Metastability and nucleation for conservative dynamics

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Abstract

In this paper we study metastability and nucleation for the two-dimensional lattice gas with Kawasaki dynamics at low temperature and low density. Let $\beta > 0$ be the inverse temperature and let $\bar\Lambda \subset \Lambda_\beta \subset \mathbb{Z}^2$ be two finite boxes. Particles perform independent random walks on $\Lambda_\beta \setminus \bar\Lambda$ and inside $\bar\Lambda$ feel exclusion as well as a binding energy $U > 0$ with particles at neighboring sites, i.e., inside $\bar\Lambda$ the dynamics follows a Metroplis algorithm with an attractive lattice gas Hamiltonian. The initial configuration is chosen such that $\bar\Lambda$ is empty, while a total of $\rho|\Lambda_\beta|$ particles is distributed randomly over $\Lambda_\beta \setminus \bar\Lambda$. That is to say, *initially the system is in equilibrium with particle density ρ conditioned on $\bar\Lambda$ being empty.* For large β, the system in equilibrium has $\bar\Lambda$ fully occupied because of the binding energy. We consider the case where $\rho = e^{-\Delta\beta}$ for some $\Delta \in (U, 2U)$ and investigate how the transition from empty to full takes place under the dynamics. In particular, we identify the size and shape of the *critical droplet* and the time of its creation in the limit as $\beta \to \infty$ for *fixed* $\bar\Lambda$ and $\lim_{\beta\to\infty} \frac{1}{\beta} \log|\Lambda_\beta| = \infty$. In addition, we obtain some information on the typical trajectory of the system prior to the creation of the critical droplet. The choice $\Delta \in (U, 2U)$ corresponds to the situation where the critical droplet has side length $\ell_c \in (1, \infty)$, i.e., the system is metastable. The side length of $\bar\Lambda$ must be much larger than ℓ_c, but is otherwise arbitrary.

Because particles are *conserved* under Kawasaki dynamics, the analysis of metastability and nucleation is more difficult than for Ising spins under Glauber dynamics. The key point is to show that at low density the gas in $\Lambda_\beta \setminus \bar\Lambda$ can be treated as a reservoir that creates particles with rate ρ at sites on the interior boundary of $\bar\Lambda$ and annihilates particles with rate 1 at sites on the exterior boundary of $\bar\Lambda$. Once this approximation has been achieved, the problem reduces to understanding the *local metastable*

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behavior inside $\bar{\Lambda}$, and standard techniques from non-conservative dynamics can be applied. Even so, the dynamics inside $\bar{\Lambda}$ is still conservative, and this difficulty has to be handled via *local geometric arguments*. Here it turns out that the Kawasaki dynamics has its own peculiarities. For instance, rectangular droplets tend to become square through a movement of particles *along the border* of the droplet. This is different from the behavior under the Glauber dynamics, where subcritical rectangular droplets are attracted by the maximal square contained in the interior, while supercritical rectangular droplets tend to grow uniformly in all directions (at least for not too large times) without being attracted by a square.

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Nunc age, res quoniam docui non posse creari
de nilo neque item genitas ad nil revocari,
ne qua forte tamen coeptes diffidere dictis
quod nequeunt oculi rerum primordia cerni

(Lucretius, De Rerum Natura)

1 Introduction and main results

In this paper we study metastability for *conservative* (C) dynamics. In particular, we study the transition to the liquid phase of a supersaturated vapour described by the two-dimensional lattice gas with Kawasaki dynamics.

Metastability is a relevant phenomenon for thermodynamic systems close to a first-order phase transition. Suppose the system is in a pure equilibrium phase, corresponding to a point in the phase diagram close to a first-order phase transition curve. Suppose we change the thermodynamic parameters to values associated with a different equilibrium phase, corresponding to a point on the opposite side of the curve. Then, in certain experimental situations, instead of undergoing a phase transition the system can remain in the old equilibrium, far from the new equilibrium, for a long time. This unstable old equilibrium, called *metastable state*, persists until an external perturbation or a spontaneous fluctuation leads the system to the stable new equilibrium.

Examples of metastable states are supersaturated vapours and solutions, supercooled liquids, and ferromagnets with a magnetization opposite to the magnetic field.

In Section 1.1 we recall some of the main features of metastability by describing some well-known results obtained for a *non-conservative* (NC) dynamics, namely Ising spins with Glauber dynamics. In Section 1.2 we introduce a conservative model, namely the lattice gas with Kawasaki dynamics, and discuss the main differences between C and NC. In Section 1.3 we propose a simplified model, where the interaction and the exclusion only act in a *finite* box, and formulate our main theorem establishing metastable behavior for this model. In Section 1.4 we give an outline of the key ideas needed to prove this theorem, which are further developed in the remainder of the paper. In Section 1.5 we collect the basic notation that is used throughout the paper.

1.1 The non-conservative case

1.1.1 Grand-canonical ensemble. Let $\Lambda \subset \mathbb{Z}^2$ be a large finite box centered at the origin. With each site $x \in \Lambda$ we associate a spin variable $\sigma(x)$, assuming the values $+1$ or -1. With each configuration $\sigma \in \mathcal{X} = \{-1,+1\}^\Lambda$ we associate an energy

$$H(\sigma) = -\frac{J}{2} \sum_{(x,y)\in\Lambda^*} \sigma(x)\sigma(y) - \frac{h}{2}\sum_{x\in\Lambda} \sigma(x), \tag{1.1}$$

where Λ^* is the set of bonds between nearest-neighbor sites in Λ, $J > 0$ is the pair interaction, $h > 0$ is the magnetic field, and we assume periodic boundary conditions on Λ. The *grand-canonical Gibbs measure* associated with the Hamiltonian H, describing the equilibrium properties of the system, is given by

$$\mu(\sigma) = \frac{e^{-\beta H(\sigma)}}{Z} \qquad (\sigma \in \mathcal{X}), \tag{1.2}$$

where Z is the partition function

$$Z = \sum_{\sigma \in \mathcal{X}} e^{-\beta H(\sigma)} \tag{1.3}$$

and $\beta > 0$ is the inverse temperature. The qualification 'grand-canonical' is used here because h plays the role of a chemical potential and the total magnetization $\sum_{x\in\Lambda} \sigma(x)$ is not constant under μ.

It is well known that for every $J, h, \beta > 0$ in the thermodynamic limit $\Lambda \to \mathbb{Z}^2$ a unique Gibbs state with a positive magnetization exists (see e.g. Ruelle [14], Sinai [20]). We will be interested in the regime where

$$\Lambda \text{ is large but finite}, \quad h \in (0, 2J), \quad \beta \to \infty. \tag{1.4}$$

Let

$$\begin{array}{rcl} \boxplus &=& \text{the configuration with } \sigma(x) = +1 \text{ for all } x \in \Lambda \\ \boxminus &=& \text{the configuration with } \sigma(x) = -1 \text{ for all } x \in \Lambda. \end{array} \tag{1.5}$$

In the regime (1.4) the Gibbs measure will be concentrated around $\boxplus$, which is the unique ground state of H. Clearly, $\boxminus$ is only a local minimum of H, and it is therefore naturally related to metastability.

For $\ell \in \mathbb{N}$, let

$$E(\ell) = H(\sigma_{\ell\times\ell}) - H(\boxminus), \tag{1.6}$$

where $\sigma_{\ell\times\ell}$ is the configuration in which the $(+1)$-spins form an $\ell \times \ell$ square, centered at the origin, in a sea of (-1)-spins. Then $e^{-\beta E(\ell)}$ is the ratio of the probabilities to see $\sigma_{\ell\times\ell}$ resp. $\boxminus$ under the equilibrium μ. It follows from (1.1) that $E(\ell) = 4J\ell - h\ell^2$, which is maximal for $\ell = \frac{2J}{h}$:

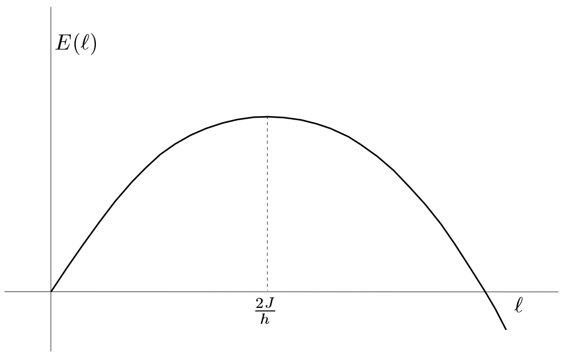

FIG. 1. The energy of an $\ell \times \ell$ droplet (NC)

This means that, even though an arbitrarily small non-vanishing magnetic field determines the phase, its effect is relevant only on sufficiently large space scales, namely $\ell \geq \ell_c$ with

$$\ell_c = \left\lceil \frac{2J}{h} \right\rceil. \tag{1.7}$$

Only on such scales the volume energy dominates the surface energy and a larger square of $(+1)$-spins is energetically favorable over a smaller square (see Fig. 1). The choice $h \in (0, 2J)$ corresponds to $\ell_c \in (1, \infty)$, i.e., to a non-trivial critical droplet size.

This describes the metastable behavior from a *static* point of view.

1.1.2 Glauber dynamics. In order to describe the metastable behavior from a *dynamic* point of view, we introduce a discrete-time stochastic dynamics by means of a Markov chain on $\mathcal{X}$ with transition probabilities $P(\sigma, \sigma')$ satisfying the *reversibility* condition

$$\mu(\sigma)P(\sigma, \sigma') = \mu(\sigma')P(\sigma', \sigma) \qquad \forall \sigma, \sigma' \in \mathcal{X}, \tag{1.8}$$

where μ is the Gibbs measure in (1.2), and the *ergodicity* condition

$$\forall \sigma, \eta \in \mathcal{X} \quad \exists t \in \mathbb{N} \text{ such that } P^t(\sigma, \eta) > 0, \tag{1.9}$$

where P^t is the t-step transition kernel. From the ergodic theorem for reversible Markov chains it follows that $P^t(\sigma, \eta)$ converges to $\mu(\eta)$ as $t \to \infty$ for all $\sigma, \eta \in \mathcal{X}$.

An explicit construction of a Markov chain satisfying the above conditions can be given, for instance, by the Glauber-Metropolis algorithm, which is defined as follows. For $\sigma \in \mathcal{X}$ and $x \in \Lambda$, let

$$\sigma^x(y) = \begin{cases} \sigma(y) & \text{if } x \neq y \\ -\sigma(y) & \text{if } x = y \end{cases} \tag{1.10}$$

and choose as transition probabilities

$$\begin{aligned} \sigma \neq \sigma' : \quad & P(\sigma, \sigma') = \begin{cases} 0 & \text{if } \sigma' \neq \sigma^x \, \forall x \in \Lambda \\ \frac{1}{|\Lambda|} e^{-\beta[H(\sigma^x) - H(\sigma)]_+} & \text{if } \sigma' = \sigma^x \, \exists x \in \Lambda \end{cases} \\ \sigma = \sigma' : \quad & P(\sigma, \sigma) = 1 - \sum_{\sigma' \neq \sigma} P(\sigma, \sigma'). \end{aligned} \tag{1.11}$$

This dynamics randomly selects a site from Λ and flips the spin at this site with a probability equal to the Boltzmann weight associated with the positive part of the energy difference caused by the flip. We emphasize that the dynamics given by (1.11) is NC, in the sense that the total magnetization is *not* a conserved quantity.

1.1.3 Metastability. Suppose we consider the typical paths of the Markov chain defined by (1.11), starting from $\boxminus$, in the regime (1.4). We can use a computer simulation and perform a large number of independent runs (see e.g. Tomita and Miyashita [22]). What we see is that in the sea of (-1)-spins small droplets of $(+1)$-spins appear, which however shrink and disappear before they are able to become large. Only after a very long time, and under the effect of a large fluctuation, a large enough droplet appears that grows without hesitation.

In order to understand this behavior, let us compare the probabilities of shrinking resp. growing for a connected cluster of $(+1)$-spins in a sea of (-1)-spins. First of all, each cluster of $(+1)$-spins becomes rectangular after a finite time (independent of β) with a probability of order one following a sequence of transitions with $H(\sigma^x)-H(\sigma)<0$. Indeed, the rectangle is the only shape such that: (i) all $(+1)$-spins have ≤ 2 nearest-neighbor (-1)-spins; (ii) all (-1)-spins have <2 nearest-neighbor $(+1)$-spins. Hence for the rectangle there are no spins that can be flipped with $H(\sigma^x)-H(\sigma)<0$.

Starting from a rectangular cluster of $(+1)$-spins, to *remove* a row or column of length ℓ costs $(\ell-1)h$:

- $[H(\sigma^x)-H(\sigma)]_+=[h]_+=h$ for each of the sites except the last one.
- $[H(\sigma^x)-H(\sigma)]_+=[h-2J]_+=0$ for the last site.

On the other hand, to *add* a row or column of length ℓ costs $2J-h$:

- $[H(\sigma^x)-H(\sigma)]_+=[2J-h]_+=2J-h$ for the first site.
- $[H(\sigma^x)-H(\sigma)]_+=[-h]_+=0$ for each of the sites except the first one.

This means that if the minimal side length ℓ of the rectangular cluster is such that $h(\ell-1)>2J-h$, i.e., $\ell\geq\ell_c$ with ℓ_c given by (1.7), then it tends to grow, while if $\ell<\ell_c$ then it tends to shrink.

The above heuristic argument has been developed in a rigorous way by Neves and Schonmann [8], [9] (see also Schonmann [15], [16], [17]). Let $(\sigma_t)_{t\in\mathbb{N}_0}$ be the Markov process on $\mathcal{X}$ with transition probabilities as in (1.11). Write $\mathbb{P}_\sigma,\mathbb{E}_\sigma$ for its probability law and expectation on path space given $\sigma_0=\sigma$. Let

$$\tau_\sigma=\min\{t\in\mathbb{N}_0\colon\sigma_t=\sigma\} \tag{1.12}$$

be the first hitting time of the configuration σ. The main result for metastability reads:

Theorem 1 (Neves and Schonmann [8],[9]) *Fix $h\in(0,2J)$, with $\frac{2J}{h}$ not integer, put $\ell_c=\lceil\frac{2J}{h}\rceil$, and suppose that Λ is sufficiently large.*
(a) Let $\mathcal{R}$ be the set of configurations where the $(+1)$-spins form a rectangle in a sea of (-1)-spins. For $\sigma\in\mathcal{R}$, let $\ell_1(\sigma)\times\ell_2(\sigma)$ be the rectangle of $(+1)$-spins in σ, and let $\ell(\sigma)=\min\{\ell_1(\sigma),\ell_2(\sigma)\}$. Then for any $\sigma\in\mathcal{R}$

$$\begin{aligned}\ell(\sigma)<\ell_c: &\quad \lim_{\beta\to\infty}\mathbb{P}_\sigma(\tau_{\boxminus}<\tau_{\boxplus})=1\\ \ell(\sigma)\geq\ell_c: &\quad \lim_{\beta\to\infty}\mathbb{P}_\sigma(\tau_{\boxplus}<\tau_{\boxminus})=1.\end{aligned} \tag{1.13}$$

(b) Let $\mathcal{R}^*$ be the set of configurations where the $(+1)$-spins form an $\ell_c\times(\ell_c-1)$ or $(\ell_c-1)\times\ell_c$ rectangle with a protuberance attached anywhere to one of the

sides of length ℓ_c. *Let* $\theta_{\boxminus,\boxplus} = \max\{t < \tau_{\boxplus}\colon\ \sigma_t = \boxminus\}$ *and* $\tau_{\boxminus,\mathcal{R}^*,\boxplus} = \min\{t > \theta_{\boxminus,\boxplus}\colon\ \sigma_t \in \mathcal{R}^*\}$. *Then*

$$\lim_{\beta\to\infty} \mathbb{P}_{\boxminus}(\tau_{\boxminus,\mathcal{R}^*,\boxplus} < \tau_{\boxplus}) = 1. \tag{1.14}$$

(c) Let $\Gamma = \Gamma(J,h) = 4J\ell_c - h(\ell_c^2 - \ell_c + 1)$. *Then*

$$\lim_{\beta\to\infty} \mathbb{P}_{\boxminus}\left(e^{(\Gamma-\delta)\beta} < \tau_{\boxplus} < e^{(\Gamma+\delta)\beta}\right) = 1 \qquad \forall\delta > 0. \tag{1.15}$$

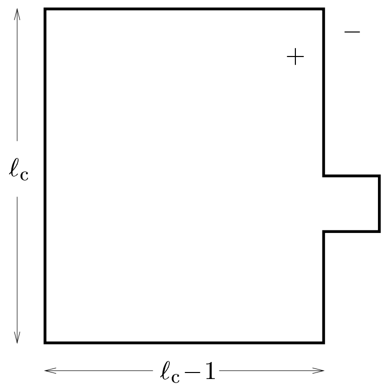

FIG. 2. The critical droplet (NC)

Theorem 1 not only identifies the size and shape of the critical droplet (see Fig. 2), it also shows that the critical droplet is the 'gate' of the transition from $\boxminus$ to $\boxplus$ and it identifies the transition time up to logarithmic equivalence in β. Note that $\Gamma(J,h)$ is the formation energy of the critical droplet under the Hamiltonian in (1.1).

1.1.4 Nucleation. The problem of identifying the typical path of nucleation, i.e., the path between $\theta_{\boxminus,\boxplus}$ and $\tau_{\boxplus}$, corresponds to the problem of the typical first exit of $(\sigma_t)_{t\in\mathbb{N}_0}$ from a suitable region in the state space $\mathcal{X}$. This problem is discussed in detail in Freidlin and Wentzell [5] Chapter 6, Schonmann [16], Olivieri and Scoppola [10], [11], Catoni and Cerf [3] under rather general hypotheses on the Markov chain. We recall here the main result for the case of the Glauber Ising model.

A sequence of configurations $\sigma_1, \ldots, \sigma_n$ ($n \in \mathbb{N}$) is called *standard* when:

1. the (+1)-spins of σ_i form a rectangular droplet $R_i = \ell_{1,i} \times \ell_{2,i}$;
2. $R_{i+1} \backslash R_i$ is a single row or a single column;
3. if $\min\{\ell_{1,i}, \ell_{2,i}\} < \ell_c$, then $|\ell_{1,i} - \ell_{2,i}| \leq 1$;
4. $R_1 = 2 \times 2$ and $R_n = \boxplus$.

The configurations in such a sequence are *stable*, since they are local minima of H, i.e., $H(\sigma_i) < \min_{\sigma' \sim \sigma_i} H(\sigma')$, where $\sigma' \sim \sigma$ if and only if $P(\sigma, \sigma') > 0$. With each σ_i it is possible to associate a *permanence set* $\mathcal{Q}_i$ (a suitable 'environment' of σ_i) and a *permanence time* $T_i = \mathbb{E}_{\sigma_i} \tau_{\mathcal{Q}_i^c}$ (the mean exit time of $\mathcal{Q}_i$ starting from σ_i). In this way we obtain a *standard sequence of permanence sets* (see Olivieri and Scoppola [12] for more precise definitions).

For each standard sequence of permanence sets and each $\epsilon > 0$ we can introduce a tube of trajectories $\mathbf{T}_{\epsilon,\beta}(\mathcal{Q}_1, \ldots, \mathcal{Q}_n)$, defined as the set of paths of configurations visiting the ordered sequence $\mathcal{Q}_1, \ldots, \mathcal{Q}_n$ and spending in each set $\mathcal{Q}_i$ a time that falls in the interval $[T_i e^{-\epsilon\beta}, T_i e^{+\epsilon\beta}]$. In terms of these quantities the main result for the path of nucleation reads:

Theorem 2 (Schonmann [16], Olivieri and Scoppola [12]) *For every $\kappa, \epsilon > 0$ there exists a $\beta_0 = \beta_0(\kappa, \epsilon)$ such that for all $\beta > \beta_0$*

$$\begin{aligned} \mathbb{P}_{\boxminus}\Big((\sigma_t)_{t \in [\theta_{\boxminus,\boxplus}, \tau_{\boxplus}]} \in \mathbf{T}_{\epsilon,\beta}(\mathcal{Q}_1, \ldots, \mathcal{Q}_n) & \textit{ for some standard} \\ \textit{sequence of permanence sets } \mathcal{Q}_1, \ldots, \mathcal{Q}_n \Big) & > 1 - e^{-\kappa\beta}. \end{aligned} \tag{1.16}$$

Theorem 2 shows that the transition from $\boxminus$ to $\boxplus$ takes place in a narrow tube around rectangular droplets that are squares or quasi-squares when the droplet is subcritical.

The main idea behind Theorem 2 is the following. The Markov chain $(\sigma_t)_{t \in \mathbb{N}_0}$ is in the *Freidlin-Wentzell regime*, i.e., its state space is finite and its transition probabilities satisfy the following estimates:

$$e^{-[V(\sigma,\sigma') + \gamma_\beta]\beta} \le P(\sigma, \sigma') \le e^{-[V(\sigma,\sigma') - \gamma_\beta]\beta} \qquad \forall \sigma \sim \sigma', \tag{1.17}$$

where $V(\cdot,\cdot)$ is a non-negative function, and $\lim_{\beta \to \infty} \gamma_\beta = 0$. Indeed, this property trivially follows from (1.11), because Λ is fixed and $V(\sigma, \sigma') = [H(\sigma') - H(\sigma)]_+$. From (1.17) it is standard to obtain estimates on $\mathbb{E}_{\sigma_i} \tau_{\mathcal{Q}_i^c}$ and $\mathbb{P}_{\sigma_i}(\sigma_{\tau_{\mathcal{Q}_i^c}} = \sigma')$ (see Freidlin and Wentzell [5] Chapter 6).

The main steps in the proof of Theorem 2 are the following:

(1) One must solve a certain sequence of variational problems defined in terms of the energy function H. These variational problems are *minimax problems* necessary to find the minimal saddle point energy between pairs of states σ, σ' defined by

$$H(\sigma, \sigma') = \min_{\phi:\ \sigma \to \sigma'} \max_{\eta \in \phi} H(\eta). \tag{1.18}$$

The output of this first step is a standard sequence of configurations.

(2) One must associate with each stable configuration a permanence set and a stability in terms of a permanence time. This can be done by using a so-called *cycle decomposition*. Cycles can be defined in the Freidlin-Wetzell regime (see Freidlin and Wentzell [5] Chapter 6, Olivieri and Scoppola [12], Trouvé [23]). In the case of the Glauber Ising model cycles turn out to be connected sets of configurations with energy below a given value.

1.2 The conservative case

1.2.1 Canonical ensemble. In the present paper we want to study the metastable behavior of conservative systems. To that end we consider a lattice gas model defined as follows. Let $\Lambda_\beta \subset \mathbb{Z}^2$ be a large finite box centered at the origin, with periodic boundary conditions. With each $x \in \Lambda_\beta$ we associate an occupation variable $\eta(x)$, assuming the values 0 or 1. A lattice gas configuration is denoted by $\eta \in \mathcal{X} = \{0,1\}^{\Lambda_\beta}$. We consider the interaction defined by the following Hamiltonian:

$$H(\eta) = -U \sum_{(x,y)\in\Lambda_\beta^*} \eta(x)\eta(y), \tag{1.19}$$

i.e., there is a binding energy $U > 0$ between neighboring occupied sites. For $A \subset \Lambda_\beta$, we let

$$N_A(\eta) = \sum_{x\in A} \eta(x). \tag{1.20}$$

We fix the particle density in Λ_β at

$$\frac{1}{|\Lambda_\beta|} \sum_{x\in\Lambda_\beta} \eta(x) = \rho = e^{-\Delta\beta}, \tag{1.21}$$

where $\Delta > 0$ is an activity parameter. This corresponds to a total number of particles in Λ_β equal to

$$N = \rho|\Lambda_\beta|. \tag{1.22}$$

On the set of configurations with N particles

$$\mathcal{N}_N = \{\eta \in \mathcal{X}\colon\ N_{\Lambda_\beta}(\eta) = N\}, \tag{1.23}$$

we define the *canonical Gibbs measure*

$$\nu_N(\eta) = \frac{e^{-\beta H(\eta)} 1_{\mathcal{N}_N}(\eta)}{Z_N} \qquad (\eta \in \mathcal{X}), \tag{1.24}$$

where

$$Z_N = \sum_{\eta\in\mathcal{N}_N} e^{-\beta H(\eta)}. \tag{1.25}$$

We see from (1.21) and (1.22) that in order to have particles at all we must pick $|\Lambda_\beta|$ at least exponentially large in β. This means that the regime where Λ_β is fixed, considered in the NC-case, has no relevance here. We will in fact be interested in the regime

$$\Delta \in (U, 2U), \quad \beta \to \infty, \quad \lim_{\beta\to\infty} \frac{1}{\beta} \log|\Lambda_\beta| = \infty, \tag{1.26}$$

which takes over the role that (1.4) played in the NC-case.

1.2.2 Kawasaki dynamics. We define a stochastic dynamics in terms of a continuous-time Markov chain $(\eta_t)_{t\geq 0}$ with state space $\mathcal{N}_N$, given by the following generator:

$$(\mathcal{L}f)(\eta) = \sum_{(x,y)\in\Lambda_\beta^*} c((x,y),\eta)[f(\eta^{(x,y)}) - f(\eta)], \tag{1.27}$$

where

$$\eta^{(x,y)}(z) = \begin{cases} \eta(z) & \text{if } z \neq x,y \\ \eta(x) & \text{if } z = y \\ \eta(y) & \text{if } z = x \end{cases} \tag{1.28}$$

and

$$c((x,y),\eta) = e^{-\beta[H(\eta^{(x,y)})-H(\eta)]_+}. \tag{1.29}$$

It is easily verified that the reversibility condition holds:

$$\nu_N(\eta)c((x,y),\eta) = \nu_N(\eta^{(x,y)})c((x,y),\eta^{(x,y)}). \tag{1.30}$$

The Markov chain $(\eta_t)_{t\geq 0}$ can be represented as follows. With each bond $b = (x,y) \in \Lambda_\beta^*$ we associate a random clock ringing at exponential times. When the clock at b rings, we consider the configuration with the particles swapped along b. This configuration is accepted with a Metropolis rate given by the Boltzmann factor in (1.29). More formally, for each bond b put $\tau_{b,0} = 0$ and let $\tau_{b,i}, i \in \mathbb{N}$, be the sequence of random times whose increments are i.i.d. exponentially distributed with mean 1. Since $|\Lambda_\beta| < \infty$, we have

$$\mathbb{P}(\exists b, b', i, i'\colon\ \tau_{b,i} = \tau_{b',i'}) = 0. \tag{1.31}$$

Now, if $t = \tau_{b,i}$ for some b and i, then we define

$$\eta_t = \begin{cases} \eta_{t-} & \text{with probability } 1 - e^{-\beta[H(\eta_{t-}^b)-H(\eta_{t-})]_+} \\ \eta_{t-}^b & \text{with probability } e^{-\beta[H(\eta_{t-}^b)-H(\eta_{t-})]_+}, \end{cases} \tag{1.32}$$

while between ringing times the configuration stays fixed.

1.2.3 Metastability. In order to see that the regime in (1.26) is metastable, let us consider the grand-canonical Gibbs measure associated with the model, i.e.,

$$\mu_\lambda(\eta) = \frac{e^{-\beta H_\lambda(\eta)}}{Z_\lambda}, \tag{1.33}$$

where

$$H_\lambda(\eta) = H(\eta) - \lambda N_{\Lambda_\beta}(\eta), \tag{1.34}$$

$\lambda \in \mathbb{R}$ is an activity parameter, and

$$Z_\lambda = \sum_{\eta\in\mathcal{X}} e^{-\beta H_\lambda(\eta)}. \tag{1.35}$$

It turns out that if $\lambda = -\Delta$, then for the description of metastability the canonical Gibbs measure is equivalent to the grand-canonical Gibbs measure in the limit of large β, provided they are suitably restricted in the following way.

Consider the lattice gas at low temperature at its condensation point. Let

$$\rho_l(\beta) = \frac{1+m^*(\beta)}{2}, \quad \rho_g(\beta) = \frac{1-m^*(\beta)}{2} \tag{1.36}$$

denote the density of the liquid resp. gas phase. Here $m^*(\beta)$ is the spontaneous magnetization in the spin language (see (1.42)). Since

$$\rho_g(\beta) = e^{-2U\beta+o(\beta)} \qquad (\beta \to \infty), \tag{1.37}$$

we see that $e^{-2U\beta}$ can be identified as the density of the saturated gas at the condensation point (in the sense of logarithmic equivalence in β). Suppose that we slightly increase the density, avoiding however the appearance of droplets of the liquid phase. Then we get a supersaturated gas that can be described in terms of a *restricted ensemble* (see Lebowitz and Penrose [7] and Capocaccia, Cassandro and Olivieri [2]), namely, the grand-canonical Gibbs measure restricted to a suitable subset of configurations, for instance, where all sufficiently large clusters are suppressed. At low temperature this supersaturated gas will stay rarified, so that its metastable state can be described as a pure gas phase with strong mixing properties.

In these conditions, let us make a rough calculation of the probability to see an $\ell \times \ell$ droplet of occupied sites centered at the origin. Under the restricted ensemble, which we denote by μ^*, we have

$$\mu^*(\ell \times \ell \text{ droplet}) = \rho^{\ell^2} e^{2\ell(\ell-1)U\beta}, \tag{1.38}$$

since ρ is the probability to find a particle at a given site and U is the binding energy between particles at neighboring sites. Substituting $\rho = e^{-\Delta\beta}$ we obtain

$$\mu^*(\ell \times \ell \text{ droplet}) = e^{-\beta E(\ell)}, \tag{1.39}$$

where

$$E(\ell) = 2U\ell - (2U - \Delta)\ell^2. \tag{1.40}$$

The maximum of $E(\ell)$ is at $\ell = \frac{U}{2U-\Delta}$:

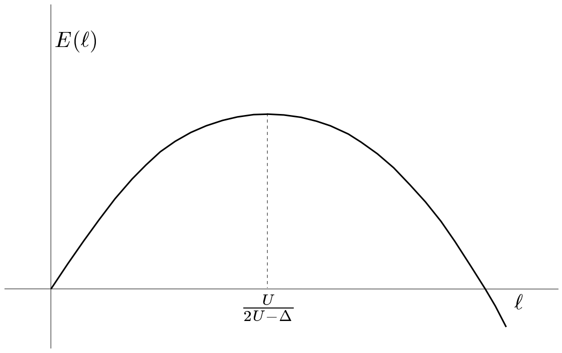

FIG. 3. The energy of an $\ell \times \ell$ droplet (C)

This means that droplets with side length $\ell < \ell_c$ have a probability decreasing in ℓ and droplets with side length $\ell \geq \ell_c$ a probability increasing in ℓ, where

$$\ell_c = \left\lceil \frac{U}{2U - \Delta} \right\rceil \tag{1.41}$$

(see Fig. 3). The choice $\Delta \in (U, 2U)$ corresponds to $\ell_c \in (1, \infty)$, i.e., to a non-trivial critical droplet size.

Another way of understanding our choice of Δ is the following. In the grand-canonical Gibbs measure the configuration can be represented in terms of spin variables. Indeed, after we make the substitution $\eta(x) = \frac{1+\sigma(x)}{2}$, where $\sigma(x) \in \{-1, +1\}$ is a spin variable, we can write

$$\begin{aligned} H_\lambda(\sigma) &= -U \sum_{(x,y)\in\Lambda_\beta^*} \tfrac{1+\sigma(x)}{2}\tfrac{1+\sigma(y)}{2} - \lambda \sum_{x\in\Lambda_\beta} \tfrac{1+\sigma(x)}{2} \\ &= -\tfrac{U}{4} \sum_{(x,y)\in\Lambda_\beta^*} \sigma(x)\sigma(y) - \tfrac{2U+\lambda}{2} \sum_{x\in\Lambda_\beta} \sigma(x) + \text{ const.} \end{aligned} \tag{1.42}$$

So if $\lambda = -\Delta$, then we have a spin Hamiltonian like (1.1) with pair interaction $J = \frac{U}{2}$ and magnetic field $h = 2U - \Delta$. By the discussion developed in Section 1.1.3, we therefore expect metastable behavior with a critical droplet size given by (1.41) (compare with (1.7)). The metastable behavior for the NC-case in the spin language occurs when $h \in (0, 2J)$. This corresponds precisely to $\Delta \in (U, 2U)$.

In physical terms, $\Delta \in (0, U)$ represents the unstable gas, $\Delta = U$ the spinodal point, $\Delta \in (U, 2U)$ the metastable gas, $\Delta = 2U$ the condensation point, and $\Delta \in (2U, \infty)$ the stable gas.

The above describes the metastable behavior from a *static* point of view. A static comparison of Glauber vs. Kawasaki dynamics in the spin language is indicated in the following figure:

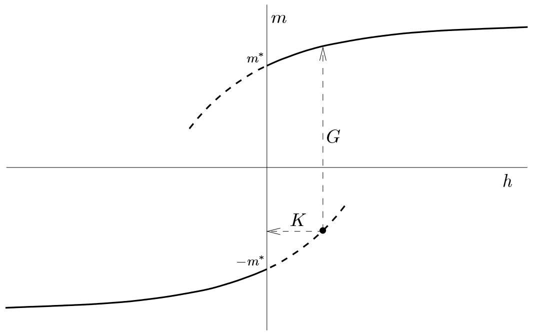

FIG. 4. Relaxation to equilibrium for Glauber (G) and Kawasaki (K) dynamics

1.2.4 Local description. Let us now consider the metastable behavior from a *dynamic* point of view and see what happens locally. As discussed in the

NC-case, we want to compare the probabilities of growing resp. shrinking for a rectangular cluster of particles. Again the argument will be very rough. Suppose we pick a large finite box $\bar{\Lambda}$, centered at the origin, and start with an $\ell \times \ell$ droplet inside $\bar{\Lambda}$. Suppose that the effect on $\bar{\Lambda}$ of the gas in $\Lambda_\beta \setminus \bar{\Lambda}$ may be described in terms of the *creation* of new particles with rate $\rho = e^{-\Delta\beta}$ at sites on the interior boundary of $\bar{\Lambda}$ and the *annihilation* of particles with rate 1 at sites on the exterior boundary of $\bar{\Lambda}$. In other words, suppose that inside $\bar{\Lambda}$ the Kawasaki dynamics may be described by a Metropolis algorithm with energy given by the *local grand-canonical Hamiltonian*:

$$\bar{H}(\eta) = H(\eta) + \Delta N_{\bar{\Lambda}}(\eta). \tag{1.43}$$

Then the energy barriers for adding resp. removing a row or column of length ℓ are given in terms of the local saddles of $\bar{H}$ (see Fig. 5):

$$\begin{array}{lcl} \text{energy barrier for adding} & = & 2\Delta - U \\ \text{energy barrier for removing} & = & (2U - \Delta)(\ell - 2) + 2U, \end{array} \tag{1.44}$$

and the balance of the two barriers indeed gives the critical size ℓ_c in (1.41).

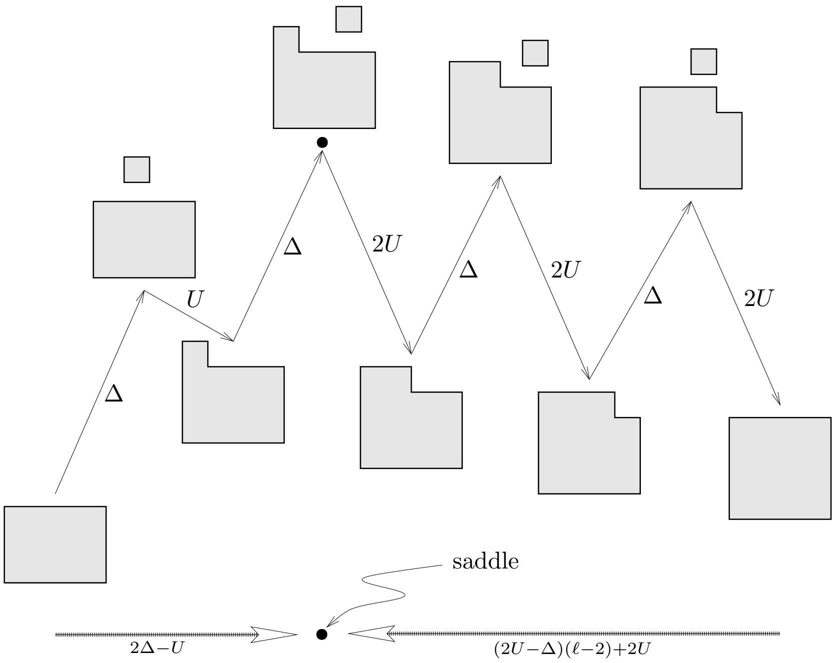

FIG. 5. Local saddles of $\bar{H}$

Let us briefly discuss the main difficulties arising in the attempt to develop the above idea rigorously and underline the main differences with the NC-case. As we already remarked, in the C-case the Markov chain $(\eta_t)_{t\geq 0}$ is not in the Freidlin-Wentzell regime, so we need new ideas. *The real difficulty is to find the correct way to treat the gas in* $\Lambda_\beta \setminus \bar{\Lambda}$. The heuristic discussion given above was based on the assumption that the dynamics inside $\bar{\Lambda}$ is effectively described by the local grand-canonical Hamiltonian $\bar{H}$ in (1.43). However, unlike the NC-dynamics, the C-dynamics is not really local: *particles must arrive from or return to the gas, which acts as a reservoir.* It is therefore not possible to decouple the dynamics of the particles inside $\bar{\Lambda}$ from the dynamics of the gas in $\Lambda_\beta \setminus \bar{\Lambda}$. This means that the gas must be controlled in some detail in order to prove that the above assumption is indeed a good enough approximation.

A second consequence of the non-local behavior of the C-dynamics is that the argument used in the NC-case, based on the stability of configurations and on the corresponding partition into cycles of the state space (see Section 1.1.4), is completely lost in the C-case. In other words, we cannot define the stability of a configuration inside $\bar{\Lambda}$, since it depends on the configuration in $\Lambda_\beta \setminus \bar{\Lambda}$. A different aspect of the same problem is the following: *What is the mechanism by which the gas remains in or close to equilibrium, so that its description in terms of $\bar{H}$ is correct, even over long time intervals during which exchange of many particles occurs?*

1.3 A simplified model

Unfortunately, we are unable to handle the model described in Section 1.2. Instead, in the present paper we solve the problem of metastability for a simplified model. Namely, we *remove the interaction outside the box* $\bar{\Lambda}_0 = \bar{\Lambda} \setminus \partial^- \bar{\Lambda}$, with $\partial^- \bar{\Lambda}$ the interior boundary of $\bar{\Lambda}$, i.e., we replace the interaction energy (1.19) by

$$H(\eta) = -U \sum_{(x,y)\in \bar{\Lambda}_0^*} \eta(x)\eta(y). \tag{1.45}$$

Moreover, we also *remove the exclusion outside* $\bar{\Lambda}$, i.e., the dynamics of the gas outside $\bar{\Lambda}$ is that of independent random walks (IRW's). These two simplifications will allow us to control the gas and to overcome the difficulties outlined in Section 1.2.4.

Our state space is

$$\mathcal{N}_N = \{\eta \in \mathcal{X}\colon\ N_{\Lambda_\beta}(\eta) = N\}, \tag{1.46}$$

where $\mathcal{X} = \{0,1\}^{\bar{\Lambda}} \times \mathbb{N}^{\Lambda_\beta \setminus \bar{\Lambda}}$ and $N = \rho|\Lambda_\beta|$ (with $\rho = e^{-\Delta\beta}$). The local grand-canonical Hamiltonian is

$$\bar{H}(\eta) = H(\eta) + \Delta N_{\bar{\Lambda}}(\eta), \tag{1.47}$$

where H is the Hamiltonian in (1.45). Throughout the remainder of this paper we assume that we are in the regime (1.26).

Our main theorem reads as follows. Let

$$\begin{aligned} \blacksquare &= \{\eta \in \mathcal{X}\colon\ \eta(x) = 1\ \forall x \in \bar{\Lambda}_0\} \\ \square &= \{\eta \in \mathcal{X}\colon\ \eta(x) = 0\ \forall x \in \bar{\Lambda}\}. \end{aligned} \tag{1.48}$$

For $\bar{\eta} \in \bar{\mathcal{X}} = \{0,1\}^{\bar{\Lambda}}$, let $\nu_{\bar{\eta}}$ denote the canonical Gibbs measure conditioned on the configuration inside $\bar{\Lambda}$ being $\bar{\eta}$, i.e.,

$$\nu_{\bar{\eta}}(\eta) = \frac{\nu(\eta) 1_{I_{\bar{\eta}}}(\eta)}{\nu(I_{\bar{\eta}})} \qquad (\eta \in \mathcal{X}), \tag{1.49}$$

where $I_{\bar\eta} = \{\eta \in \mathcal{X}\colon\ \eta|_{\bar\Lambda} = \bar\eta\}$, with $\eta|_{\bar\Lambda}$ the restriction of η to $\bar\Lambda$, and ν is the canonical Gibbs measure defined in (1.22–1.24). For $\bar\eta \in \bar{\mathcal{X}} = \{0,1\}^{\bar\Lambda}$, write $\mathbb{P}_{\nu_{\bar\eta}}$, $\mathbb{E}_{\nu_{\bar\eta}}$ to denote the probability law and expectation for the Markov process $(\eta_t)_{t\geq 0}$ on $\mathcal{X}$ following the Kawasaki dynamics with Hamiltonian (1.45) given that η_0 is chosen according to $\nu_{\bar\eta}$. Write $\bar\square$ to denote the empty configuration in $\bar\Lambda$. For $\mathcal{A} \subset \mathcal{X}$, let

$$\tau_{\mathcal{A}} = \min\{t \geq 0\colon\ \eta_t \in \mathcal{A}\} \tag{1.50}$$

be the first hitting time of the set $\mathcal{A}$.

