},\frac{1}{2}\right)^3} Q_\omega(X_1,z_1)\int_{\mathbb{R}^{3(l-1)}} \psi_l(z_2,\dots,z_l)\Psi_l(X_2,\dots,X_l)\right.$$

$$\left(\Pi_{s=2}^l Q_\omega(X_1+b^{-1/2}X_s,z_1+z_s) - \Pi_{s=2}^l Q_\omega(X_1,z_1+z_s)\right)$$

$$\left.dX_2\dots dX_l\,dz_2\dots dz_l\,dX_1dz_1\right). \tag{3.14}$$

Since Q_ω is almost surely uniformly bounded and continuous, while $\psi_l \in L^1(\mathbb{R}^{l-1})$ and $\Psi_l \in L^1(\mathbb{R}^{2(l-1)})$, we find that it follows from the dominated convergence theorem that (3.14) entail (3.13). $\diamondsuit$.

Combining Propositions 3.1, 3.2, and 3.3, we get the following

Corollary 3.1 *For each $l \geq 1$ almost surely the limits*

$$\lim_{b\to\infty}\lim_{R\to\infty} b^{-1}R^{-3}\mathrm{Tr}\; t_{b,R}(Q_\omega)^l$$

and

$$\frac{1}{2\pi}\lim_{R\to\infty} R^{-1}\int_{\left(-\frac{1}{2},\frac{1}{2}\right)^2}\mathrm{Tr}\;\tau_R(Q_\omega(X))^l dX$$

exist, coincide, and are non-random.

4 Auxiliary results

Proposition 4.1 *Let $\tilde{H}(b,\omega,\lambda,R)$ be the operator defined in (2.5). Let $\lambda + b$ with $\lambda \in \mathbb{R}$ and $b > 0$, be a continuity point of $\mathcal{D}_b$. Then (2.6) is valid.*

Proof. First, note that we have

$$N(\lambda + b; H_{0,R}(b) + V_\omega) = N(-1; H_{0,R}(b) - b + V_\omega - \lambda - 1).$$

Hence, the minimax principle implies

$$N(-1; H_{0,R}(b) - b + V_\omega - \lambda - 1) \leq N(-1; \tilde{H}(b,\omega,\lambda,R)).$$

Therefore,

$$\liminf_{R\to\infty} R^{-3}N(-1;\tilde{H}(b,\omega,\lambda,R)) \geq \liminf_{R\to\infty} R^{-3}N(\lambda+b;H_{0,R}(b)+V_\omega) =$$

$$\lim_{R\to\infty} R^{-3}N(\lambda+b;H_{0,R}(b)+V_\omega) = \mathcal{D}_b(\lambda+b). \tag{4.1}$$

Further, fix $R>0$, $R_0\in(0,R)$, put

$$\mathcal{O}_1 = \mathcal{O}_{1,R} = \left(-\frac{R}{2},\frac{R}{2}\right)^3, \quad \mathcal{O}_2 = \mathcal{O}_{2,R,R_0} = \mathbb{R}^3 \setminus \left[-\frac{R-R_0}{2},\frac{R-R_0}{2}\right]^3,$$

and pick two functions φ_1 and φ_2 satisfying the following properties:

i) $\varphi_j\in C^\infty(\mathbb{R}^3)$, $j=1,2$;
ii) supp $\varphi_j\subseteq\mathcal{O}_j$, $j=1,2$;
iii) $\varphi_1^2(\mathbf{x})+\varphi_2^2(\mathbf{x})=1$ for every $\mathbf{x}\in\mathbb{R}^3$;
iv) $|\nabla\varphi_j(\mathbf{x})|\leq c_2R_0^{-1}$ for every $\mathbf{x}\in\mathbb{R}^3$ with c_2 which is independent of $\mathbf{x}$, R, and R_0.

