Inverse Scattering on the Line for the Nonlinear Klein-Gordon Equation with a Potential*

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Abstract

In this paper we obtain a $L^p - L^{\acute{p}}$ estimate for the Klein-Gordon equation with a potential on the line. This result follows from the previously known estimates in the case without potential and from the continuity on the Sobolev spaces of the Schrödinger wave operators that we proved in a previous paper.
Using this estimate we construct the scattering operator for the nonlinear Klein-Gordon equation with a potential. Moreover, we prove that the small amplitude scattering operator uniquely determines both the potential and the nonlinearity. Our proof gives also a method for the reconstruction of the potential and of the nonlinearity.

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1 Introduction

In this paper we obtain a $L^p - L^{\acute{p}}$ estimate for the Klein-Gordon equation with a potential on the line. With the help of this estimate we construct the scattering operator for the nonlinear Klein-Gordon equation with a potential and we solve the inverse nonlinear scattering problem. Namely, we prove that the small amplitude scattering operator uniquely determines both the potential and the nonlinearity. Our proof gives also a method for the reconstruction of the potential and of the nonlinearity. The basic imput for the proof of these results is the continuity in the Sobolev spaces, $W_{k,p}$, of the Schrödinger wave operators that we proved in [45]. This fact allows us to obtain a $L^p - L^{\acute{p}}$ estimate for the Klein-Gordon equation with a potential from the corresponding estimate that was proven by [19] in the case without potential.

For any $\gamma \in \mathbf{R}$ let us denote by L^1_γ the Banach space of all complex-valued measurable functions, ϕ, defined on $\mathbf{R}$ and such that

$$\|\phi\|_{L^1_\gamma} := \int |\phi(x)| \, (1+|x|)^\gamma \, dx < \infty. \tag{1.1}$$

If $V \in L^1_1$ the differential expression $\tau := -\frac{d^2}{dx^2} + V(x)$ is essentially self-adjoint on the domain

$$D(\tau) := \left\{ \phi \in L^2_C : \phi \, \text{and} \, \frac{d}{dx}\phi \, \text{are absolutely continuous and} \, \tau\phi \in L^2 \right\} \tag{1.2}$$

where L^2_C denotes the set of all functions on L^2 that have compact support. We denote by H the unique self-adjoint realization of τ. It is known (see [8], [46]) that H has a finite number of negative eigenvalues, that it has no positive or zero eigenvalues, that it has no singular-continuous spectrum and that the absolutely-continuous spectrum is $[0, \infty)$. By H_0 we denote the unique self-adjoint realization of $-\frac{d^2}{dx^2}$ with domain the Sobolev space $W_{2,2}$. For the definition of the spaces that we use see Section 2. The wave operators are given by:

$$W_\pm := s - \lim_{t \to \pm\infty} e^{itH} \, e^{-itH_0}. \tag{1.3}$$

It is proven in [34] that the limits in (1.3) exit in the strong topology in L^2 and that Range$W_\pm = \mathcal{H}_c$, the subspace of continuity of H. The adjoints of the $W_\pm$ are given by

$$W^*_\pm = s - \lim_{t \to \pm\infty} e^{itH_0} \, e^{-itH} P_c, \tag{1.4}$$

where P_c denotes the orthogonal projector onto $\mathcal{H}_c$. For any pair u, v of solutions to the stationary Schrödinger equation:

$$-\frac{d^2}{dx^2}u + Vu = k^2 u,\ k \in \mathrm{C}, \tag{1.5}$$

let $[u, v]$ denotes the Wronskian of u, v:

$$[u, v] := \left(\frac{d}{dx}u\right) v - u\frac{d}{dx}v. \tag{1.6}$$

Let $f_j(x, k), j = 1, 2, \Im k \geq 0$, be the Jost solutions to (1.5) (see [11], [12], [8], [7] and [44]). A potential V is said to be *generic* if $[f_1(x, 0), f_2(x, 0)] \neq 0$ and V is said to be *exceptional* if $[f_1(x, 0), f_2(x, 0)] = 0$. If V is *exceptional* there is a bounded solution to (1.5) with $k^2 = 0$ (a half-bound state or a zero-energy resonance). For these definitions and related issues see [25]. Note that the trivial potential, $V = 0$, is *exceptional*. We denote: $V^{(l)} := \frac{d^l}{dx^l}V(x)$. Clearly, $V^{(0)} = V$. We proved in [45] the following theorem.

THEOREM 1.1. *Suppose that* $V \in L^1_\gamma$, *where in the* generic case $\gamma > 3/2$ *and in the* exceptional case $\gamma > 5/2$, *and that for some* $k = 1, 2, \cdots$, $V^{(l)} \in L^1$, *for* $l = 0, 1, 2, \cdots, k-1$. *Then* $W_\pm$ *and* $W_\pm^*$ *originally defined on* $W_{k,p} \cap L^2, 1 \leq p \leq \infty$, *have extensions to bounded operators on* $W_{k,p}$, $1 < p < \infty$. *Moreover, there are constants* $C_p, 1 < p < \infty$, *such that:*

$$\|W_\pm f\|_{k,p} \leq C_p \|f\|_{k,p};\ \left\|W_\pm^* f\right\|_{k,p} \leq C_p \|f\|_{k,p}, f \in W_{k,p} \cap L^2, 1 < p < \infty. \tag{1.7}$$

Furthermore, if V *is* exceptional *and* $\lim_{x \to -\infty} f_1(x, 0) = 1$, $W_\pm$ *and* $W_\pm^*$ *have extensions to bounded operators on* $W_{k,1}$ *and to bounded operators on* $W_{k,\infty}$, *and there are constants* C_1 *and* C_∞ *such that (1.7) holds for* $p = 1$ *and* $p = \infty$.

An important property of the wave operators is that they intertwine between the continuous part of H and H_0, i.e., $H\,P_c = W_\pm\, H_0\, W_\pm^*$ and $H_0 = W_\pm^*\, H P_c\, W_\pm$. It follows that for any Borel function$f$:

$$f(H)\, P_c = W_\pm\, f(H_0)\, W_\pm^*;\ f(H_0) = W_\pm^* f(H)\, P_c\, W_\pm, \tag{1.8}$$

where $f(H)$ and $f(H_0)$ are defined by functional calculus. For X, Y, Banach spaces, we denote by $\mathcal{B}(X, Y)$ the space of all bounded operators from X into Y and by $\mathcal{B}(X)$ the space of all bounded operators on X.

