Slow Blow Up in the (2+1)-dimensional S^2 Sigma Model

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Abstract

We study singularity formation in spherically symmetric solitons of the charge one sector of the (2+1) dimensional S^2 sigma model, also known as $\mathbb{C}P^1$ wave maps, in the adiabatic limit. These equations are non-integrable, and so studies are performed numerically on radially symmetric solutions using an iterative finite differencing scheme. Analytic estimates are made by using an effective Lagrangian cutoff outside a ball of fixed radius. We show the geodesic approximation is valid when the cutoff is applied, with the cutoff approaching infinity linearly as the reciprocal of the initial velocity. Additionally a characterization of the shape of a time slice $f(r, T)$ with T fixed is provided.

Mathematics Subject Classification: 35-04, 35L15, 35L70, 35Q51, 35Q60
Physics and Astronomy Classification: 02.30.Jr, 02.60.Cb

1 Introduction

In this paper we study a hyperbolic partial differential equation that develops a singularity in finite time.

The two-dimensional S^2 sigma model has been studied extensively over the past few years in [1], [4], [5],[6], [7], [8]. It is a good toy model for studying two-dimensional analogues of elementary particles in the framework of classical field theory. Elementary particles are described by classical extended solutions of this model, called solitons. This model is extended to (2+1) dimensions. The previous solitons are static or time-independent solutions, and then the dynamics of these solitons are studied. Since this model is not integrable in (2+1) dimensions studies are performed numerically, and analytic estimates are made by cutting off the model outside a radius R.

The model can also be regarded as the continuum limit of an array of Heisenberg ferromagnets.

The S^2 sigma model displays both slow blow up and fast blow up. In slow blow up, all relevant speeds go to zero as the singularity is approached. In fast blow up the relevant speeds do not go to zero as the singularity is approached . The charge 1 sector of the S^2 sigma model exhibits logarithmic slow blow up, whereas the charge 2 sector and the similar Yang Mills (4+1) dimensional model, both investigated in [3] and [2], exhibit fast blow up.

The static Lagrangian density for the S^2 sigma model is given by

$$L = \int |\nabla \vec{\phi}|^2,$$

where $\vec{\phi}$ is a unit vector field.

In the dynamic version of this problem, where $\phi : \mathbb{R}^{2+1} \to S^2$, the Lagrangian is

$$L = \int_{\mathbb{R}^2} |\partial_t \vec{\phi}|^2 - |\nabla \vec{\phi}|^2.$$

Identifying $S^2 = \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ we can rewrite this in terms of a complex scalar field u:

$$L = \int_{\mathbb{R}^2} \frac{|\partial_t u|^2}{(1+|u|^2)^2} - \frac{|\nabla u|^2}{(1+|u|^2)^2}. \tag{1}$$

The calculus of variations on this Lagrangian in conjunction with integration by parts yields the following equation of motion for the $\mathbb{C}P^1$ model:

$$(1+|u|^2)(\partial_t^2 u - \partial_x^2 u - \partial_y^2 u) = 2\bar{u}(|\partial_t u|^2 - |\partial_x u|^2 - |\partial_y u|^2) \tag{2}$$

Here $\bar{u}$ represents the complex conjugate of u.

The first thing to identify in this problem are the static solutions determined by equation (2). These are outlined in [7] among others. The entire space of static solutions can be broken into finite dimensional manifolds $\mathcal{M}_n$ consisting of the harmonic maps of degree n. If n is a positive integer, then $\mathcal{M}_n$ consists of the set of all rational functions of $z = x + iy$ of degree n. For this chapter, we restrict our attention to $\mathcal{M}_1$, the charge one sector, on which all static solutions have the form

$$u = \alpha + \beta(z + \gamma)^{-1}. \tag{3}$$

In order to simplify, consider only solutions of the form

$$\beta z^{-1}$$

The geodesic approximation says that for slow velocities solutions should evolve close to

$$\frac{\beta(t)}{z}.$$

Instead of $\beta(t)$, we look for a real radially symmetric function $f(r,t)$, and find the evolution of:

$$\frac{f(r,t)}{z}.$$

The differences between the evolution of $f(0,t)$ and that predicted in the geodesic approximation, and the deviation of $f(r,T)$ with T fixed has from a horizontal line gives us a means to gauge how good the geodesic approximation is.