Theorem 3 *Let $\Delta \in (\frac{3}{2}U, 2U)$, with $\frac{U}{2U-\Delta}$ not integer, put $\ell_c = \lceil \frac{U}{2U-\Delta} \rceil$, and suppose that $\log|\Lambda_\beta| \gg \beta$.*
(a) Let $\bar{\mathcal{R}} \subset \bar{\mathcal{X}}$ be the set of configurations inside $\bar\Lambda$ where the particles form a square or quasi-square contained in $\bar\Lambda_0$. For $\bar\eta \in \bar{\mathcal{R}}$, let $\ell_1(\bar\eta) \times \ell_2(\bar\eta)$ with $|\ell_1(\bar\eta) - \ell_2(\bar\eta)| \leq 1$ be the square or quasi-square of particles in $\bar\eta$, and let $\ell(\bar\eta) = \min\{\ell_1(\bar\eta), \ell_2(\bar\eta)\}$. Then for any $\bar\eta \in \bar{\mathcal{R}}$

$$\begin{aligned} \ell(\bar\eta) < \ell_c: &\quad \lim_{\beta\to\infty} \mathbb{P}_{\nu_{\bar\eta}}(\tau_{\square} < \tau_{\blacksquare}) = 1 \\ \ell(\bar\eta) \geq \ell_c: &\quad \lim_{\beta\to\infty} \mathbb{P}_{\nu_{\bar\eta}}(\tau_{\blacksquare} < \tau_{\square}) = 1. \end{aligned} \tag{1.51}$$

(b) Let $\bar{\mathcal{C}}^*$ be the set of configurations defined in (4.20). Let $\theta_{\square,\blacksquare} = \max\{t < \tau_{\blacksquare}\colon\ \eta_t \in \square\}$ and $\tau_{\square,\bar{\mathcal{C}}^*,\blacksquare} = \min\{t > \theta_{\square,\blacksquare}\colon\ \eta_t \in \bar{\mathcal{C}}^*\}$. Then

$$\lim_{\beta\to\infty} \mathbb{P}_{\nu_{\bar\square}}(\tau_{\square,\bar{\mathcal{C}}^*,\blacksquare} < \tau_{\blacksquare}) = 1. \tag{1.52}$$

(c) Let $\Gamma = \Gamma(U,\Delta) = -U(2\ell_c^2 - 4\ell_c + 2) + \Delta(\ell_c^2 - \ell_c + 2)$. Then

$$\lim_{\beta\to\infty} \mathbb{P}_{\nu_{\bar\square}}\left(e^{(\Gamma-\delta)\beta} < \tau_{\blacksquare} < e^{(\Gamma+\delta)\beta}\right) = 1 \qquad \forall \delta > 0. \tag{1.53}$$

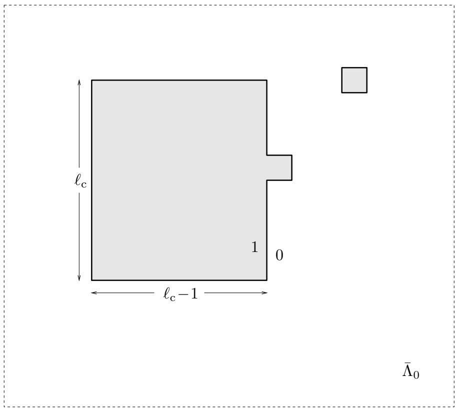

FIG. 6. A critical configuration (C)

Theorem 3 is the analogue of Theorem 1. There are, however, a number of important differences.

- The mechanisms for the evolution of clusters under the Kawasaki dynamics and the Glauber dynamics are different. In particular, under the Kawasaki dynamics there is a movement of particles *along the border* of a rectangular droplet, leading to a (more stable) square droplet on a time scale much shorter than the one needed to grow or shrink (namely $e^{\Delta\beta}$). Moreover, the sub- vs. supercriticality of a rectangle (i.e., its tendency to reach $\square$ before $\blacksquare$ or vice versa) is related to its area. In contrast, under the Glauber dynamics the sub- vs. supercriticality is related to its minimal side length: a subcritical rectangle is attracted by the maximal square contained in its interior, while a supercritical rectangle does not manifest any tendency towards a square shape.

- Theorem 3(a): We only identify the sub- vs. supercriticality of squares and quasi-squares. We believe that it should be possible to show that, starting from an $\ell_1 \times \ell_2$ rectangle with $\ell_1\ell_2 = \ell^2$ for ℓ soem integer, the system reaches an $\ell \times \ell$ square in a time of order $e^{\Delta\beta}$ and from there proceeds as described in (1.51).

- Theorem 3(b): Let $\bar{\mathcal{R}}^* \subset \bar{\mathcal{X}}$ be the set of configurations inside $\bar{\Lambda}$ where the particles form an $\ell_c \times (\ell_c - 1)$ or $(\ell_c - 1) \times \ell_c$ quasi-square with a protuberance attached anywhere to one of the sides of length ℓ_c and with a free particle anywhere else, all contained in $\bar{\Lambda}_0$ (see Fig. 6). We will see in Section 4.2 that $\bar{\mathcal{C}}^*$, the 'gate' of the transition from $\square$ to $\blacksquare$, consists of all configurations that are 'U-equivalent' to some configuration in $\bar{\mathcal{R}}^*$, i.e., can be connected to that configuration via a path with a 'maximal saddle U' and vice versa. In particular, $\bar{\mathcal{C}}^* \supset \bar{\mathcal{R}}^*$, but the full set is more complex (see Fig. 9 in Section 5.2). This complexity comes from the fact that under the Kawasaki dynamics particles can move along the border of a rectangular droplet at a cost U.

The critical configuration in Fig. 6 has the same shape as in the NC-case, but with an extra free particle. This particle signals that the 'gate' of the transition from $\square$ to $\blacksquare$ has been passed and that the droplet starts to grow without hesitation. Note that $\Gamma(U,\Delta)$ is the energy of the critical droplet under the local grand-canonical Hamiltonian in (1.47).

It is certainly feasible to also prove the analogue of Theorem 2 for the simplified model. However, in the present paper we will not address this issue for reasons of space.

Remarks:

(1) As explained above, the removal of the interaction outside $\bar{\Lambda}_0$ and the exclusion outside $\bar{\Lambda}$ allows us to mathematically control the gas. From a physical point of view this approximation seems very reasonable, because $\beta \to \infty$ corresponds to a low density limit ($\rho = e^{-\Delta\beta}$) in which the gas essentially behaves like an ideal gas.

(2) In the simplified model we are focussing on the *local* aspects of metastability and nucleation: the removal of the interaction outside $\bar{\Lambda}_0$ forces the critical

droplet to appear inside $\bar{\Lambda}_0$. In the original model with interaction and exclusion throughout Λ_β, if $\liminf_{\beta\to\infty} \frac{1}{\beta} \log |\Lambda_\beta|$ is large enough, then the decay from the metastable to the stable state is driven by the formation of many droplets far away from the origin, which subsequently grow and coalesce. This is a much harder problem, which we hope to tackle in the future (see Deghampour and Schonmann [4] for a description of this behavior for Ising spins under Glauber dynamics). In the original model also the question of the growth of large *supercritical* droplets comes up, which is absent for the simplified model because $\bar{\Lambda}$ is finite. For Kawasaki dynamics this poses new problems compared to Glauber dynamics, because large droplets deplete the gas.

1.4 Outline of the paper

Our strategy to prove Theorem 3 will be the following. In Section 2 we show that, under the measure $\nu_{\bar{\eta}}$ ($\bar{\eta} \in \bar{\mathcal{X}}$), particle densities in suitable regions around $\bar{\Lambda}$ are not too far from their expected value. With the help of large deviation estimates we show that these density properties are preserved under the dynamics over very long time intervals with a very large probability. In Section 3 we use this fact to control the gas, essentially via a series of *mixing lemmas*. Once the gas behavior is under control, we start to tackle the metastability problem inside $\bar{\Lambda}$. This is done in Sections 6–7 via *recurrence* and *reduction*. Namely, in Section 6 we show that certain subsets of configurations of increasing 'regularity'

$$\mathcal{X}_1 \supset \mathcal{X}_2 \supset \mathcal{X}_3 \tag{1.54}$$

are visited by the process on certain basic time scales

$$T_1 = e^{0\beta} \ll T_2 = e^{U\beta} \ll T_3 = e^{\Delta\beta}. \tag{1.55}$$

This fact leads us in Section 7 to define a *reduced Markov chain* with state space $\mathcal{X}_3$, whose transition probabilities we can estimate in a way that allows us to control the metastable behavior. In essence, we show that this reduced chain is 'equivalent' in its metastable behavior to a *local Markov chain* with state space $\bar{\mathcal{X}} = \{0,1\}^{\bar{\Lambda}}$ that is reversible w.r.t. the local grand-canonical Hamiltonian $\bar{H}$ defined in (1.47). This approximation is what drives the argument. In Section 5 we study the local Markov chain using general ideas from renormalization. The dynamics inside $\bar{\Lambda}$ is still conservative, and this difficulty has to be handled via *local geometric arguments*, as explained in Section 4. Here we also show that the Kawasaki dynamics has its own peculiarities, which need to be understood in order to describe the evolution of droplets. The proof of our main result in Theorem 3 comes in Section 8. Here the fact that the full Markov chain is *reversible w.r.t. the canonical Gibbs measure* plays an important role. In Section 9 we prove the equivalence of the canonical and the grand-canonical ensemble for the simplified model in the regime (1.26). This equivalence is used in some of the calculations.

1.5 Notation

Before we start the proof of Theorem 3, we collect the frequently used notation.

We use *capital* letters for subsets of $\mathbb{Z}^2$, *calligraphic* capital letters for subsets of the configuration space $\mathcal{X}$, and *boldface* capital letters for events involving the Markov process and the clocks. This style is used consistently in order to keep different types of quantities apart. We use the symbols t, T for time, ν for the canonical Gibbs measure with particle density $\rho = e^{-\Delta\beta}$ (recall (1.22–1.24)), and κ for a generic positive constant.

For $A \subset \mathbb{Z}^2$, the set of nearest-neighbor bonds in A is

$$A^* = \{b = (x,y)\colon\ x,y \in A\}. \tag{1.56}$$

For $\mathcal{A} \subset \mathcal{X}$, the base of $\mathcal{A}$ is

$$\text{BASE}(\mathcal{A}) = \min\{A \subset \mathbb{Z}^2\colon\ \eta \in \mathcal{A} \Longrightarrow \zeta \in \mathcal{A}\ \forall\zeta\colon\ \zeta|_A = \eta|_A\}, \tag{1.57}$$

i.e., the minimal set of sites on which the configuration determines the event $\mathcal{A}$. For $\eta \in \mathcal{X}$, $\eta|_{\bar\Lambda} \in \{0,1\}^{\bar\Lambda}$ denotes the restriction of η to $\bar\Lambda$. Conversely, for $\bar\eta \in \bar{\mathcal{X}}$,

$$I_{\bar\eta} = \{\eta \in \mathcal{X}\colon\ \eta|_{\bar\Lambda} = \bar\eta\}. \tag{1.58}$$

For $A \subset \mathbb{Z}^2$ and $\eta \in \mathcal{X}$, the number of particles the configuration η places in A is

$$N_A(\eta) = \sum_{x \in A} \eta(x). \tag{1.59}$$

For $A \subset \mathbb{Z}^2$, the interior resp. exterior boundary of A are

$$\begin{array}{rcl} \partial^- A &=& \{x \in A\colon\ \exists b = (x,y)\colon\ y \notin A\} \\ \partial^+ A &=& \{x \notin A\colon\ \exists b = (x,y)\colon\ y \in A\}. \end{array} \tag{1.60}$$

For $\ell \in \mathbb{N}$, the box with side length ℓ centered at the origin is denoted by Λ_ℓ. The side length of $\bar\Lambda_0$, the local box appearing in the Hamiltonian H in (1.45), is ℓ_0. All quantities that live on $\bar\Lambda$ are written with a bar on top, in order to distinguish them from quantities that live on Λ_β or other boxes. A function $\beta \mapsto f(\beta)$ is called *superexponentially small* (SES) if

$$\lim_{\beta\to\infty} \frac{1}{\beta} \log f(\beta) = -\infty. \tag{1.61}$$

We frequently round off large integers, in order to avoid a plethora of brackets like $\lceil\cdot\rceil$.

2 LD-estimates for clocks and equilibrium

In this section we formulate several large deviation estimates that will be needed later on.

2.1 LD-estimates for clocks

Let $\tau_{b,i}$, $i \in \mathbb{N}$, denote the ringing times of the clock at bond b. For $t > 0$, let $r_b(t) = \max\{i \in \mathbb{N}\colon \tau_{b,i} \le t\}$ denote the number of rings prior to time t. For $m, n \in \mathbb{N}$, put $r_b(n, n+m) = r_b(n+m) - r_b(n)$. For $A \subset \mathbb{Z}^2$, $T \ge 0$ and $\delta > 0$, define

$$\mathbf{R}_T^\delta(A) = \Big\{\forall b \in A^* \ \forall n \le T \ \forall m \ge e^{\delta\beta}\colon \ r_b(n, n+m) \in [\tfrac12 m, \tfrac32 m]\Big\}. \tag{2.1}$$

Proposition 4 below shows that clocks ring regularly over long time intervals. This proposition will be needed to switch from continuous to discrete time.

Proposition 4 *For all $A \subset \mathbb{Z}^2$, $T \ge 0$ and $\delta > 0$*

$$\mathbb{P}(\mathbf{R}_T^\delta(A)^c) \le T|A^*|\text{SES}. \tag{2.2}$$

Proof. Write

$$\begin{aligned} \mathbb{P}(\mathbf{R}_T^\delta(A)^c) &= \Big\{\exists b \in A^* \ \exists n \le T \ \exists m \ge e^{\delta\beta}\colon \ r_b(n, n+m) \notin [\tfrac12 m, \tfrac32 m]\Big\} \\ &\le T|A^*| \sum_{m \ge e^{\delta\beta}} \mathbb{P}(r_{b_0}(0,m) \notin [\tfrac12 m, \tfrac32 m]), \end{aligned} \tag{2.3}$$

where b_0 is any given bond. We have

$$\begin{aligned} r_{b_0}(0,m) < \tfrac12 m &\Longrightarrow \tau_{b_0, \lceil \frac12 m \rceil} > m \\ r_{b_0}(0,m) > \tfrac32 m &\Longrightarrow \tau_{b_0, \lceil \frac32 m \rceil} < m. \end{aligned} \tag{2.4}$$

Since $\tau_{b_0,m} = \chi_1 + \ldots + \chi_m$, with $(\chi_i)_{i\in\mathbb{N}}$ i.i.d. exponential random variables with mean 1, a standard LD-estimate gives that the summand of the last term in (2.3) is $\le e^{-\kappa m}$ for some $\kappa > 0$. Hence the claim follows. **QED**

2.2 LD-estimates for equilibrium

2.2.1 Hitting times. Proposition 5 below gives us an estimate on the hitting time, under the dynamics starting in equilibrium, of sets that have a small probability under the equilibrium measure.

Proposition 5 *Let $\mathcal{A} \subset \mathcal{X}$ and $\tau_{\mathcal{A}} = \inf\{s \ge 0\colon \eta_s \in \mathcal{A}\}$. Then for any $t \ge 0$*

$$\mathbb{P}_\nu(\tau_{\mathcal{A}} < t) = \int_{\mathcal{X}} \nu(d\eta)\mathbb{P}_\eta(\tau_{\mathcal{A}} < t) \le 3t|\text{BASE}(\mathcal{A})^*|\nu(\mathcal{A}). \tag{2.5}$$

The same holds when ν is replaced by $\nu_{\mathcal{B}} = \nu 1_{\mathcal{B}}/\nu(\mathcal{B})$ for any $\mathcal{B} \subset \mathcal{X}$.

Proof. Fix $\mathcal{A}$. For $s \ge 0$, let

$$\mathbf{F}_s = \{\eta_s \in \mathcal{A}, \eta_u \notin \mathcal{A} \ \forall 0 \le u < s\}. \tag{2.6}$$

Fix $\epsilon > 0$ and define

$$\mathbf{R}_s = \{\text{some clock in BASE}(\mathcal{A})^* \text{ rings during } [s, s+\epsilon)\}. \tag{2.7}$$

Then we have

$$\begin{aligned} \mathbb{P}_\nu(\tau_{\mathcal{A}} < t) &= \mathbb{P}_\nu(\exists s \in [0,t)\colon\ \mathbf{F}_s) \\ &= \mathbb{P}_\nu(\exists s \in [0,t)\colon\ \mathbf{F}_s \cap \mathbf{R}_s) + \mathbb{P}_\nu(\exists s \in [0,t)\colon\ \mathbf{F}_s \cap \mathbf{R}_s^c). \end{aligned} \tag{2.8}$$

The first term equals $\mathbb{P}_\nu(\tau_{\mathcal{A}} < t)[1 - e^{-\epsilon|\text{BASE}(\mathcal{A})^*|}]$, because clocks have no memory. The second term is bounded above by

$$\mathbb{P}_\nu(\exists 0 < i \leq t/\epsilon\colon\ \eta_{i\epsilon} \in \mathcal{A}) \leq \frac{t}{\epsilon}\nu(\mathcal{A}), \tag{2.9}$$

where we use that $\mathbb{P}_\nu(\eta_{i\epsilon} \in \mathcal{A})$ does not depend on i because ν is the equilibrium measure. Combining the latter two observations with (2.8) we get

$$\mathbb{P}_\nu(\tau_{\mathcal{A}} < t) \leq t\nu(\mathcal{A})\Big[\frac{1}{\epsilon}e^{\epsilon|\text{BASE}(\mathcal{A})^*|}\Big]. \tag{2.10}$$

Optimize over ϵ, i.e., pick $\epsilon = 1/|\text{BASE}(\mathcal{A})^*|$, to arrive at the claim. **QED**

2.2.2 Recurrence times. Proposition 6 below gives us control over the successive times at which the dynamics hits a certain set. This proposition will be needed later on to establish recurrence properties to certain special sets.

Proposition 6 *Let $T \leq T' \leq T''$ and let $\mathcal{A}, \mathcal{B} \subset \mathcal{X}$. Suppose that there exists an event $\mathbf{E}_T \subset \mathcal{X}^{[0,T)}$ such that*

$$\begin{array}{ll} (i) & \mathbf{E}_T \subset \{\exists t \in [0,T)\colon \eta_t \in \mathcal{A}\} \\ (ii) & \min_{\eta_0 \in \mathcal{B}} \mathbb{P}_{\eta_0}(\mathbf{E}_T) \geq p > 0. \end{array} \tag{2.11}$$

Then

$$\begin{aligned} &\mathbb{P}_\nu\Big(\exists t \in [0,T'')\colon\ \eta_s \notin \mathcal{A}\ \forall s \in [t, t+T')\Big) \\ &\qquad \leq T''\Big[3|\text{BASE}(\mathcal{B}^c)^*|\nu(\mathcal{B}^c) + (1-p)^{T'/T}\Big]. \end{aligned} \tag{2.12}$$

Proof. Pick any $t \in [0, T'')$. Split the time interval $[t, t+T')$ into pieces of length T. By (2.11)(i–ii), on the event $\{\tau_{\mathcal{B}^c} \geq T''\}$, if at the beginning of a piece the process is not in $\mathcal{A}$, then it has a probability at most $1-p$ not to enter $\mathcal{A}$ during this piece. Hence the probability not to enter $\mathcal{A}$ during the time interval $[t, t+T')$ is at most $(1-p)^{T'/T}$ by the Markov property. Consequently,

$$\mathbb{P}_\nu\Big(\exists t \in [0,T'')\colon\ \eta_s \notin \mathcal{A}\ \forall s \in [t, t+T')\Big) \leq \mathbb{P}_\nu(\tau_{\mathcal{B}^c} < T'') + T''(1-p)^{T'/T}. \tag{2.13}$$

Now use Proposition 5 to get the claim. **QED**

2.2.3 Particle density in annuli around $\bar\Lambda$. Propositions 7–9 below give us control over the number of particles in annuli around $\bar\Lambda$ with a side length that is close to the mean particle distance on an exponential scale. In the proofs we compute the estimates using the grand-canonical Gibbs measure μ on $\mathbb{Z}^2$ with particle density ρ, rather than the canonical Gibbs measure ν on Λ_β with total particle number $\rho|\Lambda_\beta|$. However, by the equivalence of ensembles proved in Section 9, the difference is SES under our assumption that $\lim_{\beta\to\infty}\frac{1}{\beta}\log|\Lambda_\beta| = \infty$ (see the remark at the end of Section 9).

Proposition 7 *Let $\gamma>0$ and $l_+ = e^{\frac{1}{2}(\Delta+\gamma)\beta}$. Then for all $\gamma'\in(0,\gamma)$*

$$\nu\Big(\{\eta\in\mathcal{X}\colon\ N_{\Lambda_{l_+}\backslash\bar\Lambda_0}(\eta)\leq e^{\gamma'\beta}\}\Big) = \text{SES}. \tag{2.14}$$

Proof. Abbreviate $M=e^{\gamma'\beta}$. Let $\mathcal{A}=\{\eta\in\mathcal{X}\colon\ N_{\Lambda_{l_+}\backslash\bar\Lambda}(\eta)\leq M\}$. Then

$$\begin{aligned}
\mu(\mathcal{A}) &\leq e^M\int_{\mathcal{X}}\mu(d\eta)\ e^{-N_{\Lambda_{l_+}\backslash\bar\Lambda}(\eta)}\\
&= e^M\big[e^{-(1-e^{-1})\rho}\big]^{|\Lambda_{l_+}\backslash\bar\Lambda|}\\
&= e^M(1+o(1))\exp[-(1-e^{-1})e^{\gamma\beta}],
\end{aligned} \tag{2.15}$$

where we use that μ outside $\bar\Lambda$ places particles according to a Poisson random field with density ρ, and we note that $|\Lambda_{l_+}| = e^{\gamma\beta}/\rho$. **QED**

Proposition 8 *Let $\gamma>0$ and $l_- = e^{\frac{1}{2}(\Delta-\gamma)\beta}$. Then*

$$\nu\Big(\{\eta\in\mathcal{X}\colon\ N_{\Lambda_{l_-}\backslash\bar\Lambda_0}(\eta)\geq \log\beta\}\Big) = \text{SES}. \tag{2.16}$$

Proof. Abbreviate $M=\log\beta$. Let $\mathcal{A}=\{\eta\in\mathcal{X}\colon\ N_{\Lambda_{l_-}\backslash\bar\Lambda}(\eta)\geq M\}$. Then

$$\begin{aligned}
\mu(\mathcal{A}) &\leq e^{-\gamma\beta M}\int_{\mathcal{X}}\mu(d\eta)\ e^{\gamma\beta N_{\Lambda_{l_-}\backslash\bar\Lambda}(\eta)}\\
&= e^{-\gamma\beta M}\big[e^{(e^{\gamma\beta}-1)\rho}\big]^{|\Lambda_{l_-}\backslash\bar\Lambda|}\\
&= e^{-\gamma\beta M}(1+o(1)),
\end{aligned} \tag{2.17}$$

where we note that $|\Lambda_{l_-}| = e^{-\gamma\beta}/\rho$. **QED**

Proposition 9 *Let $\gamma>0$ and $l_- = e^{\frac{1}{2}(\Delta-\gamma)\beta}$. Then for all $n\in\mathbb{N}$*

$$\nu\Big(\{\eta\in\mathcal{X}\colon\ N_{2^{n+1}\Lambda_{l_-}\backslash 2^n\Lambda_{l_-}}(\eta)\geq (2^{n+1}-2^n)^2\log\beta\}\Big) = (\text{SES})^{2^{2n}}. \tag{2.18}$$

Proof. Abbreviate $M=\log\beta$. For $n\in\mathbb{N}$, let $\mathcal{A}_n=\{\eta\in\mathcal{X}\colon\ N_{2^{n+1}\Lambda_{l_-}\backslash 2^n\Lambda_{l_-}}(\eta) \geq (2^{n+1}-2^n)^2 M\}$. The same estimate as in (2.17) gives

$$\mu(\mathcal{A}_n)\leq e^{-\gamma(2^{n+1}-2^n)^2\beta\log\beta}(1+o(1)) \qquad (n\in\mathbb{N}) \tag{2.19}$$

with the error term uniform in n. **QED**

Define

$$\begin{array}{lcl}\mathcal{X}_0^1 & = & \{\eta\in\mathcal{X}\colon\ N_{\Lambda_{l_+}\backslash\bar\Lambda}(\eta)>e^{\gamma'\beta}\}\\ \mathcal{X}_0^2 & = & \{\eta\in\mathcal{X}\colon\ N_{\Lambda_{l_-}\backslash\bar\Lambda}(\eta)<\log\beta\}\\ \mathcal{X}_0^{3,n} & = & \{\eta\in\mathcal{X}\colon\ N_{2^{n+1}\Lambda_{l_-}\backslash 2^n\Lambda_{l_-}}(\eta)<(2^{n+1}-2^n)^2\log\beta\}\end{array}\tag{2.20}$$

and put

$$\mathcal{X}_0=\mathcal{X}_0^1\cap\mathcal{X}_0^2\cap\Big\{\bigcap_{n\in\mathbb{N}}\mathcal{X}_0^{3,n}\Big\}.\tag{2.21}$$

Proposition 10 *Let* $\mathbf{A}_T=\{\eta_t\in\mathcal{X}_0\ \forall t\in[0,T)\}$. *Then*

$$\mathbb{P}_\nu(\mathbf{A}_T^c)=\text{SES}\quad \textit{for all } T\leq e^{C\beta} \textit{ with } C \textit{ arbitrarily large.}\tag{2.22}$$

Proof. Estimate

$$\mathbb{P}_\nu(\mathbf{A}_T^c)\leq\mathbb{P}_\nu(\tau_{\mathcal{X}_0^1}<T)+\mathbb{P}_\nu(\tau_{\mathcal{X}_0^2}<T)+\sum_{n\in\mathbb{N}}\mathbb{P}_\nu(\tau_{\mathcal{X}_0^{3,n}}<T)\tag{2.23}$$

and use Proposition 5 in combination with Propositions 7–9. Note that $|\text{BASE}(\mathcal{X}_0^1)^*|$, $|\text{BASE}(\mathcal{X}_0^2)^*|$ and $2^{-2n}|\text{BASE}(\mathcal{X}_0^{3,n})^*|$ grow only exponentially fast with β. **QED**

Proposition 10 will be crucial later on. Namely, it says that over the exponentially long intervals we are considering for the metastable behavior we may as well assume that the process $(\eta_t)_{t\geq0}$ never leaves $\mathcal{X}_0$.

3 LD-estimates for independent random walks

In this section we formulate several large deviation estimates that involve hitting times for particles performing independent random walks. These estimates will be needed later on to control the gas outside $\bar\Lambda$. We do the estimates pretending that the random walk lives on $\mathbb{Z}^2$ instead of Λ_β. However, this only causes an error that is SES because of our assumption that $\lim_{\beta\to\infty}\frac{1}{\beta}\log|\Lambda_\beta|=\infty$.

3.1 LD-estimates for a single random walk

3.1.1 Hitting times. Let

$$(\xi_t)_{t\geq0}\tag{3.1}$$

be simple random walk on $\mathbb{Z}^2$ with jump rate 1. Let $\mathbb{P}_x$ denote its law on path space given $\xi_0=x$. Let $\tau_{\bar\Lambda}=\min\{t\geq0\colon\ \xi_t\in\bar\Lambda\}$. Proposition 11 below gives us control over $\tau_{\bar\Lambda}$ when the random walk starts from $x\in\partial^+\bar\Lambda$.

Proposition 11 *There exist* $\kappa>0$ *and* $t_0>0$ *such that for all* $t>t_0$

$$\min_{x\in\partial^+\bar\Lambda}\mathbb{P}_x(\tau_{\bar\Lambda}>t)\geq\frac{\kappa}{\log t}.\tag{3.2}$$

Proof. We begin by proving the analogous estimate for discrete time.
1. Let $(\xi_n)_{n\in\mathbb{N}_0}$ be simple random walk on $\mathbb{Z}^2$. Let $\tau_{\bar\Lambda} = \min\{n > 0\colon\ \xi_n \in \bar\Lambda\}$ (which does not include $n = 0$), and put

$$\theta_n^{\bar\Lambda} = \max\{0 \le m \le n\colon\ \xi_m \in \bar\Lambda\}. \tag{3.3}$$

Pick $x \in \bar\Lambda$ and write

$$1 = \sum_{m=0}^{n} \mathbb{P}_x(\theta_n^{\bar\Lambda} = m) = \sum_{m=0}^{n} \sum_{y\in\partial^-\bar\Lambda} \mathbb{P}_x(\xi_m = y)\mathbb{P}_y(\tau_{\bar\Lambda} > n-m). \tag{3.4}$$

Split the sum over m into two parts: $0 \le m \le n(1-\frac{1}{\log n})$ and the rest. The first part can be bounded above by

$$|\partial^-\bar\Lambda| \Big[1 + \sum_{m=1}^{n(1-\frac{1}{\log n})} \frac{\kappa_1}{m}\Big] \max_{y\in\partial^-\bar\Lambda} \mathbb{P}_y\Big(\tau_{\bar\Lambda} > \frac{n}{\log n}\Big), \tag{3.5}$$

where we use that

$$\max_{z\in\mathbb{Z}^2} \mathbb{P}_0(\xi_m = z) \le \frac{\kappa_1}{m} \qquad \forall m \ge 1 \tag{3.6}$$

(see Spitzer [21] Section 7). The second part can be bounded above by

$$|\partial^-\bar\Lambda| \sum_{m=n(1-\frac{1}{\log n})+1}^{n} \frac{\kappa_1}{m}. \tag{3.7}$$

Combining the two bounds in (3.5) and (3.7) with (3.4), we obtain, for n large enough,

$$\max_{y\in\partial^-\bar\Lambda} \mathbb{P}_y\Big(\tau_{\bar\Lambda} > \frac{n}{\log n}\Big) \ge \frac{\kappa_2}{\log n}. \tag{3.8}$$

2. Since any two sites in $\partial^-\bar\Lambda$ can be connected by a path outside $\bar\Lambda$ of length at most $2(\ell_0+1)$, it follows that uniformly in n

$$\min_{y\in\partial^-\bar\Lambda} \mathbb{P}_y\Big(\tau_{\bar\Lambda} > \frac{n}{\log n}\Big) \ge \kappa_3 \max_{y\in\partial^-\bar\Lambda} \mathbb{P}_y\Big(\tau_{\bar\Lambda} > \frac{n}{\log n} - \kappa_4\Big). \tag{3.9}$$

Together with (3.8) this settles the claim in (3.2) for discrete time.
3. The extension to continuous time is now trivial, via a standard LD-estimate on the clock of the random walk. **QED**

3.1.2 Trapping times. Let A be a finite non-empty subset of $\mathbb{Z}^2$ without holes. Let

$$(\widehat{\xi}_t^A)_{t\ge0} \tag{3.10}$$

be simple random walk on $\mathbb{Z}^2 \setminus [A \setminus \partial^- A]$ with jump rate 1 with the property that when it hits $\partial^- A$ it gets trapped, in the sense that a step from $\partial^- A$ to $\partial^+ A$ occur at rate $e^{-U\beta}$. Proposition 12 below gives us control over the time this random walk spends in the trap $\partial^- A$ starting from $x \in \partial^+ A$.

Proposition 12 *There exist $\kappa = \kappa(A) > 0$ and $\beta_0 > 0$ such that for all $\delta > 0$, all $\beta > \beta_0$ and all $t \in [e^{U\beta}, e^{C\beta}]$ with $U < C < \infty$*

$$\min_{x\in\partial^+ A} \mathbb{P}_x(\widehat{\xi}_t^A \in \partial^- A) \geq \frac{1}{t} e^{(U-\delta)\beta} \frac{\kappa}{2(C\beta)^2}. \tag{3.11}$$

Proof. Again, we first prove the analogous estimate for discrete time. The proof uses the following asymptotic result for simple random walk $(\xi_n)_{n\in\mathbb{N}_0}$ on $\mathbb{Z}^2$. Let $\tau_0 = \min\{n > 0\colon\ \xi_n = 0\}$. Then there exists a $\kappa_1 > 0$ such that

$$\mathbb{P}_x(\tau_0 = n) \sim \frac{\kappa_1}{n \log^2 n} \qquad (\forall x \in \mathbb{Z}^2,\ n \to \infty) \tag{3.12}$$

(see Spitzer [21] Section 7).
1. From (3.12) it is easily deduced that for all $A \subset \mathbb{Z}^2$, finite non-empty and without holes, there exists a $\kappa = \kappa(A) > 0$ such that

$$\min_{x\in\partial^+ A} \mathbb{P}_x(\tau_{\partial^- A} = n) \geq \frac{\kappa}{n \log^2 n} \qquad (n \to \infty), \tag{3.13}$$

where $\tau_{\partial^- A} = \min\{n > 0\colon\ \xi_n \in \partial^- A\}$.
2. Let $(\widehat{\xi}_n^A)_{n\in\mathbb{N}_0}$ be the discrete-time version of (3.10). Let $n_0 = e^{(U-\delta)\beta} \ll n$. Then, for $x \in \partial^+ A$,

$$\mathbb{P}_x(\widehat{\xi}_n^A \in \partial^- A) \geq (1 + o(1))\ \mathbb{P}_x(n - n_0 < \tau_{\partial^- A} \leq n). \tag{3.14}$$

Here we throw away all the first hits of A at or prior to time $n - n_0$ and require the random walk to stay trapped for a time at least n_0. The latter costs not more than $(1 - e^{-U\beta})^{n_0} = 1 - \text{SES}$. But, by (3.13), we have

$$\min_{x\in\partial^+ A} \mathbb{P}_x(n - n_0 < \tau_{\partial^- A} \leq n) \geq \sum_{m=n-n_0+1}^{n} \frac{\kappa}{m \log^2 m} \sim \frac{\kappa n_0}{n \log^2 n} \qquad (n \to \infty), \tag{3.15}$$

and so for all $n \in [e^{U\beta}, e^{C\beta}]$ and β sufficiently large

$$\min_{x\in\partial^+ A} \mathbb{P}_x(\widehat{\xi}_n \in \partial^- A) \geq \frac{1}{n} e^{(U-\delta)\beta} \frac{\kappa}{2(C\beta)^2}. \tag{3.16}$$

3. The extension to continuous time is again trivial, via a standard LD-estimate on the clock of the random walk. **QED**

3.2 Mixing lemmas for independent random walks

In the following lemmas, $\mathcal{X}_0$ is the set of configurations defined in (2.20–2.21).

For $\eta_0 \in \mathcal{X}_0$, let $\mathbf{C}_1^\gamma(\eta_0)$ denote the event that all particles in $\eta_0 \cap (\Lambda_{\ell_-} \setminus \bar{\Lambda})$ exit Λ_{ℓ_-} within time $T = e^{\Delta\beta}$ without entering $\bar{\Lambda}$ and remain outside Λ_{ℓ_-} until time T. We recall that $\ell_- = e^{\frac{1}{2}(\Delta-\gamma)\beta}$.

Lemma 13 *For all $\gamma > 0$ there exist $\kappa > 0$ and $\beta_0 > 0$ such that for all $\beta > \beta_0$: $\min_{\eta_0\in\mathcal{X}_0} \mathbb{P}(\mathbf{C}_1^\gamma(\eta_0)) \geq (\kappa/\beta)^{\log\beta}$.*

Proof. Because outside $\bar{\Lambda}$ particles perform independent simple random walks (see (3.1)), we have

$$\min_{\eta_0\in\mathcal{X}_0}\mathbb{P}(\mathbf{C}_1^\gamma(\eta_0)) \geq \Big[\inf_{x\in\partial^+\bar{\Lambda}}\mathbb{P}_x(\tau_{\mathbb{Z}^2\setminus\Lambda_{\ell_-}}\leq T,\,\tau_{\bar{\Lambda}}>\tau_{\mathbb{Z}^2\setminus\Lambda_{\ell_-}},\,\xi_T\notin\Lambda_{\ell_-})\Big]^{\log\beta}, \tag{3.17}$$

where we use that $N_{\Lambda_{\ell_-}\setminus\bar{\Lambda}}(\eta_0)\leq\log\beta$ for all $\eta_0\in\mathcal{X}_0$, and that the probability between square brackets is minimal for $x\in\Lambda_{\ell_-}\setminus\bar{\Lambda}$ when $x\in\partial^+\bar{\Lambda}$. We have

$$\begin{aligned}&\mathbb{P}_x(\tau_{\mathbb{Z}^2\setminus\Lambda_{\ell_-}}\leq T,\,\tau_{\bar{\Lambda}}>\tau_{\mathbb{Z}^2\setminus\Lambda_{\ell_-}},\,\xi_T\notin\Lambda_{\ell_-})\\ &\quad\geq\mathbb{P}_x(\tau_{\bar{\Lambda}}>\tau_{\mathbb{Z}^2\setminus\Lambda_{\ell_-}})-\mathbb{P}_x(\tau_{\mathbb{Z}^2\setminus\Lambda_{\ell_-}}>T)-\mathbb{P}_x(\xi_T\in\Lambda_{\ell_-}).\end{aligned} \tag{3.18}$$

But, by Brownian approximation, we know that

$$\begin{array}{llllll}\min_{x\in\partial^+\bar{\Lambda}}\mathbb{P}_x(\tau_{\bar{\Lambda}}>\tau_{\mathbb{Z}^2\setminus\Lambda_{\ell_-}}) & \geq & \frac{\kappa_1}{\log l_-} & \sim & \frac{2\kappa_1}{(\Delta-\gamma)\beta}\\ \max_{x\in\partial^+\bar{\Lambda}}\mathbb{P}_x(\tau_{\mathbb{Z}^2\setminus\Lambda_{\ell_-}}>T) & \leq & \exp[-\kappa_2 T/l_-^2] & = & \mathrm{SES},\end{array} \tag{3.19}$$

while (3.6) gives

$$\max_{\partial^+\bar{\Lambda}}\mathbb{P}_x(\xi_T\in\Lambda_{\ell_-})\leq\frac{\kappa_3|\Lambda_{\ell_-}|}{T}=\kappa_3 e^{-\gamma\beta}. \tag{3.20}$$

Insert (3.19–3.20) into (3.18) to get the claim. **QED**

For $\eta_0\in\mathcal{X}_0$, let $\mathbf{C}_2^{\gamma,\delta}(\eta_0)$ denote the event that no particle in $\eta_0\cap(\mathbb{Z}^2\setminus\Lambda_{\ell_-})$ enters $\bar{\Lambda}$ within time $T=e^{(\Delta-\delta)\beta}$.

Lemma 14 *For all $\delta>\gamma>0$:* $\inf_{\eta_0\in\mathcal{X}_0}\mathbb{P}(\mathbf{C}_2^{\gamma,\delta}(\eta_0))=1-\mathrm{SES}$.

Proof. We have

$$\mathbb{P}_{\eta_0}(\mathbf{C}_2^{\gamma,\delta}(\eta_0))=\prod_{x\in\mathbb{Z}^2\setminus\Lambda_{\ell_-}}\mathbb{P}_x(\tau_{\bar{\Lambda}}>T)^{\eta_0(x)}=\prod_{x\in\mathbb{Z}^2\setminus\Lambda_{\ell_-}}[1-\mathbb{P}_x(\tau_{\bar{\Lambda}}\leq T)]^{\eta_0(x)}. \tag{3.21}$$

But, by Brownian approximation, we have

$$\mathbb{P}_x(\tau_{\bar{\Lambda}}\leq T)\leq\exp[-\kappa|x|^2/T]\ll 1 \tag{3.22}$$

uniformly in $x\in\mathbb{Z}^2\setminus\Lambda_{\ell_-}$. Hence, for β sufficiently large,

$$\mathbb{P}_{\eta_0}(\mathbf{C}_2^{\gamma,\delta}(\eta_0))\geq\exp\Big\{-\frac{1}{2}\sum_{x\in\mathbb{Z}^2\setminus\Lambda_{\ell_-}}\eta_0(x)\;\exp[-\kappa|x|^2/T]\Big\}. \tag{3.23}$$

The sum in the exponent can be estimated from above by

$$\sum_{n=0}^{\infty}N_{2^{n+1}\Lambda_{\ell_-}\setminus 2^n\Lambda_{\ell_-}}(\eta_0)\;\exp[-\kappa\tfrac{1}{2}2^{2n}e^{(\delta-\gamma)\beta}]. \tag{3.24}$$

Hence the claim follows via (2.20). **QED**

For $\eta_0 \in \mathcal{X}_0$, $k \in \mathbb{N}$, $t_1, \dots, t_k \geq 0$ and $x_1, \dots, x_k \in \partial^- \bar\Lambda$, let

$$\mathbf{C}_3^{\gamma,\delta}(\eta_0; t_1, \dots, t_k; x_1, \dots, x_k) \tag{3.25}$$

denote the event that k particles from $\eta_0 \cap (\mathbb{Z}^2 \setminus \Lambda_{\ell_-})$ enter $\bar\Lambda$ during the time intervals $[t_1, t_1+1), \dots, [t_k, t_k+1)$ at sites $x_1, \dots, x_k$ (with no restriction on what happens at other times).

Lemma 15 *For all $\gamma, \delta > 0$ there exist $\kappa > 0$ and $\beta_0 > 0$ such that for all $\beta > \beta_0$:*

$$\min_{\eta_0 \in \mathcal{X}_0} \mathbb{P}(\mathbf{C}_3^{\gamma,\delta}(\eta_0; t_1, \dots, t_k; x_1, \dots, x_k)) \leq \left[\kappa e^{-(\Delta-\delta)\beta} \log \beta\right]^k \tag{3.26}$$

uniformly in $k \in \mathbb{N}$, $0 \leq t_1, \dots, t_k \leq T = e^{C\beta}$ and $x_1, \dots, x_k \in \partial^- \bar\Lambda$, with C arbitrarily large.