Theorem 1.1 and equations (1.8) imply that $f(H_0)$ and $f(H)P_c$ have equivalent operator norms from $W_{l,p}$ into $W_{\acute{l},\acute{p}}, 0 \leq l, \acute{l} \leq k, 1 < p, \acute{p} < \infty$. More precisely,

COROLLARY 1.2. *Suppose that the assumptions of Theorem 1.1 are satisfied. Then for any* $0 \leq l, \acute{l} \leq k, 1 < p, \acute{p} < \infty$, *there is a constant* C *such that for all Borel functions* f

$$\frac{1}{C}\left\|f(H_0)\right\|_{\mathcal{B}\left(W_{l,p},W_{\acute{l},\acute{p}}\right)} \leq \left\|f(H)P_c\right\|_{\mathcal{B}\left(W_{l,p},W_{\acute{l},\acute{p}}\right)} \leq C\left\|f(H_0)\right\|_{\mathcal{B}\left(W_{l,p},W_{\acute{l},\acute{p}}\right)}. \tag{1.9}$$

Moreover, if V *is* exceptional *and* $\lim_{x\to-\infty} f_1(x,0) = 1$, *we can take* $1 \leq p, \acute{p} \leq \infty$.

We denote: $u_t := \frac{\partial}{\partial t}u$, $u_{tt} := \frac{\partial^2}{\partial t^2}u$, $u_x := \frac{\partial}{\partial x}u$, $u_{xx} := \frac{\partial^2}{\partial x^2}u$. We consider the solutions to the following linear Klein-Gordon equation:

$$u_{tt} - u_{xx} + u + Vu = 0;\ u(x,0) = 0;\ u_t(x,0) = f(x), \tag{1.10}$$

where $u := u(x,t), x,t \in \mathbf{R}$. We denote by T_t the operator $f \hookrightarrow u(\cdot,t)$ and by $\acute{T}_t := \frac{d}{dt}T_t$ (see (3.4) in Section 3). Let $P_j, j = 0,1,\cdots,6$ be the following points in $\mathbf{R}^2$:

$$P_0 = (1/2,1/2);\ P_1 = (1,0);\ P_2 = (0,0);\ P_3 = (1,1);\ P_4 = (1/2,0);\ P_5 = (1,1/2);\ P_6 = (5/6,1/6). \tag{1.11}$$

Our $L^p - L^{\acute{p}}$ estimate is the following theorem.

THEOREM 1.3. *Suppose that* $V \in L^1_\gamma$, *where in the* generic case $\gamma > 3/2$ *and in the* exceptional case $\gamma > 5/2$, *and that for some* $k = 1,2,\cdots$, $V^{(l)} \in L^1$, *for* $l = 0,1,2,\cdots,k-1$. *Then* T_tP_c *and* $\acute{T}_t\,(I+H)^{-1/2}P_c$ *are bounded operators on* $\mathcal{B}(W_{k,p},W_{k,q})$ *if* $(1/p,1/q)$ *belongs either to the open triangle with vertices* P_1,P_2,P_3 *or to the open segment* (P_2,P_3). *Furthermore, if* V *is* exceptional *and* $\lim_{x\to-\infty} f_1(x,0) = 1$, *then* T_tP_c *is a bounded operator on* $\mathcal{B}(W_{k,p},W_{k,q})$ *if* $(1/p,1/q)$ *belongs to the closed triangle with vertices* P_1,P_2,P_3 *and* $\acute{T}_tP_c(I+H)^{-1/2}$ *is a bounded operator on* $\mathcal{B}(W_{k,p},W_{k,q})$ *if* $(1/p,1/q)$ *belongs to the closed triangle with vertices* P_1,P_2,P_3 *with the exception of* P_1. *Moreover, in the cases above the following estimates hold*

$$\left\|T_tP_c\right\|_{\mathcal{B}(W_{k,p},W_{k,q})} \leq C|t|^a,\ \left\|\acute{T}_t\,(I+H)^{-1/2}\,P_c\right\|_{\mathcal{B}(W_{k,p},W_{k,q})} \leq C|t|^a,\ |t| \geq 1, \tag{1.12}$$

where a *is the following piecewise linear function of* $(1/p,1/q)$:

$$a := 2\,(\frac{1}{p} - 1)\ \textit{in the triangle}\ P_1,\ P_5,\ P_6;\ a := -\frac{2}{q}\ \textit{in the triangle}\,P_1,\ P_4,\ P_6, \tag{1.13}$$

$$a := \frac{1}{q} - \frac{1}{2}\ \textit{in the quadrilateral}\ P_0,P_3,P_5,P_6;\ a := \frac{1}{2} - \frac{1}{p}\ \textit{in the quadrilateral}\ P_0,P_2,P_4,P_6. \tag{1.14}$$

Furthermore,

$$\|T_t P_c\|_{\mathcal{B}(W_{k,p},W_{k,q})} \leq C|t|^b, \text{ where } b := 1 - \frac{1}{p} + \frac{1}{q}, \ \left\|\acute{T}_t P_c (I+H)^{-1/2}\right\|_{\mathcal{B}(W_{k,p},W_{k,q})} \leq C; \ \ |t| \leq 1. \tag{1.15}$$

Theorem 1.3 for T_t follows from Theorem 1.1, Corollary 1.2 and from the correponding results in the case $V = 0$ [19]. In Lemma 2.1 in Section 2 we briefly give the proof of these results for T_t with $V = 0$ because in [19] some of the proofs and results are only given for $n \geq 3$. Moreover, our points P_4 and P_5 are different from the ones in [19] . For this reason, our results are different from the ones in [19]. The operator $\acute{T}_t$ with $V = 0$ is not estimated in [19]. In Lemma 2.2 in Section 2 we obtain the necessary estimates along the lines of the proof of [19] for T_t.

Let us consider the following nonlinear Klein-Gordon equation with a potential

$$u_{tt} - u_{xx} + Vu + u + F(u) = 0, \ u(x,0) = f_1, \ u_t(x,0) = f_2, \tag{1.16}$$

where $u(x,t)$ is a complex-valued funcion of $x, t \in \mathbf{R}$. F is a function from $\mathbf{C}$ into $\mathbf{C}$. As usual, we say that F is a C^2 function in real sense if $\Re F(u)$ and $\Im F(u)$ have continuous derivatives up to the second order with respect to the real and the imaginary parts of u. Moreover, if $u = r+is, r, s \in \mathbf{R}, F = F_1+iF_2$, with F_1, F_2 real-valued functions, we denote

$$F^{(2)}(u) := \sum_{j=1}^{2} \left[\left|\frac{\partial^2}{\partial r^2} F_j(u)\right| + \left|\frac{\partial^2}{\partial r \partial s} F_j(u)\right| + \left|\frac{\partial^2}{\partial s^2} F_j(u)\right| \right]. \tag{1.17}$$

We write (1.16) as a first order system in the usual way:

$$\frac{\partial}{\partial t} U = AU + P(U), \text{ where } U := \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}; \ A := \begin{bmatrix} 0 & 1 \\ \frac{d^2}{dx^2} - V - 1 & 0 \end{bmatrix}; \ P(U) := \begin{bmatrix} 0 \\ -F(u_1) \end{bmatrix}, \ U(x,0) = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}. \tag{1.18}$$

