EQUILIBRIUM FLUCTUATIONS FOR A DRIVEN TRACER PARTICLE DYNAMICS

Cláudio Landim, Sérgio B. Volchan

IMPA and CNRS UPRES-A 6085
PUC-Rio, Rio de Janeiro

Abstract. We study the equilibrium fluctuations of a tagged particle driven by an external constant force in an infinite system of particles evolving in a one-dimensional lattice according to symmetric random walks with exclusion. We prove that when the system is initially in the equilibrium state, the finite dimensional distributions of the diffusively rescaled position $\sqrt{\epsilon}X(\epsilon^{-2}t)$ of the tagged particle converges, as $\epsilon \to 0$, to the finite dimensional distributions of a mean zero Gaussian process whose covariance can be expressed in terms of a diffusion process.

Introduction.

In this paper we consider the motion of a tagged particle evolving in an infinite lattice gas and which is driven by an external force. The motion occurs along the sites of the one-dimensional lattice through a local stochastic interaction. Particles perform symmetric random walks except for the special tagged particle which, due to the action of the external force, performs an asymmetric random walk, serving in this way as a probe of the system's global behavior. Interaction among particles results from imposing an exclusion (or hard-core) condition, that is, particles are allowed to jump to nearest neighbor sites only, but the jump is suppressed in case the target site is already occupied. We are interested in the long-time behavior of the tagged particle's trajectory X_t. This system can be thought of as a caricature model of the microscopic dynamics of particles of a gas or fluid and has originated in attempts to derive a rigorous mechanical explanation of Brownian Motion. Of course, in a more realistic setting particles should evolve deterministically according to Hamilton's equations. This has proven to be one of the hardest problems of Classical Statistical Mechanics (cf. [DGL1,2], [H], [PSV], [So], [Sp]). In general, progress in this path has been achieved by adding somewhat artificial simplifications. On the other hand, as past experience suggests, various aspects of the macroscopic behavior of such many-particle systems are universal, in the sense of not depending too much on the underlying microscopic dynamics. Therefore, the study of highly simplified models is not devoid of interest and can help us improve our understanding of such systems [GLP]. Moreover, simplified models like the lattice gas can be highly non-trivial while sometimes yielding to a detailed and rigorous mathematical treatment.

In a previous work [LOV] we proved a Law of Large Numbers for the tagged particle's position subject to a suitable diffusive rescaling of space and time. That is, for a large class of initial

1991 *Mathematics Subject Classification.* Primary 60K35; secondary 82A05.
Key words and phrases. Tagged particle, Equilibrium Fluctuations, Exclusion processes, Hydrodynamic limit.

distributions of the random environment, we proved that:

$$\lim_{t\to\infty} \frac{X_t - X_0}{\sqrt{t}} \;=\; v$$

in probability, where v is a real number depending only on the macroscopic density profile of the initial state through the solution of a nonlinear parabolic partial differential equation with boundary conditions. This result was based in establishing the hydrodynamic limit for an associated zero-range model from which the tagged particle's position behavior could be derived.

As usual in probabilistic and statistical mechanical contexts, after having proved a Law of Large Numbers the next step should consist in finding a corresponding Central Limit Theorem describing the fluctuations of the system around the limiting value obtained previously. This requires an analysis on a finer scale. In fact, the main result of this paper states that, in equilibrium,

01

$$\lim_{N\to\infty} \frac{X_{tN^2} - X_0}{\sqrt{N}} \;=\; Z_t\;, \tag{0.1}$$

for all finite-dimensional distributions and where Z is a mean zero Gaussian process. The result can be interpreted as indicating that the tagged particle tends to be trapped between its neighbors and the fluctuations of its position will depend directly on the fluctuations of the density of particles around it. We observe that for one dimensional nearest neighbor symmetric simple exclusion models Arratia [Ar] proved (0.1) with convergence in law.

Our result is based on the study of the equilibrium fluctuations for an associated zero-range model, as before. As in the proof of the Law of Large Numbers, we needed to adapt the standard techniques of hydrodynamic limit theory of interacting particle systems to our situation in which the system is in infinite volume and the invariant measure has nonuniform density reflecting the presence of boundary conditions.

1. Notation and Results.

We consider a family of indistinguishable particles moving according to continuous-time, symmetric, nearest neighbor random walks on $\mathbb{Z}$ with an exclusion rule that prevents more than one particle per site. Added to this system is a tagged-particle that moves according to an asymmetric random walk, having probability p $(1/2 < p < 1)$ of jumping to the right and probability $q = 1 - p$ to the left and respecting the exclusion rule. The configuration of the system is given by the pair (X, ξ), where $X \in \mathbb{Z}$ is the position of the tagged particle, and $\xi \in \{0,1\}^{\mathbb{Z}}$ is the configuration of all other particles (which is the tagged particle's environment). Note that ξ is indexed by the sites of $\mathbb{Z}$, so that $\xi(z)$ is 0 or 1 according with the site z being occupied or empty. In particular, $\xi(X) = 0$ as the site X is already occupied by the tagged particle. The system just described is a Markov process whose infinitesimal generator, acting on local functions $F : \mathbb{Z} \times \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$, is given by

$$\begin{aligned} LF(X,\xi) \;=\; (1/2) \sum_{z\neq X-1,X} & [F(X,\sigma^{z,z+1}\xi) \;-\; F(X,\xi)] \\ & + p(1 \;-\; \xi(X+1))[F(X+1,\xi) \;-\; F(X,\xi)] \\ & + q(1 \;-\; \xi(X-1))[F(X-1,\xi) \;-\; F(X,\xi)]\,, \end{aligned}$$

where $\sigma^{z,z+1}\xi$ is the configuration obtained from ξ, exchanging the occupation variables $\xi(z)$ and $\xi(z+1)$:

$$(\sigma^{z,z+1}\xi)(y) \;=\; \begin{cases} \xi(y) & \text{if } y \neq z, z+1, \\ \xi(z) & \text{if } y = z+1, \\ \xi(z+1) & \text{if } y = z. \end{cases}$$

For simplicity, we assume $X_0 = 0$, where X_t stands for the tagged particle's position at time t. For a positive integer N and a density profile $\kappa_0 : \mathbb{R} \to [0,1]$, let $\mu^N_{\kappa_0(\cdot)}$ be the Bernoulli product measure associated to κ_0:

$$\mu^N_{\kappa_0(\cdot)}\{\xi, \xi(x) = 1\} \;=\; \kappa_0(x/N),$$

which we take as the initial distribution for the random environment ξ_0. Denote by $P_{\mu^N_{\kappa_0}}$. the probability measure on the path space $D(\mathbb{R}_+, \mathbb{Z} \times \{0,1\}^{\mathbb{Z}})$ induced by the Markov process with generator L and initial measure $\delta_0 \times \mu^N_{\kappa_0(\cdot)}$. Here δ_0 stands for the Dirac measure concentrated on the origin. In [LOV] we proved that for each $t \geq 0$ and for a large class of initial density profiles κ_0, as $N \uparrow \infty$, X_{tN^2}/N converges in probability to a real number v_t that depends only on κ_0. In the present article we prove that for $t \geq 0$, and for the *equilibrium* initial distribution of the environment, as $N \uparrow \infty$, the finite dimensional distributions of $X_{tN^2}/\sqrt{N}$ converge to the finite dimensional distributions of a mean zero Gaussian process.

The strategy begins by exploiting a connexion of the present model with a zero-range model that we now describe. Start by labeling all particles. The tagged particle is labeled 0 and for $j \geq 1$, we label the j-th particle at the right (resp. left) of the tagged particle by j (resp.$-j$). Let then $\zeta(x)$, for $x \in \mathbb{Z}$, be the number of holes (empty sites) between the particle labeled x and the particle labeled $x+1$. This establishes a transformation $\mathcal{T} : \mathbb{Z} \times \{0,1\}^{\mathbb{Z}} \to \mathbb{N}^{\mathbb{Z}}$ taking each configuration of $\{0,1\}^{\mathbb{Z}}$ with a particle in some site X into a configuration $\zeta = \{\zeta(x), x \in \mathbb{Z}\} \in \mathbb{N}^{\mathbb{Z}}$.

The dynamics of the process (X_t, ξ_t) induces a dynamics for the process ζ_t that can be described informally as follows. For all $x \neq -1$, if there is at least one particle at site x, at rate $1/2$ one of them jumps to site $x+1$ and, symmetrically, if there is at least one particle at site $x+1$, at rate $1/2$ one of them jumps to site x. The picture changes between sites -1 and 0 due to the asymmetric behavior of the tagged particle. A particle jumps at rate q from site -1 to site 0 if there is at least a particle at -1 and a particle jumps from site 0 to site -1 at rate p if there is at least a particle at the origin.

This process is a so called zero range process with an asymmetry at the origin. Note that the position at time t of the tagged particle corresponds in the zero range model to the total number of jumps in $[0,t]$ from 0 to -1 minus the total number of jumps in the same time interval from -1 to 0, that is:

pme

$$X_t \;=\; \sum_{x \geq 0} [\zeta_0(x) \;-\; \zeta_t(x)]\,. \tag{1.1}$$

As stated, this is a formal sum whose precise sense we will explain later in connexion with the notion of density fields (cf. Lemma 3.1). Since in our zero range process the jumps of particles over all bonds, except bond $\{-1,0\}$, are symmetric, we expect the process to have a diffusive hydrodynamic behavior. This suggests considering the process accelerated by N^2 and to study the fluctuation behavior from equilibrium of

$$\frac{X_{tN^2}}{\sqrt{N}} \;=\; \frac{1}{\sqrt{N}} \sum_{x \geq 0} [\zeta_0(x) \;-\; \zeta_{tN^2}(x)],$$

as $N \uparrow \infty$. This "scaling" explains the factor $t^{1/4}$ mentioned previously.