Proof. Obviously,

$$\mathbb{P}_{\eta_0}(\mathbf{C}_3^{\gamma,\delta}(\eta_0; t_1, \dots, t_k; x_1, \dots, x_k)) \leq \prod_{l=1}^{k} \mathbb{P}_{\eta_0}(\mathbf{C}_3^{\gamma,\delta}(\eta_0; t_l; x_l)) \tag{3.27}$$

and so it suffices to prove the claim for $k = 1$. Consider first the case where $T_1 = e^{(\Delta-\delta)\beta} \leq t_1 \leq T = e^{C\beta}$. Let us look at the particle configuration at time $t_1 - T_1$. By Proposition 10 we know that with a probability $1 - \text{SES}$ this configuration falls in $\mathcal{X}_0$. Hence, using the Markov property at time $t_1 - T_1$, we get

$$\begin{aligned}
&\max_{\eta_0 \in \mathcal{X}_0} \mathbb{P}_{\eta_0}(\mathbf{C}_3^{\gamma,\delta}(\eta_0; t_1; x_1)) \\
&\qquad = \text{SES} + \max_{\eta_0 \in \mathcal{X}_0} \sum_{\eta \in \mathcal{X}_0} \mathbb{P}_{\eta_0}(\eta_{t_1 - T_1} = \eta) \mathbb{P}_{\eta_0}(\mathbf{C}_3^{\gamma,\delta}(\eta_0; t_1; x_1) \mid \eta_{t_1 - T_1} = \eta).
\end{aligned} \tag{3.28}$$

But, by Lemma 14 and (3.6), we have

$$\begin{aligned}
\max_{\eta \in \mathcal{X}_0} \mathbb{P}_{\eta_0}(\mathbf{C}_3^{\gamma,\delta}(\eta_0; t_1; x_1) \mid \eta_{t_1 - T_1} = \eta) &\leq \text{SES} + \max_{\eta \in \mathcal{X}_0} \sum_{x \in \Lambda_{\ell_-} \setminus \bar\Lambda} \eta(x) \mathbb{P}_x(\xi_{T_1} = x_1) \\
&\leq \text{SES} + \tfrac{\kappa_1}{T_1} \max\nolimits_{\eta \in \mathcal{X}_0} N_{\Lambda_{\ell_-} \setminus \bar\Lambda}(\eta) \\
&\leq \text{SES} + \kappa_1 e^{-(\Delta-\delta)\beta} \log \beta.
\end{aligned} \tag{3.29}$$

Substitution into (3.28) gives the claim. The case where $0 \leq t_1 < T_1$ follows from Lemma 14 and gives SES. **QED**

4 Local Markov chain: definitions and saddle points

In this section we introduce the local Markov chain that approximates our dynamics inside $\bar{\Lambda}$, and we study its geometric properties. In Section 5 we will study the recurrence properties of this Markov chain, which will be needed in Sections 6–7 to study the metastable behavior of our dynamics.

4.1 Definition of the local Markov chain

We denote by $b = (x,y)$ an *oriented* bond, i.e., an *ordered* pair of nearest-neighbour sites, and define

$$\begin{array}{rcl} \partial^*\bar{\Lambda}^{out} & = & \{b=(x,y)\colon\ x\in\bar{\Lambda}, y\notin\bar{\Lambda}\} \\ \partial^*\bar{\Lambda}^{in} & = & \{b=(x,y)\colon\ x\notin\bar{\Lambda}, y\in\bar{\Lambda}\} \end{array} \tag{4.1}$$

and $\partial^*\bar{\Lambda} = \partial^*\bar{\Lambda}^{out} \cup \partial^*\bar{\Lambda}^{in}$. Two configurations $\bar{\eta},\bar{\eta}' \in \bar{\mathcal{X}}$ with $\bar{\eta} \neq \bar{\eta}'$ are called *communicating states* if there exists a bond $b \in \bar{\Lambda}^* \cup \partial^*\bar{\Lambda}$ such that $\bar{\eta}' = T_b\bar{\eta}$, where $T_b\bar{\eta}$ is the configuration obtained from $\bar{\eta}$ as follows:

- $b \in \bar{\Lambda}^*$: $T_b\bar{\eta}$ denotes the configuration obtained from $\bar{\eta}$ by interchanging particles along b;
- $b \in \partial^*\bar{\Lambda}^{out}$ (i.e., b is exiting from $\bar{\Lambda}$):

$$T_b\bar{\eta}(z) = \begin{cases} \bar{\eta}(z) & \forall z \neq x \\ 0 & z = x, \end{cases} \tag{4.2}$$

- $b \in \partial^*\bar{\Lambda}^{in}$ (i.e., b is entering $\bar{\Lambda}$):

$$T_b\bar{\eta}(z) = \begin{cases} \bar{\eta}(z) & \forall z \neq y \\ 1 & z = y. \end{cases} \tag{4.3}$$

The *local Markov chain* $(\bar{\eta}_t)_{t\geq 0}$ is defined to be the Markov chain on $\bar{\mathcal{X}} = \{0,1\}^{\bar{\Lambda}}$ with generator

$$(\mathcal{L}f)(\bar{\eta}) = \sum_{b\in\bar{\Lambda}^*\cup\partial^*\bar{\Lambda}} c(b,\bar{\eta})[f(T_b\bar{\eta}) - f(\bar{\eta})], \tag{4.4}$$

where

$$c(b,\bar{\eta}) = e^{-\beta[\bar{H}(T_b\bar{\eta})-\bar{H}(\bar{\eta})]_+}. \tag{4.5}$$

Note that

$$\begin{array}{ll} b\in\partial^*\bar{\Lambda}^{in}: & c(b,\bar{\eta}) = e^{-\beta\Delta} \\ b\in\partial^*\bar{\Lambda}^{out}: & c(b,\bar{\eta}) = 1. \end{array} \tag{4.6}$$

Also note that an exchange of occupation numbers inside the ring $\bar{\Lambda}\setminus\bar{\Lambda}_0$ or between $\bar{\Lambda}_0$ and the ring $\bar{\Lambda}\setminus\bar{\Lambda}_0$ does not involve any change in energy.

In a standard way the above dynamics can be realized with the help of Poisson clocks. To study the transitions of the local Markov chain, we consider the discrete-time version that is obtained from the continuous-time version by looking at the process when some clock in $\bar\Lambda^* \cup \partial\bar\Lambda^*$ rings. We denote by $\bar P(\bar\eta, \bar\eta')$ the corresponding transition probabilities, i.e.,

$$\bar P(\bar\eta, \bar\eta') = \mathbb{P}_{\bar\eta}(\bar\eta_{\bar\tau_1} = \bar\eta') \tag{4.7}$$

with $\bar\tau_1$ the first ringing time. It is easy to verify that the stochastic dynamics defined by (4.4–4.5) and (4.7) is reversible w.r.t. $\bar H$. In particular, the transition probabilities $\bar P(\bar\eta, \bar\eta')$ can be written in the form

$$\bar P(\bar\eta, \bar\eta') = q(\bar\eta, \bar\eta') e^{-\beta[\bar H(\bar\eta') - \bar H(\bar\eta)]_+}, \tag{4.8}$$

where $q(\bar\eta, \bar\eta')$ is an irreducible symmetric Markov kernel characterizing the set of communicating states.

4.2 Definitions

Let us recall some definitions from Olivieri and Scoppola [10].

1. A *path* ϕ is a sequence $\phi = \phi_1, \dots, \phi_n$ ($n \in \mathbb{N}$, $\phi_i \in \bar{\mathcal{X}}$) with $\bar P(\phi_i, \phi_{i+1}) > 0$ for $i = 1, \dots, n-1$. We write $\phi: \bar\eta \to \bar\eta'$ to denote a path from $\bar\eta$ to $\bar\eta'$. A set $\mathcal{A} \subset \bar{\mathcal{X}}$ with $|\mathcal{A}| > 1$ is *connected* if and only if $\forall \bar\eta, \bar\eta' \in \mathcal{A}\ \exists \phi: \bar\eta \to \bar\eta'$ such that $\phi \subset \mathcal{A}$. Given $\mathcal{A} \subset \bar{\mathcal{X}}$, we define its *boundary*

$$\partial\mathcal{A} = \{\bar\zeta \notin \mathcal{A}:\ \bar P(\bar\zeta, \bar\eta) > 0\}. \tag{4.9}$$

2. The set of *minima* of the Hamiltonian $\bar H$ in $\mathcal{A}$ is

$$\mathcal{F}(\mathcal{A}) = \Big\{\bar\eta \in \mathcal{A}:\ \bar H(\bar\eta) = \min_{\bar\zeta \in \mathcal{A}} \bar H(\bar\zeta)\Big\}. \tag{4.10}$$

The *communication height* between $\bar\eta, \bar\eta' \in \bar{\mathcal{X}}$ is

$$\bar H(\bar\eta, \bar\eta') = \min_{\phi:\ \bar\eta \to \bar\eta'} \max_{\bar\zeta \in \phi} \bar H(\bar\zeta). \tag{4.11}$$

The set of configurations realizing the *minimal saddles* between $\bar\eta, \bar\eta' \in \bar{\mathcal{X}}$ is

$$\mathcal{S}(\bar\eta, \bar\eta') = \Big\{\bar\zeta \in \bar{\mathcal{X}}:\ \exists\phi:\ \bar\eta \to \bar\eta',\ \phi \ni \bar\zeta:\ \max_{\bar\xi \in \phi} \bar H(\bar\xi) = \bar H(\bar\zeta) = \bar H(\bar\eta, \bar\eta')\Big\}. \tag{4.12}$$

Given two sets $\mathcal{A}, \mathcal{B} \subset \bar{\mathcal{X}}$, put

$$\bar H(\mathcal{A}, \mathcal{B}) = \min_{\bar\eta \in \mathcal{A}, \bar\eta' \in \mathcal{B}} \bar H(\bar\eta, \bar\eta') \tag{4.13}$$

and

$$\mathcal{S}(\mathcal{A}, \mathcal{B}) = \Big\{\mathcal{S}(\bar\eta, \bar\eta'):\ \bar\eta \in \mathcal{A},\ \bar\eta' \in \mathcal{B},\ \bar H(\bar\eta, \bar\eta') = \bar H(\mathcal{A}, \mathcal{B})\Big\}. \tag{4.14}$$

3. Next we introduce a geometric description of the configurations in terms of contours. Given a configuration $\bar\eta \in \bar{\mathcal{X}}$, consider the set $C(\bar\eta) \subset \mathbb{R}^2$ defined as the union of the 1×1 closed squares centered at the occupied sites of $\bar\eta$ in $\bar\Lambda_0$. The maximal connected components $C_1, \dots, C_m$ ($m \in \mathbb{N}$) of $C(\bar\eta)$ are called *clusters* of $\bar{\eta}$. The centers of the unit squares of a cluster C form a *-cluster (i.e., are within distance $\sqrt{2}$). The boundary of a cluster C is a polygon connecting sites on the dual lattice $(\mathbb{Z}^2)^* = \mathbb{Z}^2 + (\frac{1}{2}, \frac{1}{2})$. At each site an even number of bonds of this polygon meet: 0, 2 or 4. When this number is 4, we use the convention to 'round off' the corners: in this way we obtain a further decomposition of the boundary of a cluster into a set $\bar{\gamma}_1, \dots, \bar{\gamma}_k$ of closed self-avoiding contours (see e.g. Gallavotti [6]).

4. Let $\bar{\eta}$ be such that $\bar{\eta}|_{\bar{\Lambda}_0}$ gives rise to a single contour $\bar{\gamma} = \bar{\gamma}(\bar{\eta})$. Define

$$n(\bar{\eta}) = N_{\bar{\Lambda}\setminus\bar{\Lambda}_0}(\bar{\eta}). \tag{4.15}$$

In this case

$$\bar{H}(\bar{\eta}) = E(\bar{\gamma}) + \Delta n(\bar{\eta}), \tag{4.16}$$

where

$$E(\bar{\gamma}) = (-2U + \Delta)\langle\bar{\gamma}\rangle + \frac{U}{2}|\gamma|, \tag{4.17}$$

with $\langle\bar{\gamma}\rangle$ the area enclosed by $\bar{\gamma}$ and $|\bar{\gamma}|$ the perimeter of $\bar{\gamma}$. Indeed, it is easy to check that $2\langle\bar{\gamma}\rangle - \frac{1}{2}|\bar{\gamma}|$ is the number of nearest-neighbor bonds enclosed by $\bar{\gamma}$. (Note that, since there is no interaction inside $\bar{\Lambda} \setminus \bar{\Lambda}_0$ nor between $\bar{\Lambda}_0$ and $\bar{\Lambda} \setminus \bar{\Lambda}_0$, for the computation of $E(\bar{\gamma})$, the 'energy' of $\bar{\gamma}$, everything is as if we had empty boundary conditions outside $\bar{\Lambda}_0$.)

5. We denote by $R(\bar{\gamma})$ the rectangle circumscribing the contour $\bar{\gamma}$, and by $\ell_1(\bar{\gamma}), \ell_2(\bar{\gamma})$ the lengths of its sides. We use the convention $\ell_1(\bar{\gamma}) \leq \ell_2(\bar{\gamma})$, but *include all the translations and rotations of the retangle.* We denote by $\mathcal{R}_{\ell_1,\ell_2}$ the set of configurations whose single contour is an $\ell_1 \times \ell_2$ rectangle. We call *monotone* a contour $\bar{\gamma}$ such that its perimeter coincides with that of the circumscribed rectangle: $|\bar{\gamma}| = 2(\ell_1(\bar{\gamma}) + \ell_2(\bar{\gamma}))$.

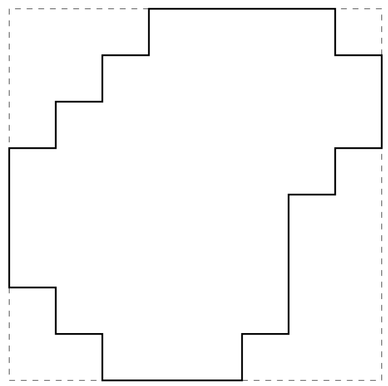

FIG. 7. A monotone contour

6. Given integers $\ell_1, \ell_2 \geq 2$, we define

– $\mathcal{D}_{\ell_1,\ell_2}$ the set of configurations where the occupied sites form an $\ell_1 \times \ell_2$ rectangle contained in $\bar{\Lambda}_0$ plus one free particle, i.e., a particle in $\bar{\Lambda}$ not touching the rectangle;

– $\widehat{\mathcal{D}}^-_{\ell_1,\ell_2}$ the set of configurations obtained from a configuration in $\mathcal{D}_{\ell_1,\ell_2}$ by attaching the free particle to one of the sides.

– $\mathcal{D}^-_{\ell_1,\ell_2}$ the set of configurations given by

$$\mathcal{D}^-_{\ell_1,\ell_2} = \{\bar\eta' \in \bar{\mathcal{X}}\colon\ \exists \bar\eta \in\ \widehat{\mathcal{D}}^-_{\ell_1,\ell_2}\colon\ \bar H(\bar\eta,\bar\eta') \le \bar H(\bar\eta) + U,\ \bar H(\bar\eta) = \bar H(\bar\eta')\}. \tag{4.18}$$

In other words, $\mathcal{D}^-_{\ell_1,\ell_2}$ is the set of configurations that can be connected to $\widehat{\mathcal{D}}^-_{l_1,l_2}$ by paths $\phi = \phi_1,\dots,\phi_n$ ($n \in \mathbb{N}$) such that

$$\phi_1 = \bar\eta, \quad \phi_n = \bar\eta', \quad \max_{1\le i<n} \bar H(\phi_i) \le \bar H(\bar\eta) + U, \quad \bar H(\bar\eta) = \bar H(\bar\eta'). \tag{4.19}$$

It is easy to see that $\mathcal{D}^-_{\ell_1,\ell_2}$ contains only configurations giving rise to a single monotone contour $\bar\gamma$ contained in $\bar\Lambda_0$ such that:

- $\langle\bar\gamma\rangle = \ell_1\ell_2 + 1$;
- $R(\bar\gamma)$ has side lengths $\ell_1(\bar\gamma) \le \ell_1 + 2$ and $\ell_2(\bar\gamma) \le \ell_2 + 2$;
- $\langle\bar\gamma\rangle$ contains a rectangle with side lengths $(\ell_1 - 4)\wedge 0, \ell_2 - 2$,

i.e., the interior of $\bar\gamma$ contains a suitably large rectangular core.

– $\mathcal{D}^0_{\ell_1,\ell_2}$ the set of configurations obtained from a configuration in $\mathcal{D}^-_{\ell_1,\ell_2}$ by adding a free particle.

– $\mathcal{D}^+_{\ell_1,\ell_2}$ the set of configurations obtained from a configuration in $\mathcal{D}^0_{\ell_1,\ell_2}$ by attaching the free particle to an external corner of the contour in $\mathcal{D}^-_{\ell_1,\ell_2}$. In particular, $\mathcal{D}^+_{\ell_1,\ell_2}$ contains:

– $\widehat{\mathcal{D}}^+_{\ell_1,\ell_2}$ the set of configurations where the occupied sites form an $\ell_1 \times \ell_2$ rectangle contained in $\bar\Lambda_0$ plus a 1×2 protuberance attached to one of the sides. Note that $\widehat{\mathcal{D}}^+_{\ell_1,\ell_2}$ can be obtained from $\widehat{\mathcal{D}}^-_{\ell_1,\ell_2}$ by attaching a 1×1 square to the 1×1 protuberance, in an external corner.

7. A for our purpose particularly important set of configurations, which play the role of 'critical configurations', is given by

$$\bar{\mathcal{C}}^* = \mathcal{D}^0_{\ell_c-1,\ell_c} \tag{4.20}$$

with

$$\ell_c = \left\lceil \frac{U}{2U-\Delta} \right\rceil. \tag{4.21}$$

Recall that in the simple static analysis developed in Section 1.2.3 this value came out as the critical droplet size. We denote by $\Gamma = \Gamma(U,\Delta)$ the energy of the critical configuration:

$$\Gamma = \bar H(\bar{\mathcal{C}}^*) = -U(2\ell_c^2 - 4\ell_c + 2) + \Delta(\ell_c^2 - \ell_c + 2). \tag{4.22}$$

We recall that $\bar\square$ denotes the configuration where $\bar\Lambda$ is empty and $\blacksquare$ the set of configurations where $\bar\Lambda_0$ is full.

4.3 Identification of saddle points

The following proposition is the key result of this section. It identifies the saddle points for the transitions between rectangular droplets for the local Markov chain (compare with (1.44)).

Proposition 16 *(i) For $2 \le \ell < \ell_c$:*

$$\begin{array}{l} \mathcal{S}(\mathcal{R}_{\ell,\ell}, \cup_{(\ell_1,\ell_2)\neq(\ell,\ell)}\mathcal{R}_{\ell_1,\ell_2}) = \mathcal{S}(\mathcal{R}_{\ell,\ell}, \mathcal{R}_{\ell-1,\ell}) = \mathcal{D}^0_{\ell-1,\ell} \\ \bar{H}(\mathcal{D}^0_{\ell-1,\ell}) - \bar{H}(\mathcal{R}_{\ell,\ell}) = (2U-\Delta)(\ell-2) + 2U \\ \\ \mathcal{S}(\mathcal{R}_{\ell,\ell+1}, \cup_{(\ell_1,\ell_2)\neq(\ell,\ell+1)}\mathcal{R}_{\ell_1,\ell_2}) = \mathcal{S}(\mathcal{R}_{\ell,\ell+1}, \mathcal{R}_{\ell,\ell}) = \mathcal{D}^0_{\ell,\ell} \\ \bar{H}(\mathcal{D}^0_{\ell,\ell}) - \bar{H}(\mathcal{R}_{\ell,\ell+1}) = (2U-\Delta)(\ell-2) + 2U. \end{array} \tag{4.23}$$

(ii) For $\ell \ge \ell_c$:

$$\begin{array}{l} \mathcal{S}(\mathcal{R}_{\ell,\ell}, \cup_{(\ell_1,\ell_2)\neq(\ell,\ell)}\mathcal{R}_{\ell_1,\ell_2}) = \mathcal{S}(\mathcal{R}_{\ell,\ell}, \mathcal{R}_{\ell,\ell+1}) = \mathcal{D}^0_{\ell,\ell} \\ \bar{H}(\mathcal{D}^0_{\ell,\ell}) - \bar{H}(\mathcal{R}_{\ell,\ell}) = 2\Delta - U \\ \\ \mathcal{S}(\mathcal{R}_{\ell,\ell+1} \cup_{(\ell_1,\ell_2),\neq(\ell,\ell+1)} \mathcal{R}_{\ell_1,\ell_2}) = \mathcal{S}(\mathcal{R}_{\ell,\ell+1}, \mathcal{R}_{\ell+1,\ell+1}) = \mathcal{D}^0_{\ell,\ell+1} \\ \bar{H}(\mathcal{D}^0_{\ell,\ell+1}) - \bar{H}(\mathcal{R}_{\ell,\ell+1}) = 2\Delta - U. \end{array} \tag{4.24}$$

(iii)

$$\begin{array}{l} \mathcal{S}(\bar{\square}, \bar{\blacksquare}) = \mathcal{D}^0_{\ell_c-1,\ell_c} \\ \bar{H}(\bar{\square}, \bar{\blacksquare}) - \bar{H}(\bar{\square}) = \Gamma. \end{array} \tag{4.25}$$

Proof. Let

$$\mathcal{N}_n = \{\bar{\eta} \in \bar{\mathcal{X}}:\ N_{\bar{\Lambda}}(\bar{\eta}) = \sum_{x\in\bar{\Lambda}} \bar{\eta}(x) = n\}. \tag{4.26}$$

We consider the foliation of $\bar{\mathcal{X}}$ into manifolds of a constant number of particles:

$$\bar{\mathcal{X}} = \cup_{n=0}^{|\bar{\Lambda}|}\mathcal{N}_n. \tag{4.27}$$

We will consider the sets $\mathcal{N}_{\ell^2}, \mathcal{N}_{\ell(\ell+1)}$ for $\ell \ge 2$ and investigate some of their geometric properties. Our proof will be based on the following lemma identifying certain sets of minima, minimal saddles and communication heights. After this lemma has been proved we givethe proof of the proposition.

Lemma 17 *(i) For $\ell \ge 2$:*

$$\mathcal{F}(\mathcal{N}_{(\ell-1)\ell}) = \mathcal{R}_{\ell-1,\ell}, \quad \mathcal{F}(\mathcal{N}_{\ell^2}) = \mathcal{R}_{\ell,\ell}. \tag{4.28}$$

(ii) For $\ell \ge 2$:

$$\begin{array}{l} \mathcal{S}(\mathcal{R}_{\ell-1,\ell}, \mathcal{N}_{\ell^2}) = \mathcal{S}(\mathcal{N}_{(\ell-1)\ell}, \mathcal{N}_{\ell^2}) = \mathcal{D}^0_{\ell-1,\ell} \\ \bar{H}(\mathcal{R}_{\ell-1,\ell}, \mathcal{N}_{\ell^2}) = \bar{H}(\mathcal{D}^0_{\ell-1,\ell}) = \bar{H}(\mathcal{D}^-_{\ell-1,\ell}) + \Delta \\ \\ \mathcal{S}(\mathcal{R}_{\ell,\ell}, \mathcal{N}_{\ell(\ell+1)}) = \mathcal{S}(\mathcal{N}_{\ell^2}, \mathcal{N}_{\ell(\ell+1)}) = \mathcal{D}^0_{\ell,\ell} \\ \bar{H}(\mathcal{R}_{\ell,\ell}, \mathcal{N}_{\ell(\ell+1)}) = \bar{H}(\mathcal{D}^0_{\ell,\ell}) = \bar{H}(\mathcal{D}^-_{\ell,\ell}) + \Delta. \end{array} \tag{4.29}$$

Proof. (i) Fix $\ell \geq 2$ and consider $n = (\ell-1)\ell$ or $n = \ell^2$. Given an $\bar{\eta} \in \mathcal{N}_n$, the energy decreases if we translate the clusters of $\bar{\eta}$ to join them into a single cluster contained in $\bar{\Lambda}_0$. It further decreases if we rearrange the 1×1 squares to get a single contour $\bar{\gamma}$ (i.e., if we fill the internal 'holes' with external 1×1 squares). Since under these operations the total number of particles remains fixed, to minimize the energy in $\mathcal{N}_n$ we just have to find the contour(s) $\bar{\gamma}$ with minimal perimeter $|\bar{\gamma}|$ among the ones with $\langle\bar{\gamma}\rangle = n$. It is clear that, starting from a contour $\bar{\gamma}'$, by rearranging the 1×1 squares inside we can construct a monotone contour $\bar{\gamma}$ with $R(\bar{\gamma}) \subseteq R(\bar{\gamma}')$ without increasing the energy. The energy associated with a monotone contour $\bar{\gamma}$ with $\langle\bar{\gamma}\rangle = n$ is

$$E(\bar{\gamma}) = (-2U + \Delta)n + U(\ell_1(\bar{\gamma}) + \ell_2(\bar{\gamma})). \tag{4.30}$$

To minimize $E(\bar{\gamma})$ in $\mathcal{N}_n$, we have to find the rectangle with minimal perimeter among those whose area is $\geq n$. From this the claim easily follows.

(ii) Fix $\ell \geq 2$.
1. We first prove the claim when starting from $\mathcal{R}_{\ell-1,\ell}$. We define a set of paths $\phi\colon \mathcal{R}_{\ell-1,\ell} \to \mathcal{N}_{\ell^2}$ as follows:

- Let

  $$\phi^{up} = \phi^{up}(\ell-1,\ell)\colon\ \mathcal{R}_{\ell-1,\ell} \to \mathcal{D}^+_{\ell-1,\ell} \tag{4.31}$$

  be defined by

  $$\phi^{up} = \phi_1, \dots, \phi_2, \dots, \phi_3, \dots, \phi_4, \dots, \phi_5 \tag{4.32}$$

  with

  $$\phi_1 \in \mathcal{R}_{\ell-1,\ell}, \quad \phi_2 \in \mathcal{D}_{\ell-1,\ell}, \quad \phi_3 \in \mathcal{D}^-_{\ell-1,\ell}, \quad \phi_4 \in \mathcal{D}^0_{\ell-1,\ell}, \quad \phi_5 \in \mathcal{D}^+_{\ell-1,\ell}, \tag{4.33}$$

  where ϕ uses $\phi_1, \phi_2, \phi_3, \phi_4, \phi_5$ as a 'skeleton' and the successive configurations in ϕ are obtained in the obvious way by successively adding or moving a suitable particle (see 6 in Section 4.2). The maximal saddle in ϕ^{up} is reached in $\mathcal{D}^0_{\ell-1,\ell}$ and is of height $\bar{H}(\mathcal{D}^0_{\ell-1,\ell}) = \bar{H}(\mathcal{D}^-_{\ell-1,\ell}) + \Delta$.

- It is easy to see that there is a path

  $$\phi^{down} = \phi^{down}(\ell-1,\ell)\colon\ \mathcal{D}^+_{\ell-1,\ell} \to \mathcal{N}_{\ell^2} \tag{4.34}$$

  such that $\max_i \bar{H}(\phi_i^{down}, \phi_{i+1}^{down}) < H(\mathcal{D}^-_{\ell-1,\ell}) + \Delta$. Indeed, to obtain this path it suffices to successively introduce into $\bar{\Lambda}$ one new particle and fill up all the corners of the contours in $\mathcal{D}^+_{\ell-1,\ell}$ until the arrival in $\mathcal{R}_{\ell,\ell} \subset \mathcal{N}_{\ell^2}$. Each time we add a particle, we have first an increase of energy by an amount Δ, but as soon as we put this particle into a corner of the cluster in $\bar{\Lambda}_0$ we have a decrease of energy by an amount $2U > \Delta$.

Thus, for each $\ell \geq 2$, the path $(\phi^{up}(\ell-1,\ell), \phi^{down}(\ell-1,\ell))$ is a candidate to realize the minimax between $\mathcal{N}_{(\ell-1)\ell}$ and $\mathcal{N}_{\ell^2}$. These paths will turn out to be useful to make a comparison with other possible candidates.

2. When starting from $\mathcal{R}_{\ell,\ell}$ we proceed exactly in the same way to construct a path $(\phi^{up}(\ell,\ell),\phi^{down}(\ell,\ell))$.
3. From the proof of (i) it is immediate to see that

$$\mathcal{F}(\mathcal{N}_{(\ell-1)\ell}\setminus\mathcal{R}_{\ell-1,\ell})=\min_{\bar\eta\in\mathcal{N}_{(\ell-1)\ell}\setminus\mathcal{R}_{\ell-1,\ell}}\bar H(\bar\eta)=\bar H(\mathcal{R}_{\ell-1,\ell})+U. \tag{4.35}$$

From this it follows that for any path passing through $\mathcal{N}_{(\ell-1)\ell}\setminus\mathcal{R}_{\ell-1,\ell}$, once it meets $\mathcal{N}_{(\ell-1)\ell+1}$ it gets an energy $\geq \bar H(\mathcal{R}_{\ell-1,\ell})+U+\Delta$, which is strictly larger than

$$\max_{i=1,\dots,5}\bar H(\phi_i)=\bar H(\mathcal{D}^0_{\ell-1,\ell})=\bar H(\mathcal{R}_{\ell-1,\ell})+2\Delta-U. \tag{4.36}$$

This, in turn, implies that any path realizing the minimax between $\mathcal{N}_{(\ell-1)\ell}$ and $\mathcal{N}_{(\ell-1)\ell+1}$ has to pass through $\mathcal{R}_{\ell-1,\ell}$. Moreover, any path realizing the minimax between $\mathcal{R}_{\ell-1,\ell}$ and $\mathcal{N}_{\ell^2}$ has to enter $\mathcal{N}_{(\ell-1)\ell+1}$ through $\mathcal{D}_{\ell-1,\ell}$, which corresponds to the saddle between $\mathcal{R}_{\ell-1,\ell}$ and $\mathcal{N}_{(\ell-1)\ell+1}$. (Similarly, any path realizing the minimax between $\mathcal{N}_{\ell^2}$ and $\mathcal{N}_{\ell(\ell+1)}$ has to pass through $\mathcal{R}_{\ell,\ell}$ and $\mathcal{D}_{\ell,\ell}$.)
4. At this point it is clear that paths realizing the minimax between $\mathcal{R}_{\ell-1,\ell}$ and $\mathcal{N}_{\ell^2}$ also have to pass through $\widehat{\mathcal{D}}^-_{\ell-1,\ell}$. Indeed, any move (with a change in energy) starting from $\mathcal{D}_{\ell-1,\ell}$ and different from attaching the free particle to the rectangle would involve an energy increment of at least U, i.e., large enough to pass over the saddle in the path $(\phi^{up}(\ell-1,\ell),\phi^{down}(\ell-1,\ell)$ because $U>\Delta-U$. Similarly, paths realizing the minimax between $\mathcal{N}_{\ell^2}$ and $\mathcal{N}_{\ell(\ell+1)}$ have to pass through $\widehat{\mathcal{D}}^-_{\ell,\ell}$.
5. Let $n=(\ell-1)\ell+1$, and consider a monotone contour $\bar\gamma$ with $\langle\bar\gamma\rangle=n$. The area of its circumscribed rectangle has to be at least $(\ell-1)(\ell+1)$, with a minimal perimeter of 4ℓ. Similarly, for $n=\ell^2+1$ the area of a circumscribed rectangle has to be at least $\ell(\ell+1)$, with a minimal perimeter of $4\ell+2$. From this it easily follows that $\mathcal{F}(\mathcal{N}_{(\ell-1)\ell+1})$ coincides with the set of configurations containing a single monotone contour inscribed in an $\ell\times\ell$ or an $(\ell-1)\times(\ell+1)$ rectangle and containing $(\ell-1)\ell+1$ particles. Similarly, $\mathcal{F}(\mathcal{N}_{\ell^2+1})$ coincides with the set of configurations containing a single monotone contour inscribed in an $\ell\times(\ell+1)$ rectangle and containing ℓ^2+1 particles. In particular,

$$\mathcal{F}(\mathcal{N}_{\ell(\ell-1)+1})\supset\mathcal{D}^-_{\ell-1,\ell},\quad \mathcal{F}(\mathcal{N}_{\ell^2+1})\supset\mathcal{D}^-_{\ell,\ell}. \tag{4.37}$$

From this the claim follows. **QED**

We can now complete the proof of Proposition 16. Parts (i) and (ii) follow from Lemma 17. To prove part (iii), note that since every path $\phi:\bar\square\to\bar\blacksquare$ has to cross all the manifolds $\mathcal{N}_{\ell^2},\mathcal{N}_{\ell(\ell+1)}$, the global saddle $\mathcal{S}(\bar\square,\bar\blacksquare)$ can be nowhere lower in energy than the saddles $\mathcal{S}(\mathcal{N}_{(\ell-1)\ell},\mathcal{N}_{\ell^2})$, $\mathcal{S}(\mathcal{N}_{\ell^2},\mathcal{N}_{(\ell+1)l})$, $\ell\geq 2$. By direct inspection we see that the saddle with maximal energy is $\mathcal{S}(\mathcal{N}_{(\ell_c-1)\ell_c},\mathcal{N}_{\ell_c^2})=\mathcal{D}^0_{\ell_c-1,\ell_c}$. On the other hand, using a comparison with the path $\phi^{up}(\ell_c-1,\ell_c),\phi^{down}(\ell_c-1,\ell_c)$ we see that $\bar H(\mathcal{S}(\bar\square,\bar\blacksquare))$ cannot exceed $\bar H(\mathcal{S}(\mathcal{N}_{(\ell_c-1)\ell_c},\mathcal{N}_{\ell_c^2}))$. From this it follows that $\mathcal{S}(\bar\square,\bar\blacksquare)=\mathcal{D}^0_{\ell_c-1,\ell_c}$. **QED**

Remarks:

(1) We want to emphasize that, contrary to what happens under the non-conservative Glauber dynamics where the global saddle is '$\mathcal{S}(\boxminus, \boxplus) = \bar{\mathcal{D}}^-_{\ell_c-1,\ell_c}$', under the Kawasaki dynamics with creation and annihilation at $\partial^* \bar\Lambda$ *the global saddle is more complex and does not correspond to a single geometric shape* (modulo translations and rotations). Indeed, after reaching $\widehat{\mathcal{D}}^-_{\ell_c-1,\ell_c}$ we can, before we add a next particle whose cost is Δ, perform all possible sequences of moves described by paths satisfying (4.19) and remain below the height of the global saddle, which is $\bar H(\mathcal{R}_{\ell_c-1,\ell_c}) + 2\Delta - U$ (see Fig. 9 in Section 5.2 for an example). This global saddle is reached when we add the next particle to the configuration containing a cluster in $\mathcal{D}^-_{\ell_c-1,\ell_c}$ (with energy $\bar H(\mathcal{R}_{\ell_c-1,\ell_c})+\Delta-U$), giving us a configuration in $\bar{\mathcal{C}}^* = \mathcal{D}^0_{\ell_c-1,\ell_c}$. This is the set that appears in Theorem 3 and that plays the role of the critical configurations.

(2) A typical $\bar\eta \in \mathcal{D}^-_{\ell_1,\ell_2}$ can in fact be quite asymmetric, i.e., quite different from square or quasi-square. However, there is always a path from $\bar\eta$ to a suitable square or quasi-square along which the energy does not exceed $\bar H(\bar\eta)$. Thus, under the Kawasaki dynamics the squares or quasi-squares act as attractors on a time scale $e^{\Delta\beta}$, which is much shorter than the time needed to grow of shrink.

(3) The specification that we included in the geometric characterization of

$$\mathcal{D}^-_{\ell_1,\ell_2}, \mathcal{D}^0_{\ell_1,\ell_2}, \mathcal{D}^+_{\ell_1,\ell_2}, \tag{4.38}$$

namely, that the interior of $\bar\gamma$ has to contain a suitably large rectangular core, is related to the fact that the above mentioned moves can only be performed on the external boundary of the rectangles circumscribing the clusters in $\widehat{\mathcal{D}}^-_{\ell_1,\ell_2}$.

5 Local Markov chain: recurrence

In this section we analyze the local Markov chain $(\bar\eta_t)_{t\geq 0}$ on $\bar\Lambda$ that was defined in Section 4.1. Since this Markov chain is non-conservative and finite, it falls in the Freidlin-Wentzel regime (recall (1.17)) and the analysis of metastability can in principle be carried out by using the general method in Olivieri and Scoppola [10]. The result obtained in Proposition 16 in Section 4.3, i.e., the solution of a certain sequence of minimax problems, is the 'model dependent' part of this method.

An alternative route is the renormalization procedure developed in Scoppola [18], which is based on the following idea:

(1) Group the configurations in $\bar{\mathcal{X}}$ into a sequence of subsets of configurations of increasing regularity: $\bar{\mathcal{X}} \supset \bar{\mathcal{X}}_1 \supset \bar{\mathcal{X}}_2 \supset \bar{\mathcal{X}}_3, \supset \dots$

(2) Prove a recurrence property of the Markov chain to these sets on an increasing sequence of time scales: $T_1 \ll T_2 \ll T_3 \ll \dots$

(3) Construct a sequence of Markov chains by observing the original Markov chain when it enters these sets, and estimate their transition probabilities.

In our case we need three sets: $\bar{\mathcal{X}}_1 \supset \bar{\mathcal{X}}_2 \supset \bar{\mathcal{X}}_3$. In Section 5.1 we define these sets, in Section 5.2 we give a geometric description of their configurations, and in Section 5.3 we prove the recurrence properties to these sets on the time scales $T_1 = e^{0\beta}, T_2 = e^{U\beta}, T_3 = e^{\Delta\beta}$. Section 5.4 contains some results on so-called cycles.

Actually, we will not follow the renormalization procedure in full detail. Rather, we make a construction that is adapted to our specific situation. The results obtained in this section will be extended to the full Markov chain $(\eta_t)_{t\geq 0}$ in Sections 6–7, and will be used in Section 8 to prove our main result in Theorem 3.

5.1 Definitions

We begin by defining a notion of reduction of a configuration that will be needed to control the dynamics.

Definition 18 *A configuration $\bar{\eta} \in \bar{\mathcal{X}}$ is* 0-REDUCIBLE *if there exists a sequence of bonds $b_1, \ldots, b_k \in \bar{\Lambda}^* \cup \partial^* \bar{\Lambda}$ $(k \in \mathbb{N})$ such that:*
(a) $\bar{H}(\bar{\eta}_{i+1}) \leq \bar{H}(\bar{\eta}_i)$ for all $0 \leq i < k$,
(b) $\bar{H}(\bar{\eta}_k) < \bar{H}(\bar{\eta})$,
where $\bar{\eta}_i = T_{b_i} T_{b_{i-1}} \cdots T_{b_1} \bar{\eta}$, $i \geq 0$.

Remarks:
(1) We can always extract a subsequence of bonds with length $k \leq t_0 = |\bar{\Lambda}^* \cup \partial^* \bar{\Lambda}|^2$, again satisfying (a) and (b), such that $\bar{\eta}_i \neq \bar{\eta}_j$ for all $i \neq j$. This is because t_0 is the maximal number of moves inside $\bar{\Lambda}$ needed to change any $\bar{\eta} \in \bar{\mathcal{X}}$ into any $\bar{\eta}' \in \bar{\mathcal{X}}$.
(2) By definition, if $\bar{\eta}$ is 0-reducible, then there exists a finite path ϕ: $\bar{\eta} \to \bar{\eta}'$ with $\bar{\eta}'$ a configuration that is not 0-reducible such that $\bar{H}(\phi_{i+1}) \leq \bar{H}(\phi_i)$ for all i. In fact, to construct such a path it suffices to glue together the paths given by the definition of 0-reducible configurations until we arrive at a configuration that is not 0-reducible. The number of 0-reductions necessary to arrive at a configuration that is not 0-reducible is finite, because with each 0-reduction the energy $\bar{H}$ decreases by at least U while $\bar{H}$ is bounded from below.

Definition 19 *The configurations $\bar{\eta}, \bar{\eta}'$ are* 0-EQUIVALENT *if there exists a sequence of bonds $b_1, \ldots, b_k \in \bar{\Lambda}^* \cup \partial^* \bar{\Lambda}$ $(k \in \mathbb{N})$ such that $\bar{H}(\bar{\eta}_{i+1}) = \bar{H}(\bar{\eta}_i)$ for all $0 \leq i < k$ and $\bar{\eta}_k = \bar{\eta}'$.*

Definition 20 *A set of configurations $\mathcal{C} \subset \bar{\mathcal{X}}$ is a* CYCLE *if it is connected and satisfies*

$$\bar{H}(\mathcal{F}(\partial\mathcal{C})) > \max_{\bar{\eta} \in \mathcal{C}} \bar{H}(\bar{\eta}), \tag{5.1}$$

where $\mathcal{F}(\partial\mathcal{C})$ is the set of minima of $\bar{H}$ in the boundary $\partial\mathcal{C}$ of $\mathcal{C}$ (recall (4.9–4.10)).

We next generalize the idea of reduction as follows:

Definition 21 *A V-PATH is a finite connected sequence $\phi_1, \ldots, \phi_k$ ($k \in \mathbb{N}$) of configurations or sets of configurations such that:*
(a) If $\phi_i = \mathcal{C}$ for some $1 < i < k$ is a set of configurations, then $\mathcal{C}$ is a cycle with $\bar{H}(\mathcal{F}(\partial\mathcal{C})) - \bar{H}(\mathcal{F}(\mathcal{C})) \leq V$, where $\mathcal{F}(\mathcal{C})$ is the set of minima of $\bar{H}$ in $\mathcal{C}$, and ϕ_{i+1}, ϕ_{i-1} are single configurations such that $\phi_{i+1} \in \mathcal{F}(\partial\mathcal{C})$ and $\phi_{i-1} \in \mathcal{C}$.
(b) If ϕ_i, ϕ_{i+1} for some $1 \leq i < k$ are single configurations, then $q(\phi_j, \phi_{j+1}) > 0$ and $\bar{H}(\phi_{j+1}) \leq \bar{H}(\phi_j)$ (recall (4.8)).

The reader should think of a V-path as a 'downhill cascade' in which a sequence of lakes of depth at most V can be present.