Introduce the selfadjoint operator

$$\tilde{H}_D(b,\omega,\lambda,R,R_0) := \left(i\nabla+\frac{\mathbf{b}\wedge\mathbf{x}}{2}\right)^2 - b + (V_\omega-\lambda-1)\mathbf{1}_{\left(-\frac{R}{2},\frac{R}{2}\right)^3}$$

whose quadratic form is defined originally on $C_0^\infty(\mathcal{O}_2)$, and then is closed in $L^2(\mathcal{O}_2)$ (cf. (2.5)). Then the "magnetic" version of the so-called ISM localization formula (see [C.F.K.S, Section 3.1]) yields

$$N(-1;\tilde{H}(b,\omega,\lambda,R)) \leq$$

$$N(-1;H_{0,R}(b)-b+V_\omega-\lambda-1-\sum_{j=1,2}|\nabla\varphi_j|^2)+N(-1;\tilde{H}_D(b,\omega,\lambda,R,R_0)-\sum_{j=1,2}|\nabla\varphi_j|^2). \tag{4.2}$$

Obviously,

$$N(-1;H_{0,R}(b)-b+V_\omega-\lambda-1-\sum_{j=1,2}|\nabla\varphi_j|^2) \leq N(\lambda+b+2c_2^2R_0^{-2};H_{0,R}(b)+V_\omega). \tag{4.3}$$

Choose a sequence $\{\varepsilon_r\}_{r\geq1}$ such that $\varepsilon_r>0$, $r\geq1$, $\lim_{r\to\infty}\varepsilon_r=0$, and $\lambda+b+\varepsilon_r$, $r\geq1$, are continuity points of $\mathcal{D}_b$. Fix $r\geq1$ and set $R_0=\sqrt{2}c_2/\sqrt{\varepsilon_r}$. Then we have

$$\limsup_{R\to\infty} R^{-3}N(\lambda+b+2c_2^2R_0^{-2};H_{0,R}(b)+V_\omega) =$$

$$\lim_{R\to\infty} R^{-3}N(\lambda+b+\varepsilon_r;H_{0,R}(b)+V_\omega) = \mathcal{D}_b(\lambda+b+\varepsilon_r),\ r\geq1. \tag{4.4}$$

The combination of (4.3) and (4.4) yields

$$\limsup_{R\to\infty} R^{-3}N(-1; H_{0,R}(b)-b+V_\omega-\lambda-1-\sum_{j=1,2}|\nabla\varphi_j|^2) \leq \mathcal{D}_b(\lambda+b+\varepsilon_r),\ r\geq 1. \quad (4.5)$$

On the other hand, the minimax principle entails

$$N(-1; \tilde{H}_D(b,\omega,\lambda,R,R_0) - \sum_{j=1,2}|\nabla\varphi_j|^2) \leq N(-1; H_0(b)-b+W) \quad (4.6)$$

where

$$W = W_{\omega,\lambda,R,R_0} = \left((V_\omega-\lambda-1)\mathbf{1}_{\left(-\frac{R}{2},\frac{R}{2}\right)^3} - \sum_{j=1,2}|\nabla\varphi_j|^2\right)\mathbf{1}_{\mathcal{O}_2}.$$

Arguing as in [R 1, Section 5], we deduce the estimate

$$N(-1; H_0(b)-b+W) \leq c_3\int_{\mathbb{R}^3}|W|^{3/2}d\mathbf{x} \leq c_4\left(R^3-(R-R_0)^3\right) \quad (4.7)$$

where the quantities c_3 and c_4 may depend on b, λ, and R_0, but are independent of R. The combination of (4.6) and (4.7) yields

$$\lim_{R\to\infty} R^{-3}N(-1; \tilde{H}_D(b,\omega,\lambda,R,R_0) - \sum_{j=1,2}|\nabla\varphi_j|^2) = 0. \quad (4.8)$$

Putting together (4.2), (4.5), and (4.8), we obtain

$$\limsup_{R\to\infty} R^{-3}N(-1; \tilde{H}(b,\omega,\lambda,R)) \leq \mathcal{D}_b(\lambda+b+\varepsilon_r),\ r\geq 1.$$

Letting $r\to\infty$ (hence, $\varepsilon_r\to 0$), and bearing in mind that $\lambda+b$ is a continuity point of $\mathcal{D}_b$, we get

$$\limsup_{R\to\infty} R^{-3}N(-1; \tilde{H}(b,\omega,\lambda,R)) \leq \mathcal{D}_b(\lambda+b). \quad (4.9)$$

Now, the combination of (4.1) and (4.9) yields (2.6). $\diamondsuit$

For $b>0$, $\omega\in\Omega$, $\lambda\in\mathbb{R}$, and $R>0$, introduce the operator

$$T_{b,\omega,\lambda,R} := (H_0(b)-b+1)^{-1/2}(V_\omega-\lambda-1)\mathbf{1}_{\left(-\frac{R}{2},\frac{R}{2}\right)^3}(H_0(b)-b+1)^{-1/2}, \quad (4.10)$$

compact and selfadjoint in $L^2(\mathbb{R}^3)$.