The equation (1.16) is equivalent to (1.18) if we take $u = u_1$, and $u_t = u_2$. Let us define the following spaces:

$$\mathcal{H} := W_{1,2} \oplus L^2 \text{ (the energy space)}, \ X := W_{1,p+1} \oplus L^{p+1}, p \geq 1, \tag{1.19}$$

and

$$M := \left\{ U \in C(\mathbf{R}, X) : \sup_{t \in \mathbf{R}} (1+|t|)^d \|U\|_X < \infty \right\}, \ \text{ with norm: } \|U\|_M := \sup_{t \in \mathbf{R}} (1+|t|)^d \|U\|_X, \tag{1.20}$$

where if $1 \leq p \leq 5$, $d := \frac{1}{2}\frac{p-1}{p+1}$ and if $5 \leq p < \infty$, $d := \frac{2}{p+1}$. Finally, we define

$$N := W_{2,1+\frac{1}{p}} \oplus W_{1,1+\frac{1}{p}}. \tag{1.21}$$

Note that by Sobolev's imbedding theorem [2], $N \subset \mathcal{H}$ with a continuous imbedding. For functions $U(x,t)$ defined on $\mathbf{R} \times \mathbf{R}$ we write $U(t)$ for $U(\cdot, t)$. In what follows we will always assume that H has no negative eigenvalues. Let us denote by $\mathcal{V}(t)$ the operator that gives the solution to the linear initial value problem (1.18) with $P = 0$, i.e., $U(t) = \mathcal{V}(t)U(0)$ (see (3.4) in Section 3). The operators $\mathcal{V}(t), t \in \mathbf{R}$, form a strongly-continuous group on $\mathcal{H}$ that is uniformly bounded in t (see Section 3). In the following theorem we construct the scattering operator.

THEOREM 1.4. *Suppose that $V \in L^1_\gamma$, where in the* generic case $\gamma > 3/2$ *and in the* exceptional case $\gamma > 5/2$, *that $V^{(1)} \in L^1$ and that H has no negative eigenvalues. Moreover, assume that F is C^2 in the real sense, that $F(0) = 0$, that all the first order derivatives in the real sense of F vanish at zero and that*

$$F^{(2)}(u) = O(|u|^{p-2}), u \to 0, \textit{ for some } \rho < p < \infty, \textit{ where } \rho \textit{ is the positive root of } \frac{1}{2}\frac{\rho-1}{\rho+1} = \frac{1}{\rho}. \tag{1.22}$$

Then, there is a $\delta > 0$ such that for all $\phi_- \in N$ with $\|\phi_-\|_N \leq \delta$ there is a unique solution, U, to (1.18) such that $U \in C(\mathbf{R}, \mathcal{H}) \cap M$ and

$$\lim_{t \to -\infty} \|U(t) - \mathcal{V}(t)\phi_-\|_{\mathcal{H}} = 0. \tag{1.23}$$

Moreover, there is a unique $\phi_+ \in \mathcal{H}$ such that

$$\lim_{t \to \infty} \|U(t) - \mathcal{V}(t)\phi_+\|_{\mathcal{H}} = 0. \tag{1.24}$$

Furthermore, $\mathcal{V}(t)\phi_\pm \in M$ and

$$\|U - \mathcal{V}(t)\phi_\pm\|_M \leq C \, \|\mathcal{V}(t)\phi_\pm\|_M^p \,, \tag{1.25}$$

and

$$\|\phi_+ - \phi_-\|_{\mathcal{H}} \leq C \, \|\phi_-\|_N^p \,. \tag{1.26}$$

The scattering operator, $S_V : \phi_- \hookrightarrow \phi_+$ is injective on N.

Note that $\rho \approx 3.56$. Moreover, observe that in Theorem 1.4 we do not need to restric F in such a way that energy is conserved.

Let us denote $B_1 := \sqrt{H+1}$ and $B_0 := \sqrt{H_0+1}$. Under the conditions of Theorem 1.4 the wave operators

$$W_{1,\pm} := s - \lim_{t \to \pm\infty} e^{itB_1}\, e^{-itB_0} \tag{1.27}$$

exit and are unitary operators on L^2. The limits in the right-hand side of (1.27) are taken in the strong topology of L^2. Moreover, the invariance principle holds and

$$W_{1,\pm} = W_{\pm}. \tag{1.28}$$

For these results see Theorem 10.4.1 in page 200 and Theorem 15.5.3 in page 295 of [34]. We denote by $\mathcal{V}_0(t)$ the solution operator for the initial value problem (1.18) with $F=0$ and $V=0$. It is given by (3.4) in Section 3 with H_0 instead of H. Let us denote by J the identification operator from $\mathcal{H}$ onto $\mathcal{H}_V$ (see (3.3)): $JU := U$. The wave operators for the linear Klein-Gordon equation with a potential are defined as

$$\Omega_{\pm} := s - \lim_{t \to \pm\infty} \mathcal{V}(-t)\mathcal{V}_0(t), \tag{1.29}$$

provided that the limits in the right-hand side of (1.29) exist in the strong topology in $\mathcal{H}_V$. It follows from Theorems XI.75 and XI.76 in [32] that $\Omega_{\pm}$ exist and that they are unitary from $\mathcal{H}$ onto $\mathcal{H}_V$. Moreover, by (1.28), the proof of Theorem XI.75 and Theorem XI.76 of [32]:

$$\Omega_{\pm} = T^{-1} \begin{bmatrix} W_{\pm} & 0 \\ 0 & W_{\mp} \end{bmatrix}; \; T := \frac{1}{\sqrt{2}} \begin{bmatrix} B_1 & i \\ B_1 & -i \end{bmatrix}. \tag{1.30}$$

The scattering operator for the linear Klein-Gordon equation with a potential is defined as:

$$S_L := \Omega_+^{-1}\, \Omega_-. \tag{1.31}$$

By (1.30),

$$S_L = \begin{bmatrix} \mathcal{S}_L & 0 \\ 0 & \mathcal{S}_L^* \end{bmatrix}, \tag{1.32}$$

where $\mathcal{S}_L$ is the scattering operator for the linear Schrödinger equation on the line:

$$\mathcal{S}_L := W_+^* W_-. \tag{1.33}$$

It is a consequence of Theorem 1.1, (1.32) and (1.33) that S_L is bounded on X and on $\mathcal{H}$. To reconstruct the potential V we introduce below the scattering operator that relates asymptotic states that are solutions to the linear Klein-Gordon equation with potential zero:

$$S := \Omega_+^{-1}\, S_V\, \Omega_-. \tag{1.34}$$

In the following Theorem we reconstruct S_L from S.

THEOREM 1.5. *Suppose that the assumptions of Theorem 1.4 are satisfied. Then for every $\phi \in N$,*

$$\frac{d}{d\epsilon} S(\epsilon\phi)\Big|_{\epsilon=0} = S_L \phi, \tag{1.35}$$

where the derivative in the left-hand side of (1.35) exists in the strong convergence in $\mathcal{H}$.

