Time Evolution of Spin Waves

Michael Moser
Justgasse 29/12/3/12, A-1210 Vienna, Austria

Alexander Prets
Institut für Theoretische Physik, Universität Wien, Boltzmanngasse 5, A-1090 Vienna, Austria

Wolfgang L Spitzer
Department of Mathematics, University of Copenhagen, Universitetsparken 5, DK-2100 Copenhagen, Denmark
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A rigorous derivation of macroscopic spin-wave equations is demonstrated. We introduce a macroscopic mean-field limit and derive the so-called Landau-Lifshitz equations for spin waves. We first discuss the ferromagnetic Heisenberg model at $T=0$ and finally extend our analysis to general spin hamiltonians for the same class of ferromagnetic ground states.

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An important problem in statistical physics is to explain the macroscopic behavior of matter from first principles, in particular equations of motion of macroscopic observables. There is by now an extensive literature on classical systems and we want to refer to a standard treatment by Spohn [1]. The general idea is that macroscopic equations emerge from microscopic equations of motion (say Newton's or Schrödinger's) for the constituents of matter by performing some kind of scaling limit. There are different types of scaling limits whose applicability is to be justified by the physical circumstances.

In this paper we study equations of motion for the magnetization in certain magnetic samples. Long ago Landau and Lifshitz [2] derived such equations. In case of ferromagnets it says that the change of the magnetization vector, $\vec{M}(r,t)^*$, is simply given by rotation, i.e.

$$\frac{\partial \vec{M}(r,t)}{\partial t} = \vec{M}(r,t) \times \vec{H}_{\text{eff}}(r,t). \tag{1}$$

Here, the angular velocity is equal to an effective magnetic field, $\vec{H}_{\text{eff}}(r,t)$, which originates from the microscopic interactions of the spins. If they, say, interact only via ferromagnetic exchange interaction [3] with coupling function J, than $\vec{H}_{\text{eff}}(r,t)$ is given by

$$\vec{H}_{\text{eff}}(r,t) = \int dr'\, J(r,r')\, \vec{M}(r',t). \tag{2}$$

Expanding $\vec{M}(r',t)$ around (r,t) and considering an isotropic potential, $J(r,r') = J(|r-r'|)$, $\vec{H}_{\text{eff}}$ becomes in lowest order proportional to the laplacian $\Delta\vec{M}(r,t)$, and (1) simplifies to

$$\frac{\partial}{\partial t}\vec{M}(r,t) = \alpha\, \vec{M}(r,t) \times \Delta\vec{M}(r,t) \tag{3}$$

with

$$\alpha = \frac{1}{6}\int dr\, J(|r|)\, r^2.$$

Usually, (3) is called the Landau-Lifshitz equation.

We are aware of only two other sources [3,4], where a derivation of equation (3) is demonstrated but, nevertheless, is partly based on non-rigorous arguments; some of them (e.g. magnon picture), however, are extremely important, influential and still a major challenge in this field.

We will derive the more general equation (1) by applying what we call a macroscopic mean-field limit [1,5,6]. In this context we mention that the Vlasov equation for a quantum mean-field system has been proven by Narnhofer and Sewell [7]. One might think that a direct derivation of equation (3) is easier than of (1), but the former does not seem to originate from a microscopic model [8]. This can be compared with a recent work on the derivation of the Cahn-Hilliard equation [9]. We want to emphasize that we did not start with a mean-field model in the first place but we were forced, both for physical and mathematical reasons, to do so. We hope that this will become clear in our presentation. Admittedly, other approaches may directly lead to equation (3).

Now we provide the set-up for the derivation of equation (1). Since magnetism is a purely quantum mechanical effect of interacting spin systems we clearly have to formulate a quantum mechanical model [10]. For mathematical convenience we consider a bounded sample, say the cube $[0,1]^d$. In order to describe the microscopic situation we enlarge the cube by a factor $\epsilon^{-1}, \epsilon > 0$. Finally, we let $\epsilon \downarrow 0$. $K(\epsilon) = [0,\epsilon^{-1}]^d \cap \mathbb{Z}^d$ represents the ferromagnet from a microscopic point of view. We use the general ferromagnetic Heisenberg-Kac hamiltonian, which is motivated by the mean-field structure of (2).