It is straightforward to calculate the evolution equation for $f(r,t)$. It is:

$$\partial_t^2 f = \partial_r^2 f + \frac{3\partial_r f}{r} - \frac{4r\partial_r f}{f^2 + r^2} + \frac{2f}{f^2 + r^2}\left(\partial_t f^2 - \partial_r f^2\right). \tag{4}$$

The static solutions for $f(r,t)$ are the horizontal lines $f(r,t) = c$. Here $c =$ length scale. In the adiabatic limit motion under small velocities should progress from line to line, i.e $f(r,t) = c(t)$. $f(r,t) = 0$ is a singularity of this system, where the instantons are not well defined. We use this to form a numerical approximation to the adiabatic limit to observe progression from $f(r,0) = c_0 > 0$ towards this singularity.

2 Numerics for the $\mathbb{C}P^1$ charge 1 sector model

A finite difference method is used to compute the evolution of (4) numerically. Centered differences are used consistently except for

$$\partial_r^2 f + \frac{3\partial_r f}{r}. \tag{5}$$

In order to avoid serious instabilities in (4) this is modeled in a special way. Let

$$\mathfrak{L} f = r^{-3}\partial_r r^3 \partial_r f = \partial_r^2 f + \frac{3\partial_r f}{r}.$$

This operator has negative real spectrum, hence it is stable. The naive central differencing scheme on (5) results in unbounded growth at the origin, but the natural differencing scheme for this operator does not. It is

$$\mathfrak{L} f \approx r^{-3}\left[\frac{\left(r+\frac{\delta}{2}\right)^3\left(\frac{f(r+\delta)-f(r)}{\delta}\right)-\left(r-\frac{\delta}{2}\right)^3\left(\frac{f(r)-f(r-\delta)}{\delta}\right)}{\delta}\right].$$

Questions arise from [5] about the stability of the solitons for this equation. They found that a soliton would shrink without perturbation from what should be the resting state. This seems to be a function of the numerical scheme used for those experiments. The numerical scheme used here has no such stability problems. Experiments show that a stationary solution is indeed stationary unless perturbed by the addition of an initial velocity.

For a full analysis of the stability of this numerical scheme see [3].

With the differencing explained, we want to derive $f(r, t+\triangle t)$. We always have an initial guess at $f(r, t+\triangle t)$. In the first time step it is $f(r, t+\triangle t) = f(r,t) + v_0 \triangle t$ with v_0 the initial velocity given in the problem. On subsequent time steps $f(r, t+\triangle t) = 2f(r,t) - f(r, t-\triangle t)$. This can be used to compute $\partial_t f(r,t)$ on the right hand side of (4). Then solve for a new and improved $f(r, t+\triangle t)$ in the differencing for the second derivative $\partial_t^2 f(r,t)$ and iterate this procedure several times to get increasingly accurate values of $f(r,t)$.

There remains the question of boundary conditions. At the origin $f(r,t)$ is presumed to be an even function, and this gives

$$f(0,t) = \frac{4}{3} f(\triangle r, t) - \frac{1}{3} f(2\triangle r, t).$$

At the $r = R$ boundary we presume that the function is horizontal so $f(R,t) = f(R - \triangle r, t)$.

3 Predictions of the Geodesic Approximation

In [4] the time evolution of the shrinking of solitons was studied. They arbitrarily cut off the Lagrangian outside of a ball of radius R, to prevent logarithmic divergence of the integral for the kinetic energy, and then analyze what happens in the $R \to \infty$ limit. In [6] the problem of the logarithmic divergence in the kinetic energy integral is solved by investigating the model on the sphere S^2. The radius of the sphere determines a parameter for the size analogous to the parameter R for the size of the ball the Lagrangian is evaluated on in [4].

Our calculations here follow those done in [4].

Equation (1) gives us the Lagrangian for the general version of this problem. In the geodesic approximation or adiabatic limit we have

$$u = \frac{\beta}{z}$$

for our evolution. If we restrict the Lagrangian to this space we get an effective Lagrangian. The integral of the spatial derivatives of u gives a constant, the Bogomol'nyi bound, and hence can be ignored. If one integrates the kinetic term over the entire plane, one sees it diverges logarithmically, so if β is a function of time, the soliton has infinite energy.