Definition 22 *A configuration $\bar{\eta}'$ is V-REACHABLE from $\bar{\eta}$ if there exists a V-path from $\bar{\eta}$ to $\bar{\eta}'$. Two configurations $\bar{\eta}$ and $\bar{\eta}'$ are V-EQUIVALENT if $\bar{\eta}$ is V-reachable from $\bar{\eta}'$ and vice versa.*

Definition 23 *A configuration $\bar{\eta}$ is V-REDUCIBLE if there exists a configuration $\bar{\eta}'$ that is V-reachable from $\bar{\eta}$ such that $\bar{H}(\bar{\eta}') < \bar{H}(\bar{\eta})$.*

With these notions we define the following sets:

$$\begin{aligned} \bar{\mathcal{X}}_1 &= \{\bar{\eta} \in \bar{\mathcal{X}} \colon \bar{\eta} \text{ is not 0-reducible}\} \\ \bar{\mathcal{X}}_2 &= \{\bar{\eta} \in \bar{\mathcal{X}} \colon \bar{\eta} \text{ is not } U\text{-reducible}\} \\ \bar{\mathcal{X}}_3 &= \{\bar{\eta} \in \bar{\mathcal{X}} \colon \bar{\eta} \text{ is not } \Delta\text{-reducible}\}. \end{aligned} \tag{5.2}$$

We note that if $V < V'$, then a configuration that is V-reducible is also V'-reducible, so $\bar{\mathcal{X}}_3 \subset \bar{\mathcal{X}}_2 \subset \bar{\mathcal{X}}_1$. We also define the following sets:

$$\begin{aligned} \mathcal{E}_1(\bar{\eta}) &= \{\bar{\eta}' \in \bar{\mathcal{X}}_1 \colon \bar{\eta} \text{ is 0-equivalent to } \bar{\eta}'\} \\ \mathcal{E}_2(\bar{\eta}) &= \{\bar{\eta} \in \bar{\mathcal{X}}_2 \colon \bar{\eta} \text{ is } U\text{-equivalent to } \bar{\eta}'\} \\ \mathcal{E}_3(\bar{\eta}) &= \{\bar{\eta} \in \bar{\mathcal{X}}_3 \colon \bar{\eta} \text{ is } \Delta\text{-equivalent to } \bar{\eta}'\}. \end{aligned} \tag{5.3}$$

5.2 Geometric description of the recurrence sets

Definition 24 *(a) For $x \in \bar{\Lambda}_0$, let $nn(x) = \{y \in \bar{\Lambda}_0 \colon |x - y| = 1\}$ be the set of nearest-neighbor sites of x in $\bar{\Lambda}_0$.*
(b) A FREE PARTICLE is a site $x \in \bar{\eta} \cap \bar{\Lambda}_0$ such that $\sum_{y \in nn(x)} \bar{\eta}(y) = 0$.
(c) A PROTUBERANCE is a site $x \in \bar{\eta} \cap \bar{\Lambda}_0$ such that $\sum_{y \in nn(x)} \bar{\eta}(y) = 1$.
(d) For $\bar{\eta} \in \bar{\mathcal{X}}$, the EXTERNAL BOUNDARY $\partial\bar{\eta}$ is the set of occupied sites in $\bar{\eta}$ that can be connected to the ring $\bar{\Lambda} \setminus \bar{\Lambda}_0$ via a path along unoccupied sites in $\bar{\eta}$.
(e) For $\bar{\eta} \in \bar{\mathcal{X}}$, an EXTERNAL CORNER is a site $x \notin \bar{\eta}$ such that there exist $y, y' \in \partial\bar{\eta} \cap nn(x)$, $y \neq y'$.
(f) $R(\ell_h, \ell_v, d_n, d_e, d_s, d_w)$ denotes the RECTANGLE in $\bar{\Lambda}_0$ of horizontal side length ℓ_h and vertical side length ℓ_v, such that its north side is a distance d_n from the north side of $\bar{\Lambda}$, and so on.
(g) R_{ℓ_1,ℓ_2} denotes a rectangle of side lengths ℓ_1, ℓ_2 anywhere in the box $\bar{\Lambda}$, including all translations and rotations. A rectangle R_{ℓ_1,ℓ_2} is called a SQUARE or a QUASI-SQUARE if $\ell_1 \geq 2$ and $0 \leq \ell_2 - \ell_1 \leq 1$ (use the convention $\ell_1 \leq \ell_2$).

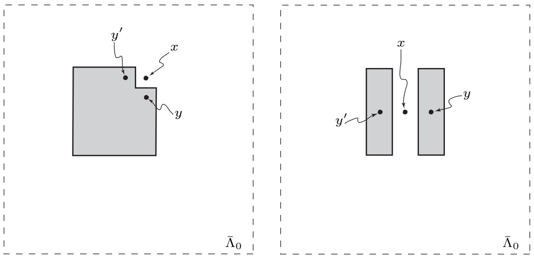

FIG. 8. Two examples of external corners

The following proposition is our main structural identification of the sets $\bar{\mathcal{X}}_1, \bar{\mathcal{X}}_2, \bar{\mathcal{X}}_3$.

Proposition 25 *(i) Configurations in $\bar{\mathcal{X}}_1$ have no free particles nor free holes. If $\bar{\eta}$ is connected with a monotone contour, then $\bar{\eta} \in \bar{\mathcal{X}}_1$.*
(ii) Configurations in $\bar{\mathcal{X}}_2$ have no protuberances. If $\bar{\eta}$ is connected with a monotone contour obtained from a square or a quasi-square by removing m sites with $0 \leq m \leq l_1 - 2$, then $\bar{\eta} \in \bar{\mathcal{X}}_2$.
(iii) Configurations in $\bar{\mathcal{X}}_3$ can be characterized as follows: $\bar{\square} \in \bar{\mathcal{X}}_3$; if $\bar{\eta} \in \bar{\mathcal{X}}_3 \setminus \{\bar{\square}\}$, then $\partial\bar{\eta}$ is the boundary of a square or a quasi-square. Moreover, if $\max\{d_n, d_s\} > l_v$ and $\max\{d_e, d_w\} > l_h$, then $\bar{\eta} = R(l_h, l_v, d_n, d_e, d_s, d_w)$.

Remarks:

(1) The characterization of $\bar{\mathcal{X}}_1$, $\bar{\mathcal{X}}_2$ in Proposition 25(i–ii)is not complete. Only those properties are given that are needed to derive the complete characterization of $\bar{\mathcal{X}}_3$ in Proposition 25(iii).

(2) We see from Proposition 25(iii) that all the elements of $\bar{\mathcal{X}}_3$ corresponding to 'small' clusters, i.e., clusters of side length $\leq \frac{\ell_0}{3}$, are square or quasi-square (ℓ_0 denotes the side length of $\bar{\Lambda}_0$ and we assume that $\ell_0 \gg \ell_c$). All other elements of $\bar{\mathcal{X}}_3$ are LACUNARY SQUARES OR QUASI-SQUARES, i.e., have holes inside. By definition, the interior structure of such configurations is not U-reducible. We denote the set of lacunary configurations by $\mathcal{L}$. These configurations need a separate treatment (see the end of Section 7.6).

(3) It follows from the proof of Proposition 25(iii) given below that if $\mathcal{E}_3(\bar{\eta})$ corresponds to a 'small' cluster, then any $\bar{\eta}' \in \mathcal{E}_3(\bar{\eta})$ can be obtained from $\bar{\eta}$ by means of a rigid motion. For this reason we will denote the elements in $\mathcal{E}_3(\bar{\eta}) \cap \bar{\mathcal{X}}_3$ corresponding to 'small' clusters by R_{ℓ_1,ℓ_2} (with $0 \leq \ell_2 - \ell_1 \leq 1$ and $\ell_1 \geq 2$).

To prove Proposition 25 we need the following lemma, which will also serve us later on. For $\bar{\eta} \in \bar{\mathcal{X}}$ and $V \in \{U, \Delta, 2U\}$, define

$$\bar{\mathcal{C}}_{\bar{\eta}}^V = \{\bar{\eta}' \in \bar{\mathcal{X}} \colon \bar{H}(\bar{\eta}, \bar{\eta}') - \bar{H}(\bar{\eta}) < V\}. \tag{5.4}$$

The structure of this set is characterized as follows.

Lemma 26 *(i) If $\bar\eta \in \bar{\mathcal{X}}_1$, then $\bar{\mathcal{C}}^U_{\bar\eta}$ is a cycle, $\bar\eta \in \mathcal{F}(\bar{\mathcal{C}}^U_{\bar\eta})$ and $\bar{\mathcal{C}}^U_{\bar\eta} \cap \bar{\mathcal{X}}_1 = \mathcal{E}_1(\bar\eta)$.*
(ii) If $\bar\eta \in \bar{\mathcal{X}}_2$, then $\bar{\mathcal{C}}^\Delta_{\bar\eta}$ is a cycle, $\bar\eta \in \mathcal{F}(\bar{\mathcal{C}}^\Delta_{\bar\eta})$ and $\bar{\mathcal{C}}^\Delta_{\bar\eta} \cap \bar{\mathcal{X}}_2 = \mathcal{E}_2(\bar\eta)$. Moreover, $\bar{\mathcal{C}}^\Delta_{\bar\eta} \setminus \mathcal{F}(\bar{\mathcal{C}}^\Delta_{\bar\eta}) \subset \bar{\mathcal{X}} \setminus \bar{\mathcal{X}}_1$.
(iii) If $\bar\eta \in \bar{\mathcal{X}}_3$, then $\bar{\mathcal{C}}^{2U}_{\bar\eta}$ is a cycle, $\bar\eta \in \mathcal{F}(\bar{\mathcal{C}}^{2U}_{\bar\eta})$ and $\bar{\mathcal{C}}^{2U}_{\bar\eta} \cap \bar{\mathcal{X}}_3 = \mathcal{E}_3(\bar\eta)$. Moreover, $\bar{\mathcal{C}}^{2U}_{\bar\eta} \setminus \mathcal{F}(\bar{\mathcal{C}}^{2U}_{\bar\eta}) \subset \bar{\mathcal{X}} \setminus \bar{\mathcal{X}}_2$.

Proof. (i) If $\bar\eta \in \bar{\mathcal{X}}_1$, then $\bar\eta \in \mathcal{F}(\bar{\mathcal{C}}^U_{\bar\eta})$. Indeed, if there exists $\bar\eta' \in \bar{\mathcal{C}}^U_{\bar\eta}$ with $\bar H(\bar\eta') < \bar H(\bar\eta)$, then $\bar\eta$ is 0-reducible, which contradicts $\bar\eta \in \bar{\mathcal{X}}_1$. Next, let $\bar{\mathcal{D}}^U_{\bar\eta}$ be the maximal connected component containing $\bar\eta$ of configurations $\bar\eta'$ such that $\bar H(\bar\eta') < \bar H(\bar\eta) + U$. By definition, $\bar{\mathcal{D}}^U_{\bar\eta}$ is a cycle. It turns out that $\bar{\mathcal{C}}^U_{\bar\eta} = \bar{\mathcal{D}}^U_{\bar\eta}$. Indeed, if $\bar\eta' \in \bar{\mathcal{C}}^U_{\bar\eta}$, then $\bar H(\bar\eta') \le \bar H(\bar\eta', \bar\eta)$, so $\bar\eta'$ is in the connected component because the trajectory realizing the minimax gives the connection. Conversely, if $\bar\eta' \in \bar{\mathcal{D}}^U_{\bar\eta}$, then there exists a path ϕ: $\bar\eta \to \bar\eta'$ such that $\max_i \bar H(\phi_i) - \bar H(\bar\eta) < U$, while by the minimax definition we have $\bar H(\bar\eta, \bar\eta') \le \max_i \bar H(\phi_i)$. Hence $\bar H(\bar\eta, \bar\eta') - \bar H(\bar\eta) < U$. Finally, if $\bar\eta' \in \bar{\mathcal{C}}^U_{\bar\eta} \cap \bar{\mathcal{X}}_1$, then it is straightforward to show that $\bar\eta' \in \mathcal{F}(\bar{\mathcal{C}}^U_{\bar\eta})$, via the same argument that was used to show that $\bar\eta \in \mathcal{F}(\bar{\mathcal{C}}^U_{\bar\eta})$. This implies $\bar H(\bar\eta) = \bar H(\bar\eta')$ and $\bar\eta' \in \mathcal{E}_1(\bar\eta)$.

(ii–iii) The proof of the first part of (ii) and (iii) can be done in the same way. The second part of (ii) follows from the following remark. If $\bar\eta' \in \bar{\mathcal{X}}_1$, then $\bar H(\bar\eta', \bar\eta'') \ge \bar H(\bar\eta') + U$ for all $\bar\eta''$ with $\bar H(\bar\eta'') < \bar H(\bar\eta')$. On the other hand, if $\bar\eta' \notin \mathcal{F}(\bar{\mathcal{C}}^\Delta_{\bar\eta})$, then $\bar H(\bar\eta') = \bar H(\bar\eta) + U$. But then $\bar H(\bar\eta', \bar\eta'') \ge \bar H(\bar\eta') + U = \bar H(\bar\eta) + 2U$, and so $\bar\eta' \notin \bar{\mathcal{C}}^\Delta_{\bar\eta}$. A similar argument works for the second part of (iii). **QED**

Proof of Proposition 25. (i) If $\bar\eta$ has a free particle or a free hole, then $\bar\eta$ is obviously 0-reducible, i.e., $\bar\eta \notin \bar{\mathcal{X}}_1$. If $\bar\eta$ is a connected configuration with a monotone contour, then there exist other configurations that are 0-equivalent to $\bar\eta$ only if $\bar\eta$ has at least one protuberance. In this case all the configurations that are 0-equivalent to $\bar\eta$ can be obtained from $\bar\eta$ by moving the protuberance along the side of the cluster. Therefore $\bar H(T_b\bar\eta) - \bar H(\bar\eta) \ge U$ for all $b \in \bar\Lambda^* \cup \partial^*\bar\Lambda$ such that $T_b\bar\eta \notin \mathcal{E}_1(\bar\eta)$.

(ii) We divide the proof into steps.
1. If $\bar\eta$ has a protuberance, then it is obvious that $\bar\eta$ is U-reducible. To prove the second claim, we first show that if $\bar\eta$ is a connected configuration with a monotone contour obtained by removing $0 \le m \le \ell_1 - 2$ particles from R_{ℓ_1,ℓ_2}, then to reach a configuration $\bar\eta'$ with $\bar H(\bar\eta') < \bar H(\bar\eta)$ we have to pass over a saddle

$$\bar H(\bar\eta, \bar\eta') \ge \bar H(\bar\eta) + \Delta. \tag{5.5}$$

The second claim follows from this inequality. Indeed, we only have to note that if there exists a U-path ϕ: $\bar\eta \to \bar\eta'$, then $\bar H(\bar\eta, \bar\eta') \le \max_i \max_{\zeta \in \phi_i} \bar H(\zeta) \le \bar H(\bar\eta) + U$, which contradicts (5.5). Hence $\bar\eta \in \bar{\mathcal{X}}_2$.
2. Equation (5.5) says that (recall (5.4))

$$\bar\eta \in \mathcal{F}(\bar{\mathcal{C}}^\Delta_{\bar\eta}). \tag{5.6}$$

Let us use the description of the energy $\bar{H}$ in terms of contours that was introduced in Section 4.2. For $\bar{\eta}' \in \bar{\Lambda}$ with contour $\bar{\gamma}' = \bar{\gamma}'(\bar{\eta}')$ we have

$$\bar{H}(\bar{\eta}') = (-2U + \Delta)n_0'(\bar{\gamma}') + \frac{U}{2}|\bar{\gamma}'| + n'(\bar{\gamma}')\Delta, \tag{5.7}$$

where $n_0' = n_0'(\bar{\gamma}')$ is the number of particles in $\bar{\Lambda}_0$ (i.e., the area inside the contour) and $n' = n'(\bar{\gamma}')$ is the number of particles in the ring $\bar{\Lambda} \setminus \bar{\Lambda}_0$. Our configuration $\bar{\eta}$ has $n_0 \in [\ell_1\ell_2 - \ell_1 + 2, \ell_1\ell_2]$, $|\bar{\gamma}| = 2(\ell_1 + \ell_2)$ and $n = 0$.
3. Denote by $\mathrm{CR}(n_0)$ the set of all configurations $\bar{\eta}'$ such that $n_0' = n_0$ and such that the circumscribed rectangle is contained in the square or quasi-square R_{ℓ_1,ℓ_2}. To prove (5.6) we first note that

$$\bar{\eta} \in \mathcal{F}(\bar{\mathcal{C}}_{\bar{\eta}}^{\Delta} \cap \mathrm{CR}(n_0)). \tag{5.8}$$

Indeed, suppose that there exists a configuration $\bar{\eta}' \in \bar{\mathcal{C}}_{\bar{\eta}}^{\Delta} \cap \mathrm{CR}(n_0)$ with $|\bar{\gamma}'| < |\bar{\gamma}|$. Let $R'_{\ell_1',\ell_2'}$ be the rectangle circumscribing $\bar{\eta}'$. Then $R'_{\ell_1',\ell_2'} \subsetneq R_{\ell_1,\ell_2}$, and we have

$$\ell_1' \leq \ell_1, \quad \ell_2' \leq \ell_2, \quad \ell_1' + \ell_2' \; < \ell_1 + \ell_2, \quad \ell_1'\ell_2' \geq n_0 = \ell_1\ell_2 - m \tag{5.9}$$

with $0 \leq m \leq \ell_1 - 2$. But $\ell_1' < \ell_1$ implies $\ell_1'\ell_2' \leq (\ell_1 - 1)\ell_2 < n_0$ and $\ell_2' < \ell_2$ implies $\ell_1'\ell_2' \leq \ell_1(\ell_2 - 1) < n_0$, which contradicts (5.9). The final step in the proof is to show that

$$\bar{\mathcal{C}}_{\bar{\eta}}^{\Delta} \cap \mathrm{CR}(n_0) = \bar{\mathcal{C}}_{\bar{\eta}}^{\Delta}. \tag{5.10}$$

But this is an immediate consequence of the fact that in order to exit from $\mathrm{CR}(n_0)$ the process has to reach an energy $\geq \bar{H}(\bar{\eta}) + \Delta$.

(iii) We divide the proof into several steps. Namely, for every $\bar{\eta} \in \bar{\mathcal{X}}_3$ we show that:

1. $\partial\bar{\eta}$ is the boundary of a union of rectangles;
2. these rectangles are squares or quasi-squares;
3. squares or quasi-squares can move;
4. $\partial\bar{\eta}$ is the boundary of a single square or quasi-square;
5. a 'small' square or quasi-square is not lacunary.

1. Suppose that $\bar{\eta}$ has an external corner x. Then $\bar{\eta}$ is Δ-reducible, because a particle can be created at cost Δ at $\partial^-\bar{\Lambda}$ and can be moved inside the external corner at x, which gives us a Δ-path (recall from Lemma 26 that $\bar{\mathcal{C}}_{\bar{\eta}}^{\Delta}$ is a cycle). The only case in which $\partial\bar{\eta}$ has no external corners is when $\partial\bar{\eta}$ is the boundary of a union of rectangles at distances > 2 (recall Fig. 8).

2. Suppose that $\bar{\eta} = R(\ell_h, \ell_v, d_n, d_e, d_s, d_w)$ with $\ell_h - \ell_v > 1$. We want to show that $\bar{\eta} \notin \bar{\mathcal{X}}_3$. Also now we can find a Δ-reduction of $\bar{\eta}$: Fig. 9 shows an example

of a Δ-path reducing $\bar{\eta}$, which is taken from Peixoto [13]. (To each monotone contour in the sequence of pictures we associate the corresponding U-cycle.)

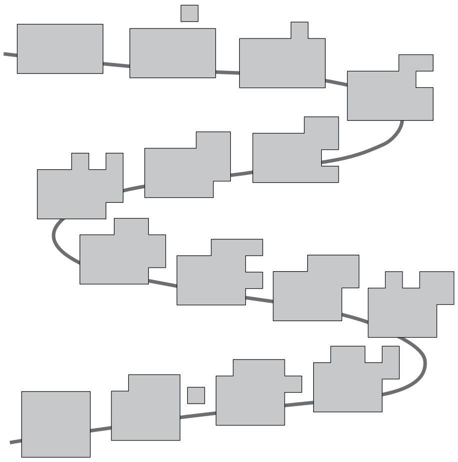

FIG. 9. Movement of particles along the border: $5 \times 3 + 1 = 4 \times 4$

3. By using the same kind of path as in Fig. 9, we can move the square or quasi-square around. The Δ-path to do so is the following:

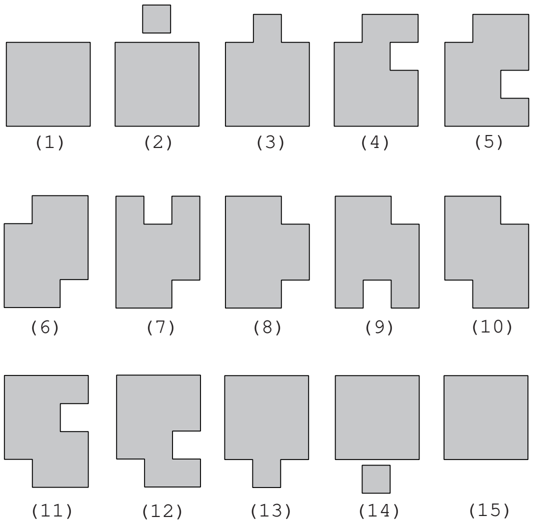

FIG. 10. Movement of a 3×3 square

4. Since, by 3, squares and quasi-squares can move around, any configuration with more than one square or quasi-square can be Δ-reduced by moving the pieces together.

5. Again by 3, for lacunary squares or quasi-squares with $\max\{d_n, d_s\} > \ell_v$ and $\max\{d_e, d_w\} > \ell_h$ there is sufficient room to move them up so as to create an opening for the internal structure. Indeed, Fig. 10 shows a motion of particles along the boundary where the holes in the cluster do not move. **QED**

5.3 Recurrence

Let $T_1 = e^{0\beta}$, $T_2 = e^{U\beta}$, $T_3 = e^{\Delta\beta}$, and let $\tau_{\bar{\mathcal{X}}_i}$ be the first hitting time of the local Markov chain $(\bar{\eta}_t)_{t\geq 0}$ to the set $\bar{\mathcal{X}}_i$ $(i = 1, 2, 3)$. The following proposition is our main recurrence result.

Proposition 27 *For every $\delta > 0$ there exists $\beta_0 > 0$ such that for all $\beta > \beta_0$*

$$\max_{\bar{\eta}\in\bar{\mathcal{X}}_i} \mathbb{P}_{\bar{\eta}}(\tau_{\bar{\mathcal{X}}_i} > T_i e^{\delta\beta}) = \text{SES} \qquad (i = 1, 2, 3). \tag{5.11}$$

Proof. 1. As noted in Remark (2) following Definition 18:

- For each $\bar{\eta} \in \bar{\mathcal{X}}\backslash\bar{\mathcal{X}}_1$ there exists a finite 0-path ϕ: $\bar{\eta} \to \bar{\eta}' \in \bar{\mathcal{X}}_1$.
- For each $\bar{\eta} \in \bar{\mathcal{X}}\backslash\bar{\mathcal{X}}_2$ there exists a finite U-path ϕ: $\bar{\eta} \to \bar{\eta}' \in \bar{\mathcal{X}}_2$.
- For each $\bar{\eta} \in \bar{\mathcal{X}}\backslash\bar{\mathcal{X}}_3$ there exists a finite Δ-path ϕ: $\bar{\eta} \to \bar{\eta}' \in \bar{\mathcal{X}}_3$.

If ϕ is a 0-path, i.e., $\bar{H}(\phi_{i+1}) \leq \bar{H}(\phi_i)$ for all i, then for every $\delta > 0$ there exist $\beta_0 > 0$ and $a = a(\delta) > 0$ such that for all $\beta > \beta_0$

$$\mathbb{P}_{\bar{\eta}}(\bar{\eta}_s = \phi_s \ \forall s \in [0, T_1]) \geq a. \tag{5.12}$$

2. We want to have a similar statement for U-paths and Δ-paths. To that end we make the following observation, valid for cycles $\mathcal{C}$:

Proposition 28 *(i) For every $\delta > 0$ there exist $\beta_0 > 0$ and $\kappa > 0$ such that for all $\beta > \beta_0$ and $\bar{\eta} \in \mathcal{C}$*

$$\mathbb{P}_{\bar{\eta}}\Big(\tau_{\partial\mathcal{C}} < e^{-[H(\mathcal{F}(\partial\mathcal{C}))-H(\mathcal{F}(\mathcal{C}))+\delta])\beta}\Big) \geq 1 - e^{-\kappa\beta}. \tag{5.13}$$

(ii) There exist $\delta_0 > 0$, $\beta_0 > 0$ and $\kappa > 0$ such that for all $\beta > \beta_0$ and $\bar{\eta}, \bar{\eta}' \in \mathcal{C}$

$$\mathbb{P}_{\bar{\eta}}\Big(\tau_{\bar{\eta}'} < \tau_{\partial\mathcal{C}},\, \tau_{\bar{\eta}'} < e^{-[H(\mathcal{F}(\partial\mathcal{C}))-H(\mathcal{F}(\mathcal{C}))-\delta_0]\beta}\Big) \geq 1 - e^{-\kappa\beta}. \tag{5.14}$$

(iii) For all $\delta > 0$, $\bar{\eta} \in \mathcal{C}$, $\bar{\eta}' \in \partial\mathcal{C}$ and β sufficiently large

$$\mathbb{P}_{\bar{\eta}}(\bar{\eta}_{\tau_{\partial\mathcal{C}}} = \bar{\eta}') \geq e^{-[H(\bar{\eta}')-H(\mathcal{F}(\partial\mathcal{C}))+\delta]\beta}. \tag{5.15}$$

Proof. See Olivieri and Scoppola [10] Proposition 3.7. **QED**

It follows from Proposition 28(i,iii) that for every $\delta > 0$ and every U-path ϕ there exist $\beta_0 > 0$ and $a = a(\delta) > 0$ such that for all $\beta > \beta_0$

$$\mathbb{P}_{\bar\eta}(\bar\eta_s = \phi \ \forall s \in [0, T_2 e^{\delta\beta})) \geq a \tag{5.16}$$

and similarly for Δ-paths with T_2 replaced by T_3. So we have indeed generalized (5.12).

3. To conclude the proof of Proposition 27 it now suffices to show that for every $\bar\eta \notin \bar{\mathcal{X}}_i$ there exists an event $\mathbf{E}_{T_i}$ such that:

- On this event the process $(\bar\eta_t)_{t\geq 0}$ starting from $\bar\eta$ exits $\bar{\mathcal{X}}_i$ prior to time $T_i e^{\frac{\delta}{2}\beta}$.
- $\min_{\bar\eta \in \bar{\mathcal{X}}} \mathbb{P}_{\bar\eta}(\mathbf{E}_{T_i}) \geq e^{-\delta'\beta}$ with $\delta' < \frac{\delta}{2}$.

Namely, by Proposition 6 we then have

$$\max_{\bar\eta \in \bar{\mathcal{X}}} \mathbb{P}_{\bar\eta}(\tau_{\bar{\mathcal{X}}_i} > T_i e^{\delta\beta}) \leq (1 - e^{-\delta'\beta})^{e^{\frac{\delta}{2}\beta}} = \text{SES}. \tag{5.17}$$

The event $\mathbf{E}_{T_i}$ is the following: The process follows the $0, U, \Delta$-path from $\bar\eta$ to $\bar{\mathcal{X}}_i$ within time $T_i e^{\frac{\delta}{2}\beta}$. By (5.12) and (5.16) we indeed have $\mathbb{P}_{\bar\eta}(\mathbf{E}_{T_i}) \geq e^{-\delta'\beta}$. **QED**

5.4 Additional results on cycles

In this section we collect some results on cycles and their relation to the recurrence sets $\mathcal{X}_i$.

Lemma 29 *(i) If $\bar\eta_1, \bar\eta_2 \in \bar{\mathcal{X}}_1$ with $\bar H(\bar\eta_1) < \bar H(\bar\eta_2)$, then $\bar H(\bar\eta_1, \bar\eta_2) - \bar H(\bar\eta_2) \geq U$. (ii) If $\bar\eta_1, \bar\eta_2 \in \bar{\mathcal{X}}_2$ with $\bar H(\bar\eta_1) < \bar H(\bar\eta_2)$, then $\bar H(\bar\eta_1, \bar\eta_2) - \bar H(\bar\eta_2) \geq \Delta$.*

Proof. Note that the two smallest positive values for $\bar H(\bar\eta_1, \bar\eta_2) - \bar H(\bar\eta_2)$ are U and Δ.
(i) If $\bar\eta_1, \bar\eta_2 \in \bar{\mathcal{X}}_1$ with $\bar H(\bar\eta_1) < \bar H(\bar\eta_2)$, then $\bar\eta_1 \notin \bar{\mathcal{C}}^U_{\bar\eta_2}$ because otherwise $\bar\eta_2$ were 0-reducible. But then $\bar H(\bar\eta_1, \bar\eta_2) - \bar H(\bar\eta_2) \geq U$.
(ii) If $\bar\eta_1, \bar\eta_2 \in \bar{\mathcal{X}}_2$ with $\bar H(\bar\eta_1) < \bar H(\bar\eta_2)$, then $\bar\eta_1 \notin \bar{\mathcal{C}}^\Delta_{\bar\eta_2}$ because otherwise $\bar\eta_2$ were U-reducible. But then $\bar H(\bar\eta_1, \bar\eta_2) - \bar H(\bar\eta_2) \geq \Delta$. **QED**

For the lacunary squares or quasi-squares in $\bar{\mathcal{X}}_3$ (recall Remark (2) in Section 5.2) we can prove the following.

Lemma 30 *Let $\bar\eta \in \mathcal{L}$ (i.e., $\bar\eta$ is a lacunary square or quasi-square). Then there exists a $\bar\eta' \in \bar{\mathcal{X}}_3 \backslash \mathcal{L}$ with $\bar H(\bar\eta') < \bar H(\bar\eta)$ such that there exists a $2U$-path from $\bar\eta$ to $\bar\eta'$.*

Proof. We define an external corner of the internal structure of an element of $\mathcal{L}$ as a site $x \in \bar{\Lambda}_0$ such that $\eta(x) = 0$, $\sum_{y \in nn(x)} \eta(y) = 2$ and such that there exists a path along occupied sites in $\bar{\eta}$ from x to the external boundary $\partial\bar{\eta}$. The following $2U$-path goes from $\bar{\eta}$ to $\bar{\eta}'$ and reduces the energy:

- $\phi_0 = \bar{\mathcal{C}}^{2U}_{\bar{\eta}}$;
- $\phi_1, \ldots, \phi_{t_1}$ is the sequence of configurations in which a hole that is an external corner of the internal structure of $\bar{\eta}$ goes to $\partial\bar{\eta}$;
- $\phi_{t_1+1}, \ldots, \phi_{t_2}$ is a Δ-path going to $\bar{\mathcal{X}}_3$;
- $\phi_{t_2+1} = \bar{\mathcal{C}}^{2U}_{\bar{\eta}_{t_2}}$;
- $\phi_{t_2+2}, \ldots, \phi_{t_3}$ is the sequence of configurations in which a hole that is an external corner of the internal structure of $\bar{\eta}$ goes to $\partial\bar{\eta}$;
- and so on, until the complete removal of the internal structure.

QED

For $\bar{\eta} \in \bar{\mathcal{X}}_3 \backslash \mathcal{L}$, $\bar{\eta} = R_{\ell_1,\ell_2}$, define

$$\bar{\mathcal{C}}_{\bar{\eta}} = \{\bar{\eta}' \in \bar{\mathcal{X}} \colon \bar{H}(\bar{\eta}', \bar{\eta}) - \bar{H}(\bar{\eta}) < r(\bar{\eta}) + \Delta\}, \tag{5.18}$$

where $r(\bar{\eta})$ is defined by

$$\begin{array}{ll} \ell_1 < \ell_c: & r(\bar{\eta}) = (2U - \Delta)(\ell_1 - 1) \\ \ell_1 \geq \ell_c: & r(\bar{\eta}) = \Delta - U. \end{array} \tag{5.19}$$

Lemma 31 *(i) $\bar{\mathcal{C}}_{\bar{\eta}}$ is the maximal connected component containing $\bar{\eta}$ of configurations $\bar{\eta}'$ such that $\bar{H}(\bar{\eta}') < \bar{H}(\bar{\eta}) + r(\bar{\eta}) + \Delta$.*
(ii) If $\bar{\eta}' \in \bar{\mathcal{C}}_{\bar{\eta}} \cap (\bar{\mathcal{X}}_3 \backslash \mathcal{L})$, then $\bar{\eta}' \in \mathcal{E}_3(\bar{\eta})$ and $\bar{H}(\bar{\eta}') = \bar{H}(\bar{\eta})$.
(iii) If $\bar{\eta}' \in \bar{\mathcal{C}}_{\bar{\eta}} \cap \mathcal{L}$, then $\bar{\mathcal{C}}^{2U}_{\bar{\eta}'} \subset \bar{\mathcal{C}}_{\bar{\eta}}$ and $\bar{H}(\bar{\eta}') > \bar{H}(\bar{\eta})$.
(iv) $\bar{\eta} \in \mathcal{F}(\bar{\mathcal{C}}_{\bar{\eta}})$.

Proof. (i) Let $\bar{\mathcal{D}}_{\bar{\eta}}$ be the maximal connected component containing $\bar{\eta}$ of configurations $\bar{\eta}'$ such that $\bar{H}(\bar{\eta}') < \bar{H}(\bar{\eta}) + r(\bar{\eta}) + \Delta$. If $\bar{\eta}' \in \bar{\mathcal{C}}_{\bar{\eta}}$, then $\bar{\eta}' \in \bar{\mathcal{D}}_{\bar{\eta}}$. Indeed, $\bar{H}(\bar{\eta}') \leq \bar{H}(\bar{\eta}', \bar{\eta})$ and $\bar{\eta}'$ is in the connected component because the trajectory realizing the minimax gives the connection. Conversely, if $\bar{\eta}' \in \bar{\mathcal{D}}_{\bar{\eta}}$, then there exists a path ϕ: $\bar{\eta} \to \bar{\eta}'$ such that $\max_i \bar{H}(\phi_i) - \bar{H}(\bar{\eta}) < r(\bar{\eta}) + \Delta$, while by the minimax definition we have $\bar{H}(\bar{\eta}, \bar{\eta}') \leq \max_i \bar{H}(\phi_i)$. Hence $\bar{H}(\bar{\eta}, \bar{\eta}') - \bar{H}(\bar{\eta}) < r(\bar{\eta}) + \Delta$.

(ii) If $\bar{\eta}, \bar{\eta}' \in \bar{\mathcal{X}}_3 \setminus \mathcal{L}$, then by the saddle point results in Proposition 16(i–ii) we have $\bar{H}(\bar{\eta}, \bar{\eta}') - \bar{H}(\bar{\eta}) \geq r(\bar{\eta}) + \Delta$.

(iii) If $\bar{\eta}' \in \bar{\mathcal{C}}_{\bar{\eta}} \cap \mathcal{L}$, then $\bar{H}(\bar{\eta}') + 2U - \bar{H}(\bar{\eta}) < r(\bar{\eta}) + \Delta$ by Lemma 26. Hence $\bar{\mathcal{C}}^{2U}_{\bar{\eta}'} \subset \bar{\mathcal{C}}_{\bar{\eta}}$ and every $2U$-path starting from $\bar{\eta}'$ is contained in $\bar{\mathcal{C}}_{\bar{\eta}}$. But then, by Lemma 30, there exists a configuration $\bar{\eta}'' \in \bar{\mathcal{X}}_3 \setminus \mathcal{L}$ that is an element of $\bar{\mathcal{C}}_{\bar{\eta}}$. By (ii), this is possible only if $\bar{\eta}'' \in \mathcal{E}_3(\bar{\eta})$, and in this case $\bar{H}(\bar{\eta}') > \bar{H}(\bar{\eta}'') = \bar{H}(\bar{\eta})$.

(iv) Suppose that there exists $\bar{\eta}' \in \bar{\mathcal{C}}_{\bar{\eta}}$ such that $\bar{H}(\bar{\eta}') < \bar{H}(\bar{\eta})$. If $\bar{\eta}' \notin \bar{\mathcal{X}}_3$, then there exists $\bar{\eta}'' \in \bar{\mathcal{X}}_3$ with $\bar{H}(\bar{\eta}'') < \bar{H}(\bar{\eta}')$ and $\bar{H}(\bar{\eta}', \bar{\eta}'') - \bar{H}(\bar{\eta}') \leq \Delta$. If $\bar{\eta}'' \in \bar{\mathcal{X}}_3 \backslash \mathcal{L}$, then this contradicts (ii). If $\bar{\eta}'' \in \mathcal{L}$, then by (iii) we have $\bar{H}(\bar{\eta}'') > \bar{H}(\bar{\eta})$, which contradicts the inequality $\bar{H}(\bar{\eta}'') < \bar{H}(\bar{\eta}') < \bar{H}(\bar{\eta})$ coming from the reduction. **QED**

Remark: Throughout Sections 4–5, Δ is a parameter in $(U, 2U)$ and $l_c(\Delta) = \lceil \frac{U}{2U-\Delta} \rceil$. If Δ and Δ' are such that $l_c(\Delta) = l_c(\Delta')$ and if we consider the two local Markov chains based on the values Δ resp. Δ' for the creation rate along $\partial^* \bar{\Lambda}^{in}$, then all the results obtained in Sections 4–5 for these two Markov chains are equal up to a correction where in all the exponents containing Δ an error term of order $|\Delta' - \Delta|$ is added. This observation is needed in Section 7, where we will need to perturb Δ. It also explains why in Theorem 3 we need to assume that $\frac{U}{2U-\Delta}$ is not integer.

6 Full Markov chain: recurrence

In this section we extend the definition of $\bar{\mathcal{X}}_1 \supset \bar{\mathcal{X}}_2 \supset \bar{\mathcal{X}}_3$, which were used in Section 5 as the recurrence sets for the local Markov chain, to $\mathcal{X}_1 \supset \mathcal{X}_2 \supset \mathcal{X}_3$ and we prove the recurrence properties of the full Markov chain $(\eta_t)_{t \geq 0}$ to these sets.

6.1 Definitions

The sets $\mathcal{X}_i$ $(i = 1, 2, 3)$ are defined as follows:

$$\begin{array}{rcl} \mathcal{X}_1 & = & \{\eta \in \mathcal{X}\colon\ \eta|_{\bar{\Lambda}} \in \bar{\mathcal{X}}_1\} \\ \mathcal{X}_2 & = & \{\eta \in \mathcal{X}\colon\ \eta|_{\bar{\Lambda}} \in \bar{\mathcal{X}}_2\} \\ \mathcal{X}_3 & = & \{\eta \in \mathcal{X}\colon\ \eta|_{\bar{\Lambda}} \in \bar{\mathcal{X}}_3 \cap N_{\Lambda_{\ell_-} \backslash \bar{\Lambda}}(\eta) = 0\}. \end{array} \tag{6.1}$$

6.2 Recurrence

Proposition 32 below shows that, up to a superexponentially small probability, the process $(\eta_t)_{t \geq 0}$ returns to $\mathcal{X}_i$ after time lapses of order T_i.

Proposition 32 ('$\mathcal{X}_i$-recurrence') *Let $T_1 = e^{0\beta}$, $T_2 = e^{U\beta}$, $T_3 = e^{\Delta\beta}$. For $i = 1, 2, 3$ the following holds: There exist $\delta = \delta(\gamma) > 0$, satisfying* $\lim_{\gamma \downarrow 0} \delta(\gamma) = 0$, *and $\beta_0 > 0$ and $\kappa = \kappa(\delta) > 0$ such that for all $\beta > \beta_0$*

$$\mathbb{P}_\nu\Big(\exists t \in [0, T'')\colon\ \eta_s \notin \mathcal{X}_i \, \forall s \in [t, t+T')\Big) = \mathrm{SES} \tag{6.2}$$

for all

$$T = T_i e^{\frac{\delta}{2}\beta} \quad \ll \quad T_i' = T_i e^{\delta\beta} c \quad \ll \quad T'' = e^{C\beta} \tag{6.3}$$

with $C > 0$ arbitrarily large.

Proof. We want to apply Proposition 6. Our task therefore is to define, for each $i = 1, 2, 3$, an event $\mathbf{E}_T^i \subset \mathcal{X}^{[0,T)}$ such that

$$\begin{aligned} &\mathbf{E}_T^i \subset \{\exists t \in [0,T]\colon\ \eta_t \in \mathcal{X}_i\} \\ &\min_{\eta_0 \in \mathcal{X}_0} \mathbb{P}_{\eta_0}(\mathbf{E}_T^i) \geq p > 0 \end{aligned} \tag{6.4}$$

with p not exponentially small in β.

$i = 1, 2$: The event $\mathbf{E}_T^i$ is the following:

- During the time interval $[0, T]$ no particle enters $\bar{\Lambda}$.
- The process $(\eta_t)_{t \geq 0}$ restricted to $\bar{\Lambda}$ follows a 0–path ($i = 1$) resp. a U-path ($i = 2$) from $\eta_0|_{\bar{\Lambda}}$ to $\bar{\mathcal{X}}_i$ within the time interval $[0, T]$.

Since $T = T_i e^{\frac{\delta}{2}\beta} \leq e^{\Delta\beta - \frac{\delta}{2}\beta}$ for $i = 1, 2$ and $\delta > 0$ sufficiently small we have, by Lemmas 13–14 and Proposition 8,

$$\min_{\eta_0 \in \mathcal{X}_0} \mathbb{P}_{\eta_0}(\text{during } [0,T] \text{ no particles enter } \bar{\Lambda}) \geq \left(\frac{\kappa}{\beta}\right)^{\log \beta}. \tag{6.5}$$

The estimate of $\mathbb{P}_{\eta_0}(\mathbf{E}_T^i)$ can therefore be completed by using (5.12), (5.16), Proposition 10 and Proposition 4.

$i = 3$: We cannot proceed in the same way as for $i = 1, 2$, since we cannot avoid the arrival of particles over a time interval of length $T = T_3 e^{\frac{\delta}{2}\beta}$. Actually, the arrival of particles is important to reach $\mathcal{X}_3$. The event $\mathbf{E}_T^3$ is the following:

(i) The process $(\eta_t)_{t \geq 0}$ restricted to $\bar{\Lambda}$ follows a Δ-path from $\eta_0|_{\bar{\Lambda}}$ to $\bar{\mathcal{X}}_3$ within time $\frac{1}{2} T_3 e^{\frac{\delta}{2}\beta}$

(ii) After that, the process within time $T_3 e^{-\frac{\gamma}{2}\beta}$ empties the annulus $\Lambda_{\ell_-} \setminus \bar{\Lambda}$, while keeping the configuration in $\bar{\Lambda}$ fixed and avoiding that particles enter $\bar{\Lambda}$.

To complete the proof we have to show that the probabilities of (i) and (ii) are not exponentially small.