Corollary 4.1 *Let $\lambda+b$ be a continuity point of $\mathcal{D}_b$. Then we have*

$$\mathcal{D}_b(\lambda+b) = \lim_{R\to\infty} R^{-3}n_-(1; T_{b,\omega,\lambda,R})$$

almost surely.

Proof. It suffices to note that the Birman-Schwinger principle (see [B, Lemma 1.1]) entails $N(-1; \tilde{H}(b,\omega,\lambda,R)) = n_-(1; T_{b,\omega,\lambda,R})$, and then to apply Proposition 4.1. $\diamondsuit$

Fix $X \in \left(-\frac{1}{2},\frac{1}{2}\right)^3$ and $\lambda \in \mathbb{R}$. For $s \in \mathbb{R}$, $s \neq 0$, set

$$\tilde{\varrho}_\lambda(s;X) := -\mathrm{sign}(s)\ \varrho\left(-\frac{\lambda+1}{s} - 1; -\frac{1}{s}V(X,.)\right)$$

Since $\varrho(\lambda, V(X,.))$ is a continuous function with respect to $\lambda \in \mathbb{R}$ (see Proposition 2.1), and V_ω is uniformly bounded (see (1.3)), we find that $\tilde{\varrho}_\lambda(s;X)$ is a continuous function with respect to $s \in \mathbb{R}\setminus\{0\}$ for any fixed $\lambda \in \mathbb{R}$. Moreover,

$$\tilde{\varrho}_\lambda(-1,X) = \varrho(\lambda, V(X,.)),\ X \in \left(-\frac{1}{2},\frac{1}{2}\right)^2.$$

For $R > 0$, $\omega \in \Omega$, $\lambda \in \mathbb{R}$, $X \in \left(-\frac{1}{2},\frac{1}{2}\right)^2$, and $s \in \mathbb{R}\setminus\{0\}$ set

$$\nu_{R,\omega,X,\lambda}(s) = \begin{cases} n_-(-s; \tau_R(V_\omega(X,.) - \lambda - 1)) & \text{if} \quad s < 0, \\ -n_+(s; \tau_R(V_\omega(X,.) - \lambda - 1)) & \text{if} \quad s > 0, \end{cases}$$

where the operators τ_R are defined in (3.12).

Proposition 4.2 *For every $\lambda \in \mathbb{R}$, $X \in \left(-\frac{1}{2},\frac{1}{2}\right)^2$, and $s \in \mathbb{R}\setminus\{0\}$, we have*

$$\tilde{\varrho}_\lambda(s;X) = \lim_{R\to\infty} R^{-1}\nu_{R,\omega,X,\lambda}(s) \tag{4.11}$$

almost surely.

Proof. We shall prove (4.11) in the case $s < 0$. In this case Proposition 2.1 implies

$$\tilde{\varrho}_\lambda(s,X) = \lim_{R\to\infty} R^{-1}N(-\frac{\lambda+1}{s} - 1; \chi_{0,R} - \frac{1}{s}V_\omega(X,.)) =$$

$$\lim_{R\to\infty} R^{-1}N(-1; \chi_{0,R} - \frac{1}{s}(V_\omega(X,.) - \lambda - 1)). \tag{4.12}$$

On $\mathrm{H}^2(\mathbb{R})$ define the operator

$$\tilde{\chi}_{s,\omega,X,\lambda,R} := -\frac{d^2}{dz^2} - \frac{1}{s}(V_\omega(X,.) - \lambda - 1)\mathbf{1}_{\left(-\frac{R}{2},\frac{R}{2}\right)},\ R > 0.$$

Applying the minimax principle, and bearing in mind that $\chi_{0,R} - \frac{1}{s}(V_\omega(X,.) - \lambda - 1)$ and $\tilde{\chi}_{s,\omega,X,\lambda,R}$ are second-order ordinary differential operators, we get

$$0 \leq N(-1; \tilde{\chi}_{s,\omega,X,\lambda,R}) - N(-1; \chi_{0,R} - \frac{1}{s}(V_\omega(X,.) - \lambda - 1)) \leq 2.$$