Nonetheless, this is what we wish to investigate. We cannot address the entire plane in our numerical procedure either, hence we presume that the evolution takes place in a ball around the origin of size R. If $\beta = f(r,t)$ shrinks to 0 in time T, we need $R > T$. Under these assumptions, up to a multiplicative constant, the effective or cutoff Lagrangian becomes

$$L = \int_0^R r dr \frac{r^2 \partial_t f^2}{(r^2 + f^2)^2}$$

which integrates to

$$L = \frac{\partial_t f^2}{2} \left[\ln\left(1 + \frac{R^2}{f^2}\right) - \frac{R^2}{f^2 + R^2} \right]$$

Since the potential energy is constant, so is the purely kinetic Lagrangian, and

$$\frac{\partial_t f^2}{2}\left[\ln\left(1+\frac{R^2}{f^2}\right)-\frac{R^2}{f^2+R^2}\right]=\frac{c^2}{2},$$

with c (and hence $c^2/2$) a constant. Solving for $\partial_t f$ we obtain

$$\partial_t f=\frac{c}{\sqrt{\left[\ln\left(1+\frac{R^2}{f^2}\right)-\frac{R^2}{f^2+R^2}\right]}}. \tag{6}$$

Since we are starting at some value f_0 and evolving toward the singularity at $f=0$ this gives:

$$\int_{f_0}^{f(0,t)} df\sqrt{\ln\left(1+\frac{R^2}{f^2}\right)-\frac{R^2}{f^2+R^2}}=\int_0^t cdt. \tag{7}$$

The integral on the right gives ct. The integral on the left can be evaluated numerically for given values of R, f_0 and $f(0,t)$. A plot can then be generated for ct vs. $f(0,t)$. What we really are concerned with is $f(0,t)$ vs. t, but once the value of c is determined this can be easily obtained. One such plot with $f_0=1.0$, $R=100$ of $f(0,t)$ vs ct is given in Figure 1. This curve is not quite linear, as seen by comparison with the best fit line to this data which is also plotted in Figure 1. The best fit line is obtained by a least squares method.

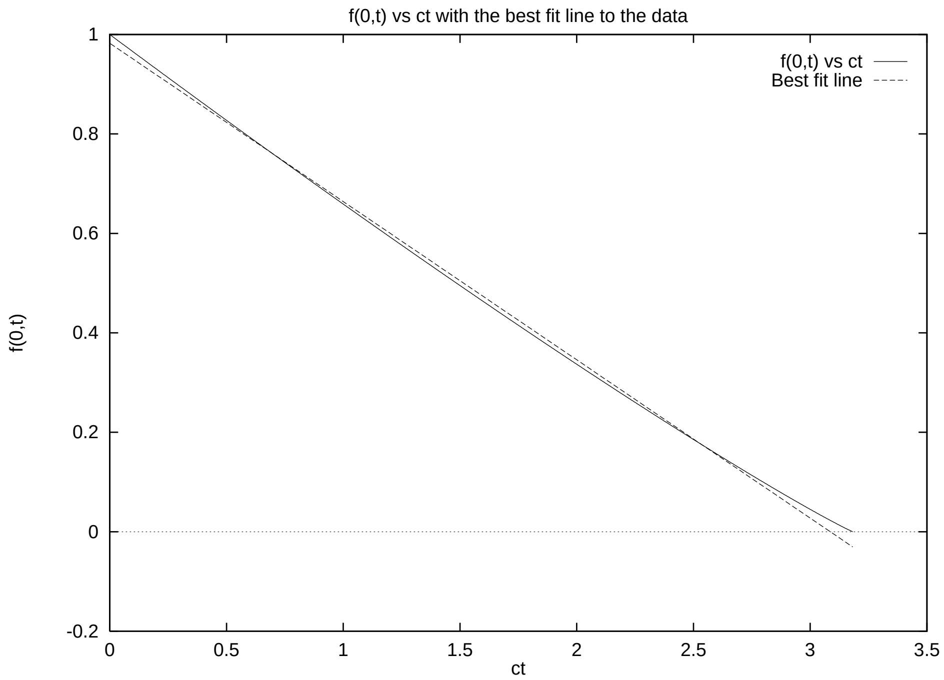

Figure 1: Example plot of $f(0,t)$ vs ct as predicted by the effective Lagrangian.

4 Evolution of the Origin Results

The computer model was run under the condition that $f(r,0) = f_0$ with various small velocities. The initial velocity is $\partial_t f(r,0) = v_0$, other input parameters are $R = r_{max}$, $\triangle r$ and $\triangle t$.