$^*\vec{M} : (r,t) \in \mathbb{R}^d \times \mathbb{R} \mapsto \vec{M}(r,t) \in \mathbb{R}^3$. We set Bohr's magneton $\mu_0 = \hbar/2$, which puts the constant on the right hand side equal to 1.

$$H^\epsilon = -\frac{1}{2}\sum_{x,y\in K(\epsilon)} J^\epsilon_{xy}\, S^\alpha_x\, S^\alpha_y, \tag{4}$$

to describe the microscopic exchange interaction energy between spins in $K(\epsilon)$. As usual, H^ϵ acts on the $|K(\epsilon)|$-fold tensor product of the space $\mathbb{C}^{2S+1}$. S is some fixed (but arbitrary) half-integer. Spin operators $\vec{S_x} = (S^1_x, S^2_x, S^3_x)$ at lattice site x (acting on $\mathbb{C}^{2S+1}$) satisfy the usual commutation relations

$$\left[S^\alpha_x, S^\beta_y\right] = i\,\delta_{xy}\,\epsilon_{\alpha\beta\gamma}\,S^\gamma_y.$$

We scale the interaction potential like $J^\epsilon_{xy} = \epsilon^{+d} J(\epsilon x, \epsilon y)$, which can be viewed as a Lebowitz-Penrose approximation [11] to the true interaction; the factor ϵ^d in front of the sum provides us with an extensive energy function (see also another discussion below). Here, the function J on $[0,1]^{2d}$ is assumed to be non-negative but not necessarily symmetric.

The general idea relating the different levels of description is to define the *macroscopic* magnetization profile $M^\gamma(r,t)$ as the ϵ-scaling limit of the expectation value of the time dependent *microscopic* spin operators $S^\gamma_x(t)$, see (6).

Thus the second main ingredient in our analysis, besides the hamiltonian (4), is to specify a class of (initial) states, denoted hereafter by $\mathfrak{F}$. It is clear that (1) can only be correct at low temperature. As is well-known spins are parallel to each other in the (highest weight) ground state of the (isotropic) ferromagnetic Heisenberg model. In our states spins will remain almost parallel over short (compared to ϵ^{-1}) distances but vary over distances of order ϵ^{-1}, i.e. we locally describe $T = 0$ equilibrium states as

$$\mathfrak{F} = \Big\{(\omega^\epsilon)_{\epsilon>0} : \omega^\epsilon = \bigotimes_{x\in K(\epsilon)} \omega^\epsilon_x : \omega^\epsilon_x(S^\gamma_x) = M^\gamma(\epsilon x)\Big\}. \tag{5}$$

ω^ϵ_x is a state on the matrix algebra over $\mathbb{C}^{2S+1}$; in the case $S = 1/2$ the states ω^ϵ_x are uniquely determined by M^γ itself.

We rescale space by ϵ but we *do not rescale time.* This particular space vs time scaling limit fits nicely with Landau's idea of classical spin waves of large wave-length, since the microscopic spin-wave length goes as $\mathcal{O}(\epsilon^{-1})$.

In what follows we require $M^\gamma(r,0), \phi(r)$ and $J(r,r')$ to be continuous functions on the cube $[0,1]^d$. We now show that at least for $0 < |t| < C$ there are analytic measures $dM^\gamma(r,t)$ such that the limits

$$\lim_{\epsilon\downarrow 0} \epsilon^d \sum_{x\in K(\epsilon)} \phi(\epsilon x)\,\omega^\epsilon(S^\gamma_x(t)) = \int_{[0,1]^d} \phi(r)\,dM^\gamma(r,t) \tag{6}$$

for states $\omega^\epsilon \in \mathfrak{F}$ exist. Further, $dM^\gamma(r,t)$ satisfy the Landau-Lifshitz equation (1-2).