Formally, we are dealing with a Markov process on $\mathbb{Z}$ whose generator, acting on cylindrical functions f, is given by

$$(\mathcal{L}f)(\zeta) \;=\; \sum_{x \in \mathbb{Z}, |y|=1} g(\zeta_x)[f(\zeta^{x,x+y}) - f(\zeta)]\, p_x(y),$$

where $\zeta^{x,y}$ is the configuration obtained from ζ by letting a particle jump from x to y:

$$(\zeta^{x,y})(z) = \begin{cases} \zeta(x) - 1 & \text{if } z = x, \\ \zeta(y) + 1 & \text{if } z = y, \\ \zeta(z) & \text{otherwise ;} \end{cases} \qquad p_x(y) \;=\; \begin{cases} p & x = 0, y = -1 \\ q & x = -1, y = 1 \\ 1/2 & \text{otherwise} \end{cases}$$

and $g : \mathbb{N} \to \mathbb{R}_+$ is a fixed jump rate given by $g(k) = \mathbf{1}_{\{k \geq 1\}}$. For each φ, with $0 \leq \varphi < p^{-1}$ this process has an invariant reversible product measure ν_φ with marginals given by:

inm

$$\nu_\varphi\{\zeta : \zeta(x) = k\} \;=\; (1 - \varphi_x)\, \varphi_x^k \,, \tag{1.2}$$

where

$$\varphi_x \;=\; \begin{cases} p\varphi & x \leq -1, \\ q\varphi & x \geq 0\,. \end{cases}$$

For $N \geq 1$, let $\eta_t = \eta_t^N$ be the Markov process on $\mathbb{N}^{\mathbb{Z}}$ with generator $N^2 \mathcal{L}$: $\eta_t = \zeta_{tN^2}$. Let $\pi^N = \pi^N(\eta)$ be the empirical measure defined as the positive Radon measure on $\mathbb{R}$ obtained by assigning mass N^{-1} to each particle:

$$\pi^N \;=\; N^{-1} \sum_{z \in \mathbb{Z}} \eta(z) \delta_{z/N}.$$

Set $\pi_t^N = \pi^N(\eta_t)$ and let $\mathcal{M}_+$ be the space of positive Radon measures on $\mathbb{R}$. We proved in [LOV] that starting from a sequence of probability measures $\{\nu^N, N \geq 1\}$, associated to an initial density profile $\rho_0 : \mathbb{R} \to \mathbb{R}_+$ in the sense that

$$\lim_{N \to \infty} \nu^N \Big\{ \Big| \frac{1}{N} \sum_{x \in \mathbb{Z}} H(x/N) \eta(x) \;-\; \int_{\mathbb{R}} H(u)\, \rho_0(u)\, du \Big| > \delta \Big\} \;=\; 0$$

for all continuous functions H with compact support and all $\delta > 0$, then for all $t \geq 0$ π_t^N converges in probability to the absolutely continuous trajectory $\pi(t, du) = \rho(t,u) du$ whose density $\rho(\cdot, \cdot)$ is the weak solution of the following non-linear parabolic partial differential equation with boundary conditions at the origin:

dife1

$$\begin{cases} \partial_t \rho = (1/2) \Delta \Phi(\rho) \\ p\Phi(\rho(t, 0+)) = q\Phi(\rho(t, 0-)) \\ \partial_u \Phi(\rho(t, 0+)) = \partial_u \Phi(\rho(t, 0-)) \\ \rho(0, \cdot) = \rho_0(\cdot)\,, \end{cases} \tag{1.3}$$

where $\Phi(\rho) = \rho/1 + \rho$.

Fix once and for all $0 < \varphi < p^{-1}$. Denote by ρ_+ (resp. ρ_-) the density at the right (resp. left) of the origin for ν_φ: $\rho_\pm = E_{\nu_\varphi}[\eta(\pm 1)]$. To investigate the equilibrium fluctuations of π_t^N, we focus on the density field $Y_t^N(\cdot)$ which acts on smooth functions H as

$$Y_t^N(H) \;=\; N^{-1/2} \sum_{x \in \mathbb{Z}} H(x/N) [\eta_t(x) \;-\; \rho_x] \,,$$

where $\rho_x \;=\; E_{\nu_\varphi}[\eta(x)]$. Our aim is then to prove that $Y_\cdot^N$ converges to a stationary generalized Ornstein-Uhlenbeck process $Y_\cdot$ which is solution of the formal stochastic differential equation:

$$dY_t \;=\; (1/2) \Phi'(\rho) \Delta\, Y_t\, dt \;+\; \sqrt{\Phi(\rho)/2}\, \nabla\, dW_t,$$

where W_t is a generalized Brownian motion. In other words, Y_t is solution of the linearized hydrodynamic equation (1.3) with a white noise added. To derive the linearization of equation (1.3), recall the definition of $\rho_\pm$ and define

roo

$$\rho(u) \;=\; \rho_+ \,\mathbf{1}_{\{u>0\}} \;+\; \rho_- \,\mathbf{1}_{\{u<0\}} \;. \tag{1.4}$$

It is clear that ρ is a *stationary* solution of the hydrodynamic equation (1.3). Let $\rho^\epsilon(\cdot)$ be an initial data which is close to that solution in the sense that $\epsilon^{-1}(\rho^\epsilon - \rho) \to \lambda$ as $\epsilon \downarrow 0$. That is, $\lambda(\cdot)$ is the "deviation" from equilibrium in first (or linear) order in ϵ. We first find the boundary conditions for λ. As $\rho(\cdot)$ and $\rho^\epsilon(\cdot,\cdot)$ are solutions of the hydrodynamic equation, they satisfy the boundary condition $p\,\Phi(\rho^\epsilon(t,0+)) = q\,\Phi(\rho^\epsilon(t,0-))$ and $p\Phi(\rho_+) = q\Phi(\rho_-)$, from which follows, subtracting term by term and expanding in ϵ, that $p\Phi'(\rho_+)\lambda(t,0+) = q\Phi'(\rho_-)\lambda(t,0-)$. On the other hand, from the second boundary condition, we have that $\partial_u\Phi(\rho^\epsilon(t,0+)) = \partial_u\Phi(\rho^\epsilon(t,0-))$. A simple Taylor expansion then gives that $\Phi'(\rho_+)\partial_u\lambda(t,0+) = \Phi'(\rho_-)\partial_u\lambda(t,0-)$. We can now find the differential equation for $\lambda(\cdot)$. We know that $\rho^\epsilon(\cdot,\cdot)$ satisfies

$$\int_{\mathbb{R}} du\, \rho^\epsilon_t(u) H(u) \;=\; \int_{\mathbb{R}} du\, \rho^\epsilon_0(u) H(u) \;-\; (1/2)\int_0^t ds \int_{\mathbb{R}} du\, \partial_u H(u)\, \partial_u \Phi(\rho^\epsilon_s)$$

for smooth functions $H(\cdot)$. Expanding in ϵ we obtain that λ satisfies the equation

$$\int_{\mathbb{R}} du\, \lambda_t(u) H(u) \;-\; \int_{\mathbb{R}} du\, \lambda_0(u) H(u) \;=\; -(1/2)\int_0^t ds \int_{\mathbb{R}} du\, \partial_u H(u)\, \Phi'(\rho)\, \partial_u \lambda_s(u)\;.$$

Notice that in this formula $\Phi'(\rho(u)) = \Phi'(\rho_+)$ (resp. $\Phi'(\rho_-)$) for $u > 0$ (resp. $u < 0$). Hence, λ is the solution of the linear second order equation with boundary conditions at the origin:

dife2

$$\begin{cases} \partial_t \lambda \;=\; (1/2)\Phi'(\rho)\Delta\lambda \\ p\Phi'(\rho_+)\,\lambda(t,0+) \;=\; q\Phi'(\rho_-)\,\lambda(t,0-) \\ \Phi'(\rho_+)\partial_u\lambda(t,0+) \;=\; \Phi'(\rho_-)\,\partial_u\lambda(t,0-) \\ \lambda(0,\cdot) \;=\; \lambda_0(\cdot)\,. \end{cases} \tag{1.5}$$

In this formula, ρ is the stationary solution (1.4) of the original equation. The aim of this article is thus to prove that the density field Y^N_t converges to the solution Y_t of (1.5) with a white noise added.

Denote by $C^2_K(\mathbb{R})$ the space of continuous functions on $\mathbb{R}$, with compact support and that are twice continuously differentiable on $\mathbb{R}_* = (-\infty,0) \cup (0,+\infty)$. Let

$$C^{2,*}_K \;=\; \Big\{ H \in C^2_K \,;\; qH'(0+) \;=\; pH'(0-) \;\;,\; \Phi'(\rho_+)\,H''(0+) \;=\; \Phi'(\rho_-)\,H''(0-) \Big\}.$$

Notice that for $H \in C^{2,*}_K$,

$$\begin{aligned} \int_{\mathbb{R}} du\, \lambda_t(u) H(u) \;-\; \int_{\mathbb{R}} du\, \lambda_0(u) H(u) \;&=\; (1/2)\int_0^t ds \int_{\mathbb{R}} du\, \partial^2_u H(u)\, \Phi'(\rho)\, \lambda_s(u) \\ &=\; (1/2)\int_0^t ds \int_{\mathbb{R}} du\, H(u)\, \Phi'(\rho)\, \partial^2_u\, \lambda_s(u) \end{aligned}$$

because of the boundary conditions of λ and H at the origin.