Evolution of $f(0,t)$

The primary concern with the evolution of the horizontal line is the way in which the singularity at $f(0,t) = 0$ is approached, because once again as $r \to \infty$ equation (4) reduces to the linear wave equation

$$\partial_t^2 f = \partial_r^2 f,$$

and so we expect the interesting behavior to occur near $r = 0$. The model is run with initial conditions that $f(r,0) = f_0$, $\partial_t f(r,0) = v_0$. Other input parameters are $R = r_{\max}$, $\triangle r$ and $\triangle t$.

The evolution of the initial horizontal line seems to remain largely flat and horizontal, although there is some slope downward as time increases. This is shown in figure 2.

We track $f(0,t)$ as it heads toward this singularity, and find that its trajectory is not quite linear, as seen in figure 3. This is suggestive of the result obtained in the predictions for this model.

We want to check the legitimacy of the result from equation (7) in chapter 3.2. This requires a determination of the parameters R, the cutoff, and c. We already have f_0 and $f(0,t)$. To determine R and c, observe from equation (6) that:

$$\frac{1}{\partial_t f^2} = \frac{\left[\ln\left(1 + \frac{R^2}{f^2}\right) - \frac{R^2}{f^2 + R^2}\right]}{c^2} \tag{8}$$

Since R is large and f is small

$$\ln\left(1 + \frac{R^2}{f^2}\right) \approx \ln\left(\frac{R^2}{f^2}\right)$$

and

$$\frac{R^2}{f^2 + R^2} \approx 1.$$

Consequently we can rewrite equation (8) as

$$\frac{1}{\partial_t f^2} \approx \frac{[\ln(R^2) - \ln(f^2) - 1]}{c^2}.$$

The plot of $\ln(f) = \ln(f(0,t))$ vs $1/\partial_t f^2 = 1/\partial_t f^2(0,t)$ should be linear with the slope $m = 2/c^2$ and the intercept $b = (2\ln(R) - 1)/c^2$. Such a plot is easily obtained from the model, and given slope and intercept, the parameters c and R are easily obtained.

Figure 4 is a plot of $\ln(f(0,t))$ vs. $1/\partial_t f^2(0,t)$, with initial conditions $\triangle r = 0.01$, $\triangle t = 0.001$, $f_0 = 1.0$ and $v_0 = -0.01$. It is easily seen that although the plot of $\ln(f(0,t))$ vs. $1/\partial_t f^2(0,t)$ is nearly straight, it is not quite a straight line. This may indicate that the values of R and c are changing with time.

The best fit line $y = mx + b$ has slope $m = -2810$ and $b = 10200$. We have

$$c = \sqrt{\frac{-2}{m}}$$

and

$$R = \exp\left(-\frac{b}{m} + \frac{1}{2}\right).$$

This gives values $c = 0.0267$ and $R = 62.1$. Using these values of c and R in the calculation of equation (7), we obtain the plot of $f(0,t)$ vs t given in Figure 5. This is overlayed with the model data for $f(0,t)$ vs. t for comparison. These two are virtually identical.

This shows that the phenomenon of cutting off the Lagrangian outside of a ball of radius R is not just an artifact of necessity because the full Lagrangian is divergent, but an inherent feature of this system.

Table 1 contains the data for c and R vs. change in the initial velocity v_0, under the initial conditions $f_0 = 1.0$, $\triangle r = 0.01$ and $\triangle t = 0.001$. The data for R varying with $1/v_0$ fits well to the line

$$y = 0.5407x + 6.032.$$

This fit is shown in figure 6. A linear fit makes sense, since as the velocity tends toward zero, we expect the cutoff to head toward infinity.

Table 2 containing the data for c and R vs. change in f_0, the initial height. The parameter R varies close to linearly with f_0, while the parameter c remains nearly constant, changing by less than 7% over the course of the runs.