Therefore, (4.12) implies

$$\tilde{\varrho}_\lambda(s, X) = \lim_{R\to\infty} R^{-1} N(-1; \tilde{\chi}_{s,\omega,X,\lambda,R}). \tag{4.13}$$

On the other hand, the Birman-Schwinger principle entails

$$N(-1; \tilde{\chi}_{s,\omega,X,\lambda,R}) = n_-(-s; \tau_R(V_\omega(X,.) - \lambda - 1)) \equiv \nu_{R,\omega,X,\lambda}(s). \tag{4.14}$$

Putting together (4.13) and (4.14), we obtain (4.11) with $s < 0$. The proof in the case $s > 0$ is completely analogous. $\diamondsuit$

For $\lambda \in \mathbb{R}$ and $s \in \mathbb{R} \setminus \{0\}$ set

$$\tilde{\delta}_\lambda(s) := \int_{\left(-\frac{1}{2},\frac{1}{2}\right)^2} \tilde{\varrho}_\lambda(s; X) dX. \tag{4.15}$$

Obviously, $\tilde{\delta}_\lambda(s)$ is continuous with respect to s. Moreover,

$$\tilde{\delta}_\lambda(-1) = \delta(\lambda), \ \lambda \in \mathbb{R}. \tag{4.16}$$

Remark. The function $\nu_{R,\omega,X,\lambda}(s)$ of the variable $s \in \mathbb{R}\setminus\{0\}$ is non-negative on $(-\infty, 0)$, non-positive on $(0,\infty)$, and non-decreasing on $(-\infty, 0)$ and $(0,\infty)$. By (4.11) and (4.15), the functions $\tilde{\varrho}_\lambda(s; X)$ and $\tilde{\delta}_\lambda(s)$ have the same properties.

Corollary 4.2 *For each $\lambda \in \mathbb{R}$ almost surely we have*

$$\lim_{b\to\infty} \lim_{R\to\infty} b^{-1} R^{-3} \mathrm{Tr}\, t_{b,R}(V_\omega - \lambda - 1)^l = \frac{1}{2\pi} \int_{\mathbb{R}} s^l d\tilde{\delta}_\lambda(s), \ l \geq 1. \tag{4.17}$$

Proof. By Corollary 3.1, we have

$$\lim_{b\to\infty} \lim_{R\to\infty} b^{-1} R^{-3} \mathrm{Tr}\, t_{b,R}(V_\omega - \lambda - 1)^l = \frac{1}{2\pi} \lim_{R\to\infty} R^{-1} \int_{\left(-\frac{1}{2},\frac{1}{2}\right)^2} \mathrm{Tr}\, \tau_R(V_\omega(X,.) - \lambda - 1)^l dX =$$

$$\frac{1}{2\pi} \lim_{R\to\infty} R^{-1} \int_{\left(-\frac{1}{2},\frac{1}{2}\right)^2} \int_{\mathbb{R}} s^l d\nu_{R,\omega,X,\lambda}(s)\, dX. \tag{4.18}$$

Proposition 4.2 easily implies that

$$\lim_{R\to\infty} R^{-1} \int_{\left(-\frac{1}{2},\frac{1}{2}\right)^2} \int_{\mathbb{R}} s^l d\nu_{R,\omega,X,\lambda}(s)\, dX = \int_{\left(-\frac{1}{2},\frac{1}{2}\right)^2} \int_{\mathbb{R}} s^l d\tilde{\varrho}_\lambda(s; X)\, dX = \int_{\mathbb{R}} s^l d\tilde{\delta}_\lambda(s), \tag{4.19}$$

and the combination of (4.18) and (4.19) yields (4.17). $\diamondsuit$

Corollary 4.3 *For each $\lambda \in R$ and $s < 0$ we have*

$$\lim_{b\to\infty}\lim_{R\to\infty} b^{-1}R^{-3}n_-(-s; t_{b,R}(V_\omega-\lambda-1)) = \frac{1}{2\pi}\tilde{\delta}_\lambda(s). \tag{4.20}$$