On $C_K^{2,*}$ define the operator $\mathcal{A}$:

op
$$\mathcal{A} \;=\; (1/2)\Phi'(\rho)\,\Delta \;=\; \begin{cases} (1/2)\Phi'(\rho_+)\Delta & \text{in } \mathbb{R}_+ ,\\ (1/2)\Phi'(\rho_-)\Delta & \text{in } \mathbb{R}_- \; . \end{cases} \tag{1.6}$$

Define m_+ and m_- by

$$m_- \;=\; q\,\Phi'(\rho_-) \;\text{ and }\; m_+ \;=\; p\,\Phi'(\rho_+)$$

and let $m(u) = m_- \mathbf{1}_{\{u<0\}} + m_+ \mathbf{1}_{\{u>0\}}$. It is not difficult to check that $-\mathcal{A}$ is a symmetric and non-negative operator on $L^2(m(u)du)$. Denote by $D(\mathcal{A})$ its domain in $L^2(m(u)du)$. In view of the Hille-Yosida theorem, to show that $\mathcal{A}$ is the generator of a Markov semigroup T_t, we only need to check that the Poisson equation

Poisson
$$\gamma f \;-\; \mathcal{A} f \;=\; g \tag{1.7}$$

has a solution $f \in D(\mathcal{A})$ for every $g \in L^2(m(u)du)$ and every $\gamma > 0$. Fix such a function g and $\gamma > 0$. Let f be defined by

$$f(u) \;=\; \begin{cases} a_- e^{b_- u} \;+\; \displaystyle\int_{-\infty}^{0} \frac{1}{2 b_- \theta_-} e^{-b_- |u-v|}\, g(v)dv, \quad u<0 \\ a_+ e^{-b_+ u} \;+\; \displaystyle\int_{0}^{+\infty} \frac{1}{2 b_+ \theta_+} e^{-b_+ |u-v|}\, g(v)dv, \quad u>0 \; . \end{cases}$$

Here $\theta_\pm = (1/2)\Phi'(\rho_\pm)$, $b_\pm = \sqrt{\gamma/\theta_\pm}$ and a_+, a_- are chosen so that $f(0+) = f(0-)$, $qf'(0+) = pf'(0-)$. It is straightforward to check that $f(\cdot)$ is a solution of (1.7). Denote by T_t (resp. W_t) the semigroup (resp. diffusion process) associated to the generator $\mathcal{A}$.

To state the main theorems of this paper, we need some more notation and concepts. Let $\mathfrak{L}$ be the linear operator $\mathfrak{L} = (u^2/2\Phi'(\rho)) - \mathcal{A}$. Denote by $< \cdot, \cdot >$ (resp. $< \cdot, \cdot >_m$) the inner product on $L^2(du)$ (resp. $L^2(m(u)du)$). For $k \ge 0$ let $\mathfrak{H}_k$ be the Sobolev space obtained by completion with respect to the inner product

$$< f \,,\, g >_{m,k} = < f \,,\, \mathfrak{L}^k g >_m \quad .$$

It is easy to check that,

$$\mathfrak{H}_k \;=\; \Big\{ f \,\in\, L^2(m(u)du);\, \sum_{j\ge 0} (2j+1)^k < f, H_j >_m^2 < \infty \Big\}.$$

Denote by $\mathfrak{H}_{-k}$ the dual of $\mathfrak{H}_k$ with respect to $L^2(m(u)du)$. We consider the density field $Y_t^N(\cdot)$ as taking values in the Sobolev space $\mathfrak{H}_{-k}$ for some sufficiently large k. Fix a time $T > 0$, a positive integer k_0 and let $D([0,T], \mathfrak{H}_{-k_0})$ be the space of $\mathfrak{H}_{-k_0}$-valued functions which are right-continuous with left limits. Denote by P_N the probability measure on $D([0,T], \mathfrak{H}_{-k_0})$ induced by the process $Y_\cdot^N$ and the product measure ν_φ defined in (1.8). Let $\mathbb{P}_{\nu_\varphi}$ be the probability measure on $D([0,T], \mathbb{N}^{\mathbb{Z}})$ induced by the probability measure ν_φ and the process η_t and let $E_{P_N}[\cdot]$ and $\mathbb{E}_{\nu_\varphi}[\cdot]$ denote the respective expectations.

mt1 **Theorem 1.1 .** *Fix a positive integer $k_0 \ge 3$. Let P be the probability measure concentrated on $C([0,T], \mathfrak{H}_{-k_0})$ corresponding to the generalized Ornstein-Uhlenbeck process Y_t with mean 0 and covariance*

$$E_P\Big[Y_t(H) Y_s(G) \Big] \;=\; \int_{\mathbb{R}} \chi(u) (T_{|t-s|} H)(u) G(u)\, du$$

for all $H, G \in \mathfrak{H}_{k_0}$. *Here* $\chi(\cdot)$ *is the variance of* $\eta(\cdot)$: $\chi(u) = \chi_+ \mathbf{1}_{\{u>0\}} + \chi_- \mathbf{1}_{\{u<0\}}$ *and* $\chi_\pm = E_{\nu_\varphi}[\eta(\pm 1)^2] - \rho_\pm^2$. *Then the sequence* P_N *converges weakly to the probability measure* P.

Fix $0 < \alpha < 1$ and denote by μ_α the product measure on $\{0,1\}^{\mathbb{Z}_*}$ with marginals given by

$$\mu_\alpha\{\xi ; \xi(x) = 1\} \;=\; \begin{cases} \alpha & x < 0 \\ 1 - (q/p)(1-\alpha) & x > 0. \end{cases}$$

It is easy to check that μ_α is a reversible invariant measure for the environment as seen from the tagged particle. Moreover, μ_α is sent to ν_φ with $\varphi = p^{-1}(1-\alpha)$ by the transformation $\mathcal{T}$ defined in the beginning of this section. For this fixed value of $\varphi = \varphi(\alpha)$, recall that W_t stands for the diffusion process with generator $\mathcal{A}$ (that depends on $\varphi = \varphi(\alpha)$).

mt2 **Theorem 1.2 .** *Let* $\delta_0 \times \mu_\alpha$ *be the initial measure of the process* (X_t, ξ_t). *The finite dimensional distributions of* $N^{-1/2} X_{tN^2}$ *converge to the finite dimensional distributions of the mean zero Gaussian process* Z_t *with covariance given by*

covar
$$E[Z_t\, Z_s] \;=\; \int_0^\infty P_u[W_t \le 0]\, du \;+\; \int_0^\infty P_u[W_s \le 0]\, du \;-\; \int_0^\infty P_u[W_{|t-s|} \le 0]\, du \tag{1.9}$$

and such that $Z_0 = 0$.

2. The density field.

In this section we prove Theorem 1.1. We follow the approach presented in [C] and we refer the reader to [S], [KL] for references and details omitted here.

The proof of Theorem 1.1 is divided in three steps. We first prove that the sequence P_N is tight and that all limit points are concentrated on continuous trajectories. Then, we show that all limit points P of the sequence P_N solve the following martingale problem. For each $t \ge 0$, denote by $\mathcal{F}_t$ the σ-algebra generated by $Y_s(H)$ for $0 \le s \le t$ and H in $\mathfrak{H}_{k_0}$. For each H in $\mathfrak{H}_{k_0}$

mp1
$$\begin{aligned} M_t^{\mathcal{A},H} \;&=\; Y_t(H) \;-\; Y_0(H) \;-\; \int_0^t Y_s(\mathcal{A}H)\, ds\,, \\ K_t^{\mathcal{A},H} \;&=\; (M_t^{\mathcal{A},H})^2 \;-\; \|\mathcal{B}H\|_2^2\, t \end{aligned} \tag{2.1}$$

are $(P, \mathcal{F}_t)$-martingales and Y_0 is a mean zero Gaussian field with covariance given by

b2
$$E_P\Big[Y_0(H) Y_0(G)\Big] \;=\; \int_{\mathbb{R}} \chi(u) H(u) G(u)\, du \tag{2.2}$$

for H, G in $\mathfrak{H}_{k_0}$. Here $\mathcal{A}$ is the differential operator defined in (1.6), $\mathcal{B}H = \sqrt{\Phi(\rho)/2}\nabla H$ and $\|G\|_2^2 = <G, G>$. We conclude the proof of Theorem 1.1 invoking Holley–Stroock [HS] theory on generalized Ornstein–Uhlenbeck processes that states that for each measure q on $\mathfrak{H}_{k_0}$, there is a unique probability measure on $C([0,T], \mathfrak{H}_{-k_0})$ that solves the martingale problem (2.1) and whose marginal at time 0 is q.