COROLLARY 1.6. *Under the conditions of Theorem 1.4 the scattering operator, S, determines uniquely the potential V.*

Proof: By Theorem 1.5 S determines uniquely S_L. Then, from (1.32) we obtain $\mathcal{S}_L$. From $\mathcal{S}_L$ we get the reflection coefficients for linear Schrödinger scattering on the line (see Section 9.7 of [26] and [44]). As H has no bound states we uniquely reconstruct V from one of the reflection coefficients by using any of the standard methods (see for example [11], [12], [8], [20], [7], [15]).

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In the case where $F(u) = F_1(|u|^2)\, u$ with F_1 analytic on a neighborhood of zero we can also reconstruct F.

LEMMA 1.7. *Suppose that the conditions of Theorem 1.4 are satisfied and that $F(u) = F_1(|u|^2)u$ with $F_1(u)$ analytic in a neighborhood of zero. Then, if $F_1(u) = \sum_{j=j_0}^{\infty} \lambda_j\, u^j$, $j_0 \geq (p-1)/2$, $|u| \leq \eta$, for some $\eta > 0$, we have that for any $\phi \in W_{1,1+1/p}$ there is an $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$:*

$$((S_V - I)(\epsilon U_0), V_0)_{\mathcal{H}_V} = \sum_{j=j_0}^{\infty} \epsilon^{2j+1} \left[\lambda_j \int_{-\infty}^{\infty} dt\ \|(\mathcal{V}(t)U_0)_1\|_{L^{2j+2}}^{2j+2} + Q_j\right], \text{ with} \tag{1.36}$$

$$U_0 := \begin{bmatrix} 0 \\ \phi \end{bmatrix}, V_0 := \begin{bmatrix} (H+I)^{-1}\phi \\ 0 \end{bmatrix}, \tag{1.37}$$

where $Q_{j_0} = 0$, and $Q_j, j > j_0$, depends only on ϕ ,and on λ_k with $k < j$.

As U_0 in (1.37) belongs to N and $d(p+1) > 1$ it follows from Theorem 1.3 that:

$$\int_{-\infty}^{\infty} \|(\mathcal{V}(t)U_0)_1\|_{L^{p+1}}^{p+1}\ dt < \infty. \tag{1.38}$$

Moreover, by Sobolev's imbedding theorem [2]

$$\left\|(\mathcal{V}(t)U_0)_1\right\|_{L^\infty} \leq C\left\|\mathcal{V}(t)U_0\right\|_{\mathcal{H}_V} = C\left\|U_0\right\|_{\mathcal{H}_V} = C\left\|\phi\right\|_{L^2} \leq C\left\|\phi\right\|_{W_{1,1+\frac{1}{p}}}. \tag{1.39}$$

Then for $j \geq j_0$:

$$\left\|(\mathcal{V}(t)U_0)_1\right\|_{2j+2}^{2j+2} \leq C\left\|(\mathcal{V}(t)U_0)_1\right\|_{L^{p+1}}^{p+1}, \tag{1.40}$$

and it follows that,

$$\int_{-\infty}^{\infty}\left\|(\mathcal{V}(t)U_0)_1\right\|_{L^{2j+2}}^{2j+2}\, dt < \infty, \text{ for} j \geq j_0. \tag{1.41}$$

COROLLARY 1.8. *Under the conditions of Lemma 1.7 the scattering operator, S, determines uniquely the potential, V, and the nonlinearity, F.*

Proof : By Corollary 1.6 S determines uniquely V. Then, the $\Omega_\pm$ are uniquely determined, and by (1.34) S_V is uniquely determined by S. Finally, by (1.36) each one of the λ_j is uniquely determined.

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Remark that we have also obtained a method for the reconstruction of V and of F. First using (1.32) and (1.35) we reconstruct $\mathcal{S}_L$ from S. Then, using any standard method for inverse Schrödinger scattering on the line we reconstruct V from $\mathcal{S}_L$. Using (1.30) and (1.34) we reconstruct S_V from S and $\Omega_\pm$. Finally, we reconstruct recursively each λ_j from (1.36). Note that each coefficient Q_j can be explicitly computed.

A result as in Corollary 1.8 in three dimensions and with $V = 0$ was proven by Morawetz and Strauss in [24] and by Bachelot in [3] and [4]. They reconstruct the λ_j from a skew form. There is a large literature in direct scattering for the nonlinear Klein-Gordon equation in the case when $V = 0$. See [35], [36], [23], [39], [5], [27], [6], [28], [14], [21], [22], [40], [29], and the references quoted there.

2 The $L^p - L^{\acute{p}}$ Estimates

We first introduce some standard notation. By $W_{k,p}, k = 0, 1, \cdots, 1 \leq p \leq \infty$, we denote the Sobolev space [2] of all functions on L^p such that all its derivatives of order up to k are functions in L^p. By $\|\cdot\|_{k,p}$ we denote the norm in $W_{k,p}$. Clearly, $W_{0,p} = L^p$. The space of functions of bounded mean oscillation is designated by BMO [33], [38]. By $\mathrm{BMO}_k, k=0,1,\cdots$, we denote the space of all functions $f \in \mathrm{BMO}$ such that $\frac{d^l}{dx^l} f(x) \in \mathrm{BMO}, 0 \leq l \leq k$, with norm:

$$\|f\|_{\mathrm{BMO},k} := \sum_{l=0}^{k} \left\| \frac{d^l}{dx^l} f(x) \right\|_{\mathrm{BMO}}. \tag{2.1}$$

For k any complex number we denote by $\Re k$ and $\Im k$, respectively, the real and the imaginary parts of k. Below we denote by C any constant whose particular value is not relevant and that is independent of t.

We denote by $T_{0,t}$ the operator T_t in the case $V=0$.

LEMMA 2.1. *The operator $T_{0,t}$ is a bounded operator on $\mathcal{B}(W_{k,p}, W_{k,q}), k=0,1,\cdots$, if $(1/p, 1/q)$ belongs to the closed triangle with vertices P_1, P_2, P_3. Moreover, the estimates below hold:*

$$\|T_{0,t}\|_{\mathcal{B}(W_{k,p},W_{k,q})} \leq C|t|^a, \ |t| \geq 1, \tag{2.2}$$

where a is the following piecewise linear function of $(1/p, 1/q)$:

$$a := 2(\frac{1}{p} - 1) \text{ in the triangle } P_1,\ P_5,\ P_6;\ a := -\frac{2}{q} \text{ in the triangle } P_1,\ P_4,\ P_6, \tag{2.3}$$

$$a := \frac{1}{q} - \frac{1}{2} \text{ in the quadrilateral } P_0, P_3, P_5, P_6;\ a := \frac{1}{2} - \frac{1}{p} \text{ in the quadrilateral } P_0, P_2, P_4, P_6. \tag{2.4}$$