Our analysis is based on a series expansion of the time evolution of spin operators in finite volume:

$$S^\gamma_x(t) = \sum_{n=0}^\infty \frac{(it)^n}{n!}\, K^n(H^\epsilon)\, S^\gamma_x,$$

where $K^n(H^\epsilon)S^\gamma_x$ means the n-th commutator of H^ϵ with S^γ_x. We split this n-th commutator into two terms, each of them can be obtained recurrently[†],

$$\begin{aligned} &K^n(H^\epsilon)\, S^\gamma_{y_1} = \\ &(-i)^n \sum_{y_2,\ldots,y_{n+1}\in K(\epsilon)}{}' P_n[J^\epsilon](\vec{y}_{n+1})^\gamma_{\vec{\alpha}_{n+1}} S^{\vec{\alpha}_{n+1}}_{\vec{y}_{n+1}} + \hat{R}^\epsilon_n. \end{aligned} \tag{7}$$

$\sum'_{y_2,\ldots,y_{n+1}}$ means summation over all pairwise disjoint variables $y_i \in K(\epsilon)$, i.e. $y_i \neq y_j$ for all $i \neq j$ with $i,j = 1,\ldots,n+1$, $\vec{y}_{n+1} = (\vec{y}_n, y_{n+1})$, $\vec{\alpha}_{n+1} = (\vec{\alpha}_n, \alpha_{n+1})$, and $S^{\vec{\alpha}_n}_{\vec{y}_n} = \prod_{k=1}^n S^{\alpha_k}_{y_k}$. The polynomials $P_n = P_n[J^\epsilon]$ are defined in the following way:

$$\begin{aligned} P_0(y_1)^\gamma_{\alpha_1} &= \delta_{\gamma\alpha_1}, \\ P_1(y_1,y_2)^\gamma_{\alpha_1\alpha_2} &= \epsilon_{\gamma\alpha_1\alpha_2}\, J^\epsilon_{y_1 y_2}, \\ P_n(\vec{y}_{n+1})^\gamma_{\vec{\alpha}_{n+1}} &= P_{n-1}(\vec{y}_n)^\gamma_{\vec{\beta}_n}\ (T^{\alpha_{n+1}}_n)^{\vec{\beta}_n}_{\vec{\alpha}_n}\quad n>1 \end{aligned} \tag{8}$$

with

$$T^\alpha_m = \sum_{k=1}^m \underbrace{1\otimes\ldots\otimes 1}_{k-1\text{ times}} \otimes \left(\epsilon^\alpha\, J_{y_k y_{m+1}}\right) \otimes \underbrace{1\otimes\ldots\otimes 1}_{m-k\text{ times}}.$$

ϵ^α denotes the matrix of the ϵ-tensor, $\epsilon_{123} = 1$. The rest term, $\hat{R}^\epsilon_n$, is given by

$$\begin{aligned} \hat{R}^\epsilon_n &= \sum_{k=0}^n K^{n-k}(H^\epsilon)\, R^\epsilon_k, \\ R^\epsilon_k &= \frac{1}{4}(-i)^k \sum_{y_2,\ldots,y_k\in K(\epsilon)}{}' \sum_{m=1}^k P_k(\vec{y}_k, y_m)^\gamma_{\vec{\alpha}_k,\alpha_m} \hat{S}^{\alpha_m}_{y_m}, \end{aligned} \tag{9}$$

with $\hat{S}^{\alpha_m}_{y_m} = \prod_{m\neq l=1}^k S^{\alpha_l}_{y_l}$ and $R^\epsilon_0 = R^\epsilon_1 = 0$. Formulas (8-9) can be proven by induction. Since there are more ϵ^d-factors than sums the operator norm of the rest term vanishes as $\epsilon \downarrow 0$,

$$\lim_{\epsilon\downarrow 0} ||\hat{R}^\epsilon_n|| = 0. \tag{10}$$

We first show that (for $\gamma = 1,2,3$) the maps

$$\phi \mapsto L^\gamma(\phi,t) := \lim_{\epsilon\downarrow 0} \epsilon^d \sum_{x\in K(\epsilon)} \phi(\epsilon x)\,\omega^\epsilon(S^\gamma_x(t))$$