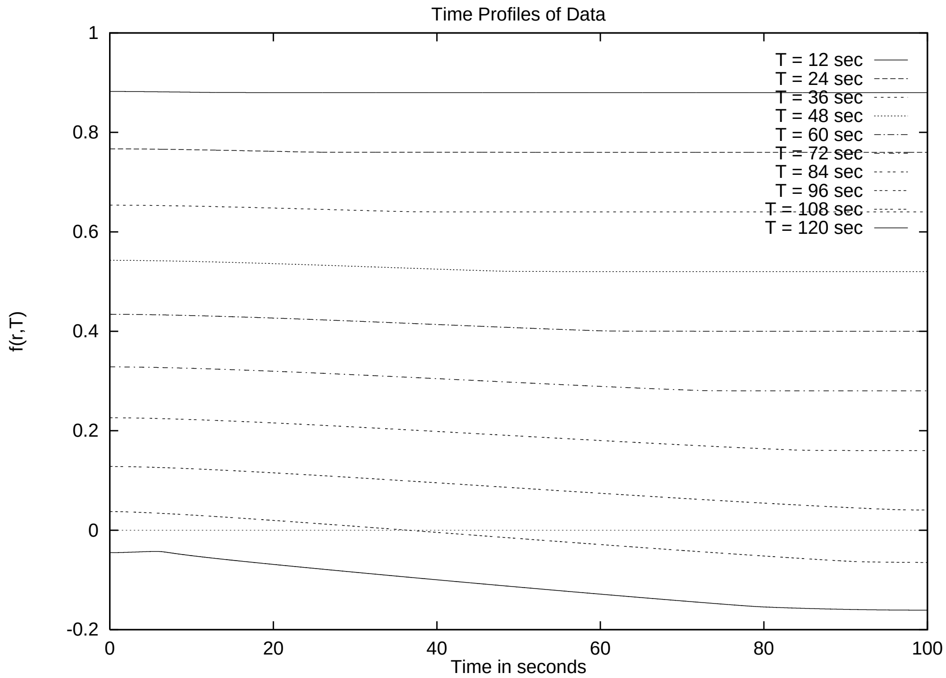

Figure 2: Time slices $f(r, T)$

Table 1: c and R vs. v_0. $f_0 = 1.0$.

v_0	c	R
-0.005	0.0145	115
-0.00667	0.0187	89
-0.01	0.0263	53
-0.0133	0.0342	49
-0.02	0.0485	34
-0.03	0.0683	25
-0.05	0.104	17
-0.06	0.121	15

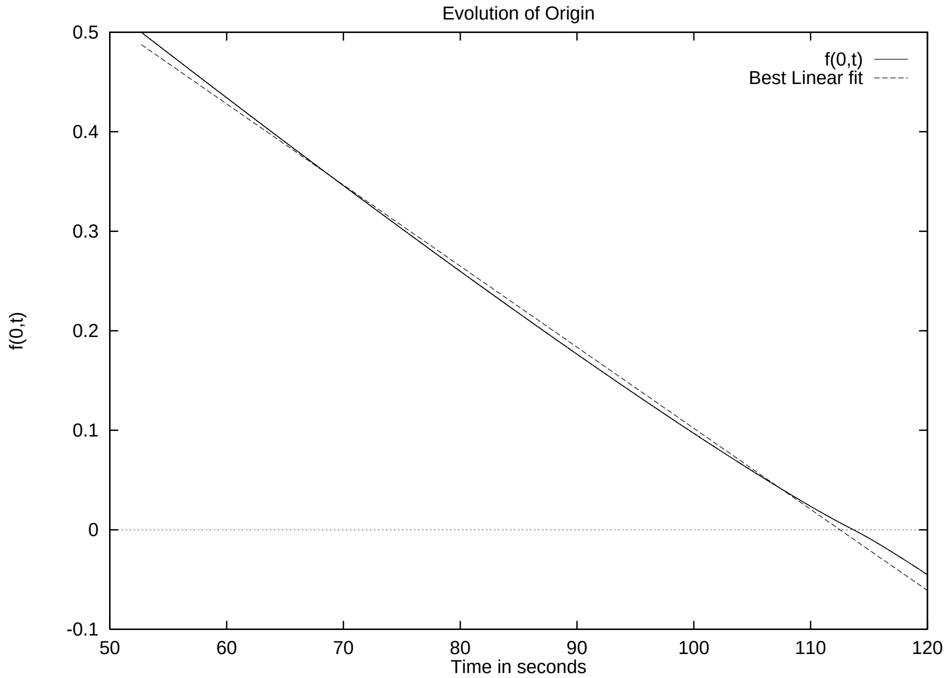

Figure 3: Evolution of $f(0,t)$ is not quite linear.