Furthermore,

$$\|T_{0,t}\|_{\mathcal{B}(W_{k,p},W_{k,q})} \leq C\,|t|^b,\ \ |t| \leq 1, \text{ where } b := 1 - \frac{1}{p} + \frac{1}{q}. \tag{2.5}$$

Proof: This result is essentially contained in [19]. However, since our choice of P_4 and of P_5 is different from the one in [19], our result is different from the one in of [19]. Moreover, as some of the proofs and results of [19] are only given for $n \geq 3$, we give a brief proof below for the reader's convenience. Note that since $T_{0,t}$ commutes with derivatives it is enough to prove the Lemma in the case $k=0$. By the Riez-Thorin and the Calderón-Lions interpolation theorems (see [31]) and by duality it is enough to prove the Lemma for the points P_0, P_1, P_3, P_5 and P_6. Note that (see [10] page 156) the intermediate interpolation space, $[BMO, L^p]_\theta$, between BMO and $L^p, 1 \leq p < \infty$, is L^q with $1/q = \theta p^{-1}$, $0 < \theta < 1$. Let us denote by F the Fourier transform

$$(Ff)(k) := \frac{1}{\sqrt{2\pi}} \int e^{-ikx}\, f(x)\, dx. \tag{2.6}$$

For $\Re\alpha \geq 0$ we denote

$$Q_t^{(\alpha)} := F^{-1}\hat{Q}_t^{(\alpha)}, \text{ where } \hat{Q}_t^{(\alpha)} := \frac{\sin(t(1+k^2)^{1/2})}{\sqrt{2\pi}\,(1+k^2)^{\alpha/2}}. \tag{2.7}$$

By Fourier transform:

$$T_{0,t}f = Q_t^{(1)} \star f. \tag{2.8}$$

where $\star$ denotes convolution. Since $|\hat{Q}_t^{(1)}| \leq \frac{1}{\sqrt{2\pi}} \min(1,|t|)$, equations (2.2) and (2.5)hold at P_0. If $f \in L^1$, we have that $Ff \in L^\infty$. It follows that $\hat{Q}_t^{(1)} Ff \in L^2$ and that (2.2) holds at P_5. Let us prove (2.2) at P_6. Clearly,

$$\|Q_t^{(0)} \star \|_{\mathcal{B}(L^2)} \leq C. \tag{2.9}$$

By Lemma 2 in page 422 of [19] :

$$\|Q_t^{(3/2)}\|_{L^\infty} \leq \frac{C}{\sqrt{|t|}}, \ |t| \geq 1. \tag{2.10}$$

Equation (2.10) implies that

$$\left\| Q_t^{(3/2)} \star \right\|_{\mathcal{B}(L^1,L^\infty)} \leq \frac{C}{\sqrt{|t|}}. \tag{2.11}$$

Since (see the comments at the end of page 150 and Corollary 2 of [10]) $F^{-1}(1+|k^2|)^{i\beta}F$ is bounded in BMO for $\beta \in \mathbf{R}$, with operator norm bounded by $(1+|\beta|)$, we have that,

$$\left\| Q_t^{(3/2+i\beta)} \star \right\|_{\mathcal{B}\left(L^1,\mathrm{BMO}\right)} \leq \frac{C((1+|\beta|)}{\sqrt{|t|}}, \ |t| \geq 1. \tag{2.12}$$

By (2.9), (2.12) and Calderón-Lions interpolation theorem [31], the estimate (2.2) follows at P_6.

It follows from formula (30) in page 26 of [9] that

$$Q_t^{(1)}(x) = \begin{cases} \frac{1}{2\pi^2} J_0\left((t^2-x^2)^{1/2}\right) & ,0<|x|<|t|, \\ 0 & ,|t|<|x|. \end{cases} \tag{2.13}$$

Then, by formulae 9.1.12 in page 360 and 9.2.1 in page 364 of [1],

$$\left\| Q_t^{(1)}(\cdot) \right\|_{L^\infty} \leq C. \tag{2.14}$$

Estimates (2.2) and (2.5) follow at P_1 from (2.8) and (2.14).Finally, we prove that (2.2) and (2.5) hold at P_3. It follows from (2.13) and formulae 9.1.12 and 9.2.1 of [1] that $Q_t^{(1)}(\cdot)$ is integrable and that its L^1 norm satisfies the required estimates.

$\blacksquare$

We denote by $\acute{T}_{0,t}$ the operator $f :\to u_t(\cdot, t)$, where $u(x,t)$ is the solution of the initial valued problem (1.10) with $V = 0$. In the following Lemma we obtain our $L^p - L^{\acute{p}}$ estimate for $\acute{T}_{0,t}\,(I + H_0)^{-1/2}$. Note that we prove that $\acute{T}_{0,t}\,(I + H_0)^{-1/2} \in \mathcal{B}\,(W_{k,p}, W_{k,q})$ for the same values of (p,q) that were obtained in [19] for T_t in the case $V = 0$, with the exception of the point P_1 where we have that $\acute{T}_{0,t}\,(I + H_0)^{-1/2} \in \mathcal{B}(W_{k,1}, \mathrm{BMO}_k)$, whereas for $V = 0$, $T_t \in \mathcal{B}(W_{k,1}, W_{k,\infty})$.

LEMMA 2.2. *The operator $\acute{T}_{0,t}(I + H_0)^{-1/2}$ is a bounded operator on $\mathcal{B}(W_{k,p}, W_{k,q}), k = 0, 1, \cdots,$ if $(1/p, 1/q)$ belongs to the closed triangle with vertices P_1, P_2, P_3 with the exception of P_1, where it is a bounded operator on $\mathcal{B}(W_{k,1}, BMO_k)$. Moreover, the estimates below hold, where in the case of P_1 we replace the norm in $\mathcal{B}(W_{k,1}, W_{k,\infty})$ by the norm in $\mathcal{B}(W_{k,1}, BMO_k)$.*

$$\left\| \acute{T}_{0,t}(I + H_0)^{-1/2} \right\|_{\mathcal{B}(W_{k,p}, W_{k,q})} \leq C|t|^a, \ |t| \geq 1, \tag{2.15}$$

where a is the following piecewise linear function of $(1/p, 1/q)$:

$$a := 2(\frac{1}{p} - 1) \text{ in the triangle } P_1,\ P_5,\ P_6;\ a := -\frac{2}{q} \text{ in the triangle } P_1,\ P_4,\ P_6, \tag{2.16}$$

$$a := \frac{1}{q} - \frac{1}{2} \text{ in the quadrilateral } P_0, P_3, P_5, P_6;\ a := \frac{1}{2} - \frac{1}{p} \text{in the quadrilateral } P_0, P_2, P_4, P_6. \tag{2.17}$$

Furthermore,

$$\left\| \acute{T}_{0,t}(I + H_0)^{-1/2} \right\|_{\mathcal{B}(W_{k,p}, W_{k,q})} \leq C, \ |t| \leq 1. \tag{2.18}$$