Estimate of (i): For each segment of the Δ-path not containing cycles of depth Δ we can prove (by using the same argument as for $i = 1, 2$) that the probability that the process follows this segment within time $T_2 e^{\frac{\delta}{2}\beta}$ is not exponentially small. We therefore only have to control the probability that the process follows the segments of the Δ-path containing cycles of depth Δ, i.e., we have to control the probability that it exits from a set

$$C_{\bar{\eta}'}^{\Delta} = \{\eta \in \mathcal{X}\colon\ \eta|_{\bar{\Lambda}} \in \bar{\mathcal{C}}_{\bar{\eta}'}^{\Delta}\} \qquad \text{for some } \bar{\eta}' \in \bar{\mathcal{X}}_2 \tag{6.6}$$

within a time larger than $T_3 e^{\frac{\delta}{3}\beta}$ and that it exits at a state in $\mathcal{F}(\partial \bar{\mathcal{C}}_{\bar{\eta}'}^{\Delta})$. This is done by the following estimate:

Lemma 33 *For $\delta > 0$ small enough and $0 < \gamma < \frac{\delta}{3}$ there exists $c > 0$, depending on γ, δ but not on β, such that*

$$\min_{\eta \in C_{\bar\eta'}^{\Delta} \cap \mathcal{X}_0} \mathbb{P}_\eta \Big(\{\tau_{(C_{\bar\eta'}^{\Delta})^c} < T_3 e^{\frac{\delta}{3}\beta}\} \cap \{\eta_{\tau_{(C_{\bar\eta'}^{\Delta})^c}}|_{\bar\Lambda} \in \mathcal{F}(\partial \bar{\mathcal{C}}_{\bar\eta'}^{\Delta})\} \Big) \geq c. \tag{6.7}$$

Proof. 1. From Lemma 26(ii) we know that if $\bar\eta \notin \mathcal{F}(\bar{\mathcal{C}}_{\bar\eta'}^{\Delta})$, then $\bar\eta \in \bar{\mathcal{X}} \setminus \bar{\mathcal{X}}_1$ and there exists a downhill path from $\bar\eta$ to $\mathcal{F}(\bar{\mathcal{C}}_{\bar\eta'}^{\Delta})$. We can therefore require that the process $(\eta_t)_{t\geq 0}$ follows this path within a time $e^{\delta\beta}$ and that during this time no particle enters $\bar\Lambda$. As before, this probability is not exponentially small in β. It is therefore sufficient to show that for all $\eta'_0 \in \mathcal{X}_0$ such that $\eta'_0|_{\bar\Lambda} \in \mathcal{F}(\bar{\mathcal{C}}_{\bar\eta'}^{\Delta})$

$$\mathbb{P}_{\eta'_0} \Big(\{\tau_{(C_{\bar\eta'}^{\Delta})^c} < T_3 e^{\frac{\delta}{3}\beta}\} \cap \{\eta_{\tau_{(C_{\bar\eta'}^{\Delta})^c}}|_{\bar\Lambda} \in \mathcal{F}(\partial \bar{\mathcal{C}}_{\bar\eta'}^{\Delta})\} \Big) \geq c. \tag{6.8}$$

2. Let τ denote the first time a particle enters $\bar\Lambda$. Then

$$\mathbb{P}_{\eta'_0} \Big(\{\tau_{(C_{\bar\eta'}^{\Delta})^c} < T_3 e^{\frac{\delta}{3}\beta}\} \cap \{\eta_{\tau_{(C_{\bar\eta'}^{\Delta})^c}}|_{\bar\Lambda} \in \mathcal{F}(\partial \bar{\mathcal{C}}_{\bar\eta'}^{\Delta})\} \Big) \geq \int_0^{T_3 e^{\frac{\delta}{3}\beta}} \mathbb{P}_{\eta'_0}(\tau \in dt,\, \mathbf{A}_t \cap \mathbf{B}_t), \tag{6.9}$$

where

$$\begin{aligned} \mathbf{A}_t &= \{\tau_{(\mathcal{C}_{\bar\eta'}^{\Delta})^c} > t\} \\ \mathbf{B}_t &= \{\eta_{t-}|_{\bar\Lambda} \in \mathcal{F}(\bar{\mathcal{C}}_{\bar\eta'}^{\Delta})\}. \end{aligned} \tag{6.10}$$

The r.h.s. of (6.9) equals

$$\int_0^{T_3 e^{\frac{\delta}{3}\beta}} [\mathbb{P}_{\eta'_0}(\tau \in dt,\, \mathbf{A}_t) - \mathbb{P}_{\eta'_0}(\tau \in dt,\, \mathbf{A}_t \cap \mathbf{B}_t^c)]. \tag{6.11}$$

Due to the recurrence property in $\mathcal{X}_1$ and due to the fact that $\bar{\mathcal{C}}_{\bar\eta'}^{\Delta} \setminus \mathcal{F}(\bar{\mathcal{C}}_{\bar\eta'}^{\Delta}) \subset \bar{\mathcal{X}} \setminus \bar{\mathcal{X}}_1$ (Lemma 26(ii)), we have that on the event $\{\tau \in dt,\, \mathbf{A}_t \cap \mathbf{B}_t^c\}$ a move of cost U occurs during the time interval $[(t - e^{\delta\beta}) \vee 0, t]$, so that

$$\int_0^{T_3 e^{\frac{\delta}{3}\beta}} \mathbb{P}_{\eta'_0}(\tau \in dt,\, \mathbf{A}_t \cap \mathbf{B}_t^c) \leq e^{-U\beta} e^{\delta\beta}. \tag{6.12}$$

On the other hand,

$$\begin{aligned} \textstyle\int_0^{T_3 e^{\frac{\delta}{3}\beta}} \mathbb{P}_{\eta'_0}(\tau \in dt,\, \mathbf{A}_t) &= \textstyle\int_0^{T_3 e^{\frac{\delta}{3}\beta}} [\mathbb{P}_{\eta'_0}(\tau \in dt) - \mathbb{P}_{\eta'_0}(\tau \in dt,\, \mathbf{A}_t^c)] \\ &\geq \textstyle\int_0^{T_3 e^{\frac{\delta}{3}\beta}} \mathbb{P}_{\eta'_0}(\tau \in dt) - e^{-(2U - \Delta - \frac{\delta}{3})\beta}, \end{aligned} \tag{6.13}$$

since on the event $\{\tau \in dt,\, \mathbf{A}_t^c\}$ a move of cost $2U$ occurs within time t.

3. Finally use that $\{\eta'_0 \in \mathcal{X}_0\} \subset \{N_{\Lambda_{\ell_+}}(\eta'_0) > 1\}$ (recall (2.20)). If $\frac{\delta}{3} > \gamma$, then $\mathbb{P}_x(\tau_{\bar\Lambda} < T_3 e^{\frac{\delta}{3}\beta}) \geq c$ for a simple random walk starting at $x \in \Lambda_{\ell_+}$. Hence

$$\int_0^{T_3 e^{\frac{\delta}{3}\beta}} \mathbb{P}_{\eta'_0}(\tau \in dt) = \mathbb{P}_{\eta'_0}(\tau \leq T_3 e^{\frac{\delta}{3}\beta}) \geq c. \tag{6.14}$$

QED

<u>Estimate of (ii)</u>: We estimate the probability of the following events:

- **F**: during the time interval $[0, T_3 e^{-\frac{\gamma}{2}\beta}]$ no particle enters $\bar{\Lambda}$;
- **G**: at time $T_3 e^{-\frac{\gamma}{2}\beta}$ there are no particles in $\Lambda_{\ell_-} \setminus \bar{\Lambda}$;
- **H**: during the time interval $[0, T_3 e^{-\frac{\gamma}{2}\beta}]$ the configuration in $\bar{\Lambda}$ does not change.

From Lemma 13 we obtain $\min_{\eta_0' \in \mathcal{X}_0} \mathbb{P}_{\eta_0'}(\mathbf{F}) \geq (\kappa/\beta)^{\log \beta}$. From Proposition 8 and (3.6), we obtain

$$\min_{\eta_0' \in \mathcal{X}_0} \mathbb{P}_{\eta_0'}(\mathbf{G}) \geq \Big(1 - \frac{|\Lambda_{\ell_-}|}{T_3 e^{-\frac{\gamma}{2}\beta}}\Big)^{\log \beta} > 1 - e^{-\frac{\gamma}{2}\beta} \log \beta. \tag{6.15}$$

Finally, $\mathbb{P}_{\eta_0'}(\mathbf{H}|\mathbf{F} \cap \mathbf{G})$ can be estimated by considering the local Markov chain, and we obtain

$$\min_{\eta_0' \in \mathcal{X}_0} \mathbb{P}_{\eta_0'}(\mathbf{H}|\mathbf{F} \cap \mathbf{G}) \geq (1 - e^{-2U\beta})^{T_3} \geq 1 - e^{-(2U - \Delta + \frac{\gamma}{2})\beta}. \tag{6.16}$$

So we arrive at

$$\begin{aligned} \min_{\eta_0' \in \mathcal{X}_0} \mathbb{P}_{\eta_0'}(\mathbf{F} \cap \mathbf{G} \cap \mathbf{H}) &= \min_{\eta_0' \in \mathcal{X}_0} \mathbb{P}_{\eta_0'}(\mathbf{H}|\mathbf{F} \cap \mathbf{G}) \mathbb{P}_{\eta_0'}(\mathbf{F} \cap \mathbf{G}) \\ &\geq \Big[1 - e^{-(2U - \Delta + \frac{\gamma}{2})\beta}\Big] \Big[\Big(\tfrac{\kappa}{\beta}\Big)^{\log \beta} - e^{-\frac{\gamma}{2}\beta} \log \beta\Big]. \end{aligned} \tag{6.17}$$

This completes the proof (i-ii) for $i = 3$ and hence of Proposition 32. **QED**

7 Full Markov chain: reduction

In this section we derive all the key estimates for the full Markov chain that are needed to study its metastable behavior. The computations are long and difficult.

7.1 Definition of the reduced Markov chain

We begin by defining the *reduced Markov chain* that is obtained by observing the process only when it enters $\mathcal{X}_3$. For $\bar{\eta} \in \bar{\mathcal{X}}_3$, let

$$\bar{\mathcal{C}}_{\bar{\eta}}^{\Delta} := \{\bar{\eta}' \in \bar{\mathcal{X}}\colon\ \bar{H}(\bar{\eta}', \bar{\eta}) - \bar{H}(\bar{\eta}) < \Delta\}. \tag{7.1}$$

Extend this definition to the configurations in $\mathcal{X}$ in the obvious way, namely for $\eta \in \mathcal{X}_3$, let

$$\mathcal{C}_{\eta}^{\Delta} := \{\eta' \in \mathcal{X}\colon\ \eta'|_{\bar{\Lambda}} \in \bar{\mathcal{C}}_{\eta|_{\bar{\Lambda}}}^{\Delta}\}. \tag{7.2}$$

Definition 34 *Let*

$$\begin{aligned} \tau_0 &= 0 \\ \sigma_0 &= \min\{t \geq 0\colon\ \eta_t \notin \mathcal{C}^{\Delta}_{\eta_0}\} \end{aligned} \tag{7.3}$$

and, for $i \in \mathbb{N}$,

$$\begin{aligned} \tau_i &= \min\{t > \sigma_{i-1}\colon\ \eta_t \in \mathcal{X}_3\} \\ \sigma_i &= \min\{t > \tau_i\colon\ \eta_t \notin \mathcal{C}^{\Delta}_{\eta_{\tau_i}}\}. \end{aligned} \tag{7.4}$$

Then

$$\eta^R_i = \eta_{\tau_i},\ i \in \mathbb{N}_0, \tag{7.5}$$

defines a Markov chain $(\eta^R_i)_{i\in\mathbb{N}_0}$ *on* $\mathcal{X}_3$ *with transition probabilities*

$$P^R(\eta, \eta') = \mathbb{P}_\eta(\eta_{\tau_1} = \eta'). \tag{7.6}$$

For $\bar\eta \in \bar{\mathcal{X}}$, as before, let $I_{\bar\eta}$ be the set of configurations $\eta' \in \mathcal{X}$ such that $\eta'|_{\bar\Lambda} = \bar\eta$. For $\eta \in \mathcal{X}_3$ and $\bar\eta' \in \bar{\mathcal{X}}_3$, let

$$P^R(\eta, I_{\bar\eta'}) := \sum_{\eta'' \in I_{\bar\eta'} \cap \mathcal{X}_3} P^R(\eta, \eta''). \tag{7.7}$$

Our main result in this section is the following proposition, which makes a comparison between the transition probabilities of the full Markov chain and the local Markov chain.

Proposition 35 *There exist* $\delta = \delta(\delta_1, \gamma_0)$, *satisfying* $\lim_{\delta_1,\gamma_0\downarrow 0} \delta(\delta_1, \gamma_0) = 0$, *and* $\beta_0 > 0$ *such that for all* $\beta > \beta_0$*:*
(i) If $\bar\eta \in \mathcal{R}_{\ell_1,\ell_2}$, $\bar\eta' \in \mathcal{R}_{\ell_1+1,\ell_2}$, *then*

$$\min_{\eta \in I_{\bar\eta} \cap \mathcal{X}_3} P^R(\eta, I_{\bar\eta'}) \geq e^{-(\Delta-U)\beta} e^{-\delta\beta}. \tag{7.8}$$

(ii) If $\bar\eta \in \mathcal{R}_{\ell_1,\ell_2}$, $\bar\eta' \in \mathcal{R}_{\ell_1,\ell_2-1}$, *then*

$$\min_{\eta \in I_{\bar\eta} \cap \mathcal{X}_3} P^R(\eta, I_{\bar\eta'}) \geq e^{-(2U-\Delta)(\ell_1-1)\beta} e^{-\delta\beta}. \tag{7.9}$$

(iii) If $\bar\eta$ *is a lacunary square or quasi-square, i.e.,* $\bar\eta \in \mathcal{L}$, *then there exists a sequence* $\bar\eta_0, \bar\eta_1, \ldots, \bar\eta_n$ *such that* $\bar\eta_0 = \bar\eta$, $\bar\eta_i \in \mathcal{L}$ *for* $i = 1, \ldots, n-1$, $\bar\eta_n \in \bar{\mathcal{X}}_3 \setminus \mathcal{L}$, *and* $|\bar\eta_i| \geq |\bar\eta_{i-1}|$ *for all* i *for which*

$$\min_{i=0,1,\ldots,n-1} \min_{\eta_i \in I_{\bar\eta_i}} P^R(\eta_i, I_{\bar\eta_{i+1}}) \geq e^{-(2U-\Delta)\beta} e^{-\delta\beta}. \tag{7.10}$$

(iv) Let

$$\begin{array}{ll} r(\bar\eta) = (2U-\Delta)(\ell_1-1) & \textit{if } \bar\eta = R_{\ell_1,\ell_2} \textit{ and } 0 \leq \ell_2 - \ell_1 \leq 1 \textit{ with } \ell_1 < \ell_c \\ r(\bar\eta) = \Delta - U & \textit{if } \bar\eta = R_{\ell_1,\ell_2} \textit{ and } 0 \leq \ell_2 - \ell_1 \leq 1 \textit{ with } \ell_1 \geq \ell_c \\ r(\bar\eta) = (2U-\Delta) & \textit{if } \bar\eta \in \mathcal{L}. \end{array} \tag{7.11}$$

Then

$$\max_{\eta \in I_{\bar\eta}} \max_{\bar\eta' \in \bar{\mathcal{X}}_3\colon\ \bar\eta' \notin \mathcal{E}_3(\bar\eta)} P^R(\eta, I_{\bar\eta'}) \leq e^{-[r(\bar\eta)+\delta]\beta}. \tag{7.12}$$

Using Proposition 35 we will in Section 8 conclude the proof of our main result, Theorem 3, by constructing an event of nucleation and by controlling its probability.

The remainder of this section is organized as follows. In Section 7.2 we prove the lower bounds (i), (ii), (iii). The proof of the upper bound (iv) is much more difficult. Indeed, to prove a lower bound it suffices tto estimate the probability of a particular realization of the event we are considering, but to prove an upper bound every possible realization of the event must be controlled. In Section 7.3 we define a coupling of our Markov chain to an auxiliary Markov chain. This coupling enables us to study separately the effect of particles entering $\bar{\Lambda}$ from the gas ('green' particles) and particles exiting from $\bar{\Lambda}$ but afterwards returning to $\bar{\Lambda}$ and thus interacting again with the local configuration in $\bar{\Lambda}$ ('red' particles).

With the help of the auxiliary Markov chain we prove Proposition 35(iv) in Section 7.4 by using the control of green and red particles obtained in Sections 7.5 resp. 7.6. Section 7.7 contains the proofs of some lemmas that are needed along the way. In section 7.8 we collect some consequence of Proposition 35 that will be used in Section 8 to prove our main theorem.

7.2 Lower bounds

In this section we prove Proposition 35(i–iii).

Estimate (i): The proof follows the same ideas as in Section 6.
1. Let $\bar{\eta}_i$, $0 \le i \le \ell_2$, be the configurations with monotone contour obtained from R_{ℓ_1+1,ℓ_2} by removing $\ell_2 - i$ particles, so that $\bar{\eta}_0 = \bar{\eta}$ and $\bar{\eta}_{\ell_2} = \bar{\eta}'$. We know from Section 5 that $\bar{\eta}_i \in \bar{\mathcal{X}}_2$ for $i = 0, 2, \dots, \ell_2$. We construct our realization of the growing transition $\bar{\eta} = R_{\ell_1,\ell_2} \to \bar{\eta}' = R_{\ell_1+1,\ell_2}$ by using the following events:

- For $i = 1$:
  - $\mathbf{A}_{1,t} := \{\tau_{(I_{\bar{\eta}})^c} \ge t\}$;
  - $\mathbf{B}_{1,t} := \{\eta_{t^-}|_{\bar{\Lambda}} = \bar{\eta}\}$;
  - for $\eta'' \in I_{\bar{\eta}}$ and $x \in \partial\bar{\Lambda}$, $\mathbf{C}_{1,\eta'',x}$ is the event where the process, starting from $\eta'' + x$, inside $\bar{\Lambda}$ follows a path that brings the particle at x to the cluster, thus reaching $I_{\bar{\eta}_1}\}$;
- For $i = 2$:
  - $\mathbf{A}_{2,t} := \{\tau_{(\mathcal{C}^{2U}_{\bar{\eta}})^c} \ge t\}$;
  - $\mathbf{B}_{2,t} := \{\eta_{t^-}|_{\bar{\Lambda}} \in \bar{\mathcal{B}}_{\bar{\eta}}\}$ with $\bar{\mathcal{B}}_{\bar{\eta}} := \bar{\mathcal{C}}^{2U}_{\bar{\eta}} \cap \bar{\mathcal{X}}_1 \cap \{\bar{\eta}'' \colon N_{\bar{\Lambda}}(\bar{\eta}'') = \ell_1\ell_2 + 1\}$;
  - for $\eta'' \in I_{\bar{\mathcal{B}}_{\bar{\eta}}}$ and $x \in \partial\bar{\Lambda}$, $\mathbf{C}_{2,\eta'',x}$ is the event where the process, starting at $\eta'' + x$, inside $\bar{\Lambda}$ follows a path that takes the particle at x to the cluster, thus reaching $I_{\bar{\eta}_2}\}$;
- For $i = 3, \dots, \ell_2$:
  - $A_{i,t} := \{\tau_{(\mathcal{C}^{\Delta}_{\bar{\eta}_{i-1}})^c} \ge t\}$;

– $B_{i,t} := \{\eta_{t^-}|_{\bar\Lambda} \in \mathcal{F}(\mathcal{C}^{\Delta}_{\bar\eta_{i-1}})\}$;

– for $\eta'' \in I_{\mathcal{F}(\mathcal{C}^{\Delta}_{\bar\eta_{i-1}})}$ and $x \in \partial\bar\Lambda$, $C_{i,\eta'',x}$ is the event where the process, starting at $\eta'' + x$, inside $\bar\Lambda$ follows a path that brings the particle at x to the cluster, thus reaching $I_{\bar\eta_i\}}$.

2. As in Section 6, we define:

– $\mathbf{F}$:= during the time interval $[0, T_3 e^{-\frac{\delta_0}{2}\beta}]$ no particle enters $\bar\Lambda$;

– $\mathbf{G}$:= at time $T_3 e^{-\frac{\delta_0}{2}\beta}$ there are no particles in $\Lambda_{\ell_-} \setminus \bar\Lambda$;

– $\mathbf{H}$:= during the time interval $[0, T_3 e^{-\frac{\delta_0}{2}\beta}]$ the in $\bar\Lambda$ does not change.

Let $\tau := \min\{t \geq 0\colon$ at time t a particle enters $\bar\Lambda\}$ and $T := \frac{T_3 e^{\delta_1\beta}}{\ell_1+1}$. For $\eta \in I_{\bar\eta}$ we have

$$\begin{aligned}
P^R(\eta, I_{\bar\eta'}) \quad \geq \quad & \int_0^T \mathbb{P}_\eta(\tau \in dt_1,\, \mathbf{A}_{1,t_1} \cap \mathbf{B}_{1,t_1}) \min_{x_1 \in \partial\bar\Lambda,\, \eta'' \in I_{\bar\eta}} \mathbb{P}_{\bar\eta''+x_1}(\mathbf{C}_{1,\eta'',x_1}) \\
& \int_{e^{U\beta}}^T \mathbb{P}_{\eta_1}(\tau \in dt_2,\, \mathbf{A}_{2,t_2} \cap \mathbf{B}_{2,t_2}) \min_{x_2 \in \partial\bar\Lambda,\, \eta'' \in I_{\bar{\mathcal{B}}_{\bar\eta}}} \mathbb{P}_{\bar\eta''+x_2}(\mathbf{C}_{2,\eta'',x_2}) \\
& \dots \\
& \int_0^T \mathbb{P}_{\eta_{i-1}}(\tau \in dt_i,\, \mathbf{A}_{i,t_i} \cap \mathbf{B}_{i,t_i}) \min_{x_i \in \partial\bar\Lambda,\, \eta'' \in I_{\mathcal{F}(\mathcal{C}^{\Delta}_{\bar\eta_{i-1}})}} \mathbb{P}_{\bar\eta''+x_i}(\mathbf{C}_{i,\eta'',x_i}) \\
& \dots \\
& \int_0^T \mathbb{P}_{\eta_{\ell_2-1}}(\tau \in dt_{\ell_2},\, \mathbf{A}_{\ell_2,t_{\ell_2}} \cap \mathbf{B}_{\ell_2,t_{\ell_2}}) \min_{x_{\ell_2} \in \partial\bar\Lambda,\, \eta'' \in I_{\mathcal{F}(\mathcal{C}^{\Delta}_{\bar\eta_{\ell_2-1}})}} \mathbb{P}_{\bar\eta''+x_{\ell_2}}(\mathbf{C}_{\ell_2,\eta'',x_{\ell_2}}) \\
& \min_{\eta'' \in I_{\bar\eta'} \cap \mathcal{X}_0} \mathbb{P}_{\eta''}(\mathbf{F} \cap \mathbf{G} \cap \mathbf{H}),
\end{aligned} \tag{7.13}$$

where $\mathcal{X}_0$ is set defined in 2.21.

3. For any i we have that the minimum of the probability of $\mathbf{C}_{i,\eta'',x_i}$ can estimated from below by a constant a not exponentially small in β. As in Section 6, for each $i = 1, 3, \dots, \ell_2$ we can write

$$\begin{aligned}
& \int_0^T \mathbb{P}_{\eta_{i-1}}(\tau \in dt_i,\, \mathbf{A}_{i,t_i} \cap \mathbf{B}_{i,t_i}) \\
& = \int_0^T [\mathbb{P}_{\eta_{i-1}}(\tau \in dt_i,\, \mathbf{A}_{i,t_i}) - \mathbb{P}_{\eta_{i-1}}(\tau \in dt_i,\, \mathbf{A}_{i,t_i} \cap \mathbf{B}^c_{i,t_i})
\end{aligned} \tag{7.14}$$

and this expression can be estimated from below by a constant c independent of β exactly as in the proof of Lemma 33. Moreover,we again have that $\min_{\eta'' \in I_{\bar\eta'} \cap \mathcal{X}_0} \mathbb{P}_{\eta''}(\mathbf{F} \cap \mathbf{G} \cap \mathbf{H}) \geq (\frac{\kappa}{\beta})^{\log\beta}$. It therefore remains to estimate the term corresponding to $i = 2$.

4. On the event $\mathbf{A}_{2,t_2}$, before the arrival of the second particle the process is in $\mathcal{C}_{\bar{\eta}}^{2U}$. This means that for each time s until that arrival we have $N_{\bar{\Lambda}}(\eta_s) \in \{\ell_1\ell_2, \ell_1\ell_2+1\}$, and if $N_{\bar{\Lambda}}(\eta_s) = \ell_1\ell_2$, then $\bar{\eta}_s = R_{\ell_1,\ell_2}$. This, in turn, implies that the process starting at η_1 during the time interval $[0,\tau)$ can be estimated as follows:

- During the time intervals in which $N_{\bar{\Lambda}}(\eta_s) = \ell_1\ell_2$ we have a particle moving as a SRW and a fixed rectangular configuration.
- During the time intervals in which $N_{\bar{\Lambda}}(\eta_s) = \ell_1\ell_2 + 1$ the particle can be trapped by the cluster, producing a configuration that inside $\bar{\Lambda}$ falls in $\bar{\mathcal{X}}\backslash\bar{\mathcal{X}}_2$. In other words, during this time interval the process behaves like a SRW with a trap at $\partial R_{\ell_1,\ell_2}$ of escape rate $e^{-U\beta}$. We denote this process by $(\hat{\xi}_s)_{s\geq 0}$.

5. We can now conclude our estimate:

$$\begin{aligned}
&\int_{e^{U\beta}}^{T} \mathbb{P}_{\eta_1}(\tau \in dt_2,\, \mathbf{A}_{2,t_2} \cap \mathbf{B}_{2,t_2}) \\
&= \int_{e^{U\beta}}^{T} \mathbb{P}_{\eta_1}(B_{2,t_2}|\tau = t_2,\, \mathbf{A}_{2,t_2})\mathbb{P}(\tau \in dt_2,\, \mathbf{A}_{2,t_2}) \\
&\geq \int_{e^{U\beta}}^{T} \mathbb{P}_{\eta_1}(\hat{\xi}_{t_2} \in \partial R_{\ell_1,\ell_2})\mathbb{P}_{\eta_1}(\tau \in dt_2,\, \mathbf{A}_{2,t_2}) \qquad (7.15)\\
&\geq \int_{e^{U\beta}}^{T} \frac{1}{t_2}e^{(U-\delta)\beta}\mathbb{P}_{\eta_1}(\tau \in dt_2,\, \mathbf{A}_{2,t_2}) \\
&\geq \frac{1}{T}e^{(U-\delta)\beta} \int_{e^{U\beta}}^{T} \mathbb{P}_{\eta_1}(\tau \in dt_2,\, \mathbf{A}_{2,t_2}),
\end{aligned}$$

where the second inequality uses proposition 12 and the last probability can be estimated as before.

Estimate (ii): **1.** The lower bound in the case of shrinking transitions can be obtained by forcing the transition to happen within a time $T_3 e^{-2\delta_0\beta}$ and requiring that during this time no particles enter $\bar{\Lambda}$. In this way the transition can be estimated as in the case of the local Markov chain. More precisely, let $\bar{\eta} = R_{\ell_1,\ell_2}$ and $\bar{\eta}' = R_{\ell_1,\ell_2-1}$. For $\eta \in I_{\bar{\eta}} \cap \mathcal{X}_3$ we have

$$P^R(\eta, I_{\bar{\eta}'}) = \mathbb{P}_\eta(\eta_{\tau_1} \in I_{\bar{\eta}'}) \geq \mathbb{P}_\eta(\mathbf{A}_{T_3e^{-2\delta_0\beta}} \cap \mathbf{D}_{T_3e^{-2\delta_0\beta}} \cap \bar{\mathbf{D}}_{T_3e^{-2\delta_0\beta}}), \qquad (7.16)$$

where

- $\mathbf{A}_{T_3e^{-2\delta_0\beta}} := \eta_{\{\tau_1} \in I_{\bar{\eta}'}\} \cap \{\tau_1 < T_3 e^{-2\delta_0\beta}\}$;
- $\mathbf{D}_{T_3e^{-2\delta_0\beta}}$ is the event where during time $T_3e^{-2\delta_0\beta}$ no particles entering $\bar{\Lambda}\}$;
- $\bar{D}_{T_3e^{-2\delta_0\beta}}$ is the event where during time $T_3e^{-2\delta_0\beta}$ no particles are created in $\partial\bar{\Lambda}\}$.

The last term in the r.h.s. of (7.16) is equal to $\mathbb{P}_{\bar\eta}(\bar{\mathbf{A}}_{T_3e^{-2\delta_0\beta}} \cap \mathbf{D}_{T_3e^{-2\delta_0\beta}} \cap \bar{\mathbf{D}}_{T_3e^{-2\delta_0\beta}})$, where $\bar{\mathbf{A}}_{T_3e^{-2\delta_0\beta}}$ is the analogue of $\mathbf{A}_{T_3e^{-2\delta_0\beta}}$ for the local Markov chain.
2. We note that the events $\mathbf{D}_{T_3e^{-2\delta_0\beta}}$ and $\bar{\mathbf{A}}_{T_3e^{-2\delta_0\beta}} \cap \bar{\mathbf{D}}_{T_3e^{-2\delta_0\beta}}$ are independent, since they involve different clocks and marks. Moreover, $\mathbb{P}(\mathbf{D}_{T_3e^{-2\delta_0\beta}}) \geq (\frac{\kappa}{\beta})^{\log\beta}$, $\mathbb{P}(\bar{\mathbf{D}}_{T_3e^{-2\delta_0\beta}}) \geq (1-p)^{T_3e^{-\delta_0\beta}} + \text{SES} \geq e^{e^{-\delta_0\beta}}$, and its easy to show that $\mathbb{P}(\bar{\mathbf{A}}_{T_3e^{-2\delta_0\beta}}|\bar{\mathbf{D}}_{T_3e^{-2\delta_0\beta}}) \geq e^{-(2U-\Delta+3\delta_0)\beta}$.

Estimate (iii): A similar argument works for the transitions involving the lacunary configurations.

7.3 Definitions

Recall that $N = \rho|\Lambda_\beta|$ is the total number of particles and that our state space is

$$\mathcal{X} = \{0,1\}^{\bar\Lambda} \times \{0,1,2,\dots,N\}^{\Lambda_\beta\setminus\bar\Lambda}. \tag{7.17}$$

Dynamics. We begin by realizing the process $(\eta_t)_{t\geq 0}$ in terms of a process in which particles are distinguishable. This means that instead of $\mathcal{X}$ we consider the space $\mathcal{X}_d = \Lambda_\beta^N$ where a configuration is given in terms of the position $\xi(n) \subset \Lambda$ of each particle $n = 1,\dots,N$. Each configuration $\eta \in \mathcal{X}$ corresponds to $N!$ different configurations in $\mathcal{X}_d$. We denote by $n_t(x)$ the set of labels of the particles that are at x at time t:

$$n_t(x) = \{n = 1,\dots,N\colon\ \xi_t(n) = x\}, \tag{7.18}$$

so $\eta_t(x) = |n_t(x)|$. We note that for any $x \in \bar\Lambda$ the set $n_t(x)$ contains at most one element.

We can define a stochastic dynamics (ξ_t) on $\mathcal{X}_d$ corresponding to our stochastic dynamics (η_t) on $\mathcal{X}$, provided we take the uniform distribution for the initial configuration ξ_0 corresponding to the initial configuration η_0, i.e., each particle initially gets a label drawn randomly from $\{1,\dots,N\}$. This rules of the dynamics are straightforward and read as follows:

- For each oriented bond $b \in \bar\Lambda^* \cup \partial^*\bar\Lambda^{out}$, define a sequence of i.i.d. random times $\tau_{b,i}$, $i \in \mathbb{N}$, exponential with mean 1, and a sequence of i.i.d. marks $\mu_{b,i}$, $i \in \mathbb{N}$, uniform on [0,1].
- For each oriented bond $b \in (\Lambda_\beta \setminus \bar\Lambda)^* \cup \partial^*\bar\Lambda^{in}$ and each label $1,\dots,N$, define a sequence of i.i.d. times $\tau(n)_{b,i}$, $i \in \mathbb{N}$, exponential with mean 1.
- Inside $\Lambda_\beta \setminus \bar\Lambda$ particle n moves as an IRW with jump times $\tau(n)_{b,i}$, $i \in \mathbb{N}$, $b \in (\Lambda \setminus \bar\Lambda)^*$.
- Inside $\bar\Lambda$ particles move with exclusion and with interaction as follows. When $t = \tau_{b,i}$ for some $b = (x,y) \in \bar\Lambda^*$, $i \in \mathbb{N}$:

- If $n_t(x) = \{n\}$, $n_t(y) = \emptyset$, then the proposed move y is given by $\xi_t^{prop}(n) = y$, $\xi_t^{prop}(j) = \xi_t(j)$ $\forall j \neq n$. This proposed move is accepted if the corresponding mark $\mu_{b,i}$ is less than $e^{-\beta[H(\eta_t^{prop} - H(\eta_t)]_+}$, where η_t^{prop} and η_t are the configuration in $\mathcal{X}$ corresponding to ξ_t^{prop} resp. ξ_t.
- If $n_t(x) = \emptyset$, $n_t(y) = \{n\}$, then the proposed move y is given by $\xi_t^{prop}(n) = x$, $\xi_t^{prop}(j) = \xi_t(j)$ $\forall j \neq n$. This proposed move is accepted if the corresponding mark $\mu_{b,i}$ is less than $e^{-\beta[H(\eta_t^{prop} - H(\eta_t)]_+}$.
- In all other cases there is no move.

- When $t = \tau_{b,i}$ for some $b = (x, y) \in \partial^* \bar{\Lambda}^{out}$, $i \in \mathbb{N}$:

  - If $n_t(x) = n$, then $\xi_{t+}(n) = y$, $\xi_{t+}(j) = \xi_t(j)$ $\forall j \neq n$.
  - If $n_t(x) = \emptyset$, then there is no move.

- When $t = \tau(n)_{b,i}$ for some $b = (x, y) \in \partial^* \bar{\Lambda}^{in}$, $i \in \mathbb{N}$:

  - If $n_t(x) = n$ and $n_t(y) = \emptyset$, then $\xi_{t+}(n) = y$, $\xi_{t+}(j) = \xi_t(j)$ $\forall j \neq n$.
  - Otherwise there is no move.

Coloring of particles. Next we assign a color to each particle. This color depends on time and will later be used to distinguish between particles arriving from the gas and particles returning to $\bar{\Lambda}_0$ after leaving it.

Definition 36 *(a) Every particle in $\xi_t \cap \bar{\Lambda}$ is* WHITE *at time t.*
(b) Every particle in $\xi_t \cap \Lambda_\beta \setminus \Lambda_{\ell_-}$ is GREEN *at time t.*
(c) Particles in $\Lambda_{\ell_-} \setminus \bar{\Lambda}$ are GREEN *or* RED *depending on their past in the following way. For $n \in n_t(x)$, let*

$$\theta = \theta(n, t) = \max\{0 \leq s < t \colon \xi_s(n) \notin \Lambda_{\ell_-} \setminus \bar{\Lambda}\}. \tag{7.19}$$

If $\xi_\theta(n) \in \bar{\Lambda}$, then n is a RED *particle at time t. If $\xi_\theta(n) \in \Lambda_\beta \setminus \Lambda_{\ell_-}$, then n is a* GREEN *particle at time t.*

It is easy to see that if the process starts from a configuration $\eta \in \mathcal{X}_3$, then the time θ is well-defined for each particle $\xi_t(n) \in \Lambda_{\ell_-} \setminus \bar{\Lambda}$.

The colors at different times are obviously correlated, since when a particle leaves $\bar{\Lambda}$ it become red. Green particles become white when they enter $\bar{\Lambda}$. Red particles become white or green when they leave $\Lambda_{\ell_-} \setminus \bar{\Lambda}$.

In order to control the behavior of green and red particles, we separate their effects by introducing an auxiliary Markov chain $(\tilde{\eta}_t)_{t \geq 0}$ in which the arrival in $\bar{\Lambda}$ of green particles is simulated by a process of creation at $\partial^- \bar{\Lambda}$ with a rate of order $e^{-\beta\Delta}$ and a process of annihilation at $\partial^+ \bar{\Lambda}$ with rate 1.

Auxiliary chain. We define the dynamics $(\tilde{\eta}_t)_{t \geq 0}$ by means of a process $(\tilde{\xi}_t)_{t \geq 0}$ of distinguishable particles. Fix a parameter

$$p := \frac{1}{NT_3} e^{\gamma_0 \beta} \qquad \text{with } T_3 = e^{\Delta\beta}, \tag{7.20}$$

where γ_0 is small and will be chosen later. Let

$$\tilde{\mathcal{X}}_d = (\Lambda_{\ell_-} \cup \{g\})^N. \tag{7.21}$$

This means that each particle in the auxiliary dynamics can be in a site of Λ_{ℓ_-} or in a state called g (for green).

We can use the same ingredients as for the Markov chain $(\xi_t)_{t\geq 0}$ for the bonds in $\Lambda^*_{\ell_-} \cup \partial^*\Lambda^{out}_{\ell_-}$, but we need to add for each bond $b \in \partial^*\bar{\Lambda}^{in}$ a sequence of i.i.d. marks $\mu^c_{b,i}$, $i \in \mathbb{N}$, uniform on $[0,1]$. The stochastic dynamics $(\tilde{\xi}_t)_{t\geq 0}$ can be realized exactly as $(\xi_t)_{t\geq 0}$ for each bond $b \notin \partial^*\bar{\Lambda}^{in}$. If $t = \tau(n)_{b,i}$ for some $b = (x,y) \in \partial^*\bar{\Lambda}^{in}$, $i \in \mathbb{N}$, then we consider an additional move of creation of a particle at site y. The details are again straightforward and read:

- Inside $\Lambda_{\ell_-} \setminus \bar{\Lambda}$ particle n moves as an IRW with jump times $\tau(n)_{b,i}$.
- Inside $\bar{\Lambda}$ particles move with exclusion and with interaction as for $(\xi_t)_{t\geq 0}$.
- When $t = \tau_{b,i}$ for some $b = (x,y) \in \partial^*\bar{\Lambda}^{out}$, $i \in \mathbb{N}$:
  - If $n_t(x) = \{n\}$, then $\tilde{\xi}_{t+}(n) = y$, $\tilde{\xi}_{t+}(j) = \tilde{\xi}_t(j)$ $\forall j \neq n$.
  - If $n_t(x) = \emptyset$, then there is no move.
- When $t = \tau(n)_{b,i}$ for some $b = (x,y) \in \partial^*\bar{\Lambda}^{in}$, $i \in \mathbb{N}$:
  - If $n_t(x) = n$ and $n_t(y) = \emptyset$, then $\tilde{\xi}_{t+}(n) = y$.
  - If $\tilde{\xi}_t(n) = g$ and $n_t(y) = \emptyset$, then $\tilde{\xi}_{t+}(n) = y$ when $\mu^c_{b,i} < p$, otherwise there is no move.
- When $t = \tau(n)_{b,i}$ for some $b = (x,y) \in \partial^*\Lambda^{out}_{\ell_-}$, $i \in \mathbb{N}$:
- If $n_t(x) = n$, then $\tilde{\xi}_{t+}(n) = g$, otherwise there is no move.

We will consider the discrete-time Markov chains corresponding to $(\xi_t)_{t\geq 0}$ and $(\tilde{\xi}_t)_{t\geq 0}$ by observing these processes when a clock rings in Λ_β. From now on we will consider only these discrete versions. By Proposition 4 we know that the control of the discrete-time Markov chain enables us to control the continuous-time Markov chain.

Now that the process $(\tilde{\xi}_t)_{t\geq 0}$ is defined, the definition of $\tilde{\eta}_t$ in terms of $\tilde{\xi}_t$ is given exactly as η_t in terms of ξ_t. The state space $\tilde{\mathcal{X}}$ is

$$\tilde{\mathcal{X}} = \{0,1\}^{\bar{\Lambda}} \times \{0,1,2,\dots,N\}^{(\Lambda_{\ell_-} \setminus \bar{\Lambda} \cup \{g\})}. \tag{7.22}$$

Coupling. We can define a coupling between the Markov chains $(\eta_t)_{t\geq 0}$ and $(\tilde{\eta}_t)_{t\geq 0}$ by using the same clocks and marks for common bonds. Two events will be important:

- $\mathbf{B}_\eta(t_1,\dots,t_k; b_1,\dots,b_k; n_1,\dots,n_k; t)$ is the event for $(\eta_t)_{t\geq 0}$ where during the time interval $[0,t]$ green particles enter $\bar{\Lambda}$ through the bonds $b_1,\dots,b_k \in \partial^*\bar{\Lambda}^{in}$ at times $t_1,\dots,t_k$ and their labels are $n_1,\dots,n_k$.

– $\mathbf{B}_{\tilde\eta}(t_1,\dots,t_k;b_1,\dots,b_k;n_1,\dots,n_k;t)$ is the event for $(\tilde\eta_t)_{t\geq 0}$ where during the time interval $[0,t]$ particles are created through the bonds $b_1,\dots,b_k\in\partial^*\bar\Lambda^{in}$ at times $t_1,\dots,t_k$ and their labels are $n_1,\dots,n_k$.