Table 2: c and R vs. f_0. $v_0 = -0.01$.

f_0	c	R
1.0	0.0267	62
2.0	0.0263	108
3.0	0.0260	150
4.0	0.0259	190

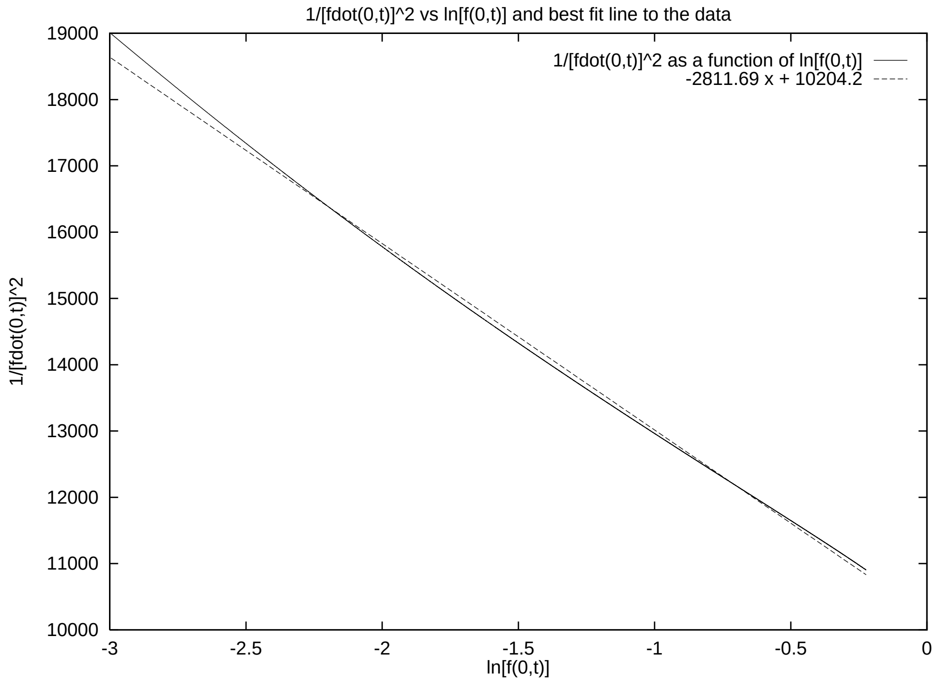

Figure 4: Plot of $1/(\partial_t f)^2$ vs $\ln(f)$ and the best fit line to this data.

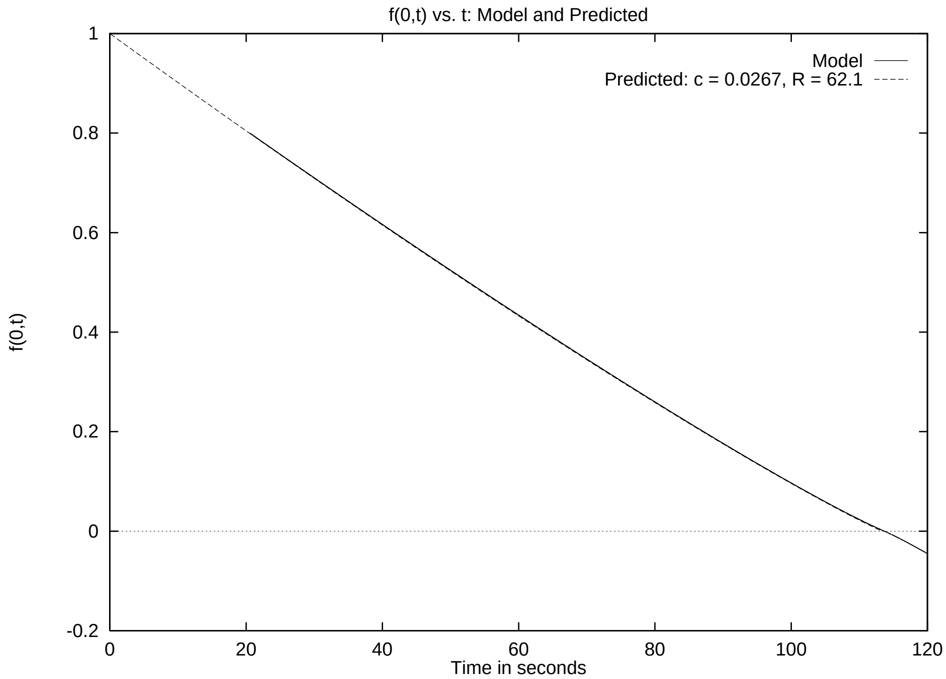

Figure 5: Predicted course of $f(0,t)$ from equation (7) and actual course of $f(0,t)$ vs. t.