Proof: We follow the proof given in [19] in the case of T_t with $V = 0$. As in Lemma 2.1 it is enough to prove the results for $k = 0$ and for the points P_0, P_1, P_3, P_5, and P_6. For $\Re\alpha \geq 0$ we denote:

$$K_t^{(\alpha)} := F^{-1}\,\hat{K}_t^{(\alpha)}; \text{ where } \hat{K}_t^{(\alpha)} := \frac{\cos(t(1 + |k|^2)^{1/2})}{\sqrt{2\pi}(1 + |k|^2)^{(\alpha/2)}}. \tag{2.19}$$

Then, by Fourier transform:

$$\acute{T}_t(I + H_0)^{-1/2} f = K_t^{(1)} \star f. \tag{2.20}$$

Since, $\left|\hat{K}_t^{(1)}\right| \leq C$, (2.15) and (2.18) hold at P_0. Let us prove (2.15) and (2.18) at P_5. If $f \in L^1$, we have that $Ff \in L^\infty$. Then, $\hat{K}_t^{(1)}\,Ff(k) \in L^2$, and (2.15) and (2.18) hold at P_5. We now prove (2.15) at P_6. Clearly,

$$\| K_t^{(0)} \star \|_{\mathcal{B}(L^2)} \leq C. \tag{2.21}$$

It follows from the proof of Lemma 2 in [19] that:

$$\|K_t^{(3/2)}\|_{L^\infty} \leq \frac{C}{\sqrt{|t|}}, \ |t| \geq 1. \tag{2.22}$$

Equation (2.22) implies that

$$\left\| K_t^{(3/2)} \star \right\|_{\mathcal{B}(L^1, L^\infty)} \leq \frac{C}{\sqrt{|t|}}. \tag{2.23}$$

Since (see the comments at the end of page 150 and Corollary 2 of [10]) $F^{-1}(1+|k^2|)^{i\beta}F$ is bounded in BMO for $\beta \in \mathbf{R}$, with operator norm bounded by $(1+|\beta|)$, we have that,

$$\left\| K_t^{(3/2+i\beta)} \star \right\|_{\mathcal{B}(L^1, \mathrm{BMO})} \leq \frac{C((1+|\beta|)}{\sqrt{|t|}}, \ |t| \geq 1. \tag{2.24}$$

By (2.21), (2.24) and Calderón-Lions interpolation theorem [31], the estimate (2.15) follows at P_6. Estimates (2.15) and (2.18) at P_1 follow from the estimate

$$\|K_t^{(1)}(\cdot)\|_{\mathrm{BMO}} \leq C. \tag{2.25}$$

We now prove (2.25). By formula (34) in page 26 of [9]

$$K_t^{(1)}(x) = \frac{1}{\pi} \begin{cases} \frac{-\pi}{2} Y_0\left((t^2-x^2)^{1/2}\right) & , 0 < |x| < |t|, \\ K_0((x^2-t^2)^{1/2}) & , |t| < |x|. \end{cases} \tag{2.26}$$

We denote: $D_t := \{x \in \mathbf{R} : |x^2 - t^2| \leq 1/2\}$ and by χ_t the characteristic function of D_t. Then, by formula (9.2.2) in page 364 and formula (9.7.2) in page 378 of [1]

$$\|(1-\chi_t(x))\, K_t^{(1)}(x)\|_{L^\infty} \leq C. \tag{2.27}$$

Moreover, by formulae (9.1.13) in page 360 and (9.6.13) in page 375 of [1]:

$$\chi_t K_t^{(1)}(x) = \frac{-1}{\pi} \chi_t(x) \ln(|t^2-x^2|^{1/2}) + C_t(x), \text{with } \|C_t(\cdot)\|_{L^\infty} \leq C. \tag{2.28}$$

We will prove that $f_t(x) := \chi_t(x) \ln|t^2-x^2| \in \mathrm{BMO}$. For every open segment, Q, we have that:

$$\{x \in Q : |f_t(x)| > \alpha\} \subset \left\{x \in Q : |t^2-x^2| \leq \min(1/2, e^{-\alpha})\right\}. \tag{2.29}$$

Then, denoting by $m(\cdot)$ the Lebesgue measure, we have that

$$\nu_Q(\alpha) := m(x \in Q : |f_t(x)| > \alpha) \leq C \frac{e^{-\alpha}}{|t|} \leq C\, m(Q)\, e^{-\alpha}, \text{ if } m(Q) \geq \frac{1}{N|t|}, \tag{2.30}$$

$N > 0$. It follows that,

$$\int_Q |f_t(x)| = \int_0^\infty \nu_Q(\alpha) d\alpha \leq Cm(Q), \text{ if } m(Q) \geq \frac{1}{N|t|}. \tag{2.31}$$

For $Q \subset D_t$ we denote

$$f_{t,Q} := \frac{1}{m(Q)} \int_Q f_t(x)\, dx = \frac{1}{m(Q)} \int_Q \ln(|x^2 - t^2|)\, dx. \tag{2.32}$$

Hence,

$$\int_Q |f_t(x) - f_{t,Q}|\, dx = \int_Q |\ln(|x^2 - t^2|) - f_{t,Q}|\, dx \leq Cm(Q), \text{ if } Q \subset D_t, \tag{2.33}$$

where we used that $\ln(|x^2 - t^2|) = \ln(|x - t|) + \ln(|x + t|)$, that $\ln(|x \pm t|)$ are in BMO (see [17]) and that the BMO norm is invariant under translation. Note that we can take N so large that for all $|t| \geq 1$ and some α_0:

$$\left\{x \in \mathbf{R} : |x^2 - t^2| \leq \frac{1}{e^\alpha}\right\} \cap Q = \emptyset \text{ if } Q \not\subset D_t,\ m(Q) \leq \frac{1}{N|t|} \text{ and } \alpha \geq \alpha_0. \tag{2.34}$$

Then,

$$\nu_Q(\alpha) = 0, \text{ if } Q \not\subset D_t,\ m(Q) \leq \frac{1}{N|t|},\ |t| \geq 1 \text{ and } \alpha \geq \alpha_0. \tag{2.35}$$

It follows that

$$\int_Q |f_t(x)|\, dx = \int_0^\infty \nu_Q(\alpha)\, d\nu \leq Cm(Q), \text{ if } Q \not\subset D_t,\ m(Q) \leq \frac{1}{N|t|},\ |t| \geq 1, \tag{2.36}$$

where we used that $\nu_Q(\alpha) \leq m(Q)$. By (2.27), (2.28),(2.31), (2.33), (2.36) and Lemma 3.3 of [33], (2.25) holds for $|t| \geq 1$. To prove (2.25) for $|t| \leq 1$ we decompose f_t as follows:

$$f_t = f_{t,+} + f_{t,-}; \text{ with } f_{t,\pm}(x) := \chi_t(x) \ln|x \mp |t||. \tag{2.37}$$