We postpone the proof of the tightness to section 4. We prove here that all limit points of the sequence P_N solve the martingale problem (2.1), (2.2). Fix a limit point P and assume without loss of generality that P_N converges to P. Recall the definition of the filtration $\mathcal{F}_t$. We first prove that the restriction of P to $\mathcal{F}_0$ is a Gaussian field with covariance matrix given by (2.2).

lemaux **Lemma 2.1 .** *For every continuous function H with compact support and every $t > 0$,*

$$\lim_{N\to\infty} \log \mathbb{E}_{\nu_\varphi}\Big[\exp\{iY_t(H)\}\Big] \;=\; -(1/2) < \chi H, H > .$$

Proof. Recall that the initial state is the product invariant measure ν_φ. Hence, for each fixed continuous function with compact support,

$$\log \mathbb{E}_{\nu_\varphi}\Big[\exp\Big\{iY_t(H)\Big\}\Big] \;=\; \sum_{x\in\mathbb{Z}} \log E_{\nu_\varphi}\Big[\exp\Big\{iN^{-1/2}H(x/N)[\eta(x)-\rho_x]\Big\}\Big].$$

By Taylor expansion, the last expectation is equal to

$$1 \;-\; (1/2)N^{-1}\chi(x/N)H(x/N)^2 \;+\; O(N^{-3/2}).$$

Therefore, summing over x and using that $\log(1+x) = x + O(x^2)$ we obtain that

$$\log \mathbb{E}_{\nu_\varphi}\Big[\exp\{iY_t(H)\}\Big] \;=\; -(1/2)N^{-1}\sum_{x\in\mathbb{Z}}\chi(x/N)H(x/N)^2 \;+\; O(N^{-1/2})$$

which converges, as $N\uparrow\infty$, to $-(1/2) < \chi H, H >$. $\square$

From this lemma and the linearity of Y_t it is not difficult (cf. [KL], Chap. 11) to prove that P restricted to $\mathcal{F}_0$ is a Gaussian field:

colo **Corollary 2.2 .** *Restricted to $\mathcal{F}_0$, P is a Gaussian field with covariance given by*

$$E_P\Big[Y_0(G)Y_0(H)\Big] \;=< \chi H, G > .$$

To prove that P solves the martingale problem (2.1), for each H in $C_K^{2,*}$ define the martingales $M_t^{N,H}$, $K_t^{N,H}$ by

$$\begin{aligned} M_t^{N,H} \;&=\; Y_t^N(H) \;-\; Y_0^N(H) \;-\; \int_0^t N^2\mathcal{L}Y_s^N(H)\,ds\,, \\ K_t^{N,H} \;&=\; \Big(M_t^{N,H}\Big)^2 \;-\; N^2\int_0^t \Big\{\mathcal{L}(Y_s^N(H))^2 \;-\; 2Y_s^N(H)\,\mathcal{L}Y_s^N(H)\Big\}\,ds\,. \end{aligned}$$

Denote by ∇_N and Δ_N the discrete derivative and the discrete Laplacian: $(\nabla_N f)(x) = N[f(x+1)-f(x)]$ and $(\Delta_N f)(x) = N^2[f(x+1)+f(x-1)-2f(x)]$. Note that $N^2\mathcal{L}\eta(x) = (1/2)\Delta_N g(\eta(x))$ for $x \neq -1,\, 0$. A simple computation taking into account the boundary terms shows that $N^2\mathcal{L}Y^N(H)$ is equal to

$$\begin{aligned} &(1/2)N^{-1/2}\sum_{x\neq -1,0} H(x/N)\,\Delta_N g(\eta(x)) \\ &\quad +\; N^{3/2}H(-1/N)\Big\{(1/2)g(\eta(-2)) + pg(\eta(0)) - [(1/2)+q]g(\eta(-1))\Big\} \\ &\qquad +\; N^{3/2}H(0)\Big\{(1/2)g(\eta(1)) + qg(\eta(-1)) - [(1/2)+p]g(\eta(0))\Big\}\,. \end{aligned}$$

Let $\beta_x = E_{\nu_\varphi}[g(\eta(x))]$ and notice that $\Delta\beta_x = 0$ for $x \neq -1, 0$. We may therefore replace $\Delta_N g(\eta(x))$ by $\Delta_N[g(\eta(x)) - \beta_x]$ in the first expression. A double summation by parts in this term gives that $N^2 \mathcal{L} Y^N(H)$ is equal to

$$
\begin{aligned}
&(1/2)N^{-1/2} \sum_{x\neq -1,0} (\Delta_N H)(x/N)(g_x - \beta_x) \;+\; (1/2)N^{3/2}(g_{-1} - \beta_{-1})\big\{H(-2/N) - H(-1/N)\big\} \\
&\quad +\; (1/2)N^{3/2}(g_0 - \beta_0)\big\{H(1/N) - H(0)\Big\} \;+\; N^{3/2}\,[pg_0 - qg_{-1}]\big\{H(-1/N) - H(0)\big\}\;.
\end{aligned}
$$

Here we denoted $g(\eta(x))$ by g_x and we used the fact that $\beta_{-1} = \beta_{-2}$, $\beta_0 = \beta_1$. Recall that H belongs to $C^{2,*}_K$ so that $pH'(0-) = qH'(0+)$. On the other hand, an elementary computation shows that $q\beta_{-1} = qE_{\nu_\varphi}[g(\eta(-1))] = qp\varphi$ which is also equal to $p\beta_0$. In particular, expanding the last three terms in the previous formula, we obtain that they are equal to

$$
\sqrt{N} p^{-1}[q + (1/2)]H'(0+)(pg_0 - qg_{-1}) \;+\; O(N^{-1/2})\;.
$$

For x in $\mathbb{Z}$ denote by $W_{x,x+1}$ the instantaneous current over the bond $\{x, x+1\}$, i.e., the rate at which a particle jumps from x to $x+1$ minus the rate at which a particle jumps from $x+1$ to x. In particular, for each $x \in \mathbb{Z}$, $\mathcal{L}\eta(x) = W_{x-1,x} - W_{x,x+1}$, $W_{x,x+1} = (1/2)[g(\eta(x)) - g(\eta(x+1))]$ for $x \neq -1$ and $W_{-1,0} = qg(\eta(-1)) - pg(\eta(0))$. Therefore we end up with

f1
$$
\begin{aligned}
M^{N,H}_t \;&=\; Y^N_t(H) \;-\; Y^N_0(H) \;-\; \int_0^t (1/2)N^{-1/2} \sum_{x\neq -1,0} (\Delta_N H)(x/N)[g(\eta_s(x)) - \beta_x]\, ds \\
&\quad +\; \int_0^t \alpha_N W_{-1,0}(s)\, ds \;+\; O(N^{-1/2})\;,
\end{aligned} \tag{2.3}
$$

where $\alpha_N = \sqrt{N} p^{-1}[q + (1/2)]\, H'(0+)$.

Observe that there is an homogeneous part involving differences of type $g(\eta(x))) - \beta_x$, for all sites $x \neq -1, 0$, and a boundary term $W_{-1,0}$ involving only the sites $x = 0, -1$. In the next lemma we show that the boundary term converges to zero and in the proposition which follows we state the Boltzmann–Gibbs principle that allows to express the homogeneous part in terms of the density field $Y^N_s(H)$, closing the equation of the martingale $M^{N,H}_t$ in terms of the density field.

lem2.1 **Lemma 2.3 .** *Let $\{\alpha_N, N \geq 1\}$ be a sequence of numbers such that* $\limsup_{N\to\infty} \alpha_N/N = 0$. *Then, for each $t \geq 0$*

$$
\limsup_{N\to\infty} \; \mathbb{E}_{\nu_\varphi}\Big[\Big(\int_0^t \alpha_N W_{-1,0}(s)\, ds\Big)^2\Big] \;=\; 0\;.
$$

Proof. Let $V(\eta_s) = \alpha_N W_{-1,0}(s)$. Since ν_φ is reversible, by Proposition A.1.6.1 in [KL], there exists a finite constant C_0 such that

$$
\begin{aligned}
\mathbb{E}_{\nu_\varphi}\Big[\Big(\int_0^t V(\eta_s)\, ds\Big)^2\Big] \;&\leq\; C_0 t < V, (-N^2\mathcal{L})^{-1}\, V > \\
&:=\; C_0 t\, \sup_h \Big\{ 2 < V, h > \;-\; N^2 < h, \mathcal{L}\, h > \Big\}\;.
\end{aligned}
$$

In this formula $< \cdot, \cdot >$ stands for the inner product in $L^2(\nu_\varphi)$ and the supremum is taken over all bounded cylinder functions h. For $x \in \mathbb{Z}$, denote by $\mathfrak{d}_x$ the configuration with just one particle at x. Also, the summation of two configurations is to be understood componentwise. Fix a bounded cylinder function h. Making the change of variables $\tilde{\eta} = \eta - \mathfrak{d}_0, \eta' = \eta - \mathfrak{d}_{-1}$ we obtain that

ibp
$$\int W_{-1,0} h \, d\nu_\varphi \;=\; pq\varphi \int [h(\eta + \mathfrak{d}_{-1}) \;-\; h(\eta + \mathfrak{d}_0)] \, d\nu_\varphi \; . \tag{2.4}$$

The same change of variables gives that

$$< h, -\mathcal{L}h > \geq \; pq\varphi \int \Big[h(\eta + \mathfrak{d}_{-1}) - h(\eta + \mathfrak{d}_0)\Big]^2 \, d\nu_\varphi \; .$$

Therefore,

$$\begin{aligned} & 2\alpha_N \; < W_{-1,0}, h > \; - N^2 < h, \mathcal{L}h > \\ & \leq \; q\,p\,\varphi \int \Big\{ 2\alpha_N [h(\eta + \mathfrak{d}_0) - h(\eta + \mathfrak{d}_{-1})] \;-\; N^2 [h(\eta + \mathfrak{d}_{-1}) - h(\eta + \mathfrak{d}_0)]^2 \Big\} \, d\nu_\varphi \\ & \leq \; q\,p\,\varphi \, \frac{\alpha_N^2}{N^2} \; . \end{aligned}$$