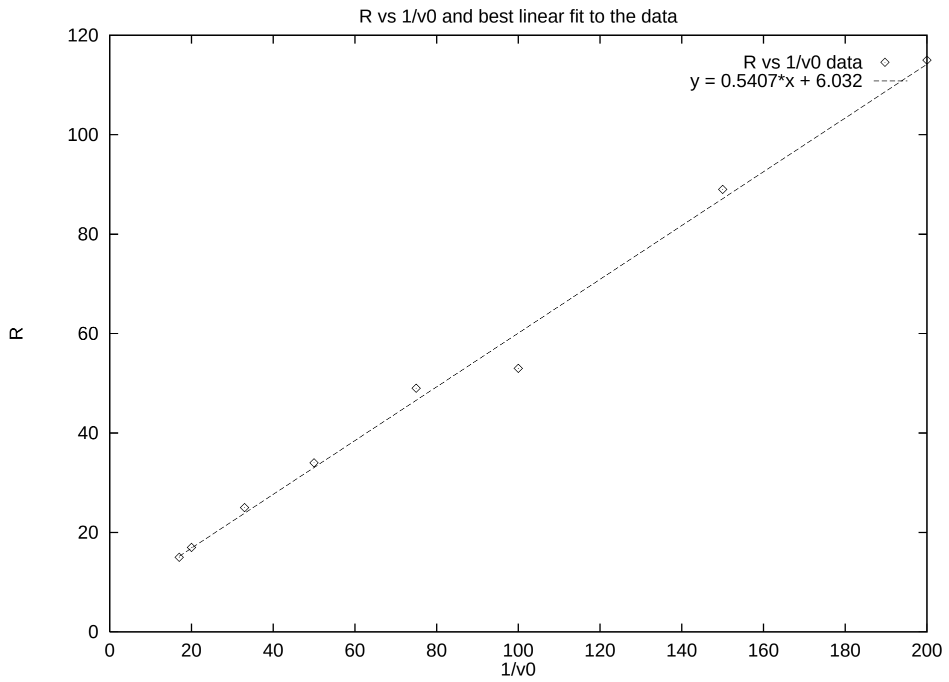

Figure 6: R vs 1/v0 and the best fit line.

Characterization of time slices $f(r,T)$

Making a closer inspection of the time profiles $f(r,T)$ with T fixed as in Figure 2, one may observe that the initial part of the data is close to a hyperbola as seen in Figure 7. The best hyperbolic fit is determined by a least squares method.

The equation for the hyperbola is

$$\frac{(y-k)^2}{b^2} - \frac{x^2}{a^2} = 1.$$

One would naturally ask about the evolution of the hyperbolic parameters a and b with time, however, neither of these is particularly edifying. A simple calculation shows that k should follow $f(0,t)$ closely if b is small, as it is.

The evolution of $-b/a$ gives the slope of the asymptotic line to the hyperbola, and this evolution is close to linear as seen in Figure 8.