We estimate $f_{t,\pm}$ adapting to this case the argument given in [17] for $\ln|x - y|$. We give the proof for $f_{t,+}$. Let us denote by Ξ_ϵ the characteristic function of $E_\epsilon := \{x : |x - |t|| \leq \epsilon\}$, where $\epsilon > 0$ is so small that $E_\epsilon \subset D_t$ for all $|t| \leq 1$.Then

$$f_{t,+} = g_{t,1} + g_{t,2}, \text{ where } g_{t,1} = \Xi_\epsilon f_{t,+} \text{ and } g_{t,2} := (1 - \Xi) f_{t,+}. \tag{2.38}$$

Clearly,

$$\|g_{t,2}\|_{L^\infty} \leq C, \text{ for } |t| \leq 1. \tag{2.39}$$

Suppose that $Q \cap E_\epsilon$ is not empty. We designate:

$$a_Q := \ln |\xi - |t||, \text{ where } \xi \text{ is a solution to } |\xi - |t|| = \max_{x \in Q \cap E_\epsilon} |x - |t||. \tag{2.40}$$

It follows from the argument given in [17] that

$$\int_{Q \cap E_\epsilon} |g_{t,1} - a_Q| \leq C m(Q), \text{if} \, Q \cap E_\epsilon \neq \emptyset. \tag{2.41}$$

Furthermore, if $Q \not\subset E_\epsilon$, we have that $|a_Q| \leq C$, for a constant C independent of Q and of $|t| \leq 1$. Then, we have that

$$\int_{Q \cap \tilde{E}_\epsilon} |g_{t,1} - a_Q| \, dx \leq a_Q \, m(Q) \leq C \, m(Q), \text{ if} \, Q \not\subset E_\epsilon, \text{ and} \, Q \cap E_\epsilon \neq \emptyset, \tag{2.42}$$

where $\tilde{E}_\epsilon$ denotes the complement of E_ϵ. Since,

$$\int_Q |g_{t,1}| = 0, \text{ if} \, Q \cap E_\epsilon = \emptyset, \tag{2.43}$$

we have proven that

$$\|g_{t,1}\|_{\text{BMO}} \leq C, \, |t| \leq 1, \tag{2.44}$$

and (2.38), (2.39) imply that

$$\|f_{t,+}\|_{\text{BMO}} \leq C, \, |t| \leq 1. \tag{2.45}$$

Then, by (2.27), (2.28) equation (2.25)holds for $|t| \leq 1$.

It only remains to prove that (2.15) and (2.18) hold at P_3. But it follows from (2.26) and formulas (9.1.13) in page 360, (9.2.2) in page 364, (9.6.13) in page 375 and (9.7.2) in page 378 of [1] that $K_t^{(1)}(\cdot)$ is integrable and that its L^1 norm satisfies the required estimates.

■

Proof of Theorem 1.3: The Theorem follows from Theorem 1.1, Corollary 1.2, Lemma 2.1 and Lemma 2.2.

3 Scattering

In this Section we will always assume that $V \in L^1_\gamma$, where in the *generic case* $\gamma > 3/2$ and in the *exceptional case* $\gamma > 5/2$, that $V^{(1)} \in L^1$, and that that H has no eigenvalues. It follows that $H > 0$.

By Theorem 3 in page 135 of [37],

$$\|F^{-1}(1+q^2)^{k/2}(Ff)(q)\|_{L^p}, \tag{3.1}$$

is a norm that is equivalent to the norm of $W_{k,p}, 1 < p < \infty$. Then, by Corollary 1.2,

$$\|(I+H)^{k/2} f\|_{L^p}, \tag{3.2}$$

defines a norm that is equivalent to the norm of $W_{k,p}, k = 1, 2, 1 < p < \infty$. Below we will use this equivalence without further comments. We denote by $\mathcal{H}_V$ the space $\mathcal{H}$ endowed with the scalar product

$$(U, D)_{\mathcal{H}_V} := (\sqrt{I+H}U_1, \sqrt{I+H}D_1)_{L^2} + (U_2, D_2)_{L^2}. \tag{3.3}$$

The norm defined by (3.3) is equivalent to the norm of $\mathcal{H}$. The solution operator, $\mathcal{V}(t)$, to the initial value problem (1.18) with $F = 0$ is given by

$$\mathcal{V}(t) \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} \cos(t\sqrt{I+H}) f_1 + \frac{\sin(t\sqrt{I+H})}{\sqrt{I+H}} f_2 \\ \\ -\sin(t\sqrt{I+H})\sqrt{I+H} f_1 + \cos(t\sqrt{I+H}) f_2 \end{bmatrix}. \tag{3.4}$$

The $\mathcal{V}(t)$ are a strongly-continuous unitary group of operators on $\mathcal{H}_V$. It follows that they are uniformly bounded on $\mathcal{H}$ for $t \in \mathbf{R}$.

Proof of Theorem 1.4: As by Sobolev imbedding theorem L^∞ is continuously imbedded in $W_{1,2}$, P is locally Lipschitz continuous on $\mathcal{H}$. It follows from standard arguments that $U \in C(\mathbf{R}, \mathcal{H}) \cap M$ is a solution to (1.18) with $\lim_{t\to-\infty} \|U(t) - \mathcal{V}(t)\phi\|_{\mathcal{H}} = 0$, for some $\phi \in \mathcal{H}$, if and only if U is a solution to the following integral equation

$$U = \mathcal{V}(t)\phi + \int_{-\infty}^{t} \mathcal{V}(t-\tau) P(U(\tau))\, d\tau. \tag{3.5}$$

As we will see below the integral in the right-hand side of (3.5) converges absolutely in $\mathcal{H}$ and in M. For $U \in M$ we denote,

$$\mathcal{Q}U(t) := \int_{-\infty}^{t} \mathcal{V}(t-\tau) P(U(\tau))\, d\tau. \tag{3.6}$$

It follows from Theorem 1.3, (1.22), from the generalized Young inequality (see [31]) and since L^∞ is continuously imbedded in $W_{1,p+1}$ that

$$\|\mathcal{Q}U(t) - \mathcal{Q}D(t)\|_X \leq C\,(1+|t|)^{-d} (\|U\|_M + \|D\|_M)^{p-1} \|U - D\|_M, \tag{3.7}$$

where we used that $p\,d > 1$. The constants C in (3.7) can be taken uniform in closed balls in M. Then, by (3.7) with $D = 0$ and since $\mathcal{V}(t)$ is unitary on $\mathcal{H}_V$:

$$\|\mathcal{Q}U(t)\|^2_{\mathcal{H}_V} = 2\Re \int_{-\infty}^t d\tau\ (P(U(\tau)), \mathcal{Q}U(\tau))_{\mathcal{H}_V} \leq C \int_{-\infty}^t d\tau\ \|P(U)(\tau)\|_{L^{1+1/p}}\,(1+|\tau|)^{-d}\,\|U\|^p_M$$
$$\leq C \int_{-\infty}^t d\tau\ \|U_1\|^p_{L^{1+p}}\,(1+|\tau|)^{-d}\|U\|^p_M \leq C \int_{-\infty}^t d\tau\,(1+|\tau|)^{-d(p+1)}\,\|U\|^{2p}_M$$
$$\leq C(1+\max[0,-t])^{-(d+dp-1)}\,\|U\|^{2p}_M. \tag{3.8}$$