This concludes the proof of the lemma. □

We now close the equation for the martingale $M_t^{N,H}$ in terms of the density field Y_t^N. This is the content of the Boltzmann-Gibbs Principle stated below. The proof presented in [KL], Chapter 11 is easily adapted to our context. This result was first proved by Brox and Rost [BR] in the context of homogeneous zero-range processes.

prop2.2 **Proposition 2.4 .** *For all continuous functions $G(\cdot)$ on $\mathbb{R}_*$ of compact support and every $t > 0$,*

$$\lim_{N \to \infty} \mathbb{E}_{\nu_\varphi} \Big[\Big(\int_0^t N^{-1/2} \sum_{x \neq 0, -1} G(x/N) \Big\{ g(\eta_s(x)) - \beta_x - \Phi'(\rho_x)[\eta_s(x) - \rho_x] \Big\} \, ds \Big)^2 \Big] \;=\; 0 \; .$$

It follows from this proposition applied to $G = \Delta_N H$ and Lemma 2.3 that $M_t^{N,H}$ is equal to

$$Y_t^N(H) \;-\; Y_0^N(H) \;-\; \int_0^t ds \, (1/2) N^{-1/2} \sum_{x \neq -1, 0} (\Delta_N H)(x/N) \Phi'(\rho_x) [\eta_s(x) - \rho_x] \;+\; A_N,$$

where A_N is a remainder term that converges to 0 in $L^2(\mathbb{P}_{\nu_\varphi})$. Since H belongs to $C_K^{2,*}$, we may replace the discrete Laplacian by the continuous one. In this case, since $\Phi'(\rho)\Delta H$ is continuous, we may replace summation over $x \neq -1, 0$ by a summation over $x \in \mathbb{Z}$ to obtain that

bolz
$$M_t^{N,H} \;=\; Y_t^N(H) \;-\; Y_0^N(H) \;-\; \int_0^t Y_s^N(\mathcal{A}H) \, ds \;+\; A_N \; . \tag{2.5}$$

The expression of A_N changed but still converges to 0 in $L^2(\mathbb{P}_{\nu_\varphi})$. It is not difficult to deduce from the previous identity that $M_t^{\mathcal{A},H}$ defined in (2.1) is a $(P, \mathcal{F}_t)$-martingale for each H in $\mathfrak{H}_{k_0}$. This is the content of the next proposition.

prop2.4 **Proposition 2.5 .** *For each H in $\mathfrak{H}_{k_0}$, $M_t^{\mathcal{A},H}$ is a $(P,\mathcal{F}_t)$-martingale.*

Proof. Fix first H in $C_K^{2,*}\cap\mathfrak{H}_{k_0}$. By definition, $M_t^{\mathcal{A},H}$ is measurable with respect to the filtration $\mathcal{F}_t$. In order to prove that $M_t^{\mathcal{A},H}$ is a martingale it suffices to check that

$$E_P\Big[M_t^{\mathcal{A},H}U\Big] \;=\; E_P\Big[M_s^{\mathcal{A},H}U\Big]$$

for all $0\le s\le t\le T$ and all $U=\mathbf{1}_{\{Y_{s_i}(H_i)\in A_i,\,1\le i\le n\}}$, where n is a positive integer, $0\le s_1\le \cdots\le s_n\le s$ and, for each $1\le i\le n$, H_i belongs to $C_K^{2,*}$ and A_i is some measurable subset of $\mathbb{R}$. Since $M_t^{N,H}$ is a martingale, $\mathbb{E}_{\nu_\varphi}[M_t^{N,H}U]=\mathbb{E}_{\nu_\varphi}[M_s^{N,H}U]$. Hence, we only need to show that these expectations converge respectively to $E_P[M_t^{\mathcal{A},H}U]$ and $E_P[M_s^{\mathcal{A},H}U]$.

By (2.5), since U is bounded,

$$\lim_{N\to\infty}\mathbb{E}_{\nu_\varphi}\Big[M_t^{N,H}\,U\Big] \;=\; \lim_{N\to\infty}E_{P_N}\Big[\Big\{Y_t(H)-Y_0(H)-\int_0^t ds\,Y_s(\mathcal{A}H)\Big\}U\Big].$$

A simple computation shows that the term inside braces is square integrable, uniformly in N. In particular, since U is bounded and P_N converges weakly to P and since, by Proposition 4.1 below, P is concentrated on continuous paths, $E_{P_N}[M_t^{N,H}U]$ converges to $E_P[M_t^{\mathcal{A},H}U]$ as $N\uparrow\infty$. Since this statement remains in force with s in place of t, $M_t^{\mathcal{A},H}$ is a $(P,\mathcal{F}_t)$-martingale for each $H\in C_K^{2,*}\cap\mathfrak{H}_{k_0}$. Fix now H in $\mathfrak{H}_{k_0}$. With a sequence H_n of functions in $C_K^{2,*}\cap\mathfrak{H}_{k_0}$ such that H_n, $\Phi'(\rho)\Delta H_n$ converges to H, $\Phi'(\rho)\Delta H$ in $L^2(P)$, it is easy to show that $M_t^{\mathcal{A},H}$ is a $(P,\mathcal{F}_t)$-martingale. This concludes the proof of the lemma. □

We now turn to the martingale $K_t^{N,H}$. An elementary computation gives that $N^2\mathcal{L}Y^N(H)^2$ $-2N^2Y^N(H)\,\mathcal{L}Y^N(H)$ is equal to

$$\begin{aligned}&(1/2)N^{-1}\sum_{x\neq 0,-1} g(\eta(x))[(\nabla_N H)(x/N)]^2\\ &\qquad +\;N^{-1}pg_{-1}[(\nabla_N H)(-1/N)]^2\;+\;N^{-1}qg_0[(\nabla_N H)(0)]^2\,.\end{aligned}$$

lem2.2 **Lemma 2.6 .** *For every continuous function G on $\mathbb{R}_*$ of compact support*

$$\lim_{N\to\infty}\mathbb{E}_{\nu_\varphi}\Big[\Big(\int_0^t N^{-1}\sum_{x\neq 0,-1}G(x/N)[g(\eta_s(x))-\beta_x]\Big)^2\Big] \;=\; 0\,.$$

The proof is elementary because ν_φ is a product stationary measure. It follows from this result applied to $G=\nabla_N H$ and from the previous computation that

$$K_t^{N,H} \;=\; (M_t^{N,H})^2\;-\;\frac{t}{2N}\sum_{x\neq -1,0}\Big[(\nabla_N H)(x/N)\Big]^2\beta_x\;+\;\tilde{A}_N\,,$$

where $\tilde{A}_N$ converges to 0 in $L^2(\mathbb{P}_{\nu_\varphi})$. Here again we may replace the discrete gradient by the continuous one and extend the summation to $\mathbb{Z}$. Repeating the arguments presented in the proof of Proposition 2.5, we show that $K_t^{\mathcal{A},H}$ is a $(P,\mathcal{F}_t)$-martingale for each $H\in\mathfrak{H}_{k_0}$.

Proof of Theorem 1.1. By Proposition 4.1 the sequence P_N is tight and all its limit points are concentrated on continuous paths. Fix a limit point P of the sequence P_N. We proved in this section that P solves the martingale problem (2.1), (2.2). The proof of Theorem (1.4) in [HS] can be adapted to show that there exists at most one solution of (2.1), (2.2). This concludes the proof of the theorem. □

3. Fluctuation of the asymmetric tagged particle.

In this section we use the results proven for the zero-range model to deduce a Central Limit Theorem for the asymmetric tagged particle in the nearest-neighbor symmetric simple exclusion process. We follow the approach presented by [RV] for the case of a symmetric tagged particle.

Recall from section 2 that (X_t, ξ_t) is a one-dimensional nearest neighbor symmetric exclusion process with an asymmetric tagged particle that sits at the origin at $t = 0$. We defined in section 1 a correspondence between this exclusion process and a symmetric zero range process with an asymmetry at the bond $\{-1, 0\}$. In this correspondence the position of the tagged particle at time t is given by the total number of jumps from 0 to -1 minus the total number of jumps from -1 to 0 in the time interval $[0, t]$.

Fix $0 \le \alpha \le 1$ and recall the definition of the product measure μ_α given in section 1. Let $\varphi = p^{-1}(1-\alpha)$. Since $\eta_t = \eta_t^N$ is the zero range process with generator $N^2\mathcal{L}$, it is accelerated by N^2, in contrast with the process (X_t, ξ_t) which is not. Denote by $S_+^N(t)$ (resp. $S_-^N(t)$) the total number of jumps from -1 to 0 (resp. 0 to -1) in the time interval $[0, t]$ for the process η. The position of the tagged particle is such that

$$X_{tN^2} \;=\; S_-^N(t) \;-\; S_+^N(t)\ .$$

For $n \ge 1$, let $G_n(u) = [1 - (u/n)]^+ \mathbf{1}_{\{u \ge 0\}}$. The next result gives a precise meaning to identity (1.1).

lem3.1 **Lemma 3.1 .** *There exists a finite constant c such that*

$$\sup_{N\ge 1} \mathbb{E}_{\nu_\varphi}\Big[\Big(\frac{X_{tN^2}}{\sqrt{N}} \;-\; \frac{1}{\sqrt{N}}\sum_{x\ge 0} G_n(x/N)\,[\eta_0(x) \;-\; \eta_t(x)]\Big)^2\Big] \;\le\; \frac{ct}{n}$$

for all $n \ge 1$ and $t \in [0, T]$.