Proof. We have $\|t_{R,b}(V_\omega-\lambda-1)\| \leq c_1$ (see (3.2)). Moreover, $\tilde{\delta}_\lambda(s) = 0$ if $|s| > c_1$. Hence we can apply the Kac-Murdock-Szegö theorem (see [R 1, Section 3]) and, taking into account the continuity of $\tilde{\delta}_\lambda(.)$, to conclude that (4.17) implies (4.20). $\diamondsuit$

5 Proof of Theorem 2.1

In order to prove (2.2), it suffices to show that for each sequence $\{b_j\}_{j\geq 1}$ such that $b_j \to \infty$ as $j \to \infty$, we have

$$\lim_{j\to\infty} b_j^{-1}\mathcal{D}_{b_j}(\lambda+b_j) = \frac{1}{2\pi}\delta(\lambda),\ \lambda\in\mathbb{R}. \tag{5.1}$$

Fix two sequences $\{\lambda_m^\pm\}_{m\geq 1}$ such that $\lambda_m^- < \lambda < \lambda_m^+$, $m \geq 1$, $\lim_{m\to\infty}\lambda_m^\pm = \lambda$, and $\lambda_m^\pm + b_j$ are continuity points of $\mathcal{D}_{b_j}$ for all $m \geq 1$ and $j \geq 1$. Then by Corollary 4.1 we have

$$\limsup_{j\to\infty} b_j^{-1}\mathcal{D}_{b_j}(\lambda+b_j) \leq \limsup_{j\to\infty}\lim_{R\to\infty} b_j^{-1}R^{-3}n_-(1; T_{b_j,\omega,\lambda_m^+,R}), \tag{5.2}$$

$$\liminf_{j\to\infty} b_j^{-1}\mathcal{D}_{b_j}(\lambda+b_j) \geq \liminf_{j\to\infty}\lim_{R\to\infty} b_j^{-1}R^{-3}n_-(1; T_{b_j,\omega,\lambda_m^-,R}), \tag{5.3}$$

where the operators $T_{b,\omega,\lambda,R}$ are defined in (4.10). The minimax principle entails

$$\liminf_{j\to\infty}\lim_{R\to\infty} b_j^{-1}R^{-3}n_-(1; T_{b_j,\omega,\lambda_m^-,R}) \geq \liminf_{j\to\infty}\liminf_{R\to\infty} b_j^{-1}R^{-3}n_-(1; p_{b_j}T_{b_j,\omega,\lambda_m^-,R}\, p_{b_j}). \tag{5.4}$$

Note that the operator $p_bT_{b,\omega,\lambda,R}\, p_b$ coincides with the operator $t_{b,R}(V_\omega-\lambda-1)$ (see (3.3)). Hence Corollary 4.3 and (4.16) imply

$$\liminf_{j\to\infty}\liminf_{R\to\infty} b_j^{-1}R^{-3}n_-(1; p_{b_j}T_{b_j,\omega,\lambda_m^-,R}p_{b_j}) \geq$$

$$\lim_{b\to\infty}\lim_{R\to\infty} b^{-1}R^{-3}n_-(1; t_{R,b}(V_\omega-\lambda_m^--1)) = \frac{1}{2\pi}\tilde{\delta}_{\lambda_m^-}(-1) = \frac{1}{2\pi}\delta(\lambda_m^-). \tag{5.5}$$

The combination of (5.3)-(5.5) yields

$$\liminf_{j\to\infty} b_j^{-1}\mathcal{D}_{b_j}(\lambda+b_j) \geq \frac{1}{2\pi}\delta(\lambda_m^-). \tag{5.6}$$

On the other hand, we have

$$T_{b_j,\omega,\lambda_m^+,R} = p_{b_j} T_{b_j,\omega,\lambda_m^+,R}\, p_{b_j} + (\mathrm{Id} - p_{b_j}) T_{b_j,\omega,\lambda_m^+,R} (\mathrm{Id} - p_{b_j}) + 2\mathrm{Re}\, p_{b_j} T_{b_j,\omega,\lambda_m^+,R} (\mathrm{Id} - p_{b_j}). \tag{5.7}$$