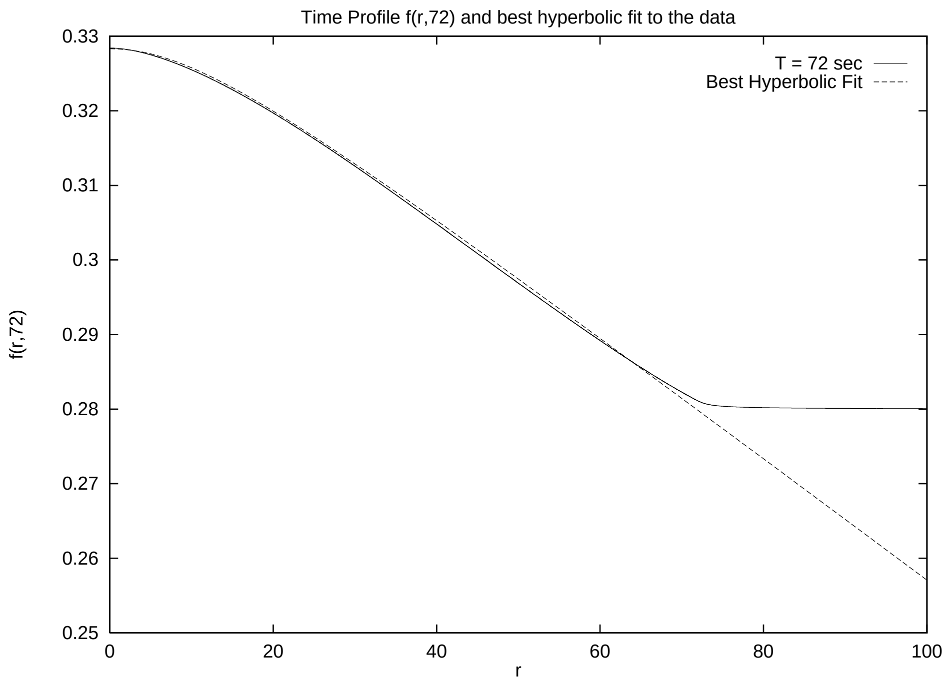

Figure 7: Time slices evolve hyperbolic bump at origin.

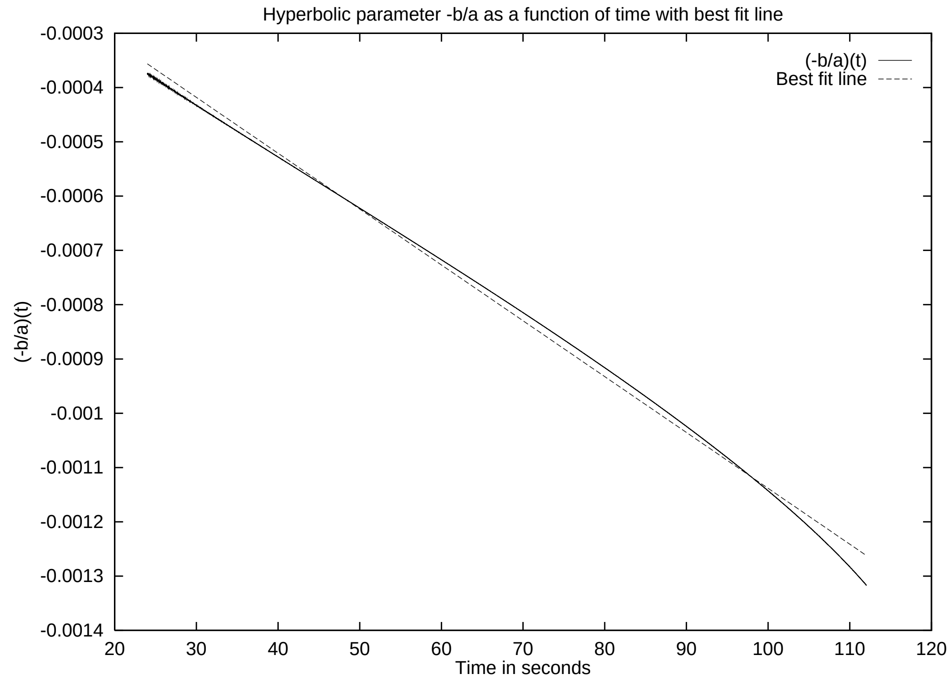

Figure 8: Plot of hyperbolic parameter $-b/a$ vs time, $f_0 = 1.0$, $v_0 = -0.01$.

5 Conclusions

In [5], solitons are found to be numerically unstable. A solution of the form

$$\frac{\beta}{z}$$

shrinks spontaneously under their numerical procedure. This does not occur in our numerical implementation of the S^2 sigma model. The static solutions do not evolve in time unless given an initial rate of shrinking. Further, stability and convergence analysis of two of the numerical procedures is provided in [3].

We use a method analogous to that in [4] of cutting off the Lagrangian outside of a ball of radius R, and we find an explicit integral for the shrinking of the soliton, dependent on two parameters: c which is a function of the kinetic energy, and R which is the size of the ball on which we evaluate the Lagrangian. Once these are specified, this integral gives the theoretical trajectory of the soliton. We find an explicit integral for the shrinking of a soliton where the Lagrangian is cut off outside of the ball of radius R, and when R is calculated (along with the parameter c), the shrinking seen in the numerical model matches that predicted. The dependence of R on the initial conditions appears to be linear with the initial velocity. The validity of the geodesic approximation with a cutoff R is shown by this, and the cutoff is an essential part of this system.

In addition to this, the shape of a time slice $f(r, T)$, with T fixed, is characterized by hyperbolic bumps at the origin.

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