Let us take two initial configurations η_0 and $\tilde\eta_0$, such that $\eta_0(x)=0$ for all $x\in\Lambda_{\ell_-}\setminus\bar\Lambda$ and $\eta_0(x)=\tilde\eta_0(x)$ for all $x\in\Lambda_{\ell_-}$. If both

$$\begin{array}{l}\mathbf{B}_{\eta}(t_1,\dots,t_k;b_1,\dots,b_k;n_1,\dots,n_k;t)\\ \mathbf{B}_{\tilde\eta}(t_1,\dots,t_k;b_1,\dots,b_k;n_1,\dots,n_k;t)\end{array}\tag{7.23}$$

occur, then we have that for all $s\in[0,t]$ the red and the white particles of η_s coincide with the particles of $\tilde\eta_s$, i.e., if

$$n_s^{w,r}(x)=\{n=1,\dots,N\colon\ \xi_s(n)=x\text{ and }n\text{ is a white or red particle at time }s\},\tag{7.24}$$

then

$$\begin{aligned}\mathbb{P}\Big(\forall s\in[0,t]\colon\ &\{\xi_s(n)=\tilde\xi_s(n)\,\forall n\in n_s^{w,r}\}\cap\{\tilde\xi_s=g\,\forall n\notin n_s^{w,r}\}n\in n_s^{w,r}\}\ \Big|\\ &\mathbf{B}_{\xi}(t_1,\dots,t_k;b_1,\dots,b_k;n_1,\dots,n_k;t)\\ &\mathbf{B}_{\tilde\eta}(t_1,\dots,t_k;b_1,\dots,b_k;n_1,\dots,n_k;t)\Big)=1.\end{aligned}\tag{7.25}$$

In a trivial way we can define a coupling between $(\tilde\eta_t)_{t\geq 0}$ and the local Markov chain $(\bar\eta_t)_{t\geq 0}$ defined in Section 4.1 by using the same clocks and marks for bonds in $\Lambda_0^*\cup\partial^*\bar\Lambda^{out}$ and the same clocks and creation marks for bonds in $b\in\partial^*\bar\Lambda^{in}$. In particular, also for the Markov chain $(\bar\eta_t)_{\tau\geq 0}$ we can realize the creation at the boundary of $\bar\Lambda$ by using N clocks and the same rate p in (7.20) used for $(\tilde\eta_t)_{t\geq 0}$.

If $\gamma_0>0$ in the definition of p is such that $\ell_c(\Delta+\gamma_0)=\ell_c(\Delta)$, then we can apply the remark given at the end of Section 5. If $\Delta\in(U,2U)$ is such that $\frac{U}{2U-\Delta}$ is not integer, then there exists such a γ_0.

It is further obvious that if no red particles enter $\bar\Lambda$ during the time interval $[0,t]$, then $\tilde\eta_s|_{\bar\Lambda}=\bar\eta_s|_{\bar\Lambda}$ for all $s\in[0,t]$.

As in the case of the full Markov chain $(\eta_t)_{t\geq 0}$, we can define recurrence sets $\tilde{\mathcal{X}}_1,\tilde{\mathcal{X}}_2,\tilde{\mathcal{X}}_3$ and we can prove also for $(\tilde\eta_t)_{t\geq 0}$ the recurrence properties to these sets. In the same way we can define the reduced Markov chains $(\tilde\eta_t^R)_{t\geq 0}$ and $(\bar\eta_t^R)_{t\geq 0}$ (recall Definition 34). The only difference is that for $(\tilde\eta_t)_{t\geq 0}$ we can only prove the recurrence to $\tilde{\mathcal{X}}_3$ in a time $p^{-1}e^{\delta_1\beta}$.

7.4 Upper bounds

The key to Proposition 35(iv) is the following (recall (7.11)):

Proposition 37 *There exist $\delta=\delta(\delta_1)$, satisfying $\lim_{\delta_1\downarrow 0}\delta(\delta_1)=0$, and $\beta_0>0$ such that for all $\beta>\beta_0$ and $\bar\eta\in\bar{\mathcal{X}}_3$*

$$\max_{\eta\in I_{\bar\eta}}\mathbb{P}_\eta(\tau_{C^c_{\bar\eta}}<T_3e^{\delta_1\beta})\leq e^{-[r(\bar\eta)-\delta]\beta},\tag{7.26}$$

where

$$\mathcal{C}_{\bar\eta} := \{\eta' \in \mathcal{X}\colon \eta'|_{\bar\Lambda} \in \bar{\mathcal{C}}_{\bar\eta}\} \tag{7.27}$$

and

$$\bar{\mathcal{C}}_{\bar\eta} := \{\bar\eta' \in \bar{\mathcal{X}}\colon \bar H(\bar\eta', \bar\eta) - \bar H(\bar\eta) < r(\bar\eta) + \Delta\}. \tag{7.28}$$

Proof of proposition 35(iv). Let $\mathbf{M}_\eta$ denote the event that the transition $\eta^R\colon \eta \to \eta'$ occurs within time $T_3 e^{\delta_1\beta}$. By the recurrence property of the set $\mathcal{X}_3$, we have

$$\mathbb{P}(\mathbf{M}^c_\eta) \le \text{SES}. \tag{7.29}$$

Hence

$$\begin{aligned}
&\max_{\eta\in I_{\bar\eta}} \max_{\bar\eta'\in\bar{\mathcal{X}}_3\colon \bar\eta'\notin\mathcal{E}_3(\bar\eta)} \mathbb{P}(\eta^R\colon \eta \to I_{\bar\eta'}) \\
&\le \max_{\eta\in I_{\bar\eta}} \max_{\bar\eta'\in\bar{\mathcal{X}}_3\colon \bar\eta'\notin\mathcal{E}_3(\bar\eta)} \mathbb{P}(\eta^R\colon \eta \to I_{\bar\eta'} \cap \mathbf{M}_\eta) + \text{SES} \\
&\le \max_{\eta\in I_{\bar\eta}} \mathbb{P}(\tau_{C_{\bar\eta}} < T_3 e^{\delta_1\beta}) + \text{SES}.
\end{aligned} \tag{7.30}$$

In fact, if $\bar\eta' \in \bar{\mathcal{X}}_3$ and $\bar\eta' \notin \mathcal{E}_3(\bar\eta)$, then $\bar\eta' \notin \bar{\mathcal{C}}_{\bar\eta}$ by Lemma 31. **QED**

Proof of Proposition 37. Abbreviate for $(\tilde\eta_t)_{t\ge0}$

$$\mathbf{A}_\eta = \{\tau_{\mathcal{C}^c_{\bar\eta}} < T_3 e^{\delta_1\beta}\} \tag{7.31}$$

and analogously for $(\tilde\eta_t)_{t\ge0}$

$$\mathbf{A}_{\tilde\eta} = \{\tau_{\tilde{\mathcal{C}}^c_{\bar\eta}} < T_3 e^{\delta_1\beta}\}, \tag{7.32}$$

where

$$\tilde{\mathcal{C}}_{\bar\eta} := \{\tilde\eta' \in \tilde{\mathcal{X}}\colon \tilde\eta'|_{\bar\Lambda} \in \bar{\mathcal{C}}_{\bar\eta}\}. \tag{7.33}$$

Proposition 38 *There exist $\delta = \delta(\delta_1)$, satisfying $\lim_{\delta_1\downarrow0}\delta(\delta_1) = 0$, and $\beta_0 > 0$ such that for all $\beta > \beta_0$ and $\eta' \in \mathcal{X}_3$*

$$\mathbb{P}_{\eta'}(\mathbf{A}_\eta) \le \mathbb{P}(\mathbf{A}_{\tilde\eta}) e^{\delta\beta}. \tag{7.34}$$

Proposition 39 *There exist $\delta = \delta(\delta_1)$, satisfying $\lim_{\delta_1\downarrow0} d(\delta_1) = 0$, and $\beta_0 > 0$ such that for all $\beta > \beta_0$ and $\eta' \in \mathcal{X}_3$*

$$\mathbb{P}_{\eta'}(\mathbf{A}_{\tilde\eta}) \le e^{-r(\bar\eta)\beta + \delta\beta}. \tag{7.35}$$

Propositions 38 and 39 imply Proposition 37. **QED**

The proof of Propositions 38 and 39 is given in Section 7.5 resp. Section 7.6.

7.5 Control of green particles

Proof of proposition 38. The problem is to compare the effect of green particles versus the creation mechanism in the auxiliary dynamics.

1. Fix $k \in \mathbb{N}$, $t_1, \ldots, t_k \geq 0$, $b_1, \ldots, b_k \in \partial^* \bar{\Lambda}^{in}$, and abbreviate

$$\begin{aligned} \mathbf{B}_\eta &= \mathbf{B}_\eta(t_1, ..., t_k; b_1, ..., b_k; n_1, ..., n_k; T_3 e^{\delta_1 \beta}) \\ \mathbf{B}_{\tilde{\eta}} &= \mathbf{B}_{\tilde{\eta}}(t_1, ..., t_k; b_1, ..., b_k; n_1, ..., n_k; T_3 e^{\delta_1 \beta}). \end{aligned} \tag{7.36}$$

We begin by proving the following:

Lemma 40 *There exist $\delta = \delta(\delta_1, \gamma_0)$, satisfying $\lim_{\delta_1, \gamma_0 \downarrow 0} d(\delta_1, \gamma_0) = 0$, and $\beta_0 > 0$ such that for all $\beta > \beta_0$ and $\eta' \in \mathcal{X}_3$*

$$\begin{aligned} \mathbb{P}_{\eta'}(\mathbf{A}_{\tilde{\eta}} \cap \mathbf{B}_{\tilde{\eta}}) \left(\tfrac{e^{-(\Delta-\delta)\beta}}{N}\right)^k &\geq \mathbb{P}_{\eta'}(\mathbf{A}_{\tilde{\eta}} \cap \mathbf{B}_\eta \cap \mathbf{B}_{\tilde{\eta}}) \\ &\geq \mathbb{P}_{\eta'}(\mathbf{A}_\eta \cap \mathbf{B}_\eta \cap \mathbf{B}_{\tilde{\eta}}) \geq \mathbb{P}_{\eta'}(\mathbf{A}_\eta \cap \mathbf{B}_\eta) \left(\tfrac{e^{-(\Delta+\delta)\beta}}{N}\right)^k. \end{aligned} \tag{7.37}$$

Proof. (1) Third inequality: Let $\mathbf{C}_{\tilde{\eta}}$ be the event that the number of rings by clocks on the bonds in $\partial^* \bar{\Lambda}^{in}$ within time $T_3 e^{\delta_1 \beta}$ is less than $N T_3 e^{2\delta_1 \beta}$. Then

$$\mathbb{P}(\mathbf{A}_\eta \cap \mathbf{B}_\eta \cap \mathbf{B}_{\tilde{\eta}}) \geq \mathbb{P}(\mathbf{A}_\eta \cap \mathbf{B}_\eta \cap \mathbf{B}_{\tilde{\eta}} \cap \mathbf{C}_{\tilde{\eta}}), \tag{7.38}$$

and

$$\mathbb{P}(\mathbf{C}_{\tilde{\eta}}) = 1 - \text{SES}. \tag{7.39}$$

We have

$$\begin{aligned} \mathbb{P}(\mathbf{A}_\eta \cap \mathbf{B}_\eta \cap \mathbf{B}_{\tilde{\eta}} \cap \mathbf{C}_{\tilde{\eta}}) &\geq \mathbb{P}(\mathbf{B}_{\tilde{\eta}} | \mathbf{A}_\eta \cap \mathbf{B}_\eta \cap \mathbf{C}_{\tilde{\eta}}) \mathbb{P}(\mathbf{A}_\eta \cap \mathbf{B}_\eta \cap \mathbf{C}_{\tilde{\eta}}) \\ &\geq p^k (1-p)^{N T_3 e^{2\delta_1 \beta}} \mathbb{P}(\mathbf{A}_\eta \cap \mathbf{B}_\eta \cap \mathbf{C}_{\tilde{\eta}}) \end{aligned} \tag{7.40}$$

If γ_0 in the definition of p in (7.20) satisfies $\gamma_0 > 2\delta_1$, then

$$\mathbb{P}(\mathbf{A}_\eta \cap \mathbf{B}_\eta \cap \mathbf{B}_{\tilde{\eta}} \cap \mathbf{C}_{\tilde{\eta}}) \geq p^k e^{-\delta_1 \beta} \mathbb{P}(\mathbf{A}_\eta \cap \mathbf{B}_\eta) + \text{SES} \geq p^k e^{-2\delta_1 \beta} \mathbb{P}(\mathbf{A}_\eta \cap \mathbf{B}_\eta). \tag{7.41}$$

(2) Second inequality: This immediately follows from the coupling between $(\eta_t)_{t \geq 0}$ and $(\tilde{\eta}_t)_{t \geq 0}$.
(3) First inequality: This follows from Lemma 15. The factor $\frac{1}{N^k}$ comes from the fact that Lemma 15 gives an estimate for a process with indistinguishable particles and the probability that a given particle has a given label is $\frac{1}{N}$. **QED**

2. Continuing the proof of Proposition 38, we write out

$$\begin{aligned} \mathbb{P}(\mathbf{A}_\eta) = & \textstyle\sum_{k=0}^\infty \sum_{b_1, \ldots, b_k} \sum_{n_1, \ldots, n_k} \int_0^{T_3 e^{\delta_1 \beta}} \cdots \int_0^{T_3 e^{\delta_1 \beta}} \\ & d\mathbb{P}(\mathbf{A}_\eta \cap \mathbf{B}_\eta(t_1, \ldots, t_k; b_1, \ldots, b_k; n_1, \ldots, n_k; T_3 e^{\delta_1 \beta})), \end{aligned} \tag{7.42}$$

and a similar expansion for $\mathbb{P}(\mathbf{A}_{\tilde{\eta}})$. By Lemma 40 we have

$$\mathbb{P}(\mathbf{A}_\eta \cap \mathbf{B}_\eta) \leq \mathbb{P}(\mathbf{A}_{\tilde{\eta}} \cap \mathbf{B}_{\tilde{\eta}}) e^{2\delta k \beta}, \tag{7.43}$$

and so we obtain

$$\begin{aligned} \mathbb{P}(\mathbf{A}_\eta) &\le \sum_{k=0}^\infty e^{2\delta k\beta}\mathbb{P}(\mathbf{A}_{\tilde\eta}\cap\{\nu=k\}) \\ &= \sum_{k=0}^\infty e^{2\delta k\beta}\mathbb{P}(\mathbf{A}_{\tilde\eta}|\{\nu=k\})\mathbb{P}(\{\nu=k\}), \end{aligned} \tag{7.44}$$

where ν is the number of particles created by the process $(\tilde\eta_t)_{t\geq 0}$ during the transition we are considering. It is easy to see that there exists a $K>0$ such that

$$\mathbb{P}(\nu=k)\le e^{-Kk} \qquad \forall k\in\mathbb{N}_0 \tag{7.45}$$

for β sufficiently large.

3. Let $a=4\frac{\delta}{K}$, $f=\mathbb{P}(\mathbf{A}_\eta|\{\nu=k\})\mathbb{P}(\{\nu=k\})^{1-a}$ and $g=e^{2\delta k\beta}\mathbb{P}(\{\nu=k\})^a$. By applying Hölder's inequality with $p=\frac{1}{1-a}$ and $q=\frac{1}{a}$, we obtain

$$\begin{aligned} \mathbb{P}(\mathbf{A}_\eta) &\le \left(\sum_{k=0}^\infty f^p\right)^{\frac1p}\left(\sum_{k=0}^\infty g^q\right)^{\frac1q} \\ &= \left(\sum_{k=0}^\infty \mathbb{P}(\mathbf{A}_{\tilde\eta}|\{\nu=k\})^{\frac{1}{1-a}}\mathbb{P}(\{\nu=k\})\right)^{1-a}\left(\sum_{k=0}^\infty e^{\frac{2\delta k\beta}{a}}\mathbb{P}(\{\nu=k\})\right)^a \\ &\le C(a,K,\delta,\beta)\mathbb{P}(A_{\tilde\eta})^{1-a}, \end{aligned} \tag{7.46}$$

where the last inequality uses (7.45). But rough estimate gives $P(\mathbf{A}_{\tilde\eta})>e^{K\beta}$ with $K=\Delta\ell_2+\delta\ell_1$, obtained by creating ℓ_2 particles and bringing them to the cluster in a time of order one. Hence we get

$$\mathbb{P}(\mathbf{A}_\eta)\le\mathbb{P}(\mathbf{A}_{\tilde\eta})e^{\delta'\beta}, \tag{7.47}$$

which completes the proof. **QED**

7.6 Control of red particles

We now come to the hardest part of the argument, because we have to control the effect of red particles.

Proof of Proposition 39. In order to control every possible mechanism of exit from the set $\tilde{\mathcal{C}}_{\bar\eta}$, we will analyze the final exit move. More precisely, we will consider a partition of the set $\bar{\mathcal{C}}_{\bar\eta}$ (essentially in terms of its intersection with the sets $\bar{\mathcal{X}}_1,\bar{\mathcal{X}}_2,\bar{\mathcal{X}}_3$), and we will compute the cost (in term of the difference of energy) of the final exiting move starting from an element of this partition. This final move can be obtained by the arrival of a red particle inside $\bar\Lambda$ or by a move inside $\bar\Lambda_0$. We will estimate the probability of the event $\mathbf{A}_{\bar\eta}$ in terms of the probability of this final move. As far as the probability of the arrival of a red particle is concerned, we will show that red particles essentially behave as IRW's.

We divide this proof into several lemmas, the proof of which is deferred to Section 7.7. We first consider the case $\bar\eta\in\bar{\mathcal{X}}_3\setminus\mathcal{L}$.

1. We recall (5.7) that for a configuration $\bar\eta'$ in $\bar\Lambda$ with contour $\gamma' = \gamma'(\bar\eta')$:

$$\bar H(\bar\eta') = (-2U + \Delta)n_0' + \frac{U}{2}|\gamma'| + \Delta n', \tag{7.48}$$

where n_0' is the total number of particles in $\bar\Lambda_0$, n' is the total number of particles in the ring $\bar\Lambda \setminus \bar\Lambda_0$. We will use this expression to classify the configurations in $\bar{\mathcal C}_{\bar\eta}$ in terms of the sets $\bar{\mathcal X}_1$, $\bar{\mathcal X}_2$ and in terms of the cost to exit from $\bar{\mathcal C}_{\bar\eta}$ in one move, defined by

$$\mathrm{EC}(\bar\eta') := \min_{\bar\eta''\notin\bar{\mathcal C}_{\bar\eta}:\, q(\bar\eta',\bar\eta'')>0} \bar H(\bar\eta'') - \bar H(\bar\eta') \qquad (\bar\eta' \in \bar{\mathcal C}_{\bar\eta}), \tag{7.49}$$

where we put $\mathrm{EC}(\bar\eta') = \infty$ if $q(\bar\eta',\bar\eta'') = 0$ for all $\bar\eta'' \notin \bar{\mathcal C}_{\bar\eta}$.

Note that, by the definition of the set $\bar{\mathcal C}_{\bar\eta}$ in (7.28), the exit cost is strictly positive for all $\bar\eta' \in \bar{\mathcal C}_{\bar\eta}$ and thus $\mathrm{EC}(\bar\eta')$ assumes the values U, Δ, $2U$, $3U$.

Lemma 41 *For all $\bar\eta \in \bar{\mathcal X}_3 \setminus \mathcal L$ the following hold:*
(i) If $\bar\eta' \in \bar{\mathcal C}_{\bar\eta}$, then its contour is larger than or equal to that of $\bar\eta$ (i.e, $|\gamma'| \ge |\gamma|$) and its number of internal particles n_0' belongs to the interval $[n_0-\ell_1+2, n_0+1]$, where n_0 is the number of internal particles of $\bar\eta$ and ℓ_1 is the minimal side length of $\bar\eta$.
(ii) Let $\bar\eta' \in \bar{\mathcal C}_{\bar\eta}$, $|\gamma'| = |\gamma|$ and $n' = 0$. Then

$$n_0' = n_0 - l' \quad \textit{for some } \ell' = 0, 1, \dots, \ell_1 - 2. \tag{7.50}$$

Moreover, $\bar\eta' \in \bar{\mathcal X}_2$ and

$$\begin{array}{rcll} \mathrm{EC}(\bar\eta') & = & 3U & \textit{if } \ell' = 0, 1, \dots, \ell_1 - 3 \\ \mathrm{EC}(\bar\eta') & = & 2U & \textit{if } \ell' = \ell_1 - 2. \end{array} \tag{7.51}$$

(iii) Let $\bar\eta' \in \bar{\mathcal C}_{\bar\eta}$, $|\gamma'| = |\gamma| + 2$ and $n' = 0$. If $n_0' \ge n_0 - \ell_1 + 3$, then $\bar\eta' \notin \bar{\mathcal X}_2$ and $\mathrm{EC}(\bar\eta') = \Delta$, while if $n_0' = n_0 - \ell_1 + 2$, then $\bar\eta' \in \bar{\mathcal X} \setminus \bar{\mathcal X}_1$ and $\mathrm{EC}(\bar\eta') = U$.
(iv) Let $\bar\eta' \in \bar{\mathcal C}_{\bar\eta}$, $|\gamma'| = |\gamma| + 4$ and $n' = 0$. Then $\bar\eta' \in \bar{\mathcal X} \setminus \bar{\mathcal X}_1$ and $\mathrm{EC}(\bar\eta') = U$.
(v) Let $\bar\eta' \in \bar{\mathcal C}_{\bar\eta}$ and $n' = 1$. Then $\bar\eta' \in \bar{\mathcal X} \setminus \bar{\mathcal X}_1$ and $|\gamma'| = |\gamma|$ and $\mathrm{EC}(\bar\eta') = U$.
(vi) If $|\gamma'| > |\gamma| + 4$ or $n' > 1$, then $\bar\eta' \notin \bar{\mathcal C}_{\bar\eta}$.

2. We next consider the following partition of the set $\bar{\mathcal C}_{\bar\eta}$:

$$\bar{\mathcal C}_{\bar\eta} = \bar{\mathcal C}_{\bar\eta,0} \cup \bar{\mathcal C}_{\bar\eta,1a} \cup \bar{\mathcal C}_{\bar\eta,1b} \cup \bar{\mathcal C}_{\bar\eta,2}, \tag{7.52}$$

where

$$\begin{array}{rcl} \bar{\mathcal C}_{\bar\eta,0} & = & \{\bar\eta' \in \bar{\mathcal C}_{\bar\eta} \cap \bar{\mathcal X}_2\} \\ \bar{\mathcal C}_{\bar\eta,1a} & = & \{\bar\eta' \in \bar{\mathcal C}_{\bar\eta} \cap (\bar{\mathcal X} \setminus \bar{\mathcal X}_1)\colon |\gamma'| = |\gamma| + 2,\ n' = 0\} \\ \bar{\mathcal C}_{\bar\eta,1b} & = & \{\bar\eta' \in \bar{\mathcal C}_{\bar\eta} \cap (\bar{\mathcal X} \setminus \bar{\mathcal X}_1)\colon |\gamma'| = |\gamma| + 4,\ n' = 0 \text{ or } |\gamma'| = |\gamma|,\ n' = 1\} \\ \bar{\mathcal C}_{\bar\eta,2} & = & \{\bar\eta' \in \bar{\mathcal C}_{\bar\eta} \cap (\bar{\mathcal X}_1 \setminus \bar{\mathcal X}_2)\}. \end{array} \tag{7.53}$$

Definition 42 *(1) An interval of time* $[t_1, t_2]$ *in* $[0, \tau_{\bar{\mathcal{C}}^c_{\bar\eta}} \wedge T_3]$ *is an* INSTABILITY INTERVAL *if* $\tilde\eta_t|_{\bar\Lambda} \notin \bar{\mathcal{X}}_2$ *for all* $t \in [t_1, t_2]$.
(2) An interval $[t_1, t_2]$ *contained in an instability interval is of* TYPE 1 *if* $\tilde\eta_t|_{\bar\Lambda} \in \bar{\mathcal{X}} \setminus \bar{\mathcal{X}}_1$ *for all* $t \in [t_1, t_2]$ *and of* TYPE 2 *if* $\tilde\eta_t|_{\bar\Lambda} \in \bar{\mathcal{X}}_1 \setminus \bar{\mathcal{X}}_2$ *for all* $t \in [t_1, t_2]$. *A type 1 interval is of* TYPE 1A *if* $\tilde\eta_t|_{\bar\Lambda} \in \bar{\mathcal{C}}_{\bar\eta,1a}$ *and of* TYPE 1B *if* $\tilde\eta_t|_{\bar\Lambda} \in \bar{\mathcal{C}}_{\bar\eta,1b}$.
(3) An instability interval can be the union of intervals of different types (i.e., 1a,1b and 2).
(4) The remaining intervals of time in $[0, \tau_{\bar{\mathcal{C}}^c_{\bar\eta}} \wedge T_3]$ *that are not instability intervals are called intervals of* TYPE 0.

An immediate consequence of the recurrence property of $(\tilde\eta_t)_{t\geq 0}$ to the sets $\tilde{\mathcal{X}}_1, \tilde{\mathcal{X}}_2$ is the following:

Lemma 43 *With probability* $1 -$ SES *the instability intervals of type 1 are shorter than* T_1 *and the instability intervals of type 2 are shorter than* T_2.

3. Let us now return to the estimate of $\mathbb{P}(\mathbf{A}_{\tilde\eta})$. We have

$$\mathbf{A}_{\tilde\eta} = \mathbf{A}^r_{\tilde\eta} \cup \mathbf{A}^m_{\tilde\eta}, \tag{7.54}$$

where

- $\mathbf{A}^r_{\tilde\eta} = \mathbf{A}_{\tilde\eta} \cap \{$the final exit move is due to the arrival of a red particle$\}$;
- $gA^m_{\tilde\eta} = \mathbf{A}_{\tilde\eta} \cap \{$the final exit move is due to a move inside $\bar\Lambda\}$.

By Lemma 41, we can estimate $\mathbb{P}(\mathbf{A}^m_{\tilde\eta})$ after decomposing it according to the different kinds of exit through the set

$$\tilde{\mathcal{C}}_{\bar\eta} := \{\tilde\eta' \in \tilde{\mathcal{X}} \colon \tilde\eta'|_{\bar\Lambda} \in \bar{\mathcal{C}}_{\bar\eta}\} \tag{7.55}$$

by considering the starting point of the final exit move. More precisely, if we abbreviate $\bar\eta_1 := \tilde\eta_{\tau_{\tilde{\mathcal{C}}^c_{\bar\eta}}-1}|_{\bar\Lambda}$, $\bar\eta_2 := \tilde\eta_{\tau_{\tilde{\mathcal{C}}^c_{\bar\eta}}}|_{\bar\Lambda}$ and $\Delta\bar H_{1,2} := \bar H(\bar\eta_2) - \bar H(\bar\eta_1)$, then we have

$$\begin{aligned}
\mathbb{P}(\mathbf{A}^m_{\tilde\eta}) \;=\; & \mathbb{P}(\mathbf{A}^m_{\tilde\eta} \cap \{\bar\eta_1 \in \bar{\mathcal{C}}_{\bar\eta,0}\} \cap \{\Delta\bar H_{1,2} = 3U\}) \\
& + \mathbb{P}(\mathbf{A}^m_{\tilde\eta} \cap \{\bar\eta_1 \in \bar{\mathcal{C}}_{\bar\eta,1a}\} \cap \{\Delta\bar H_{1,2} \geq \Delta\}) \\
& + \mathbb{P}(\mathbf{A}^m_{\tilde\eta} \cap \{\bar\eta_1 \in \bar{\mathcal{C}}_{\bar\eta,1b}\} \cap \{\Delta\bar H_{1,2} \geq U\}) \\
& + \mathbb{P}(\mathbf{A}^m_{\tilde\eta} \cap \{\bar\eta_1 \in \bar{\mathcal{C}}_{\bar\eta,2}\} \cap \{\Delta\bar H_{1,2} \geq \Delta\}) \\
& + \mathbb{P}(\mathbf{A}^m_{\tilde\eta} \cap \{\bar\eta_1 \in \bar{\mathcal{C}}_{\bar\eta,0}\} \cap \{\Delta\bar H_{1,2} = 2U\}) \\
& + \mathbb{P}(\mathbf{A}^m_{\tilde\eta} \cap \{\bar\eta_1 \in \bar{\mathcal{C}}_{\bar\eta,1a}\} \cap \{\Delta\bar H_{1,2} = U\}).
\end{aligned} \tag{7.56}$$

The first term in the r.h.s. of (7.56) can be easily estimated, since on the event $\mathbf{A}^m_{\bar\eta} \cap \{\Delta \bar H_{1,2} = 3U\}$ we have a move of cost $3U$ in the interval $[0, T_3]$, i.e.,

$$\mathbb{P}(\mathbf{A}^m_{\bar\eta} \cap \{\bar\eta_1 \in \bar{\mathcal{C}}_{\bar\eta,0}\} \cap \{\Delta \bar H_{1,2} = 3U\}) \le e^{-3U\beta} T_3. \tag{7.57}$$

The control of the last two terms is postponed to Lemma 49. As far as the estimate of the remaining three terms is concerned, we can proceed in a similar way, provided we can estimate the total length of the instability intervals of type 1a, 1b, 2. This means that we need to control the number of instability intervals of the different types via a priori SES probability estimates.

4. Let us next consider the random times $\tau_i < T_3 \wedge \tau_{\tilde{\mathcal{C}}^c_{\bar\eta}}$, $i \in \mathbb{N}$, of arrival in $\bar\Lambda$ of red particles. By definition, each τ_i is the *initial point* of an instability interval of type 1b. Indeed, by Lemma 41, $\tau_i < T_3 \wedge \tau_{\tilde{\mathcal{C}}^c_{\bar\eta}}$ implies that $\tilde\eta_{\tau_i - 1}|_{\bar\Lambda} \in \bar{\mathcal{C}}_{\bar\eta,0}$ and $\tilde\eta_{\tau_i}|_{\bar\Lambda} \in \bar{\mathcal{C}}_{\bar\eta,1b}$. Namely, the arrival of a red particle during an instability interval produces the exit from the cycle $\tilde{\mathcal{C}}_{\bar\eta}$.

In a similar way, let us denote by σ_j, $j \in \mathbb{N}$, the random times corresponding to the exit of particles from $\bar\Lambda$, (i.e., the appearance of red particles) before the exit from the cycle. These times must be the *final point* of an instability interval of type 1b, and $\tilde\eta_{\sigma_j+1}|_{\bar\Lambda} \in \bar{\mathcal{C}}_{\bar\eta,0}$.

The interaction between red particles and particles in $\bar\Lambda$ is active only during instability intervals containing subintervals of type 1b. Now consider a realization of our process in the time interval $[0, T_3 \wedge \tau_{\tilde{\mathcal{C}}^c_{\bar\eta}}]$. This means, in particular, that we have a realization of instability intervals and of random times τ_i and σ_j. Let us look at the process from the point of view of the red particles: this is a system of independent *quasi random walks* (QRW) given by the following rules:

- When a red particle enters $\bar\Lambda$ at a time τ_i, it disappears as red particle during the instability interval starting at τ_i. During this interval the red particle can be killed if the final time of this instability interval is not a time σ_j. Otherwise the particle reappears at a time σ_i at some point in $\partial\bar\Lambda$.

- A new particle appears at a time σ_j, which is the final point of an instability interval not starting with a τ_i. Call the starting point of such an instability interval a *coloration time* α_k.

- Outside $\bar\Lambda$ red particles move like IRW's.

Thus, the difference between our process and a process of IRW's not only comes from the fact that particles can be created and annihilated (with a random law), but also from the fact that particles can disappear for random intervals of times when they touch $\bar\Lambda$ and can reappear again at a different point in $\partial\bar\Lambda$.

5. More precisely, for each coloration time α_k define a quasi random walk $(\mathrm{QRW}(\alpha_k)_t)_{t\ge 0}$ and a delay time $\mathrm{DT}(t, \alpha_k)$ as follows:

– Look at the particle exiting from $\bar\Lambda$ at the end, say σ_j, of the instability interval starting with α_k. Let n_1 be its label, $(\xi(n_1)_t)_{t\geq 0}$ its path, and $\tau_{\bar\Lambda\cup\partial\Lambda_{\ell\_}}(n_1)$ its first hitting time of the set $\bar\Lambda\cup\partial\Lambda_{\ell\_}$. If $\tau_{\bar\Lambda\cup\partial\Lambda_{\ell\_}}(n_1)\geq T_3\wedge\tau_{\mathcal{C}^c_{\bar\eta}}$, then the QRW$(\alpha_k)$ associated with α_k is $(\xi(n_1)_t)_{t\geq 0}$, i.e.,

$$\text{QRW}(\alpha_k)_s=\xi(n_1)_{s+\sigma_j}\qquad\forall s\in[0,(T_3\wedge\tau_{\mathcal{C}^c_{\bar\eta}})-\sigma_j],\tag{7.58}$$

and its delay time is

$$\text{DT}(t,\alpha_k)=(t\wedge\sigma_j)-\alpha_k,\tag{7.59}$$

corresponding to the time spent in $\bar\Lambda$ up to time t.

– If $\tau_{\bar\Lambda\cup\partial\Lambda_{\ell\_}}(n_1)<T_3\wedge\tau_{\mathcal{C}^c_{\bar\eta}}$ and the particle n_1 hits $\partial\Lambda_{\ell\_}$ before $\bar\Lambda$, then the particle is annihilated when exiting $\Lambda_{\ell\_}$. So in this case QRW$(\alpha_k)_s=\xi(n_1)_{s+\sigma_j}$ $\forall s\in[0,\tau_{\partial\Lambda_{\ell\_}}(n_1)-\sigma_j]$ and DT$(t,\alpha_k)=(t\wedge\sigma_j)-\alpha_k$.

– If $\tau_{\bar\Lambda\cup\partial\Lambda_{\ell\_}}(n_1)<T_3\wedge\tau_{\mathcal{C}^c_{\bar\eta}}$ and the particle n_1 hits $\bar\Lambda$ before $\partial\Lambda_{\ell\_}$, then $\tau_{\bar\Lambda\cup\partial\Lambda_{\ell\_}}=\tau_{i_1}$ for some index i_1 and we look at the end of the instability interval starting at τ_{i_1}. If this end time is not a time $\sigma_{j'}$, then the red particle is annihilated in this instability interval, so QRW$(\alpha_k)_s=\xi(n_1)_{s+\sigma_j}$ $\forall s\in[0,\tau_{i_1}-\sigma_j]$ and DT$(t,\alpha_k)=(t\wedge\sigma_j)-\alpha_k$.

– If the end of the instability interval starting with τ_{i_1} is a time σ_{j_1}, then we look at the red particle exiting at this time from $\bar\Lambda$. We let n_2 be its label (not necessarily equal to n_1), and we follow $\xi_t(n_2)$ for $t>\sigma_{j_1}$. As before, we have to distinguish between different possibilities. If $\tau_{\bar\Lambda\cup\partial\Lambda_{\ell\_}}(n_2)\geq T_3\wedge\tau_{\mathcal{C}^c_{\bar\eta}}$, then

$$\begin{aligned}&\text{QRW}(\alpha_k)_s\\&=\begin{cases}\xi_{s+\sigma_j}(n_1) & \text{for } s\in[0,\tau_{i_1}-\sigma_j]\\ \xi_{s+\sigma_{j_1}-(\tau_{i_1}-\sigma_j)}(n_2) & \text{for } s\in[\tau_{i_1}-\sigma_j,(T_3\wedge\tau_{\mathcal{C}^c_{\bar\eta}})-\sigma_{j_1}+\tau_{i_1}-\sigma_j]\end{cases}\end{aligned}\tag{7.60}$$

and

$$\text{DT}(t,\alpha_k)=\begin{cases}(t\wedge\sigma_j)-\alpha_k & \text{for } t\in[\alpha_k,\tau_{i_1}]\\ \sigma_j-\alpha_k+(t\wedge\sigma_{j_1})-\tau_{i_1} & \text{for } t\in[\tau_{i_1},T_3\wedge\tau_{\mathcal{C}^c_{\bar\eta}}]\end{cases}.\tag{7.61}$$

Similarly, we can iterate the previous construction in the other cases.

Roughly speaking, the process $(\text{QRW}(\alpha_k)_t)_{t\geq 0}$ is obtained by glueing together the pieces of random walk performed by red particles outside $\bar\Lambda$, where the paths of two red particles n_i and n_{i+1} are glued together if particle n_i hits $\bar\Lambda$ at the beginning of an instability interval ending with the exit of particle n_{i+1} from $\bar\Lambda$. The delay time DT(t,α_k) is defined as the total length of the union of the instability intervals cut out in this glueing procedure up to time t. By this construction it is clear that, starting from a configuration in $\tilde{\mathcal{X}}_3$, to each red particle we can associate a creation time α_k.

6. By using the above construction of QRW's, we will be able to control:

– the number of instability intervals;

– the probability of the arrival of a red particle in a given time interval.

To do so, we start with the following observation. For any pair of types s, s', $s \neq s'$, we define the transformation $\text{TC}(s,s')$:

$$\text{TC}(s,s') := \min_{\bar\eta_1 \in \bar{\mathcal{C}}_{\bar\eta,s'}, \bar\eta_2 \in \bar{\mathcal{C}}_{\bar\eta,s}} [\bar H(\bar\eta_1) - \bar H(\bar\eta_2)]_+ \tag{7.62}$$

for $s = 0, 1a, 1b, 2$ and $s' = 1a, 1b, 2$.

Lemma 44 *The following hold:*

$$\begin{array}{lll} \text{TC}(0,1a) = U, & \text{TC}(0,1b) = \Delta, & \text{TC}(0,2) = \infty, \\ \\ \text{TC}(1a,1b) = U, & \text{TC}(1a,2) = \infty, & \\ \text{TC}(1b,1a) = 0, & \text{TC}(1b,2) = 0, & \\ \text{TC}(2,1a) = \infty, & \text{TC}(2,1b) = U. & \end{array} \tag{7.63}$$

(The transitions that are not possible in one step get transition cost ∞.)

Lemma 44 implies that, starting from $\tilde{\mathcal{C}}_{\bar\eta}$, the initial time of an interval of type 1b is either a time τ_i or a time corresponding to a move of probability $\leq e^{-\Delta\beta}$. An instability interval containing an interval of type 1b can also be realized by an interval of type 1a followed by an interval of type 1b. Also in this case Lemma 44 implies, by (7.63) and Lemma 43, that we have two moves of probability $e^{-U\beta}$ within a time interval of length $e^{\delta\beta}$. Thus we may conclude that each coloration time α_k corresponds to a move (or a couple of moves) of probability smaller than $e^{-\Delta\beta}$.

7. We have the following a priori estimates:

Lemma 45 *Let $n(T_3, \Delta)$ be the number of moves in $[0, T_3 e^{\gamma_1\beta}]$ of probability $\leq e^{-\Delta\beta}$. Then for all $\delta > 0$*

$$\mathbb{P}(n(T_3,\Delta) > e^{(\delta_1+\delta)\beta}) = \text{SES}. \tag{7.64}$$

Lemma 45 implies that the number of red particles created in $[0, T_3]$ is less than $e^{2\delta\beta}$ with probability $1 - \text{SES}$.

Lemma 46 *Let $\nu(t)$ be the number of visits to $\bar\Lambda$ of a* QRW *describing a red particle during a time t. Then*

$$\mathbb{P}(\nu(t) > (\log t)^3) < e^{\kappa(\log t)^2}. \tag{7.65}$$

for some $\kappa > 0$.

From Lemmas 45 and 46 we obtain that with probability $1 - \text{SES}$ the number of instability intervals containing intervals of type 1b is less than $e^{(2\delta+\delta')\beta}$. In a similar way we can prove the following:

Lemma 47 *There exist $\delta_3''(\delta_0,\delta_1,\delta_2,\gamma)$, satisfying $\lim_{\delta_0,\delta_1,\delta_2,\gamma\downarrow 0}\delta_3''(\delta_0,\delta_1,\delta_2,\gamma)=0$, and $\beta_0>0$ such that for all $\beta>\beta_0$ with probability $1-\mathrm{SES}$: (i) The total number of instability intervals of type 1b in $[0,t\wedge\tau_{\bar{\mathcal{C}}^c_{\bar\eta}}]$ is less than $e^{\delta_3\beta}$.*
(ii) The total number of instability intervals of type 1a in $[0,t\wedge\tau_{\bar{\mathcal{C}}^c_{\bar\eta}}]$ is less than $e^{(\Delta-U+\delta_3)\beta}$.
(iii) The total number of instability intervals $[t_1,t_2]$ of type 1a in $[0,t\wedge\tau_{\bar{\mathcal{C}}^c_{\bar\eta}}]$ such that in $[t_1-e^{(\Delta-U+\delta)\beta}]$ there exists an interval of type 1b is less than $e^{\delta_3\beta}$.
(iv) The total number of instability intervals of type 2 in $[0,t\wedge\tau_{\bar{\mathcal{C}}^c_{\bar\eta}}]$ is less than $e^{\delta_3\beta}$.

8. We now return to the estimate of the 2-nd, 3-rd, 4-th term in the r.h.s. of (7.56). By applying Lemmas 43 and 47 we obtain:

$$\begin{aligned}
\mathbb{P}(\mathbf{A}^m_{\tilde\eta}\cap\{\bar\eta_1\in\bar{\mathcal{C}}_{\bar\eta,1a}\}\cap\{\Delta\bar H_{1,2}\ge\Delta\}) &\le e^{(\Delta-U)\beta+\delta''\beta}e^{\delta\beta}e^{-\Delta\beta}+\mathrm{SES}\\
\mathbb{P}(\mathbf{A}^m_{\tilde\eta}\cap\{\bar\eta_1\in\bar{\mathcal{C}}_{\bar\eta,1b}\}\cap\{\Delta\bar H_{1,2}\ge U\}) &\le e^{\delta''\beta}e^{\delta\beta}e^{-U\beta}+\mathrm{SES}\\
\mathbb{P}(\mathbf{A}^m_{\tilde\eta}\cap\{\bar\eta_1\in\bar{\mathcal{C}}_{\bar\eta,2}\}\cap\{\Delta\bar H_{1,2}\ge\Delta\}) &\le e^{\delta''\beta}e^{U\beta+\delta\beta}e^{-\Delta\beta}+\mathrm{SES}.
\end{aligned}\tag{7.66}$$

We have now to control the term $\mathbb{P}(\mathbf{A}^r_{\tilde\eta})$. Let us denote by $\tau_i\le T_3\wedge\tau_{\tilde{\mathcal{C}}^c_{\bar\eta}}$ the times of arrival of red particles in $\bar\Lambda$ up to time $T_3\wedge\tau_{\tilde{\mathcal{C}}^c_{\bar\eta}}$ (including $\tau_{\tilde{\mathcal{C}}^c_{\bar\eta}}$ if the exit is due to the arrival before T_3 of a red particle inside an instability interval). We observe that, by Lemmas 43 and 47, for each τ_i the delay time $\mathrm{DT}(T_3\wedge\tau_{\tilde{\mathcal{C}}^c_{\bar\eta}},\alpha_k)$ spent by the red particle inside $\bar\Lambda$ is less than $e^{U\beta+(\delta_1+\delta_3)\beta}$ with probability $1-\mathrm{SES}$, where α_k denotes the coloration time of the red particle.