Set

$$\tilde{T}_{b,\omega,\lambda,R} :=$$
$$(\mathrm{Id} - p_b)(H_0(b) - b + 1)^{-1/2} |V_\omega - \lambda - 1| \mathbf{1}_{(-\frac{R}{2},\frac{R}{2})^3} (H_0(b) - b + 1)^{-1/2} (\mathrm{Id} - p_b).$$

Applying the elementary operator inequalities

$$(\mathrm{Id} - p_{b_j}) T_{b_j,\omega,\lambda_m^+,R} (\mathrm{Id} - p_{b_j}) \geq -\tilde{T}_{b_j,\omega,\lambda_m^+,R},$$

$$2\mathrm{Re}\, p_{b_j} T_{b_j,\omega,\lambda_m^+,R} (\mathrm{Id} - p_{b_j}) \geq -\varepsilon^2 t_{b_j,R}(|V_\omega - \lambda_m^+ - 1|) - \varepsilon^{-2} \tilde{T}_{b_j,\omega,\lambda_m^+,R},\ \varepsilon > 0,$$

we find that (5.7) entails

$$n_-(1; T_{b_j,\omega,\lambda_m^+,R}) \leq n_-(1; t_{b_j,R}(V_\omega - \lambda_m^+ - 1) - \varepsilon^2 t_{b_j,R}(|V_\omega - \lambda_m^+ - 1|)) +$$
$$n_+(1; (1+\varepsilon^{-2}) \tilde{T}_{b_j,\omega,\lambda_m^+,R}),\ \varepsilon > 0. \tag{5.8}$$

It easy to verify the estimate

$$\|\tilde{T}_{b_j,\omega,\lambda_m^+,R}\| \leq (b_j + 1)^{-1}(c_0 + |\lambda_m^+ + 1|),\ R > 0. \tag{5.9}$$

Fix $\varepsilon > 0$ and assume that b_j is so big that we have

$$(1+\varepsilon^{-2})(b_j + 1)^{-1}(c_0 + |\lambda_m^+| + 1) < 1.$$

Then (5.9) entails

$$n_+(1; (1+\varepsilon^{-2}) \tilde{T}_{b_j,\omega,\lambda_m^+,R}) = 0. \tag{5.10}$$

The Weyl inequalities for the eigenvalues of compact selfadjoint operators imply

$$n_-(1; t_{b_j,R}(V_\omega - \lambda_m^+ - 1) - \varepsilon^2 t_{b_j,R}(|V_\omega - \lambda_m^+ - 1|)) \leq$$
$$n_-(1-\varepsilon; t_{b_j,R}(V_\omega - \lambda_m^+ - 1)) + n_+(1; \varepsilon t_{b_j,R}(|V_\omega - \lambda_m^+ - 1|)), \varepsilon \in (0,1). \tag{5.11}$$

The estimate (3.4) implies

$$n_+(1; \varepsilon t_{b_j,R}(|V_\omega - \lambda_m^+ - 1|)) \leq c_5 \varepsilon b_j R^3 \tag{5.12}$$

with $c_5 := (c_0 + |\lambda_m^+ + 1|)/4\pi$. Now, the combination of (5.8), (5.10), (5.11), and (5.12) yields

$$\limsup_{j\to\infty} \lim_{R\to\infty} b_j^{-1} R^{-3} n_-(1; T_{b_j,\omega,\lambda_m^+,R}) \leq$$
$$\limsup_{j\to\infty} \limsup_{R\to\infty} b_j^{-1} R^{-3} n_-(1-\varepsilon; t_{b_j,R}(V_\omega - \lambda_m^+ - 1)) + c_5\varepsilon,\ \varepsilon \in (0,1). \tag{5.13}$$

Corollary 4.3 implies

$$\limsup_{j\to\infty}\limsup_{R\to\infty} b_j^{-1}R^{-3}n_-(1-\varepsilon; t_{b_j,R}(V_\omega-\lambda_m^+-1)) =$$

$$\lim_{b\to\infty}\lim_{R\to\infty} b^{-1}R^{-3}n_-(1-\varepsilon; t_{b,R}(V_\omega-\lambda_m^+-1)) = \frac{1}{2\pi}\tilde{\delta}_{\lambda_m^+}(-1+\varepsilon),\ \varepsilon\in(0,1). \tag{5.14}$$