[†] This is strictly true only for points y_1 such that the l_1-distance to the boundary of $K(\epsilon)$, $||y_1 - \partial K(\epsilon)||_1 \geq n$. But, for fixed n, we can always choose ϵ to be small enough.

are continuous linear functionals on the Banach space $C([0,1]^d, \|\cdot\|_\infty)$ (of continuous functions equiped with the maximum norm) for sufficiently small $|t|$ (depending on the interaction function J and on the spin S), which guarantees the existence of measures $dM^\gamma(r,t)$. This is accomplished by roughly estimating $\epsilon^d \sum_{x\in K(\epsilon)} |\phi(\epsilon x)\,\omega^\epsilon(S_x^\gamma(t))|$ uniformly in ϵ by $\|\phi\|_1 \cdot \|M\|_\infty (1-2t\,\|J\|_\infty\,\|M\|_\infty)^{-1}$ (up to $\mathcal{O}(\epsilon)$) which holds at least for $|t|^{-1} > \|J\|_\infty\,\|M\|_\infty =: C^{-1}$; note that there are $2^n n!$ contributing terms. This also shows analyticity for $|t| < C$, so we are allowed to interchange the ϵ-limit with the Taylor expansion in time.

By using the formulas (7-9) for the commutator and the product property of the states ω^ϵ, we can then identify the limit

$$\lim_{\epsilon\downarrow 0} \epsilon^d \sum_{x\in K(\epsilon)} \phi(\epsilon x)\,\frac{\partial^{n+1}}{\partial t^{n+1}}\Big|_{t=0} \omega^\epsilon(S_x^\gamma(t)) = \int_{[0,1]^d} dr\,\phi(r)\,\frac{\partial^{n+1}}{\partial t^{n+1}}\Big|_{t=0} M^\gamma(r,t) \qquad (11)$$

with the n-th derivative of the right hand side of (1).

Note, that we can neglect "surface" points (see second footnote) in the sum in (11) close to $\partial K(\epsilon)$, since they are of order ϵ^{-d+1} and do not contribute in the limit.

We certainly believe that everything is true for all times but we couldn't rigorously prove it.

We now want to discuss some generalizations of the Landau-Lifshitz equations (1-3). First of all they do appear and secondly, we can fully exploit our approach to these general cases. This also gives us insights in the relation between symmetries of the microscopic hamiltonian and those of the spin-wave equation.

Before changing the spin interaction we mention that we can replace the regular (hyper)cubic lattice structure $\mathbb{Z}^d$ itself by other crystal structures such as polyatomic Bravais lattices. One could also immediately start with an infinitely extended sample instead of the cube (the test function ϕ as well as the interaction J should then be Schwartz functions), or more rigorously by additionally performing a thermodynamic limit.

We have just shown how the term $\vec{M}\times\Delta\vec{M}$ descends from a 2-point spin interaction. In general, we might have contributions (in raising complexity) from an exterior magnetic field, $\vec{M}\times\vec{B}$, magnetic-anisotropy energy (due to the presence of an axis of easiest magnetization), $(0,0,M^3)\times\vec{M}$, and relativistic corrections like $\vec{M}\times(\vec{M}\times\Delta\vec{M})$, which determine the time derivative of $\vec{M}$. We now explain how these terms emerge from general N-point spin interactions. To this purpose we start with a hamiltonian

$$H_N^\epsilon = \sum_{m=1}^{N} \sum_{y_1,\ldots,y_m} (J_m^\epsilon)_{y_1\ldots y_m}^{\alpha_1\ldots\alpha_m}\, S_{y_1}^{\alpha_1}\cdots S_{y_m}^{\alpha_m} \qquad (12)$$

which includes multiple spin interactions up to a maximal, but otherwise arbitrary order N. Whereas Akhiezer et al derive for instance $\vec{M}\times\vec{M}\times\Delta\vec{M}$ by taking into account higher orders in the Holstein-Primakoff expansion [3] we directly consider general N-point spin interactions.