Lemma 48 *There exist $\delta_4(\delta_1,\delta_3)$, satisfying $\lim_{\delta_1,\delta_3\downarrow 0}\delta_4(\delta_1,\delta_3)=0$, and $\beta_0>0$ such that for all $\beta>\beta_0$, $t_0>0$ and $T>0$*

$$\mathbb{P}(\exists\tau_i\in[t_0,t_0+T])\le\Big[e^{-(\Delta-U)\beta}+e^{-\Delta\beta}T\log t_0\Big]e^{\delta_4\beta}.\tag{7.67}$$

9. Let $\mathbb{P}(1a)$, $\mathbb{P}(1b)$, $\mathbb{P}(2)$ be the probabilities of the events where a red particle arrives during an instability interval of type $1a$, $1b$, 2, respectively. Obviously,

$$\mathbb{P}(\mathbf{A}^r_{\tilde\eta})\le\mathbb{P}(1a)+\mathbb{P}(1b)+\mathbb{P}(2).\tag{7.68}$$

By using Lemmas 43, 47 and 48, we have

$$\begin{aligned}
\mathbb{P}(1b) &\le \mathrm{SES}+\mathbb{P}\Big(\exists\text{ an interval }[t_1,t_2]\text{ of type 1b}\\
&\qquad\qquad\text{with }t_2-t_1\le e^{\delta\beta}\text{ and }\exists\tau_i\in(t_1,t_2)\Big)\\
&\le \mathrm{SES}+e^{4\delta\beta}e^{-(\Delta-U)\beta}e^{\delta_3\beta}\\
\mathbb{P}(2) &\le \mathrm{SES}+\mathbb{P}\Big(\exists\text{ an interval }[t_1,t_2]\text{ of type 2}\\
&\qquad\qquad\text{with }t_2-t_1\le e^{U\beta+\delta\beta}\text{ and }\exists\tau_i\in(t_1,t_2)\Big)\\
&\le \mathrm{SES}+e^{\delta_3\beta}2e^{-(\Delta-U)\beta+2\delta\beta}.
\end{aligned}\tag{7.69}$$

10. As far as the term $\mathbb{P}(1a)$ is concerned, we have to distinguish between two cases depending on the existence of an interval of type 1b at a distance less than $e^{(U+\delta)\beta}$ from the interval of type 1a we are considering:

$$\begin{aligned}\mathbb{P}(1a\cap\{\exists \text{ interval of type } 1b \text{ in } [t_1-e^{U\beta+\delta\beta},t_1]\}) &\leq e^{\delta''\beta}e^{-(\Delta-U-4\delta)\beta}\\ \mathbb{P}(1a\cap\{\nexists \text{ interval of type } 1b \text{ in } [t_1-e^{U\beta+\delta\beta},t_1]\}) &\leq e^{(U+\delta'')\beta}e^{-(\Delta_-\delta)\beta}.\end{aligned} \tag{7.70}$$

Indeed, in this last case the first term in the estimate in Lemma 48 is absent, since it comes from the event that there exists a creation time α_k at a distance less than $e^{(U+\delta)\beta}$ from the interval of type 1a we are considering and this is forbidden by the event $\{\nexists$ interval of type $1b$ in $[t_1-e^{U\beta+\delta\beta},t_1]\}$. (See (7.91) in the proof of Lemma 48 below.) It remains to estimate the probability of exit by contraction. This is given by the following:

Lemma 49

$$\mathbb{P}\Big(\mathbf{A}^m_{\bar\eta}\cap\{\bar\eta_1\in\bar{\mathcal{C}}_{\bar\eta,0}\}\cap\{\Delta\bar H_{1,2}=2U\}\Big)$$
$$+\ \mathbb{P}\Big(\mathbf{A}^m_{\bar\eta}\cap\{\bar\eta_1\in\bar{\mathcal{C}}_{\bar\eta,1a}\}\cap\{\Delta\bar H_{1,2}=U\}\Big)\leq e^{-[(2U-\Delta)(\ell_1-1)-\delta]\beta}. \tag{7.71}$$

This completes the proof of Proposition 39 for the case $\bar\eta\in\bar{\mathcal{X}}_3\setminus\mathcal{L}$. **QED**

For the case $\bar\eta\in\mathcal{L}$ we can proceed in a similar way. We indicate here only the differences with the case $\bar\eta\in\bar{\mathcal{X}}_3\setminus\mathcal{L}$.

The characterization of the set $\bar{\mathcal{C}}^{2U}_{\bar\eta}$ can be done as follows. By Lemma 26(iii), we know that $\mathcal{F}(\bar{\mathcal{C}}^{2U}_{\bar\eta})=\bar{\mathcal{C}}^{2U}_{\bar\eta}\cap\bar{\mathcal{X}}_3=\bar{\mathcal{C}}^{2U}_{\bar\eta}\cap\bar{\mathcal{X}}_2$. A direct check shows that $\bar H(\bar\eta')\in\{\bar H(\bar\eta),\bar H(\bar\eta)+\Delta-U,\bar H(\bar\eta)+U,\bar H(\bar\eta)+\Delta\}$ for any $\bar\eta'\in\bar{\mathcal{C}}^{2U}_{\bar\eta}$. Moreover, configurations with $\bar H(\bar\eta')\geq\bar H(\bar\eta)+U$ are not in $\bar{\mathcal{X}}_1$. Indeed, if such an $\bar\eta'$ is not 0-reducible, then this inequality implies that for each $\bar\eta''$ with $\bar H(\bar\eta'')<\bar H(\bar\eta')$ we have $\bar H(\bar\eta',\bar\eta'')\geq\bar H(\bar\eta')+U$, and so if we choose $\bar\eta''=\bar\eta$, then we obtain $\bar H(\bar\eta',\bar\eta)\geq\bar H(\bar\eta)+U+U$, which contradicts $\bar\eta'\in\bar{\mathcal{C}}^{2U}_{\bar\eta}$.

Thus the partition of the set $\bar{\mathcal{C}}^{2U}_{\bar\eta}$ uses

$$\begin{aligned}\bar{\mathcal{C}}^{2U}_{\bar\eta,0} &= \{\bar\eta'\in\bar{\mathcal{C}}^{2U}_{\bar\eta}\cap\bar{\mathcal{X}}_3\}\\ \bar{\mathcal{C}}^{2U}_{\bar\eta,1a} &= \{\bar\eta'\in\bar{\mathcal{C}}^{2U}_{\bar\eta}\cap(\bar{\mathcal{X}}\setminus\bar{\mathcal{X}}_1)\text{ with }\bar H(\bar\eta')=\bar H(\bar\eta)+U\}\\ \bar{\mathcal{C}}^{2U}_{\bar\eta,1b} &= \{\bar\eta'\in\bar{\mathcal{C}}^{2U}_{\bar\eta}\cap(\bar{\mathcal{X}}\setminus\bar{\mathcal{X}}_1)\text{ with }\bar H(\bar\eta')=\bar H(\bar\eta)+U\}\\ \bar{\mathcal{C}}^{2U}_{\bar\eta,2} &= \{\bar\eta'\in\bar{\mathcal{C}}^{2U}_{\bar\eta}\cap(\bar{\mathcal{X}}_1\setminus\bar{\mathcal{X}}_2)\text{ with }\bar H(\bar\eta')=\bar H(\bar\eta)+\Delta-U\}.\end{aligned} \tag{7.72}$$

The exit costs in this case are given by

$$\begin{aligned}\bar\eta'\in\bar{\mathcal{C}}^{2U}_{\bar\eta,0}: &\quad \mathrm{EC}(\bar\eta')=2U\\ \bar\eta'\in\bar{\mathcal{C}}^{2U}_{\bar\eta,1a}: &\quad \mathrm{EC}(\bar\eta')=U\\ \bar\eta'\in\bar{\mathcal{C}}^{2U}_{\bar\eta,1b}: &\quad EC(\bar\eta')=U\\ \bar\eta'\in\bar{\mathcal{C}}^{2U}_{\bar\eta,2}: &\quad \mathrm{EC}(\bar\eta')=\Delta,\end{aligned} \tag{7.73}$$

where we use that $2U - \Delta < \frac{U}{2}$. The transition costs are given by

$$\begin{array}{lll} \text{TC}(0,1a) = U, & \text{TC}(0,1b) = \Delta, & \text{TC}(0,2) = \infty \\ \text{TC}(1a,1b) = \infty, & \text{TC}(1a,2) = \infty, & \text{TC}(1b,1a) = \infty \\ \text{TC}(1b,2) = 0, & \text{TC}(2,1a) = \infty, & \text{TC}(2,1b) = U. \end{array} \tag{7.74}$$

We can prove also for this case the result in Lemma 47, and the rest of the proof follows exactly the same calculations performed for the case $\bar{\eta} \in \bar{\mathcal{X}}_3 \setminus \mathcal{L}$.

7.7 Proof of the lemmas

In this section we prove the lemmas that were used in Section 7.6.

Proof of Lemma 41. (i) Let ℓ_1, ℓ_2 be the side lengths of η, with $0 \le \ell_2 - \ell_1 \le 1$, and let γ be its contour. For any $\bar{\eta}'$ such that $|\gamma'| < |\gamma|$, we have $n_0' \le n_0 - \ell_1$. On the other hand, by the results in Section 4 we know that $\bar{H}(\mathcal{N}_{\ell_1\ell_2}, \mathcal{N}_{\ell_1(\ell_2-1)}) \ge r(\bar{\eta}) + \Delta + \bar{H}(\bar{\eta})$, so that such a configuration $\bar{\eta}'$ is not in $\bar{\mathcal{C}}_{\bar{\eta}}$. In the same way, by using the results in Section 4, we immediately see that if $n_0' \ge n_0 + 2$ or $n_0' \le n_0 - \ell_1 + 1$, then the configuration $\bar{\eta}'$ is not in $\bar{\mathcal{C}}_{\bar{\eta}}$. (ii) Equation (7.50) follows from (i) and the fact that n_0 is the maximal number of internal particles given a contour of length $|\gamma|$. To prove that $\eta' \in \bar{\mathcal{X}}_2$, we only have to observe that γ' must be a monotone contour, after which we can apply Proposition 25. To evaluate the exit cost of η', note that

$$\bar{H}(\eta') = \bar{H}(\bar{\eta}) + \ell'(2U - \Delta), \tag{7.75}$$

while, by the definition of EC in (7.49),

$$\bar{H}(\bar{\eta}') + \text{EC}(\bar{\eta}') - \bar{H}(\bar{\eta}) \ge r(\bar{\eta}) + \Delta + \ge (\ell_1 - 1)(2U - \Delta) + \Delta. \tag{7.76}$$

So we obtain

$$\text{EC}(\bar{\eta}') \ge (\ell_1 - 1 - \ell')(2U - \Delta) + \Delta. \tag{7.77}$$

(iii) We argue by contradiction. Suppose that $|\gamma'| = |\gamma| + 2$, $n' = 0$ and $\bar{\eta}' \in \bar{\mathcal{C}}_{\bar{\eta}} \cap \bar{\mathcal{X}}_2$. Then, since $n_0' \le n_0 + 1$, we have

$$\bar{H}(\bar{\eta}') \ge \bar{H}(\bar{\eta}) + U - (2U - \Delta). \tag{7.78}$$

But

$$\bar{H}(\bar{\eta}', \bar{\eta}) - \bar{H}(\bar{\eta}') \ge \Delta. \tag{7.79}$$

Hence

$$\bar{H}(\bar{\eta}', \bar{\eta}) \ge \Delta + \Delta - U + \bar{H}(\bar{\eta}) \ge r(\bar{\eta}) + \bar{H}(\bar{\eta}), \tag{7.80}$$

which contradicts the hypothesis $\bar{\eta}' \in \bar{\mathcal{C}}_{\bar{\eta}}$. So $\bar{\eta}' \notin \bar{\mathcal{X}}_2$. To compute $\text{EC}(\bar{\eta}')$, we note that

$$\text{EC}(\bar{\eta}') \ge \bar{H}(\bar{\eta}) - \bar{H}(\bar{\eta}') + r(\bar{\eta}) + \Delta, \tag{7.81}$$

and since

$$\bar{H}(\bar{\eta}) - \bar{H}(\bar{\eta}') = (-2U + \Delta)(n_0 - n_0') - U, \tag{7.82}$$

we obtain the result.

(iv) In a similar way, $\bar{H}(\bar{\eta}') \geq \bar{H}(\bar{\eta}) + 2U - (2U - \Delta)$, and if $\bar{\eta}' \in \bar{\mathcal{X}}_1$, then

$$\bar{H}(\bar{\eta}', \bar{\eta}) \geq \Delta + U + \bar{H}(\bar{\eta}) \geq r(\bar{\eta}) + \bar{H}(\bar{\eta}). \tag{7.83}$$

Now $\mathrm{EC}(\bar{\eta}') = U$, since

$$\bar{H}(\bar{\eta}') + U - \bar{H}(\bar{\eta}) \geq \Delta + U \geq r(\bar{\eta}) + \Delta. \tag{7.84}$$

(v) Note that

$$\bar{H}(\bar{\eta}') = (-2U+\Delta)n_0' + \frac{U}{2}|\gamma'| + \Delta \geq (-2U+\Delta) + \frac{U}{2}(|\gamma'|-|\gamma|) + \Delta + \bar{H}(\bar{\eta}), \tag{7.85}$$

so if $|\gamma'| > |\gamma|$, then

$$\bar{H}(\bar{\eta}') \geq -2U + \Delta + U + \Delta + \bar{H}(\bar{\eta}) = 2\Delta - U + \bar{H}(\bar{\eta}) \geq r(\bar{\eta}) + \bar{H}(\bar{\eta}), \tag{7.86}$$

while if $|\gamma'| = |\gamma|$ and $\bar{\eta}' \in \bar{\mathcal{X}}_1$, then

$$\bar{H}(\bar{\eta}', \bar{\eta}) \geq U + \bar{H}(\bar{\eta}') \geq U + 2\Delta - 2U + \bar{H}(\bar{\eta}) \geq r(\bar{\eta}) + \bar{H}(\bar{\eta}). \tag{7.87}$$

Also in this case we have that $\mathrm{EC}(\bar{\eta}') = U$, since $\bar{H}(\bar{\eta}') + U - \bar{H}(\bar{\eta}) \geq r(\bar{\eta}) + \Delta$.

(vi) If $\gamma'| > |\gamma| + 4$ or $n' > 1$, then $\bar{H}(\bar{\eta}') \geq U + \Delta + \bar{H}(\bar{\eta}) \geq r(\bar{\eta}) + \bar{H}(\bar{\eta})$ or $\bar{H}(\bar{\eta}') \geq 3\Delta - 2U + \bar{H}(\bar{\eta}) \geq r(\bar{\eta}) + \bar{H}(\bar{\eta})$. **QED**

Proof of Lemma 43. The claim is an immediate consequence of Proposition 32. **QED**

Proof of Lemma 44.
1. Let $s = 0$, $s' = 1a$. Then $|\gamma_2| = |\gamma_1| + 2$ and $n_2 = n_1$, since the move $\bar{\eta}_1 \to \bar{\eta}_2$ has to be inside $\bar{\Lambda}_0$. Therefore $\bar{H}(\bar{\eta}_2) - \bar{H}(\bar{\eta}_1) = U$. Analogously, let $s = 0$, $s' = 1b$. Then $|\gamma_2| = |\gamma_1| + 4$ and $n_2 = n_1$ or $|\gamma_2| = |\gamma_1|$ and $n_2' = 1$, so that $\bar{H}(\bar{\eta}_2) - \bar{H}(\bar{\eta}_1) = 2U \wedge \Delta = \Delta$.
2. The transition $\bar{\eta}_1 \in \bar{\mathcal{C}}_{\bar{\eta},0} \to \bar{\eta}_2 \in \bar{\mathcal{C}}_{\bar{\eta},2}$ is not possible. Indeed, we have $\bar{H}(\bar{\eta}_1) \neq \bar{H}(\bar{\eta}_2)$, since $\bar{\eta}_1$ is not 0-equivalent to $\bar{\eta}_2$, and if $\bar{H}(\bar{\eta}_1) < \bar{H}(\bar{\eta}_2)$, then the move $\bar{\eta}_2 \to \bar{\eta}_1$ shows that $\bar{\eta}_2 \notin \bar{\mathcal{X}}_1$, while if $\bar{H}(\bar{\eta}_1) > \bar{H}(\bar{\eta}_2)$, then the move $\bar{\eta}_1 \to \bar{\eta}_2$ shows that $\bar{\eta}_1 \notin \bar{\mathcal{X}}_1$.
3. If $s = 1a$, $s' = 1b$, then we have two possibilities:
(1) $|\gamma_2| = |\gamma_1| + 2$ and $n_1' = n_2' = 0, \nu_1 = n_2$;
(2) $|\gamma_2| = |\gamma|, |\gamma_1| = |\gamma| + 2$ and $n_2' = 1$, $n_1' = 0$.
In the first case $\bar{H}(\bar{\eta}_2) - \bar{H}(\bar{\eta}_1) = U$, in the second case the move must be between $\bar{\Lambda}_0$ and the ring $\bar{\Lambda} \setminus \bar{\Lambda}_0$, so $n_2 = n_1 - 1$ and $\bar{H}(\bar{\eta}_2) - \bar{H}(\bar{\eta}_1) = (-2U + \delta)(n_2 - n_1) + \frac{U}{2}(|\gamma_2| - |\gamma_1|) + \Delta = U$.
4. The transition $\bar{\eta}_1 \in \bar{\mathcal{C}}_{\bar{\eta},1a} \to \bar{\eta}_2 \in \bar{\mathcal{C}}_{\bar{\eta},2}$ is not possible. Indeed. we have $\bar{H}(\bar{\eta}_1) \neq \bar{H}(\bar{\eta}_2)$, since $\bar{\eta}_1$ is not 0-equivalent to $\bar{\eta}_2$, and $|\gamma_1| = |\gamma| + 2 = |\gamma_2|$ and $n_1 = n_2$, since $n_1' = n_2' = 0$, but this is a contradiction.
5. If $s = 1b, s' = 2$, then we have a creation cost zero (as given, for instance, by the transition in which a particle reaches the square or quasi-square cluster). A similar argument holds for the case $s = 1b, s' = 1a$.
6. The transition $2 \to 1a$ is forbidden by an argument equal to that used in

the transition $1a \to 2$.
7. If $s = 2, s' = 1b$, then we have two possibilities:
(1) $|\gamma_2| = |\gamma| + 4$, $|\gamma_1| = |\gamma| + 2$ and $n_1' = n_2' = 0$;
(2) $|\gamma_2| = |\gamma|$, $|\gamma_1| = |\gamma| + 2$ and $n_2' = 1$, $n_1' = 0$.
In both cases we have $\bar{H}(\bar{\eta}_2) - \bar{H}(\bar{\eta}_1) = U$. **QED**

Proof of Lemma 45. This is an elementary large deviation estimate for a binomial distribution obtained by applying an exponential Chebyshev inequality. **QED**

Proof of Lemma 46. For any $t > 1$

$$\begin{aligned}
&\mathbb{P}(\nu(t) > (\log t)^3) \\
&\quad \le \mathbb{P}(\exists\, (\log t)^3 \text{ IRW's starting at } \partial\bar{\Lambda} \text{ and returning to } \bar{\Lambda} \text{ within time } t) \\
&\quad \le (1 - \tfrac{\kappa}{\log t})^{(\log t)^3} \sim e^{-\kappa(\log t)^2},
\end{aligned} \tag{7.88}$$

where we use Proposition 11. **QED**

Proof of Lemma 47.
1. Let $N'(1b)$ be the number of instability intervals containing at least an interval of type 1b within time $T_3 e^{\delta_1 \beta} \wedge \tau_{\bar{\mathcal{C}}^c_{\bar{\eta}}}$. We have

$$\begin{aligned}
\mathbb{P}(N'(1b) > e^{(2\delta+\delta')\beta}) \;&\le\; \mathbb{P}(n(T_3, \Delta) > e^{(\delta_1+\delta)\beta}) + \mathbb{P}(\nu(T_3) > e^{\delta'\beta}) \\
&+ \mathbb{P}\Big(\{N'(1b) > e^{(2\delta+\delta'')\beta}\} \cap \{n(T_3,\Delta) \le e^{(\delta_1+\delta)\beta}\} \cap \{\nu(T_3) > e^{\delta'\beta}\}\Big).
\end{aligned} \tag{7.89}$$

By using Lemmas 45 and 46, we get $\mathbb{P}(N'(1b) > e^{(2\delta+\delta'')\beta}) \le \mathrm{SES}$.
2. In any instability interval containing intervals of type 1b, the maximal number of intervals of type 1b is $e^{\bar{\delta}\beta}$ with probability $1 - \mathrm{SES}$. This follows from Lemma 44 and the analogue of Lemma 45 for the quantity $n(T_2, U)$. Thus we have that the total number $N(1b)$ of intervals of type 1b within time $T_3 \wedge \tau_{\bar{\mathcal{C}}^c_{\bar{\eta}}}$ is smaller than $e^{3\delta''\beta}$ with probability $1 - \mathrm{SES}$.
3. The number $N(2)$ of intervals of type 2 is estimated by $N(2) \le N(1b)$, again by Lemma 44. With similar arguments we can conclude that the number $N'(1a)$ of intervals of type 1a starting from an interval of type 0 is, with probability $1 - \mathrm{SES}$, smaller than $e^{(\Delta-U)\beta+\delta\beta}$, and the number of intervals of type 1a contained in an instability interval containing intervals of type 1b is less than $e^{\bar{\delta}\beta}$, so that $N(1a) \le N'(1a) + N'(1b)e^{\bar{\delta}\beta}$. **QED**

Proof of Lemma 48. Let α_k be a coloration time. If $t_0 < e^{U\beta+(\delta_1+\delta_2)\beta}$, then

$$\begin{aligned}
\mathbb{P}(\exists \tau_i \in [t_0, t_0 + T]) &\le \mathbb{P}\Big(\exists \alpha_k \in [0, e^{U\beta+(\delta_1+\delta_2)\beta} + T]\Big) \\
&\le \sum_{k=1}^{e^{\delta_3\beta}} \sum_{s=0}^{e^{U\beta+(\delta_1+\delta_2)\beta}+T} \mathbb{P}(\alpha_k = s) + \mathrm{SES} \\
&\le e^{-\Delta\beta+\delta_3\beta}(e^{U\beta+(\delta_1+\delta_2)\beta} + T) + \mathrm{SES}.
\end{aligned} \tag{7.90}$$

On the other hand, if $t_0 > e^{U\beta+(\delta_1+\delta_2)\beta}$, then

$$\begin{aligned}
&\mathbb{P}(\exists\tau_i \in [t_0, t_0+T])\\
&= \mathbb{P}\Big(\exists\tau_i \in [t_0, t_0+T] \cap \{\alpha_k\ in[t_0 - e^{U\beta+(\delta_1+\delta_2)\beta}, t_0+T]\}\Big)\\
&\quad + \mathbb{P}\Big(\exists\tau_i \in [t_0, t_0+T] \cap \{\alpha_k < t_0 - e^{U\beta+(\delta_1+\delta_2)\beta}\}\Big)\\
&\le \sum_{k=1}^{e^{2\delta\beta}} \sum_{s=t_0-e^{U\beta+2\delta\beta}}^{t_0+T} \mathbb{P}(\alpha_k = s)\\
&+ \sum_{k=1}^{e^{\delta_3\beta}} \sum_{s=0}^{t_0-e^{U\beta+(\delta_1+\delta_2)\beta}} \sum_{t=t_0}^{t_0+T} \mathbb{P}\Big(\{\alpha_k = s\} \cap \{\mathrm{QRW}_{t-s-\mathrm{DT}(t,\alpha_k)}(\alpha_k) \in \bar{\Lambda}\}\Big) + \mathrm{SES}.
\end{aligned} \tag{7.91}$$

2. For QRW similar estimates hold as for SRW. namely,

$$\max_x \mathbb{P}(\mathrm{QRW}_t = x) \le \frac{\kappa \log t}{t}. \tag{7.92}$$

Indeed, we can write

$$\mathrm{QRW}_t = \mathrm{SRW}_t + \mathrm{JP}_t \tag{7.93}$$

where JP_t is a sum of jumps

$$\mathrm{JP}_t = \sum_{n=0}^{\nu(t)} J_n \tag{7.94}$$

with $|J_n| < 2l_0$ and with $\nu(t)$ estimated in Lemma 46.
3. From (7.92) we obtain

$$\begin{aligned}
&\mathbb{P}(\exists\tau_i \in [t_0, t_0+T])\\
&\le e^{-\Delta\beta} e^{U\beta+(\delta_1+2\delta_3)\beta} + \sum_{t=t_0}^{t_0+T} \sum_{s=0}^{t_0-e^{U\beta+(\delta_1+\delta_2)\beta}} e^{-\Delta\beta+\delta_3\beta} \frac{1}{t-s-e^{U\beta+\delta\beta}} + \mathrm{SES}\\
&\le e^{-\Delta\beta} e^{U\beta+(\delta_1+2\delta_3)\beta} + e^{-\Delta\beta+\delta_3\beta} T \log t_0 + \mathrm{SES}.
\end{aligned} \tag{7.95}$$

QED

Proof of Lemma 49.
1. For $m = n_0 - \ell_1 + 2, \ldots, n_0 + 1$, define

$$\bar{\mathcal{C}}_{\bar{\eta}}(m) = \bar{\mathcal{C}}_{\bar{\eta}} \cap \mathcal{N}_m, \tag{7.96}$$

where $\mathcal{N}_m = \{\bar{\eta}' \in \bar{\mathcal{X}}\colon n_0' + n' = m\}$. By Lemma 41, $\bar{\mathcal{C}}_{\bar{\eta}} = \cup_{m=n_0-\ell_1+2}^{n_0+1} \bar{\mathcal{C}}_{\bar{\eta}}(m)$. We note that the sets $\bar{\mathcal{C}}_{\bar{\eta}}(m)$ are not necessarily connected. We write

$$\bar{\mathcal{C}}_{\bar{\eta}}(> m) = \bigcup_{m'=m+1}^{n_0+1} \bar{\mathcal{C}}_{\bar{\eta}}(m'). \tag{7.97}$$

Let $\zeta_j(m)$, $j\in\mathbb{N}$, denote the return times to the set $\bar{\mathcal{C}}_{\bar\eta}(m)$ within $T_3\wedge\tau_{\bar{\mathcal{C}}^c_{\bar\eta}}$, i.e.,

$$
\begin{aligned}
&\zeta_1(m)=(T_3\wedge\tau_{\bar{\mathcal{C}}^c_{\bar\eta}})\wedge\min\{t\geq 0\colon\, \tilde\eta_t\in\bar{\mathcal{C}}_{\bar\eta}(m)\}\\
&\zeta'_j(m)=(T_3\wedge\tau_{\bar{\mathcal{C}}^c_{\bar\eta}})\wedge\min\{t>\zeta_j(m)\colon\, \tilde\eta_t\notin\bar{\mathcal{C}}_{\bar\eta}(m)\}\\
&\zeta_{j+1}(m)=(T_3\wedge\tau_{\bar{\mathcal{C}}^c_{\bar\eta}})\wedge\min\{t>\zeta'_j(m)\colon\, \tilde\eta_t\in\bar{\mathcal{C}}_{\bar\eta}(m)\}.
\end{aligned}
\tag{7.98}
$$

2. Let m_0 be the minimal index m, i.e., $m_0=n_0-\ell_1+2$. We say that the process $(\tilde\eta_t)_{t\geq0}$ exits by contraction from the set $\bar{\mathcal{C}}_{\bar\eta}(m_0)$ if it exits without increasing the number of particles in $\bar\Lambda$, i.e., if $\sum_{x\in\bar\Lambda}\tilde\eta_t(x)=m_0$ for all $t\leq\tau_{\bar{\mathcal{C}}^c_{\bar\eta}(m_0)}$. On the events

$$
\begin{aligned}
&\{\mathbf{A}^m_{\bar\eta}\cap\{\bar\eta_1\in\bar{\mathcal{C}}_{\bar\eta,0}\}\cap\{\Delta\bar H_{1,2}=2U\}\}\\
&\{\mathbf{A}^m_{\bar\eta}\cap\{\bar\eta_1\in\bar{\mathcal{C}}_{\bar\eta,1a}\}\cap\{\Delta\bar H_{1,2}=U\}\}
\end{aligned}
\tag{7.99}
$$

the process $(\tilde\eta_t)_{t\geq0}$ exits from $\tilde{\mathcal{C}}_{\bar\eta}$ by visiting $\bar{\mathcal{C}}_{\bar\eta}(m_0)$ and then leaving it by contraction. So we have the estimate

$$
\begin{aligned}
&\mathbb{P}(\mathbf{A}^m_{\bar\eta}\cap\{\bar\eta_1\in\bar{\mathcal{C}}_{\bar\eta,0}\}\cap\{\Delta\bar H_{1,2}=2U\})\\
&\qquad\mathbb{P}(\mathbf{A}^m_{\bar\eta}\cap\{\bar\eta_1\in\bar{\mathcal{C}}_{\bar\eta,1a}\}\cap\{\Delta\bar H_{1,2}=U\})\\
&\leq\sum_{i=1}^{e^{\delta''\beta}}\sum_{y\in\bar{\mathcal{C}}_{\bar\eta}(m_0)}\mathbb{P}(\zeta_i(m_0)<T_3\wedge\tau_{\bar{\mathcal{C}}^c_{\bar\eta}}\cap\tilde\eta_{\zeta_i}=y)\times\\
&\qquad\qquad\mathbb{P}_y\Big((\tilde\eta_t)_{t\geq0}\text{ exits by contraction from }\bar{\mathcal{C}}_{\bar\eta}(m_0)\Big)\\
&+\mathbb{P}\Big(\max\{i\geq1\colon\,\zeta_i<T_3\wedge\tau_{\bar{\mathcal{C}}^c_{\bar\eta}}\}>e^{\delta''\beta}\Big).
\end{aligned}
\tag{7.100}
$$

3. By using the coupling between $(\tilde\eta_t)_{t\geq0}$ and $(\bar\eta_t)_{t\geq0}$ we obtain

$$
\begin{aligned}
&\sup_{y\in\bar{\mathcal{C}}_{\bar\eta}(m_0)}\mathbb{P}_y\Big((\tilde\eta_t)_{\tau\geq0}\text{ exits by contraction from }\bar{\mathcal{C}}_{\bar\eta}(m_0)\Big)\\
&\leq\sup_{y\in\bar{\mathcal{C}}_{\bar\eta}(m_0)}\mathbb{P}_y\Big((\bar\eta_t)_{t\geq0}\text{ exits by contraction from }\bar{\mathcal{C}}_{\bar\eta}(m_0)\Big).
\end{aligned}
\tag{7.101}
$$

On the other hand, by using reversibility we can estimate

$$
\mathbb{P}_y\Big((\tilde\eta_t)_{t\geq0}\text{ exits by contraction from }\bar{\mathcal{C}}_{\bar\eta}(m_0)\Big)\leq e^{-(2U-\Delta-\delta)\beta}.
\tag{7.102}
$$

Moreover,

$$
\mathbb{P}(\zeta_i(m_0)<T_3\wedge\tau_{\bar{\mathcal{C}}^c_{\bar\eta}})\leq\mathbb{P}(\tau_{\bar{\mathcal{C}}_{\bar\eta}(m_0)}<T_3\wedge\tau_{\bar{\mathcal{C}}^c_{\bar\eta}}),
\tag{7.103}
$$

and thus we obtain

$$
\begin{aligned}
\text{r.h.s. of (7.100)}\ &\leq e^{\delta''\beta}\mathbb{P}(\tau_{\bar{\mathcal{C}}_{\bar\eta}(m_0)}<T_3\wedge\tau_{\bar{\mathcal{C}}^c_{\bar\eta}})\ e^{-(2U-\Delta-\delta)\beta}\\
&+\mathbb{P}\Big(\max\{i\geq1\colon\,\zeta_i<T_3\wedge\tau_{\bar{\mathcal{C}}^c_{\bar\eta}}\}>e^{\delta''\beta}\Big).
\end{aligned}
\tag{7.104}
$$

4. Each entrance into the set $\bar{\mathcal{C}}_{\bar\eta}(m_0)$, i.e., each $\zeta_i(m_0)$, is a time at which a particle exits from $\bar\Lambda$ and so it is the endpoint of an interval of type 1b. So, by Lemma 47, we have that the last term in (7.104) is SES. The proof of the lemma can now be concluded by iteration after we note that on the event $\{\tau_{\bar{\mathcal{C}}_{\bar\eta}(m_0)} < T_3 \wedge \tau_{\bar{\mathcal{C}}^c_{\bar\eta}}\}$ the process $(\tilde\eta_t)_{t\geq 0}$ exits from the set $\tilde{\mathcal{C}}_{\bar\eta}(\geq m_0+1)$ by visiting $\tilde{\mathcal{C}}_{\bar\eta}(m_0+1)$ and then leaving it by contraction. **QED**

7.8 Additional results

We close by collecting some consequences of Proposition 35 that will be needed in Section 8.

Proposition 50 *There exist $\delta = \delta(\delta_0,\delta_1) > 0$, satisfying $\lim_{\delta_0,\delta_1\downarrow 0}\delta(\delta_0,\delta_1) = 0$, and $\beta_0 > 0$ such that for all $\beta > \beta_0$*

$$\mathbb{P}_\square(\tau_{\mathcal{R}_{2,2}} < e^{(4\Delta-2U)\beta+\delta\beta}) > e^{-\delta\beta}. \tag{7.105}$$

Proof. We need the following lemma.

Lemma 51 *Let $\mathcal{N}_n := \{\eta\in\mathcal{X}\colon\ N_{\bar\Lambda}(\eta) = n\}$. Then*

$$\min_{\eta\in N_{\leq 3}} \mathbb{P}_\eta(\tau_{\mathcal{N}_4} < T_3 e^{2\delta_1\beta}) \geq e^{-(3\Delta-2U)\beta-\delta\beta}. \tag{7.106}$$

This result implies Proposition 50 as follows. For every $\eta\in\mathcal{N}_4$ we have $\mathbb{P}_\eta(\tau_{\mathcal{R}_{2,2}} < T_2 e^{\delta_1\beta}) > e^{-\delta\beta}$, and so

$$\mathbb{P}_\square(\tau_{\mathcal{R}_{2,2}} < e^{(4\Delta-2U)\beta+\delta\beta}) \geq \mathbb{P}_\square(\tau_{\mathcal{N}_4} < e^{(4\Delta-2U)\beta+\delta'\beta})e^{-\delta\beta} \qquad \delta' < \delta \tag{7.107}$$

and the probability in the r.h.s. of (7.107) is $1-$ SES if $\delta > 2\delta_1$. Indeed, if $\tau_{\mathcal{N}_4} > e^{(4\Delta-2U)\beta+\delta'\beta}$, then in any subinterval of length $T_3 e^{2\delta_1\beta}$ contained in the interval $[0, e^{(4\Delta-2U)\beta+\delta'\beta}]$ we have not yet reached $\mathcal{N}_4$. By (7.106), the probability of this event is SES if $\delta' > 2\delta_1$. **QED**

Proof of Lemma l7b.1.
1. It suffices to show that $\mathbb{P}_\square(\tau_{\mathcal{N}_4} < T_3 e^{2\delta_1\beta}) \geq e^{-(3\Delta-2U)\beta-\delta\beta}$. Indeed, by using the recurrence to $\mathcal{X}_3$ within time $T_3 e^{\delta_1\beta}$ we have

$$\begin{aligned}\mathbb{P}_\eta(\tau_{\mathcal{N}_4} < T_3 e^{2\delta_1\beta}) &\geq \mathbb{P}_\eta(\tau_{\mathcal{N}_4} < T_3 e^{2\delta_1\beta} \cap \tau_{\mathcal{N}_4} > \tau_{\mathcal{X}_3})\\ &\geq \mathbb{P}_\square(\tau_{\mathcal{N}_4} < T_3 e^{2\delta_1\beta} - T_3 e^{\delta_1\beta}) - \text{SES}.\end{aligned} \tag{7.108}$$

2. Let

$$\mathcal{A} = \{\eta\in\mathcal{X}_0\colon\ N_{\Lambda_{\ell_-}\setminus\bar\Lambda}(\eta) \geq 4\}. \tag{7.109}$$

Then, for $\delta' < \delta_0$,

$$\begin{aligned}&\mathbb{P}_\square(\tau_{\mathcal{N}_4} < T_3 e^{2\delta_1\beta})\\ &\geq \mathbb{P}_\square\Big(\{\tau_{\mathcal{A}} < \tfrac12 T_3 e^{2\delta_1\beta}\} \cap \{\tau_{\mathcal{N}_{\geq 2}} > \tfrac12 T_3 e^{2\delta_1\beta}\}\Big)\\ &\quad\times \min_{\eta\in\mathcal{A}\cap\mathcal{N}_{<2}\cap\mathcal{X}_0} \mathbb{P}_\eta(\tau_{\mathcal{N}_4} < T_3 e^{-\delta'\beta}).\end{aligned} \tag{7.110}$$

The first probability in r.h.s. of (7.110) can be estimated by using (3.6):

$$\mathbb{P}_\square\Big(\{\tau_{\mathcal{A}} < \tfrac{1}{2}T_3 e^{2\delta_1\beta}\} \cap \{\tau_{\mathcal{N}_{\geq 2}} > \tfrac{1}{2}T_3 e^{2\delta_1\beta}\}\Big)$$

$$\geq \Big(\frac{|\Lambda_{\ell_-}\backslash\bar\Lambda|}{\frac{1}{2}T_3 e^{2\delta_1\beta}}\Big)^4 - \mathbb{P}(\tau_{\mathcal{N}_{\geq 2}} \leq \tfrac{1}{2}T_3 e^{2\delta_1\beta}) \geq e^{-(\gamma_0+2\delta_1)4\beta} - \Big(\frac{|\bar\Lambda|}{\frac{1}{2}T_3 e^{2\delta_1\beta}}\Big)^2. \tag{7.111}$$

The second probability in r.h.s. of (7.110) can be estimated by similar standard estimates on SRW. Indeed, the fact that $\eta \in \mathcal{A}$ allows us to estimate from below by $e^{-4\Delta\beta-\delta\beta}$ the probability that the first 4 particles arrive in $\bar\Lambda$ during given interval of times $[t_1,t_1+1],[t_2,t_2+1],[t_3,t_3+1],[t_4,t_4+1]$, provided $t_1 > \frac{1}{3}T_3 e^{-\delta'\beta}$ and $t_4 < T_3 e^{-\delta'\beta}$. If we denote by $\tau_1',\tau_2',\tau_3',\tau_4'$ the random times corresponding to the arrival of the first 4 particles, then we obtain

$$\begin{aligned}
&\min_{\eta\in\mathcal{A}\cap\mathcal{N}_{<2}\cap\mathcal{X}_0} \mathbb{P}_\eta(\tau_{N_4} < T_3 e^{-\delta'\beta} \\
&\geq \min_{\eta\in\mathcal{A}\cap\mathcal{N}_{<2}\cap\mathcal{X}_0} \int_{\frac{1}{3}T_3e^{-\delta'\beta}}^{\frac{2}{3}T_3e^{-\delta'\beta}} \mathbb{P}_\eta(\tau_1' \in dt_1) \int_{t_1}^{t_1+1} \mathbb{P}_\eta(\tau_2' \in dt_2) \\
&\int_{t_2}^{t_2+e^{U\beta-\delta\beta}} \mathbb{P}_\eta(\tau_3' \in dt_1) \int_{t_3}^{t_3+e^{U\beta-\delta\beta}} \mathbb{P}_\eta(\tau_4' \in dt_1) \\
&\geq e^{-(3\Delta-2U)\beta-\delta\beta}.
\end{aligned} \tag{7.112}$$

QED

Finally, let τ_i, $i \in \mathbb{N}_0$ be as in Definition 34 and let $\bar\tau_1 = \min\{\tau_i\colon \eta_{\tau_i} \notin \mathcal{E}^3(\eta_{\tau_0})\}$, i.e., the first time τ_i at which the reduced Markov chain changes configuration.