The combination of (5.2), (5.13), and (5.14) yields

$$\limsup_{j\to\infty} b_j^{-1}\mathcal{D}_{b_j}(\lambda+b_j) \le \frac{1}{2\pi}\tilde{\delta}_{\lambda_m^+}(-1+\varepsilon)+c_5\varepsilon,\ \varepsilon\in(0,1).$$

Letting $\varepsilon\downarrow 0$, we get

$$\limsup_{j\to\infty} b_j^{-1}\mathcal{D}_{b_j}(\lambda+b_j) \le \frac{1}{2\pi}\tilde{\delta}_{\lambda_m^+}(-1) = \frac{1}{2\pi}\delta(\lambda_m^+). \tag{5.15}$$

Letting $m\to\infty$ (hence, $\lambda_m^\pm\to\lambda$) in (5.6) and (5.15), we arrive at (5.1).

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References

[A.H.S] J.Avron, I.Herbst, B.Simon, *Schrödinger operators with magnetic fields. I. General interactions*, Duke. Math. J. **45** (1978), 847-883.

[B] M.Š.Birman, *On the spectrum of singular boundary value problems*, Mat. Sbornik **55** (1961) 125-174 (Russian); Engl. transl. in Amer. Math. Soc. Transl., (2) **53** (1966), 23-80.

[Br.G.I] E.Brézin, D.J.Gross, C.Itzykson, *Density of states in the presence of a strong magnetic field and random impurities*, Nucl.Phys. B **235** (1984)

[Bro.H.L] K.Broderix, D.Hundertmark, H.Leschke *Self-averaging, decomposition and asymptotic properties of the density of states for random Schrödinger operators with constant magnetic field*, In: Path Integrals from meV to MeV, Tutzing '92, World Scientific, Singapore (1993), p.98.

[CdV] Y.Colin de Verdière, *L'asymptotique de Weyl pour les bouteilles magnétiques*, Commun.Math.Phys. **105** (1986), 327-335.

[C.F.K.S] H.L.Cycon, R.G.Froese, W.Kirsch, B.Simon, *Schrödinger operators, with application to quantum mechanics and global geometry.* Texts and Monographs in Physics. Springer-Verlag. Berlin etc.(1987).

[K] W.Kirsch, *Random Schrödinger operators.* In: Schroedinger operators, Proc. Nord. Summer Sch. Math., Sandbjerg Slot, Soenderborg/Denmark 1988, Lect. Notes Phys. **345**, (1989) 264-370.

[E.K.Sch.S] H.Englisch, W.Kirsch, M.Schröder, B.Simon *Random Hamiltonians ergodic in all but one direction*, Commun.Math.Phys. **128** (1990), 613-625.

[M.Pu] N.Macris, J.V.Pulé, *Density of states of random Schrödinger operators with a uniform magnetic field.* Lett. Math. Phys. **24** (1992), 307-321.

[Ma] H.Matsumoto *On the integrated density of states for the Schrödinger operators with certain random electromagnetic potentials* J. Math. Soc. Japan **45** (1993), 197-214.

[P.Fi] L.Pastur, A.Figotin, *Spectra of Random and Almost-Periodic Operators.* Grundlehren der Mathematischen Wissenschaften **297**. Berlin etc.: Springer-Verlag, Berlin etc. (1992).

[Pu.Sc] J.V.Pulé, M.Scrowston, *Infinite degeneracy for a Landau Hamiltonian with Poisson impurities.* J. Math. Phys. **38** (1997), 6304-6314.

[R 1] G.D.Raikov, *Eigenvalue asymptotics for the Schrödinger operator in strong constant magnetic fields*, Commun. P.D.E. **23** (1998), 1583-1620.

[R 2] G.D.Raikov, *Eigenvalue asymptotics for the Dirac operator in strong constant magnetic fields* Math.Phys.Electr.J., **5** (1999) No.2, 22 pp.

[R 3] G.D.Raikov, *Eigenvalue asymptotics for the Pauli operator in strong non-constant magnetic fields.* (to appear in Ann.Inst.Fourier)

[U] N.Ueki, *On spectra of random Schrödinger operators with magnetic fields* Osaka J. Math. **31** (1994), 177-187.

[W] W.-M.Wang, *Asymptotic expansion for the density of states of the magnetic Schrödinger operator with a random potential*, Commun. Math. Phys. **172** (1995), 401-425.