As before the tensor-valued functions $(J_m^\epsilon)_{y_1\ldots y_m}^{\alpha_1\ldots\alpha_m}$ are slowly varying functions w.r.t. spatial coordinates and scale in magnitude as $\epsilon^{(m-1)d}$ such that the energy is extensive; we continue to write now $(J_m^\epsilon)_{y_1\ldots y_m}^{\alpha_1\ldots\alpha_m} = \epsilon^{(m-1)d}\, J_m^{\alpha_1\ldots\alpha_m}(\epsilon y_1,\ldots,\epsilon y_m)$.

Surprisingly, one can calculate the time evolution generated by the more general hamiltonian in (12) and analyze the scaling limit applied to states in $\mathfrak{F}$. Even more, the structure of the spin-wave equation is the same as in equation (1), only the effective magnetic field has to be generalized to

$$\vec{H}_{\text{eff}}(r,t) = J_1^{\alpha_0}(r) + \sum_{m=1}^{N-1} \int dr_1'\cdots dr_m' \; J_{m+1}(r,r_1',\ldots,r_m')^{\alpha_0\alpha_1\ldots\alpha_m}\, M^{\alpha_1}(r_1',t)\cdots M^{\alpha_m}(r_m',t). \qquad (13)$$

The formulas for the time evolution generated by H_N^ϵ are analogous to (7-9) but more complicated in detail and will be omited here.

We remark that the use of product-states as initial states for the Heisenberg model is perfectly justified by its local $T=0$ equilibrium properties, but in general cases we consider them just as convenient states, which seem (without proof) to be close to experimentally realizable initial states.

All kinds of Landau-Lifshitz equations which originate from hamiltonian dynamics (and therefore conserve the magnetization) can now be derived by properly choosing the interaction potentials. In order to see how this works we now apply this to the interesting case of 3-point interactions,

$$H_3^\epsilon = \sum_{x,y,z} (J_3^\epsilon)_{xyz}^{\alpha\beta\gamma}\, S_x^\alpha S_y^\beta S_z^\gamma, \qquad (14)$$

which after all is *the* motivation for our generalizations. From the invariance under rotations w.r.t. spin indices, we choose $(J_3^\epsilon)_{xyz}^{\alpha\beta\gamma} =: (J_3^\epsilon)_{xyz}\,\epsilon^{\alpha\beta\gamma}$. It is fair to ask for spatial translation invariance of J_3, such that $(J_3^\epsilon)_{xyz} =: J(\epsilon(y-x),\epsilon(z-x))$. Further, it is reasonable that J has the properties $J(y,z) = -J(z,y) = J(-y,-z)$. [Any odd function, f, provides an example by setting $J(y,z) = f(|y|^2-|z|^2)$.] After expanding the integrand in $\vec{H}_{\text{eff}}$ at r we find the desired equation

$$\frac{\partial}{\partial t}\vec{M}(r,t) = \alpha_{ij}\,\vec{M}(r,t)\times\left(\vec{M}(r,t)\times\partial_i\partial_j\vec{M}(r,t)\right)$$

with $\alpha_{ij} = \int dydz\,J(y,z)\,y_i z_j$.

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[1] H. Spohn, *Large Scale Dynamics of Interacting Particles*, Springer 1991.
[2] L.D. Landau and E.M. Lifshitz, Physikalische Zeitschrift der Sowjetunion **108**, 153 (1935), or in "Course in Statistical Physics", vol 9.
[3] A.I. Akhiezer, V.G. Bar'yakhtar, and S.V. Peletminskii, *Spin Waves*, Vol. 1, North-Holland Amsterdam (1968).
[4] C. Herring and C. Kittel, Phys. Rev. 81, (5), 869 (1951).
[5] A. De Masi and E. Presutti, LNM 1501, 1991.
[6] H. Spohn, *On Three Levels*, Plenum Press, 1994.
[7] H. Narnhofer and G.L. Sewell, CMP 79, 9 (1981)
[8] A. Prets, PhD thesis, University of Vienna, 1998.
[9] G. Giacomin and J.L. Lebowitz, JSP, 87, 37 (1997).
[10] W.L. Spitzer, PhD thesis, University of Vienna, 1996.
[11] J.L. Lebowitz and O. Penrose, J. Math. Phys. 7, 98 (1966).