Proof. First observe that

$$S_+^N(t) \;-\; qN^2\int_0^t g(\eta_s(-1))\, ds\ , \qquad S_-^N(t) \;-\; pN^2\int_0^t g(\eta_s(0))\, ds$$

are $\mathbb{P}_{\nu_\varphi}$-martingales. Hence, so is $M_t = X_{tN^2} + N^2\int_0^t W_{-1,0}(s)\, ds$. For $n \ge 1$, let $Y_t^n = \sum_{x\ge 0} G_n(x/N)\,(\eta_t(x) - \rho_x)$ and $V_t^n = Y_0^n - Y_t^n$. Then

$$M_t^n \;=\; V_t^n \;+\; \int_0^t N^2\sum_{x\ge 0} G_n(x/N)\,\mathcal{L}\eta_s(x)\, ds$$

is a $\mathbb{P}_{\nu_\varphi}$-martingale. A straightforward computation shows that the integral term of the previous expression is equal to

$$N^2\int_0^t \Big\{W_{-1,0}(s) \;+\; \frac{1}{nN}\sum_{x=0}^{nN-1} W_{x,x+1}(s)\Big\}\, ds\ .$$

Thus, $X_{tN^2} - V_t^n$ is equal to

$$M_t \;-\; M_t^n \;+\; \frac{N}{n}\int_0^t \sum_{x=0}^{nN-1} W_{x,x+1}(s)\, ds\ .$$

We claim that this expression converges to 0 in $L^2(\mathbb{P}_{\nu_\varphi})$ as $n \uparrow \infty$. Notice that the expression appearing in the statement of Lemma 2.3 is just the current $W_{-1,0}$. Repeating the arguments presented there, we deduce that

$$\mathbb{E}_{\nu_\varphi}\Big[\Big(\int_0^t \frac{N}{n}\sum_{x=0}^{nN-1} W_{x,x+1}(s)\,ds\Big)^2\Big]\,ds \;\le\; \frac{CtN}{n}$$

for some finite constant C.

For each x denote by $J_t^{x,x+1}$ the total number of jumps from x to $x+1$ minus the total number of jumps from $x+1$ to x in the time interval $[0,t]$ for the process η. Thus, $j_t^{x,x+1} = J_t^{x,x+1} - N^2\int_0^t W_{x,x+1}(s)ds$ is a martingale for each x in $\mathbb{Z}$ and $M_t = -j_t^{-1,0}$, $M_t^n = -j_t^{-1,0} + (nN)^{-1}\sum_{x=0}^{nN-1} j_t^{x,x+1}$. In particular,

$$\begin{aligned}\mathbb{E}_{\nu_\varphi}\Big[\Big(M_t^n - M_t\Big)^2\Big] \;&=\; \mathbb{E}_{\nu_\varphi}\Big[\Big(\frac{1}{nN}\sum_{x=0}^{nN-1} j_t^{x,x+1}\Big)^2\Big]\\ &=\; \frac{1}{(nN)^2}\sum_{x=0}^{nN-1} tN^2\beta_x \;=\; \frac{N\beta_0 t}{n}\cdot\end{aligned}$$

Here we used the fact that the martingales $\{j_t^{x,x+1},\ x\in\mathbb{Z}\}$ are mutually orthogonal because they don't have common jumps and that $\mathbb{E}_{\nu_\varphi}[(j_t^{x,x+1})^2] = tN^2\beta_x$. $\square$

It follows from this lemma that for each fixed t the sequence $Y_0^N(G_n) - Y_t^N(G_n)$ is Cauchy in $L^2(\mathbb{P}_{\nu_\varphi})$, uniformly over $N \ge 1$. In consequence, the sequence $Y_0(G_n) - Y_t(G_n)$ is Cauchy in $L^2(P)$ (Since $G_n - G_m$ does not belong to $\mathfrak{H}_{k_0}$, $Y_t(G_n - G_m)$ is understood as the limit of $Y_t(H_k^{m,n})$, where $H_k^{m,n}$ belongs to $\mathfrak{H}_{k_0}$ and converges in $L^2(du)$ to $G_n - G_m$).

Let Z_t be the $L^2(P)$ limit of $Y_0(G_n) - Y_t(G_n)$. Recall that W_t is the diffusion process with generator $\mathcal{A}$. For $0 \le s \le t$, the covariance $E_P[Z_sZ_t]$ is given by (1.9). Indeed, since Z_t is the $L^2(P)$ limit of $Y_0(G_n) - Y_t(G_n)$,

$$E_P[Z_sZ_t] \;=\; \lim_{n\to\infty} E_P\Big[\Big\{Y_0(G_n) - Y_s(G_n)\Big\}\Big\{Y_0(G_n) - Y_t(G_n)\Big\}\Big].$$

Recall that T_t stands for the semigroup associated to the generator $\mathcal{A}$. By Theorem 1.1, for each fixed n, the previous expectation is equal to

$$\begin{aligned}&\int_{\mathbb{R}} G_n(u)\Big[G_n(u) - T_tG_n(u)\Big]\,du \;+\; \int_{\mathbb{R}} G_n(u)\Big[G_n(u) - T_sG_n(u)\Big]\,du\\ &\quad -\int_{\mathbb{R}} G_n(u)\Big[G_n(u) - T_{|t-s|}G_n(u)\Big]\,du.\end{aligned}$$

Since $\int_{\mathbb{R}} G_n(u)[G_n(u) - T_tG_n(u)]\,du$ converges, as $n\uparrow\infty$, to $\int_0^\infty P_u[W_t \le 0]\,du$, (1.9) is proven.

We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2. Since $Y_0^N(G_n) - Y_t^N(G_n)$ converges, as $n\uparrow\infty$, to $N^{-1/2}X_{tN^2}$ in $L^2(P_N)$ uniformly in N and since $Y_0(G_n) - Y_t(G_n)$ converges, as $n\uparrow\infty$, to Z_t in $L^2(P)$, it is not hard to show that

$$\mathbb{E}_{\nu_\varphi}\Big[f\Big(N^{-1/2}X_{t_1N^2},\,\dots,N^{-1/2}X_{t_kN^2}\Big)\Big]$$

converges to $E_P[f(Z_{t_1},...,Z_{t_k})]$ for every bounded Lipschitz-continuous function f and every $0 \le t_1 < ... < t_k \le T$. $\square$

4. Tightness.

We prove in this section that the sequence P_N defined in section 1 is tight.

justa **Proposition 4.1 .** *The sequence $\{P_N,\ N \ge 1\}$ is tight. Moreover, all its limit points are concentrated on continuous paths.*

We adapt the arguments exposed in [KL], Chapter 11. We prove in this section that the sequence is tight with respect to the uniform topology. This in turn implies that the limit points are concentrated on continuous trajectories.

Recall the definition of the operator $\mathfrak{L}$. Let $\{H_j\,,\, j \ge 0\}$ be the $L^2(m(u)du)$ orthonormal basis of $\mathfrak{L}$-eigenfunctions obtained in the appendix. Notice that

$$\mathfrak{L} f \;=\; \sum_{j\ge 0} < f\,,\, H_j >_m \, (2j+1)\, H_j$$

for all f in $D(\mathcal{A})$. Recall that for $k \ge 1$, $\mathfrak{H}_k$ stands for the Hilbert space obtained by completion of $D(\mathfrak{L}^k)$ with respect to the inner product $< \cdot,\cdot >_{m,k}$ defined by $< f\,,\, g >_{m,k} = < f\,,\, \mathfrak{L}^k g >_m$ for f, g in $D(\mathfrak{L}^k)$. It can be shown that $\mathfrak{H}_k \;=\; \{f \in L^2\,;\, \sum_{j\ge 0} (2j+1)^k \,< f\,,\, H_j >_m^2 \,< \infty\}$.

Let $\mathfrak{H}_{-k}$ be the dual of $\mathfrak{H}_k$ with respect to $L^2(m(u)du)$. It is again easy to check that $\mathfrak{H}_{-k}$ consists of all sequences $\{< f\,,\, H_j >_m, j \ge 0\}$ such that $\sum_{j\ge 0} (2j+1)^{-k} \,< f\,,\, H_j >_m^2 < \infty$. In particular,

$$\dots \supseteq \mathfrak{H}_{-k} \supseteq \dots \supseteq \mathfrak{H}_0 \supseteq \mathfrak{H}_1 \supseteq \dots \supseteq \mathfrak{H}_k \supseteq \dots$$

Fix $k \ge 1$ and denote by $w_\delta(Y)$ the modulus of continuity defined by

$$w_\delta(Y) \;=\; \sup_{\substack{|s-t|\le\delta \\ 0\le s,t\le T}} \|Y_t \,-\, Y_s\|_{-k} \;.$$

A family $\{P_N, N \ge 0\}$ of probability measures on $D([0,T]\,,\,\mathfrak{H}_{-k})$ is tight if

f3
$$\begin{aligned} \lim_{A\to\infty} \limsup_{N\to\infty} P_N\Big[\sup_{0\le t\le T} \|Y_t\|_{-k} \;\ge\; A\Big] \;&=\; 0, \\ \lim_{\delta\to 0} \limsup_{N\to\infty} P_N[w_\delta(Y_t) \;\ge\; \epsilon] \;&=\; 0\,, \forall\, \epsilon > 0. \end{aligned} \tag{4.1}$$

The next result provides an estimate that allows us to prove (4.1).

lem5.1 **Lemma 4.2 .** *For each $T > 0$, there exists a finite constant $C(T)$ such that*

$$\mathbb{E}_{\nu_\varphi}\Big[\sup_{0\le t\le T} |Y_t(H_j)|^2 \Big] \;\le\; C(T) \Big\{ \frac{1}{N} \sum_{x\in\mathbb{Z}} H_j(x/N)^2 \;+\; \frac{1}{N}\sum_{x\in\mathbb{Z}} [(\nabla H_j)(x/N)]^2 \Big\}$$

for all $N \ge 1$.