By (3.7) with $D = 0$, the integral in the right-hand side of (3.5) converges in M and by (3.8) it converges in $\mathcal{H}$. We first prove the uniqueness. Let U and D be two solutions to (3.5) that satisfy (1.23). Then,

$$U(t) - D(t) = \mathcal{Q}U(t) - \mathcal{Q}D(t). \tag{3.9}$$

Let us denote $U_T := \chi_{(-\infty,T)}(t)\,U(t)$, where $\chi_{(-\infty,T)}(t)$ is the characteristic function of $(-\infty,T), T \in \mathbf{R}$. Arguing as in (3.7) it follows from (3.9) that

$$\|U_T(t) - D_T(t)\|_M < 1/2\|U_T(t) - D_T(t)\|_M, \text{ for some } T \text{ negative enough.} \tag{3.10}$$

It follows that $U(t) = D(t)$ for $t \leq T$. Hence, as P is locally Lipschitz in $\mathcal{H}$, the standard uniqueness result implies that $U = D$. The uniqueness of ϕ_+ is obvious by the unitarity of $\mathcal{V}(t)$ in $\mathcal{H}_V$. By Theorem 1.3 if $\phi_- \in N$,

$$\|\mathcal{V}(t)\phi_-\|_M \leq C\|\phi_-\|_N. \tag{3.11}$$

For $R > 0$ let us denote: $M_R := \{U \in M : \|U\|_M \leq R\}$. Let us take R so small that $C(2R)^{p-1} \leq 1/2$, with C as in (3.7), and $\delta > 0$ so small that $C\delta \leq R/4$, with C as in (3.11). Then, by (3.7) and (3.11) the map $U \hookrightarrow \mathcal{V}(t)\phi_- + \mathcal{Q}P(U)$ is a contraction from M_R into M_R for all ϕ_- with $\|\phi_-\|_N \leq \delta$. By the contraction mapping theorem (see Theorem V.18 in page 151 of [30]) this map has a unique fixed point that is a solution to (3.5) in M_R, and moreover,

$$\|U\|_M \leq C\,\|\mathcal{V}(t)\phi_-\|_M + \frac{1}{2}\,\|U\|_M\,. \tag{3.12}$$

Then,

$$\|U\|_M \leq C\,\|\mathcal{V}(t)\phi_-\|_M\,. \tag{3.13}$$

Equation(1.25) for ϕ_- follows from (3.5), (3.7) with $D = 0$ and (3.13). By (3.5) and (3.8) $U \in C(\mathbf{R}, \mathcal{H})$ and (1.23) holds.

We define

$$\phi_+ = \phi_- + \int_{-\infty}^{\infty} \mathcal{V}(-\tau)P(U(\tau))\, d\tau. \tag{3.14}$$

By Theorem 1.3

$$\left\| \int_{-\infty}^{\infty} \mathcal{V}(-\tau)P(U(\tau))\, d\tau \right\|_X \leq C \int_{-\infty}^{\infty} (1+|\tau|)^{-d(p+1)} \|U\|_M^p \, d\tau \leq C \|U\|_M^p. \tag{3.15}$$

Hence, $\phi_+ \in X$. Estimating as in (3.8) we prove that $\phi_+ \in \mathcal{H}$ and that

$$\|(S_V - I)\phi_-\|_{\mathcal{H}} \leq C \, \|U\|_M^p \, . \tag{3.16}$$

Equation (1.26) follows from (3.13) and (3.16). By (3.14) and (3.5):

$$U(t) = \mathcal{V}(t)\phi_+ - \int_t^{\infty} \mathcal{V}(t-\tau)P(U(\tau))\, d\tau, \tag{3.17}$$

and we prove (1.24) estimating as in (3.8). Moreover, estimating as in (3.7) we obtain that

$$\left\| \int_t^{\infty} \mathcal{V}(t-\tau)P(U(\tau))\, d\tau \right\|_M \leq C \, \|U\|_M^p \, . \tag{3.18}$$

By (3.17) and (3.18), $\mathcal{V}(t)\phi_+ \in M$. Using again (3.17) and arguing as in the proof of (3.13) we obtain the following estimate:

$$\|U\|_M \leq C \, \|\mathcal{V}(t)\phi_+\|_M \, . \tag{3.19}$$

Equation (1.25) for ϕ_+ follows from (3.17), (3.18) and (3.19).

If $\phi_+ = S\phi_- = 0$, it follows from (3.17) that

$$U(t) = -\int_t^{\infty} \mathcal{V}(t-\tau)P(U(\tau))\, d\tau. \tag{3.20}$$

Let us denote: $U_T(t) := \chi_{(T,\infty)}(t)U(t)$. Arguing as in (3.10) it follows from (3.20) that

$$\|U_T(t)\|_M \leq \frac{1}{2} \, \|U_T(t)\|_M \, , \text{ for some } T \text{ positive enough.} \tag{3.21}$$

Hence, (3.21) implies that $U(t) = 0$ for $t \geq T$, and then by standard uniqueness in $C(\mathbf{R}, \mathcal{H})$ (recall that P is locally Lipschitz continuous on $\mathcal{H}$), $U(t) = 0$ and by (1.23) $\phi_- = 0$, proving that S_V is injective.

Proof of Theorem 1.5: As $S(0) = 0$ it is enough to to prove that

$$s - \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \left(S_V(\epsilon\phi) - \epsilon\phi\right) = 0. \tag{3.22}$$

By (3.11) and (3.13) with ϕ_- replaced by $\epsilon\phi$:

$$\|U\|_M \leq C\epsilon\|\phi\|_N. \tag{3.23}$$

Hence, by (3.14) and (3.15)

$$\left\|\frac{1}{\epsilon}\left(S_V(\epsilon\phi)-\epsilon\phi\right)\right\|_X \leq \frac{C}{\epsilon}\|U\|_M^p \leq C\epsilon^{p-1}\|\phi\|_N^p, \tag{3.24}$$

and (3.22) follows in X. We prove (3.22) in $\mathcal{H}$ estimating the integral in the right-hand side of (3.14) as in (3.8) and using (3.23).

Proof of Lemma 1.7 By the contraction mapping theorem, [30]

$$U(t) = \mathcal{V}(t)\epsilon U_0 + \sum_{n=1}^{\infty} \mathcal{Q}^n \mathcal{V}(t)\epsilon U_0. \tag{3.25}$$

Then, (1.36) follows from (3.14) and 3.25).

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