Proof. Fix $j \ge 0$ and denote by $M_t(j) = M_t^N(j)$ the martingale defined by

$$M_t(j) \;=\; Y_t^N(H_j) \;-\; Y_0^N(H_j) \;-\; \int_0^t \Gamma_{1,j}(s)\, ds\;,$$

where $\Gamma_{1,j}(s) \;=\; N^2 \mathfrak{L} Y_s^N(H_j)$. This identity permits us to write $Y_t^N(H_j)$ in terms of $Y_0^N(H_j)$, the martingale $M_t(j)$ and the time integral of $\Gamma_{1,j}(\cdot)$. We shall estimate these three terms separately.

A simple computation shows that

$$\mathbb{E}_{\nu_\varphi}\Big[(Y_0(H_j))^2\Big] \;=\; \frac{1}{N}\sum_{x\in\mathbb{Z}} H_j(x/N)^2\chi(x/N)\;.$$

On the other hand, since $M_t(j)$ is a martingale, by Doob's inequality,

$$\mathbb{E}_{\nu_\varphi}\Big[\sup_{0\le t\le T}|M_t(j)|^2\Big] \;\le\; 4\,\mathbb{E}_{\nu_\varphi}\Big[M_T^2(j)\Big].$$

The computations performed just before Lemma 2.6 show that this expectation is bounded by $CtN^{-1}\sum_{x\in\mathbb{Z}}[\nabla_N H_j(x/N)]^2$ for some finite constant C.

It remains to consider the integral term $\int_0^t \Gamma_{1,j}(s)ds$. Fix $a_0>0$. Let $g(t)=\int_0^t\Gamma_{1,j}(s)ds$ and $\Psi(u)=e^{a_0Nu}-1$. By the Garsia-Rodemich-Rumsey inequality,

$$\sup_{0\le t\le T}|g(t)|\le 4\int_0^T \Psi^{-1}(4B/u^2)u^{-1/2}\,du\;,$$

where $B=\int_0^T ds\int_0^T dt\,\Psi\big(|g(t)-g(s)|/\sqrt{t-s}\big)$. With our choice of Ψ, $\Psi^{-1}(u)=(a_0N)^{-1}\log(1+u)$. By an integration by parts,

$$\int_0^T u^{-1/2}\log\{1+\frac{4B}{u^2}\}\,du \;\le\; 2\sqrt{T}\Big\{\log\Big(1+\frac{4B}{T^2}\Big)+4\Big\}$$

because $B/(4B+u^2)\le 1/4$. Therefore, by the two previous estimates,

bb
$$\mathbb{E}_{\nu_\varphi}\Big[\exp\Big\{(a_0N/8\sqrt{T})\sup_{0\le t\le T}|g(t)|\Big\}\Big] \;\le\; e^4\Big\{1+\frac{4}{T^2}\mathbb{E}_{\nu_\varphi}[B]\Big\}. \tag{4.2}$$

Recall the definition of B. By Lemma 4.4 below, with $a=a_0N/8\sqrt{T}$, $\mathbb{E}_{\nu_\varphi}[B]$ is bounded above by $2T^2\exp\Big\{a_0^2N\sum_{x\in\mathbb{Z}}(\nabla_N H_j(x/N))^2\Big\}-T^2$. From this estimate and (4.2) we obtain that

$$\mathbb{E}_{\nu_\varphi}\Big[\exp\Big\{(a_0N/8\sqrt{T})\sup_{0\le t\le T}|g(t)|\Big\}\Big] \;\le\; 8e^4\,\exp\Big\{a_0^2N\sum_{x\in\mathbb{Z}}(\nabla_N H_j(x/N))^2\Big\}.$$

Maximizing over a_0, taking logarithms in both sides and applying Jensen's inequality, we deduce that

$$\mathbb{E}_{\nu_\varphi}\big[\sup_{0\le t\le T}|g(t)|^2\big] \;\le\; c_0TN^{-1}\sum_{x\in\mathbb{Z}}(\nabla_N H_j(x/N))^2$$

for some universal constant c_0. This concludes the proof of the lemma. □

col **Corollary 4.3 .** *For $k\ge 3$,*

$$\limsup_{N\to\infty}\mathbb{E}_{\nu_\varphi}\big[\sup_{0\le t\le T}\|Y_t\|_{-k}^2\big] \;<\;\infty,$$

$$\lim_{n\to\infty}\limsup_{N\to\infty}\mathbb{E}_{\nu_\varphi}\big[\sup_{0\le t\le T}\sum_{j\ge n}(2j+1)^{-k}<Y_t,H_j>_m^2\big] \;=\; 0.$$

Proof. By definition of $\|Y_t\|_{-k}^2$,

$$\|Y_t\|_{-k}^2 \;=\; \sum_{j\ge 0}(2j+1)^{-k}\; < Y_t, H_j >_m^2 .$$

In particular, by the previous lemma,

$$\lim_{N\to\infty} \mathbb{E}_{\nu_\varphi}\Big[\sup_{0\le t\le T}\|Y_t\|_{-k}^2\Big] \;\le\; C(T)\sum_{j\ge 0}(2j+1)^{-k}\Big\{ < H_j, H_j >_m \;+\; < H_j', H_j' >_m \Big\}.$$

Here and below we used the fact that there exists a positive constant A such that $A < H, H >_m \le\; < H, H > \;\le A^{-1} < H, H >_m$. Since $\{H_j, j\ge 0\}$ is an orthonormal basis, $< H_j\,,\, H_j >_m \;=\; 1$ for all $j\ge 0$. On the other hand, by an integration by parts, $< H_j', H_j' >_m \le c < -\mathcal{A}H_j, H_j >_m$ for some finite constant c. Since $-\mathcal{A}\le \mathfrak{L}$ and $\mathfrak{L}H_j = (2j+1)H_j$, $< H_j', H_j' >$ is bounded by $c(2j+1)$. The right hand side of the previous expression is thus finite for $k\ge 3$.

To prove the second statement of the corollary, observe that the expectation is bounded above by

$$\sum_{j\ge n}(2j+1)^{-k}\,\mathbb{E}_{\nu_\varphi}\Big[\sup_{0\le t\le T}\; < Y_t\,,\, H_j >_m^2\Big].$$

By the estimates obtained in the previous lemma for fixed N, this expression is summable for $k\ge 3$ and thus converges to 0 as $n\uparrow\infty$. □

Proof of Proposition 4.1. We have seen that in order to prove tightness, one needs to check (4.1). The first condition in (4.1) follows directly from the previous corollary. To prove the second one, in view of the previous corollary, we need to show that for all $\epsilon > 0$ and $n\ge 1$

$$\lim_{\delta\to 0}\limsup_{N\to\infty} \mathbb{P}_{\nu_\varphi}\Big[\sup_{\substack{|s-t|\le\delta\\ 0\le s,t\le T}}\sum_{j\le n} < Y_t - Y_s\,,\, H_j >^2 (2j+1)^{-k} \;>\; \epsilon\Big] \;=\; 0\,.$$

We observe that $Y_t^N(H_j) - Y_s^N(H_j) = M_t(j) - M_s(j) + \int_s^t \Gamma_{1,j}(r)dr$. The martingale part is handled as in [KL], Chap.11. For the integral part one uses the proof in [L] for nongradient systems. We leave the details to the reader. □

We conclude this section proving an estimate used in Lemma 4.2.

lem5.3 **Lemma 4.4 .** *For all $0\le t\le T$ and $a > 0$,*

$$\mathbb{E}_{\nu_\varphi}\Big[\exp\Big\{a\Big|\int_s^t \Gamma_{1,j}(u)du\Big|\Big\}\Big] \;\le\; 2\exp\Big\{a^2(t-s)N^{-1}\sum_{x\in\mathbb{Z}}(\nabla_N H_j(x/N))^2\Big\}.$$

Proof. Since $e^{|x|}\le e^x + e^x$, it is enough to prove that the left hand side without the absolute value is bounded by one half of the right hand side. Since ν_φ is an invariant measure, by Feynman-Kac formula,

$$\mathbb{E}_{\nu_\varphi}\Big[\exp\Big\{\int_0^{s-t} a\Gamma_{1,j}(u)\,du\Big\}\Big] \;\le\; \exp\Big\{(t-s)\sup_f\Big\{\int a\Gamma_{1,j}f\,d\nu_\varphi \;-\; N^2 D(f)\Big\}\Big\},$$

where the supremum is taken over all densities with respect to ν_φ and $-D(f) =< \sqrt{f}, \mathcal{L}\sqrt{f} >_{\nu_\varphi}$. Recall the definition of the current $W_{x,x+1}$ introduced in section 2. We have that

$$a\,\Gamma_{1,j} \;=\; a\sqrt{N}\sum_{x\in\mathbb{Z}}(\nabla_N H_j)(x/N)W_{x,x+1}.$$

The integration by parts formula (2.4) for the current shows that $a \int \Gamma_{1,j} f \, d\nu_\varphi$ is bounded above by

$$\frac{2a^2}{N} \sum_{x \in \mathbb{Z}} \Big[(\nabla_N H_j)(x/N) \Big]^2 \; + \; N^2 D(f).$$

Therefore,

$$\sup_f \Big\{ \int a\Gamma_{1,j} f \, d\nu_\varphi \; - \; N^2 D(f) \Big\} \; \le \; (a^2/N) \sum_{x \in \mathbb{Z}} \Big[(\nabla_N H_j)(x/N) \Big]^2 ,$$

which concludes the proof of the lemma. □

Appendix.

Consider the linear operator $\mathfrak{L} = (u^2/2\Phi'(\rho)) - \mathcal{A}$. In this appendix we construct an orthonormal basis of $\mathfrak{L}$-eigenfunctions $\{H_j, j \ge 0\}$ for $L^2(m(u)du)$ such that

req
$$H_j \in D(\mathcal{A}) \quad \text{and} \quad \mathfrak{L} H_j = (2j+1) H_j \tag{5.1}$$

for each $j \ge 0$.

Recall that $\theta(u) = \theta_- \mathbf{1}_{\{u<0\}} + \theta_+ \mathbf{1}_{\{u>0\}}$, where $\theta_\pm = \Phi'(\rho_\pm)$. Let $J_0(u) = \exp(-u^2/2\theta)$ and notice that it belongs to the space

$$C_0^{\infty,*} = \Big\{ H \in C_0^\infty(\mathbb{R}) ; qH'(0+) = pH'(0-) \, , \, \theta_+ H''(0+) = \theta_- H''(0-) \Big\},$$

where $C_0^\infty(\mathbb{R})$ stands for the space of smooth functions on $\mathbb{R}_*$ that are rapidly decaying and continuous on $\mathbb{R}$. For $n \ge 1$, let $\psi_{2n}(u) = (u/\sqrt{\theta})^{2n}$, $\psi_{2n-1}(u) = b(u)(u/\sqrt{\theta})^{2n-1}$, where $b(u) = \sqrt{\theta_+} p \mathbf{1}_{\{u>0\}} + \sqrt{\theta_-} q \mathbf{1}_{\{u<0\}}$. Set $J_n(u) = \psi_n(u) e^{-u^2/2\theta}$ for $n \ge 1$. Notice that $J_n(\cdot)$ belong to $C_0^{\infty,*}$ for $n \ge 1$.

We claim that the set $\{J_n(\cdot), n \ge 0\}$ is complete in $L^2(m(u)du)$ or, equivalently, that the set $\{\psi_n(u), n \ge 0\}$ is complete in $L^2(e^{-u^2/2\theta} m(u)du)$. So let $f \in L^2(e^{-u^2/2\theta} m(u)du)$ and suppose that $< f, \psi_n >_\theta = 0$ for all $n \ge 0$. Here $< \cdot, \cdot >_\theta$ stands for the inner product in $L^2(e^{-u^2/2\theta} m(u)du)$.

Let $f_0(u) = f(u)m(u)$, which belongs to $L^2(e^{-u^2/2\theta} du)$. For the even terms, we have

$$\int_{\mathbb{R}} (u/\sqrt{\theta})^{2n} \, f_0(u) \, e^{-u^2/2\theta} \, du \; = \; 0$$

from which follows, after a change of variables, that

$$\int_{\mathbb{R}} (t^{2n}/2n!) \, u^{2n} f_0(u\sqrt{\theta}) \, e^{-u^2/2} \sqrt{\theta} \, du \; = \; 0$$

for each $t \in \mathbb{R}$. Thus, summing over $n \ge 0$ we get

v1
$$\int_{\mathbb{R}} \cos(tu) \, A_\theta(u) \, du \; = \; 0 \tag{5.2}$$

for all $t \in \mathbb{R}$, where $A_\theta(u) = \sqrt{\theta} \, f_0(u\sqrt{\theta}) \, e^{-u^2/2}$.

Repeating the same argument for the odd terms, we get that

$$\int_{\mathbb{R}} \sin(tu)\, b(u)\, A_\theta(u)\, du \;=\; 0$$

for all $t \in \mathbb{R}$.

Decomposing $A_\theta(\cdot)$ in its even and odd parts, denoted respectively by $A_e(\cdot)$ and $A_o(\cdot)$, we get that

$$\int_{\mathbb{R}} A_e(u) \sin(tu)\, du = 0 \;\text{ and }\; \int_{\mathbb{R}} A_e(u) \cos(tu)\, du = 0.$$

The second identity follow from (5.2) and the fact that $\cos(\cdot)$ is an even function. Therefore, the Fourier transform of the $A_e(\cdot)$ vanishes in $\mathbb{R}$ so that $A_e(u) = 0$ almost everywhere.

Now, let $B_\theta(u) = b(u)A_\theta(u)$ and denote by $B_o(\cdot)$ its odd part. A similar argument shows that $B_o(u) = 0$ almost everywhere. Therefore, for almost all $u > 0$, $b(u)A_\theta(u) - b(-u)A_\theta(-u) = 0$ and $A_\theta(u) + A_\theta(-u) = 0$. From these two identities it follows that $A_\theta(u) = 0$ almost everywhere in $\mathbb{R}$. This proves that $\{J_n, n \geq 0\}$ is complete in $L^2(m(u)du)$.

Recall that $\mathfrak{L} = (u^2/\theta) - \mathcal{A}$. A simple computation shows that $\mathfrak{L}J_0 = J_0$, $\mathfrak{L}J_1 = 3J_1$ and, for $n \geq 2$, $\mathfrak{L}J_n = (2n+1)J_n - n(n-1)J_{n-2}$. From these relations it is easy to construct an orthonormal basis $\{H_j, j \geq 0\}$ of $\mathfrak{L}$-eigenfunctions for $L^2(m(u)du)$, satisfying the requirements (5.1).

Acknowledgments. C. L. was partially supported by CNPq grant 300358/93-8. C.L. and S.V. acknowledge partial support of PRONEX 41.96.0923.00 “Fenômenos Críticos em Probabilidade e Processos Estocásticos”.

References.

[Ar] Arratia R.; The motion of a tagged particle in the simple symmetric exclusion system in $\mathbb{Z}$, Ann.Prob. **11** 362-373 (1983).

[BR] T. Brox and H. Rost; Equilibrium fluctuations of stochastic particle systems: the role of conserved quantities. Ann Prob **12** 742-759 (1984).

[C] C. C. Chang; Equilibrium fluctuations of gradient reversible particle systems. Prob. Th. Rel. Fields **100** 269–283 (1994).

[DGL1] Dürr D., Goldstein S., Lebowitz J. L.; Asymptotics of particle trajectories in infinite one-dimensional systems with collisions. Commun. Pure App. Math. **38** 573–597 (1985).

[DGL2] Dürr D., Goldstein S., Lebowitz J. L.; Self diffusion in a nonuniform model system. Phys. Rev. Let. **57** 2481–2484 (1986).

[GLP] Giacomin G., Lebowitz J. L. and Presutti E.; Deterministic and Stochastic Hydrodynamic Equations Arising from Simple Microscopic Model Systems in *Stochastic PDEs: six perspectives*, AMS, lecture notes of the USC school in stochastic PDEs, january 1996.

[H] Harris. T. E.; Diffusions with collisions between particles. J. Appl. Prob. **2** 323–338 (1965).

[HS] R. A. Holley and D. W. Stroock; Generalized Ornstein-Uhlenbeck processes and infinite branching Brownian motions. Kyoto Univ. R.I.M.S. **14** 741-814 (1978).

[KL] Kipnis C., Landim C.; *Scaling Limit of Interacting Particle Systems.*, Springer Verlag, Berlin 1999.

[LOV] Landim C., Olla S. and Volchan S.; Driven Tracer particle in one dimensional symmetric simple exclusion. Commun. Math. Phys. **192**, 287–307 (1998)

[L] S. L. Lu; Equilibrium fluctuations of a one-dimensional nongradient Ginzburg–Landau model, Ann. Prob. **22** 1252–1272, (1994).

[PSV] Pellegrinotti A., Sidoravicius V., Vares M. E.; Stationary state and diffusion for a charged particle in a one dimensional medium with lifetimes. preprint (1996)

[RV] Rost H. and Vares M. E.; Hydrodynamics of a one dimensional nearest neighbor model. Contemp. Math. **41** 329–342 (1985).

[So] Soloveitchick M.; Mechanical background of Brownian motion. Amer. Math. Soc. Transl. **171** 233–247 (1996).

[Sp] Spitzer, F.; Uniform motion with elastic collisions of an infinite particle system; J. math. Mech. **18** 973–989 (1968).

[S] H. Spohn; *Large scale dynamics of interacting particles*. Springer-Verlag, New York (1991).

IMPA-Instituto de Matemática Pura e Aplicada, Estrada Dona Castorina 110, Jardim Botanico, CEP 22460, Rio de Janeiro, Brasil and CNRS UPRES-A 6085, Université de Rouen, 76.128 Mont Saint Aignan Cedex, France.
E-mail address: `landim@impa.br`

Departamento de Matemática, PUC-Rio-Pontifícia Universidade Católica do Rio de Janeiro, Rua Marques de São Vicente, Gávea, CEP 22453-900 , Rio de Janeiro, Brasil.
E-mail address: `volchan@saci.mat.puc-rio.br`