Proposition 52 *(i) If $\bar\eta \in \mathcal{R}_{\ell_1,\ell_2}$ with $\ell_1 < \ell_c$ and $\bar\eta' \in \mathcal{R}_{\ell_1+1,\ell_2}$, then*

$$\min_{\eta\in I_{\bar\eta}} \mathbb{P}_\eta\Big(\{\eta_{\bar\tau_1} \in I_{\bar\eta'}\} \cap \{\bar\tau_1 < e^{(2\Delta-U)\beta+\delta\beta}\}\Big) \geq e^{-(\Delta-U-r(\bar\eta))\beta-2\delta\beta}. \tag{7.113}$$

(ii) If $\bar\eta \in \mathcal{R}_{\ell_1,\ell_2}$ with $\ell_1 \geq \ell_c$ and $\bar\eta' \in \mathcal{R}_{\ell_1+1,\ell_2}$, then

$$\min_{\eta\in I_{\bar\eta}} \mathbb{P}_\eta\Big(\{\eta_{\bar\tau_1} \in I_{\bar\eta'}\} \cap \{\bar\tau_1 < e^{(2\Delta-U)\beta+\delta\beta}\}\Big) \geq e^{-2\delta\beta}. \tag{7.114}$$

(iii) If $\bar\eta$ is a lacunary set and $\bar\eta_j$, $j = 0,1,\ldots,n$, is the sequence of configurations defined in Proposition 35(iii), then

$$\min_{j=0,\ldots,n-1} \min_{\eta_j\in I_{\bar\eta_j}} \mathbb{P}_{\eta_j}\Big(\{\eta_{\bar\tau_1} \in I_{\bar\eta_{j+1}}\} \cap \{\bar\tau_1 < e^{(2\Delta-U)\beta+\delta\beta}\}\Big) \geq e^{-2\delta\beta}. \tag{7.115}$$

Proof. (i) Estimate

$$\mathbb{P}_\eta(\eta_{\bar\tau_1} \in I_{\bar\eta'}) = \frac{P^R(\eta, I_{\bar\eta'})}{1-P^R(\eta, I_{\bar\eta})} \geq \frac{P^R(\eta, I_{\bar\eta'})}{C(|\bar\Lambda|) \max_{\eta\in I_{\bar\eta}} \max_{\eta''\in\mathcal{X}_3,\eta''\notin\mathcal{E}^3(\bar\eta)} P^R(\eta, I_{\bar\eta''})}, \tag{7.116}$$

where $C(|\bar{\Lambda}|) > 0$ is some constant depending only on $|\bar{\Lambda}|$. From Proposition 35(i,iv) we have

$$\mathbb{P}_\eta(\eta_{\bar{\tau}_1} \in I_{\bar{\eta}'}) \geq e^{-(\Delta - U - r(\bar{\eta}))\beta - 2\delta\beta} \tag{7.117}$$

By Proposition 35(i) we also have

$$\begin{aligned} \mathbb{P}_\eta(\bar{\tau}_1 > e^{(2\Delta-U)\beta+\delta\beta}) &\leq \mathrm{SES} + \Big[P^R(\eta, I_{\bar{\eta}})\Big]^{\frac{e^{(2\Delta-U)\beta+\delta\beta}}{T_3 e^{\delta_1\beta}}} \\ &= \mathrm{SES} + \Big[1 - \sum_{\bar{\eta}'' \notin \mathcal{E}^3(\bar{\eta})} P^R(\eta, I_{\bar{\eta}''})\Big]^{\frac{e^{(2\Delta-U)\beta+\delta\beta}}{T_3 e^{\delta_1\beta}}} \\ &\leq \mathrm{SES} + \Big[1 - P^R(\eta, I_{\bar{\eta}'})\Big]^{\frac{e^{(2\Delta-U)\beta+\delta\beta}}{T_3 e^{\delta_1\beta}}} = \mathrm{SES}. \end{aligned} \tag{7.118}$$

Parts (ii) and (iii) follow in the same way. **QED**

8 Proof of the main theorem

In this section we collect the results from Sections 4–7 and prove Theorem 3.

8.1 Lower bound for the nucleation time

We begin by proving the lower bound in Theorem 3(c), i.e.,

$$\lim_{\beta\to\infty} \mathbb{P}_{\nu_{\square}}(\tau_{\blacksquare} < T_-) = 0 \qquad \forall \delta > 0 \qquad \text{with } T_- = T_-(\beta, \delta) = e^{(\Gamma-\delta)\beta}. \tag{8.1}$$

1. Let

$$\mathcal{A} = \{\eta' \in \mathcal{X} : \bar{H}(\square, \eta') - \bar{H}(\square) < \Gamma\}. \tag{8.2}$$

Note that $\partial\mathcal{A}$ can be viewed as a subset of $\bar{\mathcal{X}}$ and that $\bar{H}(\square) = 0$. Since Γ is the communication height between $\square$ and $\blacksquare$, as was shown in Proposition 16(iii), we have $\tau_{\partial\mathcal{A}} < \tau_{\blacksquare}$ and so

$$\mathbb{P}_{\nu_{\square}}(\tau_{\blacksquare} < T_-) \leq \mathbb{P}_{\nu_{\square}}(\tau_{\partial\mathcal{A}} < T_-). \tag{8.3}$$

2. To estimate the r.h.s. of (8.3) we use reversibility. For that it is convenient to pass to the discrete-time setup. Let T_i, $i \in \mathbb{N}_0$ be the successive times at which some clock in $\bar{\Lambda}^* \cup \partial\bar{\Lambda}^*$ rings. Let $P^*(\eta, \eta')$ denote the transition probabilities of the Markov chain that is obtained by observing our process at these times:

$$P^*(\eta, \eta') = \mathbb{P}_\eta(\eta_{T_1} = \eta'). \tag{8.4}$$

Let

$$\begin{aligned} i^* &= \inf\{i \in \mathbb{N}_0\colon\ \eta_{T_i} \in \partial\mathcal{A}\} \\ N_1 &= \#\{0 \leq i < i^*\colon\ T_{i+1} - T_i < a\} \\ N_2 &= \#\{0 \leq i < i^*\colon\ T_{i+1} - T_i \geq a\}, \end{aligned} \tag{8.5}$$

where $a > 0$ is a constant that will be chosen shortly. Since $i^* = N_1 + N_2$ and $\{\tau_{\partial\mathcal{A}} < T_-\} \subset \{N_2 < T_-/a\}$, we have for any $M \in \mathbb{N}$

$$\mathbb{P}_{\nu_{\bar{\square}}}(\tau_{\partial\mathcal{A}} < T_-) \leq \mathbb{P}(N_1 > M, N_2 < T_-/a) + \mathbb{P}_{\nu_{\bar{\square}}}(i^* < M + T_-/a). \tag{8.6}$$

Moreover,

$$\mathbb{P}(N_1 > M, N_2 < T_-/a) \leq \mathbb{P}\Big(\frac{N_1}{N_1 + N_2} \geq \frac{M}{M + T_-/a}\Big). \tag{8.7}$$

Now pick a, M such that

$$\mathbb{P}(T_1 < a) \leq \frac{1}{3}, \qquad M = T_-/a. \tag{8.8}$$

Then the probability in the r.h.s. of (8.7) equals the probability that among the first $N_1 + N_2$ of a sequence of Bernoulli trials with success probability $\frac{1}{3}$ a fraction at least $\frac{1}{2}$ is successful. However, this probability is SES, and so we get

$$\mathbb{P}_{\nu_{\bar{\square}}}(\tau_{\partial\mathcal{A}} < T_-) = \text{SES} + \mathbb{P}_{\nu_{\bar{\square}}}(i^* < 2T_-/a). \tag{8.9}$$

3. Let us next consider the Markov chain obtained from our process when some clock in the large volume Λ_β rings. Let $\hat{P}(\eta, \eta')$ denote its transition probabilities. This Markov chain is easily seen to be reversible w.r.t. the same invariant measure ν as the continuous-time process. From this observation we deduce the reversibility of the Markov chain with transition probabilities given by $P^*(\eta, \eta')$ in (8.4). Indeed,

$$\begin{aligned}
&P^*(\eta, \eta') \\
&= \hat{P}(\eta, \eta') + \sum_{t=2}^{\infty} \sum_{\substack{\eta_1, \dots, \eta_{t-1} \\ \eta_i|_{\bar{\Lambda}} = \eta|_{\bar{\Lambda}} \ (i=1,\dots,t-1)}} \hat{P}(\eta, \eta_1) \times \cdots \times P(\eta_{t-1}, \eta') \\
&= \tfrac{\nu(\eta)}{\nu(\eta')} \Big[\hat{P}(\eta', \eta) + \sum_{t=2}^{\infty} \sum_{\substack{\eta_1, \dots, \eta_{t-1} \\ \eta_i|_{\bar{\Lambda}} = \eta|_{\bar{\Lambda}} \ (i=1,\dots,t-1)}} \hat{P}(\eta', \eta_{t-1}) \times \cdots \times P(\eta_1, \eta)\Big] \\
&= \tfrac{\nu(\eta)}{\nu(\eta')} P^*(\eta', \eta).
\end{aligned} \tag{8.10}$$

Hence we get

$$\begin{aligned}
&\mathbb{P}_{\nu_{\bar{\square}}}(i^* < 2T_-/a) \\
&= \tfrac{1}{\nu(I_{\bar{\square}})} \textstyle\sum_{l=1}^{2T_-/a} \sum_{\substack{\eta_1, h_2, \dots, \eta_{l-1} \in \mathcal{A} \\ \eta_l \in \partial\mathcal{A}}} P^*(\eta_1, \eta_2) \times \cdots \times P(\eta_{l-1}, \eta_l) \\
&\leq \tfrac{2T_-}{a} \sup_{\xi \in \partial\mathcal{A}} \tfrac{\nu(\xi)}{\nu(I_{\bar{\square}})}.
\end{aligned} \tag{8.11}$$

From (8.2), (8.9), (8.11), Proposition 16(iii) and Proposition 53 we get the result. Namely, we replace ν by μ, making an error that is SES because $\lim_{\beta\to\infty} \frac{1}{\beta} \log |\Lambda_\beta| = \infty$, and we use that $\sup_{\xi \in \partial\mathcal{A}} \mu(\xi)/\mu(I_{\bar{\square}}) \leq e^{\Gamma\beta}$ because of (8.2) (recall 4.9).

8.2 Upper bound for the nucleation time

Next we prove the upper bound in Theorem 3(c), i.e.,

$$\lim_{\beta\to\infty} \mathbb{P}_{\nu_{\bar\square}}(\tau_{\blacksquare} > T_+) = 0 \qquad \forall\delta > 0 \qquad \text{with } T_+ = T_+(\beta,\delta) = e^{\beta(\Gamma+\delta)}. \quad (8.12)$$

1. The idea is to construct an event $\mathbf{E}_{\eta,T}$ leading from any $\eta \in \mathcal{X}$ to $\blacksquare$ in an appropriate time $T = T(\beta,\delta)$ and having a sufficiently large probability. Let us first describe this event in words. The time T will be chosen of the form $T = e^{(4\Delta-2U+\frac{\delta}{2})\beta}$ for a suitably small $\delta > 0$. Note that, for β sufficiently large, $\frac{T}{3} > T_3^+$ with $T_3^+ = e^{(\Delta+\frac{\delta}{2})\beta}$. But, given $\eta \in I_{\bar\square}$, we know from Proposition 27 that within time T_3^+ our process visits $\bar{\mathcal{X}}_3$ with a large probability. Then, by Proposition 25(iii) and Remark (2) following it, either of the two following situations prevails:

1. There exist $\ell,\ell' \in \mathbb{N}\setminus\{1\}$, $|\ell-\ell'| \leq 1$ such that our process visits $\mathcal{R}_{\ell,\ell'}$ within time T_3^+.

2. The process passes through a configuration containing a large lacunary square or quasi-square (recall Remark (2) in Section 5.2).

In case 2, within another time $\frac{T}{3}$ with a large probability our process goes to a large (highly supercritical) square or quasi-square. Indeed, it follows from the results in Section 7 that for this to happen for β sufficiently large it suffices to have a time $e^{(2U+\frac{\delta}{2})\beta}$ for any $\delta > 0$. But for $\delta > 0$ small we have $\frac{T}{3} > e^{(2U+\frac{\delta}{2})\beta}$. From the square or quasi-square present in each configuration in $\mathcal{R}_{\ell,\ell'}$ we start growing following a sequence of squares or quasi-squares of increasing side lengths, leading eventually to $\blacksquare$. Of course, it may happen that the initial square or quasi-square is empty ($\ell = \ell' = 0$). i.e., the configuration on which we fall during the first part of our event is just $\bar\square$.

2. Let us now give precise definitions. We set

$$T_3^+ = e^{(\Delta+\frac{\delta}{2})}, \qquad T_4^+ = e^{(2U+\frac{\delta}{2})\beta}. \quad (8.13)$$

For $\eta \in \mathcal{X}$, define the set of trajectories

$$\mathbf{E}^{(1)}_{\eta,T_3^+} = \{\phi\colon\ \phi_0 = \eta,\ \exists\ 0 \leq t \leq T_3^+\colon\ \phi_t \in \mathcal{X}_3\}. \quad (8.14)$$

Let $\mathcal{Q}$ be the set of square or quasi-square configurations:

$$\mathcal{Q} = \cup_{\ell,\ell'\neq 1\colon\ |\ell-\ell'|\leq 1}\mathcal{R}_{\ell,\ell'}, \quad (8.15)$$

and recall that $\mathcal{Q} = \mathcal{X}_3 \setminus \mathcal{L}$. For $\eta \in \mathcal{X}_3$, define the set of trajectories

$$\mathbf{E}^{(2)}_{\eta,T_4^+} = \{\phi\colon\ \phi_0 = \eta,\ \exists\ 0 \leq t \leq T_4^+\colon\ \phi_t \in \mathcal{Q}\} \quad (8.16)$$

and put

$$\bar\tau_2 = \bar\tau_2(\phi) = \inf\{t \geq 0\colon\ \phi_t \in \mathcal{Q}\}. \quad (8.17)$$

Note that for any $\eta \in \mathcal{X}_3 \setminus \mathcal{Q}$ the interval $\bar{\tau}_2 - \tau_{\mathcal{X}_3}$ is strictly positive, while for any $\eta \in \mathcal{X}_3 \cap \mathcal{R}_{\ell,\ell'}$ with $\ell, \ell' \in \mathbb{N}\setminus\{1\}$, $|\ell - \ell'| \leq 1$ it is zero and the corresponding set $\mathbf{E}^{(2)}_{\eta, T_4^+}$ is trivial.

3. Given $\ell, \ell' \in \mathbb{N} \setminus \{1\}$, $|\ell - \ell'| \leq 1$, we next introduce $\mathbf{E}^{(3)}_{R_{\ell,\ell'},T}$ as the set of trajectories starting from $\mathcal{R}_{\ell,\ell'}$ and passing (at the successive times $\tau_1, \tau_2, \ldots$ of return to $\mathcal{X}_3$ to a different configuration in $\bar{\Lambda}$) through the following sequence $\psi = \psi_1, \psi_2, \ldots$ of pairs of integers:

- $(\ell, \ell') = (\ell, \ell)$, $\ell \geq 2$: $\psi = (\ell, \ell), (\ell+1, \ell), (\ell+1, \ell+1), \ldots, (\ell_0, \ell_0)$.
- $(\ell, \ell') = (\ell, \ell+1)$, $\ell \geq 2$: $\psi = (\ell, \ell+1), (\ell+1, \ell+1), (\ell+1, \ell+2), \ldots, (\ell_0, \ell_0)$.
- $(\ell, \ell') = (0, 0)$: $\psi = (2, 2), (2, 3), (3, 3), \ldots, (\ell_0, \ell_0)$.

More precisely, put $\tau_0 = 0$ and, for $i \in \mathbb{N}$,

$$\tau_i = \min\{t > \tau_{i-1} \colon\ \eta_t \in \mathcal{X}_3 \setminus I_{\bar{\eta}_t}\}, \tag{8.18}$$

where $\bar{\eta}_t = \eta_t|_{\bar{\Lambda}}$. (The τ_i are the random times at which the reduced Markov chain changes configuration inside $\bar{\Lambda}$; see Definition 34.) Let $\mathcal{R}_\psi$ be the sequence of sets of configurations $\mathcal{R}_{\ell_1,\ell_2}$ with (ℓ_1, ℓ_2) following the sequence ψ. Then we define

$$\begin{aligned} \mathbf{E}^{(3)}_{R_{\ell,\ell'},T} \;=\; \Big\{\phi\colon\ & \phi_{\tau_0} \in \mathcal{R}_{\ell,\ell'}, \phi_{\tau_i} \in \mathcal{R}_{\psi_i}\ \forall i \in \mathbb{N}, \\ & \max_i(\tau_{i+1} - \tau_i) < e^{(4\Delta - 2U + \frac{\delta}{2})\beta}\Big\}. \end{aligned} \tag{8.19}$$

4. Our event $\mathbf{E}_{\eta,T}$ can now be defined as

$$\mathbf{E}_{\eta,T} = \bigcup_{\eta_1 \in \mathcal{X}_3, \eta_2 \in \mathcal{Q}} \Big\{\mathbf{E}^{(1)}_{\eta,T_3^+} \cap \{\eta_{\tau_1} = \eta_1\}\Big\} \cap \Big\{\mathbf{E}^{(2)}_{\eta_1,T_4^+} \cap \{\eta_{\bar{\tau}_2} = \eta_2\}\Big\} \cap \mathbf{E}^{(3)}_{\eta_2,T}. \tag{8.20}$$

We will estimate the probability of each of the parts.

5. We have, for β sufficiently large,

$$\inf_{\ell_1,\ell_2} \inf_{\eta \in \mathcal{R}_{\ell_1,\ell_2}} \mathbb{P}_\eta(\mathbf{E}^{(3)}_{\eta,T}) \geq e^{-(\Gamma - 4\Delta + 2U - \frac{\delta}{4})\beta}. \tag{8.21}$$

Indeed, suppose first that $(\ell_1, \ell_2) = (0, 0)$. Then it follows from Proposition 35 that

$$\begin{aligned} \mathbb{P}_{\nu_{\square}}\Big(\tau_1 \in \mathcal{R}_{2,2},\, \tau_1 < e^{-(4\Delta - 2U + \frac{\delta}{2})\beta}\Big) &\asymp 1 \\ \min_{\eta \in \psi_i} \mathbb{P}_{\psi_i}\Big(\eta_{\tau_1} \in \mathcal{R}_{\psi_{i+1}},\, \tau_1 < e^{(r_i + \Delta + \frac{\delta}{2})\beta}\Big) &\asymp e^{-(\Delta - U - r_i)\beta}, \end{aligned} \tag{8.22}$$

where $\asymp$ denotes logarithmic equivalence in β and, for $\psi_i = (\ell, \ell+1)$ or $\psi_i = (\ell, \ell)$ with $\ell < \ell_c$, we put $r_i = (2U - \Delta)(\ell - 1)$. If, on the other hand, $\ell \geq \ell_c$, then we have

$$\mathbb{P}_{\psi_i}\Big(\eta_{\tau_1} \in \mathcal{R}_{\psi_{i+1}},\, \tau_1 < e^{(2\Delta - U + \frac{\delta}{2})\beta}\Big) \asymp 1. \tag{8.23}$$

The case $(\ell_1, \ell_2) \neq (0,0)$ can be treated in a similar way: it turns out that the worst lower estimate corresponds to the case $(\ell_1, \ell_2) = (0,0)$. Equation (8.21) follows from an immediate computation.

6. It follows from the results in Section 7.8 that for all $\eta \in \mathcal{X}$ and $\eta_1 \in \mathcal{X}_3$

$$\mathbb{P}(\mathbf{E}^{(1)}_{\eta,T_3^+}) \asymp 1, \qquad \mathbb{P}(\mathbf{E}^{(2)}_{\eta_1,T_4^+}) \asymp 1. \tag{8.24}$$

From (8.20), (8.21) and (8.24) we thus get

$$\mathbb{P}(\mathbf{E}_{\eta,T}) \geq e^{-(\Gamma - 4\Delta + 2U + \frac{\delta}{2})\beta}. \tag{8.25}$$

We can now apply Proposition 6 with

$$\begin{aligned} &\nu = \nu_{\bar{\square}}, \\ &T = e^{(4\Delta - 2U + \frac{\delta}{2})\beta},\ T' = T'' = e^{(\Gamma+\delta)\beta}, \\ &\mathcal{A} = \blacksquare,\ \mathcal{B} = \mathcal{X},\ p = e^{-(\Gamma - 4\Delta + 2U + \frac{\delta}{4})\beta} \end{aligned} \tag{8.26}$$

to complete the proof.

8.3 The gate for the nucleation

In this section we prove Theorem 3(b).

1. Abbreviate

$$N(\eta) = N_{\bar{\Lambda}}(\eta) = \sum_{x \in \bar{\Lambda}} \eta(x) \tag{8.27}$$

and consider the sets

$$\begin{aligned} \mathcal{G} &= \{\eta \in \mathcal{X}\colon\ N(\eta) < \ell_c(\ell_c - 1) + 2\} \\ \mathcal{G}^- &= \{\eta \in \mathcal{X}\colon\ N(\eta) = \ell_c(\ell_c - 1) + 1\} \\ \mathcal{G}^0 &= \{\eta \in \mathcal{X}\colon\ N(\eta) = \ell_c(\ell_c - 1) + 2\}. \end{aligned} \tag{8.28}$$

Given a path $\phi = \phi_1, \ldots, \phi_m$ $(m \in \mathbb{N})$ with $\phi_1 = \square, \phi_m = \blacksquare$, let $i^0 = i^0(\phi)$ be the first hitting time of $\mathcal{G}^0$:

$$i^0 = \inf\{i \in \mathbb{N}\colon\ \phi_i \in \mathcal{G}^0\}. \tag{8.29}$$

Then

$$\bar{H}(\phi_{i^0}) = H(\phi_{i^0 - 1}) + \Delta, \tag{8.30}$$

since $\bar{H}$ increases by Δ when we add a particle to $\bar{\Lambda}$.

2. We have

$$\lim_{\beta \to \infty} \mathbb{P}_{\nu_{\bar{\square}}}(\eta_{\tau_{\mathcal{G}^0}} \notin \mathcal{D}^0_{\ell_c - 1, \ell_c}) = 0. \tag{8.31}$$

Indeed, since $\mathbb{P}_{\nu_{\bar{\square}}}(\tau_{\mathcal{G}^0} < \tau_{\blacksquare}) = 1$, it follows from Theorem 3(c) that

$$\lim_{\beta \to \infty} \mathbb{P}_{\nu_{\bar{\square}}}\left(\tau_{\mathcal{G}^0} > e^{(\Gamma+\delta)\beta}\right) = 0 \qquad \forall \delta > 0. \tag{8.32}$$

On the other hand, from the results of Section 4 we know that

$$\bar{H}(\square, \mathcal{G}^- \setminus \mathcal{D}^-_{\ell_c - 1, \ell_c}) \leq \bar{H}(\mathcal{D}^-_{\ell_c - 1, \ell_c}) + U. \tag{8.33}$$

Using (8.33) we can deduce, via an argument based on reversibility similar to the one used to prove (8.11), that

$$\lim_{\beta\to\infty} \mathbb{P}_{\nu_{\bar{\square}}}\Big(\theta_{\square,\blacksquare} < \tau_{\mathcal{G}^-\setminus\mathcal{D}^-_{\ell_c-1,\ell_c}} < e^{(\Gamma+\frac{U}{2})\beta}\Big) = 0. \tag{8.34}$$

3. The claim now follows from (8.32) and (8.34) after choosing δ sufficiently small.

8.4 Criticality for squares and quasi-squares

In this section we prove Theorem 3(a).

1. For ℓ_1,ℓ_2 with $0\le\ell_1\le\ell_2$ and $|\ell_1-\ell_2|\le 1$, let $\mathcal{R}^{>(\ell_1,\ell_2)}$ denote the set of all configurations whose restriction to $\bar{\Lambda}$ gives rise to a single square or quasi-square strictly larger than R_{ℓ_1,ℓ_2}, i.e.,

$$\mathcal{R}^{>(\ell_1,\ell_2)} = \bigcup_{(\ell',\ell'')>(\ell_1,\ell_2)} \mathcal{R}_{\ell',\ell''}, \tag{8.35}$$

where $(\ell',\ell'')>(\ell_1,\ell_2)$ (with $\ell'\le\ell'',\ell_1\le\ell_2$) means either $\ell'>\ell_1,\ell''\ge\ell_2$ or $\ell'=\ell_1,\ell''>\ell_2$. Let

$$\mathcal{R}^{\le(\ell_1,\ell_2)} = \Big\{\bigcup_{(\ell',\ell'')} \mathcal{R}_{\ell',\ell''}\Big\} \setminus \Big\{\bigcup_{(\ell',\ell'')>(\ell_1,\ell_2)} \mathcal{R}_{\ell',\ell''}\Big\}. \tag{8.36}$$

Similarly, we define $\mathcal{R}^{<(\ell_1,\ell_2)}$ and $\mathcal{R}^{\ge(\ell_1,\ell_2)}$.

2. Let us first consider the subcritical case. With the help of reversibility, like in (8.11), we can prove that, for every $\delta>0$ and every ℓ_1,ℓ_2 such that $0\le\ell_1\le\ell_2$, $|\ell_1-\ell_2|\le 1$ and $\ell_1\le\ell_c$,

$$\lim_{\beta\to\infty} \mathbb{P}_{\nu_{R_{\ell_1,\ell_2}}}\Big(\tau_{\mathcal{R}^{>(\ell_1,\ell_2)}} < e^{(2\Delta-U-\delta)\beta}\Big) = 0. \tag{8.37}$$

Indeed, it follows from Proposition 16(ii) that the saddle of exit from $\mathcal{R}^{\le(\ell_1,\ell_2)}$, i.e., the configurations realizing the communication height between $\mathcal{R}^{\le(\ell_1,\ell_2)}$ and $\mathcal{R}^{>(\ell_1,\ell_2)}$, is $2\Delta-U$.

3. For ℓ_1,ℓ_2 such that $0\le\ell_1\le\ell_2\le\ell_1+1$ and $\ell_1\le\ell_c-1$, we define a shrinking event $\mathbf{E}^s_{\ell_1,\ell_2}$ containing the set of trajectories starting from $\mathcal{R}_{\ell_1,\ell_2}$ and passing, at the successive times $\tau_1,\tau_2,\dots$ defined in (8.18), through the following sequence $\psi^s=\psi^s_1,\psi^s_2,\dots$ of pairs of integers:

- $(\ell_1,\ell_2)=(\ell,\ell)$, $\ell\ge 2$: $\psi^s=(\ell,\ell),(\ell-1,\ell),(\ell-1,\ell-1),\dots,(0,0)$.
- $(\ell,\ell')=(\ell,\ell+1)$, $\ell\ge 2$: $\psi^2=(\ell,\ell+1),(\ell,\ell),(\ell-1,\ell),(\ell 1,\ell 2),\dots,(0,0)$.
- $(\ell,\ell')=(1,2)$: $\psi^s=(2,1),(1,1),(0,0)$.
- $(\ell,\ell')=(1,1)$: $\psi^s=(1,1),(0,0)$.
- $(\ell,\ell')=(0,0)$: ψ^s is the trivial path with no move.

More precisely, we denote by $\mathcal{R}_{\psi^s}$ the sequence of sets of configurations $\mathcal{R}_{\ell,\ell'}$ with (ℓ,ℓ') following the sequence ψ^s, and define

$$\begin{aligned}\mathbf{E}^s_{(\ell_1,\ell_2)} \;&=\; \Big\{\phi\colon\ \phi_{\tau_0}\in\mathcal{R}_{\ell_1,\ell_2}, \phi_{\tau_i}\in\mathcal{R}_{\psi^s_i}\ \forall i\in\mathbb{N},\\ &\qquad \max_i(\tau_{i+1}-\tau_i) < e^{[(2U-\Delta)(\ell_1-2)+2U+\delta]\beta}\Big\}.\end{aligned} \tag{8.38}$$

4. Let us next consider the supercritical case. Again, with the help of reversibility like in (8.11), we can prove that, for every $\delta>0$ and every ℓ_1,ℓ_2 such that $\ell_c\le\ell_1\le\ell_2\le\ell_1+1$,

$$\lim_{\beta\to\infty}\mathbb{P}_{\nu_{R_{\ell_1,\ell_2}}}\Big(\tau_{\mathcal{R}^{<(\ell_1,\ell_2)}} < e^{[(2U-\Delta)(\ell_c-2)+2U-\delta]\beta}\Big) = 0. \tag{8.39}$$

Indeed, it follows from Proposition 16(i) that the saddle of exit from $\mathcal{R}^{\ge(\ell_1,\ell_2)}$ is $e^{[(2U-\Delta)(\ell_c-2)+2U]\beta}$.

5. For $l_c\le\ell_1\le\ell_2\le\ell_1+1$ we define a growing event $\mathcal{E}^g_{(\ell_1,\ell_2)}$ containing the set of trajectories starting from $\mathcal{R}_{\ell_1,\ell_2}$ and passing, at the successive times $\tau_1,\tau_2,\ldots$, through the following sequence $\psi^g=\psi^g_1,\psi^g_2,\ldots$ of pairs of integers:

- $(\ell_1,\ell_2)=(\ell,\ell)$: $\psi^g=(\ell,\ell),(\ell+1,\ell),(\ell+1,\ell+1),\ldots,(\ell_0,\ell_0)$.
- $(\ell,\ell')=(\ell,\ell+1)$: $\psi^g=(\ell,\ell+1),(\ell+1,\ell+1),\ldots,(\ell_0,\ell_0)$.
- $(\ell,\ell')=(\ell_0,\ell_0)$: ψ^g is the trivial path with no move.

More precisely, we denote by $\mathcal{R}_{\psi^g}$ the sequence of sets of configurations $\mathcal{R}_{l,\ell'}$ with (ℓ,ℓ') following the sequence ψ^g, and define

$$\begin{aligned}\mathbf{E}^g_{(\ell_1,\ell_2)} \;&=\; \Big\{\phi\colon\ \phi_{\tau_0}\in\mathcal{R}_{\ell_1,\ell_2}, \phi_{\tau_i}\in\mathcal{R}_{\psi^g_i}\ \forall i\in\mathbb{N},\\ &\qquad \max_i(\tau_{i+1}-\tau_i) < e^{(2\Delta-U+\delta)\beta}\Big\}.\end{aligned} \tag{8.40}$$

6. In the following we abbreviate

$$\begin{aligned}\delta_s &= \tfrac12\Big(2\Delta-U-[(2U-\Delta)(\ell_c-3)+2U]\Big)\\ \delta_g &= \tfrac12\Big([(2U-\Delta)(\ell_c-2)+2U]-2\Delta-U\Big).\end{aligned} \tag{8.41}$$

7. In the subcritical case $0\le\ell_1\le\ell_2\le\ell_1+1$, $\ell_1\le\ell_c$ we have, for β sufficiently large,

$$\mathbb{P}_{\nu_{R_{\ell_1,\ell_2}}}(\tau_{\blacksquare}<\tau_{\square}) \le \mathbb{P}_{\nu_{R_{\ell_1,\ell_2}}}(\tau_{\mathcal{R}^{>(\ell_1,\ell_2)}}<\tau_{\square}). \tag{8.42}$$

On the other hand, by (8.37) we have

$$\mathbb{P}_{\nu_{R_{\ell_1,\ell_2}}}(\tau_{\mathcal{R}^{>(\ell_1,\ell_2)}}<\tau_{\square}) \le \mathbb{P}_{\nu_{R_{\ell_1,\ell_2}}}\Big(e^{(2\Delta-U-\frac{\delta_s}{2})\beta}<\tau_{\mathcal{R}^{>(\ell_1,\ell_2)}}<\tau_{\square}\Big)+o(1). \tag{8.43}$$

Moreover,

$$\begin{aligned}&\mathbb{P}_{\nu_{R_{l_1,l_2}}}\Big(e^{(2\Delta-U-\frac{\delta_s}{2})\beta}<\tau_{\mathcal{R}^{>(l_1,l_2)}}<\tau_{\square}\Big)\\ &\quad\le \mathbb{P}_{\nu_{R_{\ell_1,\ell_2}}}\Big(e^{(2\Delta-U-\frac{\delta_s}{2})\beta}<\tau_{\mathcal{R}^{>(\ell_1,\ell_2)}},\ e^{[(2U-\Delta)(\ell_1-2)+2U+\frac{\delta_s}{2}]\beta}<\tau_{\square}\Big).\end{aligned} \tag{8.44}$$

We know from Proposition 35 that if $0 \leq \ell_1 \leq \ell_2 \leq \ell_1 + 1$, $\ell_1 \leq \ell_c$, then for all $\delta > 0$ and β sufficiently large

$$\mathbb{P}\nu_{\mathcal{R}_{\ell_1,\ell_2}}(\mathbf{E}^s_{\ell_1,\ell_2}) > e^{-\delta\beta}. \tag{8.45}$$

The first claim in Theorem 3(a) now follows from (8.42), (5.13), (5.15), (8.45) and Proposition 6 with

$$\begin{aligned} &\nu = \nu_{\mathcal{R}_{\ell_1,\ell_2}}, \\ &T = e^{[(2U-\Delta)(\ell_1-2)+2U+\frac{\delta_s}{2}]\beta},\ T' = T'' = e^{(2\Delta-U-\frac{\delta_s}{2})\beta}, \\ &\mathcal{A} = \square,\ \mathcal{B} = \mathcal{R}^{>(\ell_1,\ell_2)},\ p = e^{-\frac{\delta_s}{4}\beta}. \end{aligned} \tag{8.46}$$

8. In the supercritical case $\ell_c \leq \ell_1 \leq \ell_2 \leq \ell_1 + 1$, we proceed in a similar way. We have

$$\mathbb{P}_{\nu_{R_{\ell_1,\ell_2}}}(\tau_\square < \tau_\blacksquare) \leq \mathbb{P}_{\nu_{R_{\ell_1,\ell_2}}}(\tau_{\mathcal{R}^{<(\ell_1,\ell_2)}} < \tau_\blacksquare). \tag{8.47}$$

On the other hand, by (8.39) we have

$$\begin{aligned} &\mathbb{P}_{\nu_{R_{\ell_1,\ell_2}}}(\tau_{\mathcal{R}^{<(\ell_1,\ell_2)}} < \tau_\blacksquare) \\ &\quad \leq \mathbb{P}_{\nu_{R_{\ell_1,\ell_2}}}\left(e^{[(2U-\Delta)(\ell_c-2)+2U-\frac{\delta_g}{2}]\beta} < \tau_{\mathcal{R}^{<(\ell_1,\ell_2)}} < \tau_\blacksquare\right) + o(1). \end{aligned} \tag{8.48}$$

Moreover,

$$\begin{aligned} &\mathbb{P}_{\nu_{R_{l_1,\ell_2}}}\left(e^{[(2U-\Delta)(\ell_c-2)+2U-\frac{\delta_g}{2}]\beta} < \tau_{\mathcal{R}^{<(\ell_1,\ell_2)}} < \tau_\blacksquare\right) \\ &\leq \mathbb{P}_{\nu_{R_{\ell_1,\ell_2}}}\left(e^{[(2U-\Delta)(\ell_c-2)+2U-\frac{\delta_g}{2}]\beta} < \tau_{\mathcal{R}^{<(\ell_1,\ell_2)}},\ e^{[(2D-U)+\frac{\delta_g}{2}]\beta} < \tau_\blacksquare\right). \end{aligned} \tag{8.49}$$

We know from Proposition 35 that if $\ell_c \leq \ell_1 \leq \ell_2 \leq \ell_1 + 1$, then for all $\delta > 0$ and β sufficiently large

$$\mathbb{P}_{\nu_{\mathcal{R}_{\ell_1,\ell_2}}}(\mathbf{E}^g_{\ell_1,\ell_2}) > e^{-\delta\beta}. \tag{8.50}$$

The second claim in Theorem 3(a) now follows from (5.17), (8.48), (8.50) and Proposition 6 with

$$\begin{aligned} &\nu = \nu_{\mathcal{R}_{\ell_1,\ell_2}}, \\ &T = e^{[(2\Delta-U)+\frac{\delta_g}{2}]\beta},\ T' = T'' = e^{[(2U-\Delta)(\ell_c-2)+2U-\frac{\delta_g}{2}]\beta}, \\ &\mathcal{A} = \blacksquare,\ \mathcal{B} = \mathcal{R}^{<(\ell_1,\ell_2)},\ p = e^{-\frac{\delta_g}{4}\beta}. \end{aligned} \tag{8.51}$$

9 Appendix: comparison of ensembles

The *grand-canonical* Gibbs measure for our system on a torus $\Lambda = \Lambda_\beta \subset \mathbb{Z}^2$ with activity z is:

$$\mu_{\Lambda,z}(\eta_\Lambda) = \left[\prod_{x\in\Lambda} \frac{z^{\eta(x)}}{\eta(x)!}\right] \frac{\exp[-\beta H(\eta_{\bar{\Lambda}})]}{Z(\Lambda,z,\beta)} \qquad (\eta_\Lambda \in \mathcal{X}_\Lambda), \tag{9.1}$$

where $\mathcal{X}_\Lambda = \{0,1\}^{\bar\Lambda} \times \mathbb{N}^{\Lambda\setminus\bar\Lambda}$, $\eta_{\bar\Lambda}$ is the restriction of η_Λ to $\bar\Lambda$,

$$H(\eta_{\bar\Lambda}) = -U \sum_{(x,y)\in\bar\Lambda_0} \eta(x)\eta(y) \qquad (\eta_{\bar\Lambda} \in \mathcal{X}_{\bar\Lambda}), \tag{9.2}$$

$\mathcal{X}_{\bar\Lambda} = \{0,1\}^{\bar\Lambda}$, and

$$Z(\Lambda, z, \beta) = e^{z|\Lambda\setminus\bar\Lambda|} \sum_{\eta_{\bar\Lambda}\in\mathcal{X}_{\bar\Lambda}} z^{\sum_{x\in\bar\Lambda}\eta(x)} \exp[-\beta H(\eta_{\bar\Lambda})]. \tag{9.3}$$

The *canonical* Gibbs measure with n particles is

$$\nu_{\Lambda,n}(\eta_\Lambda) = \left[\prod_{x\in\Lambda\setminus\bar\Lambda} \frac{1}{\eta(x)!}\right] \frac{\exp[-\beta H(\eta_{\bar\Lambda})]}{Z(\Lambda,n,\beta)} \qquad 1_{(\eta_\Lambda\in\mathcal{X}_{\Lambda,n})}, \tag{9.4}$$

where $\mathcal{X}_{\Lambda,n} = \{\eta_\Lambda \in \mathcal{X}_\Lambda\colon\ \sum_{x\in\Lambda}\eta(x) = n\}$ and

$$Z(\Lambda, n, \beta) = \sum_{\eta_\Lambda\in\mathcal{X}_{\Lambda,n}} \left[\prod_{x\in\Lambda\setminus\bar\Lambda} \frac{1}{\eta(x)!}\right] \exp[-\beta H(\eta_{\bar\Lambda})]. \tag{9.5}$$

It is straightforward to verify that both these measures are reversible with respect to the Kawasaki dynamics with Hamiltonian H in (1.45).

We want to compare the expected values with respect to the above two measures of a cylindrical function f with support in $\bar\Lambda$. In what follows, we give an elementary estimate showing that the difference between the two expectations is inversely proportional to the total volume $|\Lambda|$. With the help of asymptotic expansions, like the ones used to get a local central limit theorem, it would be possible to get better estimates, even in more complicated situations (see Yau [24], Bertini, Cirillo and Olivieri [1]). However, for our purpose it suffices to have a rough estimate.

Proposition 53 *There exists $c = c(\bar\Lambda) > 0$ such that for all $f\colon \mathcal{X}_{\bar\Lambda} \to \mathbb{R}$ with $\|f\|_\infty \le \frac{1}{4}$, all $n \in \mathbb{N}$ and all $\Lambda \supset \bar\Lambda$*

$$|\nu_{\Lambda,n}(f) - \mu_{\Lambda,z}(f)| \le \frac{c}{|\Lambda|} \qquad \textit{when } z = \frac{n}{|\Lambda|}. \tag{9.6}$$

Proof. 1. Put $\nu = \nu_{\Lambda,n}$ and $\mu = \mu_{\Lambda,z}$. We have

$$|\nu(f) - \mu(f)| = \frac{\mu^g(1_{N=n})}{\mu(1_{N=n})} - 1, \tag{9.7}$$

where $N = N(\eta) = \sum_{x\in\Lambda}\eta(x)$, $g = f - \mu(f)$ and $\mu^g(h) = \mu((1+g)h)$. Note that μ^g is a probability measure: $\mu^g(\eta) \ge 0$ and $\mu^g(1) = 1$, $\|g\|_\infty \le 2\|f\|_\infty \le \frac{1}{2}$ and g has mean zero under μ.
2. Let $\bar N = \bar N(\eta) = \sum_{x\in\bar\Lambda}\eta(x)$ and $\hat N = \hat N(\eta) = \sum_{x\in\Lambda\setminus\bar\Lambda}\eta(x)$. From (9.1) we see that the grand-canonical measure factorizes:

$$\begin{aligned} \mu(\eta_\Lambda) &= \mu_{\bar\Lambda}(\eta_{\bar\Lambda})\mu_{\Lambda\setminus\bar\Lambda}(\eta_{\Lambda\setminus\bar\Lambda}) \\ \mu^g(\eta_\Lambda) &= \mu^g_{\bar\Lambda}(\eta_{\bar\Lambda})\mu_{\Lambda\setminus\bar\Lambda}(\eta_{\Lambda\setminus\bar\Lambda}), \end{aligned} \tag{9.8}$$

where

$$\begin{aligned}\mu_{\bar\Lambda}(\eta_{\bar\Lambda}) &= \frac{\exp[-\beta H(\eta_{\bar\Lambda})]\ z^{\bar N(\eta_{\bar\Lambda})}}{\sum_{\eta_{\bar\Lambda}\in\bar{\mathcal{X}}}\exp[-\beta H(\eta_{\bar\Lambda})] z^{\bar N(\eta_{\bar\Lambda})}}\\ \mu^g_{\bar\Lambda}(\eta_{\bar\Lambda}) &= \frac{(1+g(\eta_{\bar\Lambda}))\exp[-\beta H(\eta_{\bar\Lambda})] z^{\bar N(\eta_{\bar\Lambda})}}{\sum_{\eta_{\bar\Lambda}\in\bar{\mathcal{X}}}\exp[-\beta H(\eta_{\bar\Lambda})] z^{\bar N(\eta_{\bar\Lambda})}}\end{aligned} \tag{9.9}$$

and

$$\mu_{\Lambda\backslash\bar\Lambda}(\eta_{\Lambda\backslash\bar\Lambda}) = e^{-z|\Lambda\backslash\bar\Lambda|}\prod_{x\in\Lambda\backslash\bar\Lambda}\frac{\lambda^{\eta(x)}}{\eta(x)!}. \tag{9.10}$$

From (9.8) we have

$$\begin{aligned}\mu(1_{N=n}) &= \mu(N=n) &= \sum_{\bar n=0}^{|\bar\Lambda|}\mu_{\bar\Lambda}(\bar N=\bar n)\mu_{\Lambda\backslash\bar\Lambda}(\hat N = n-\bar n)\\ \mu^g(1_{N=n}) &= \mu^g(N=n) &= \sum_{\bar n=0}^{|\bar\Lambda|}\mu^g_{\bar\Lambda}(\bar N=\bar n)\mu_{\Lambda\backslash\bar\Lambda}(\hat N = n-\bar n).\end{aligned} \tag{9.11}$$

3. Set $k=|\Lambda\setminus\bar\Lambda|$ and $m=n-\bar n$. Then from (9.10) we get

$$\mu_{\Lambda\backslash\bar\Lambda}(\hat N = m) = \frac{(zk)^m e^{-zk}}{m!}. \tag{9.12}$$

Thus

$$\mu_{\Lambda\backslash\bar\Lambda}(\hat N = n-\bar n) = \mu_{\Lambda\backslash\bar\Lambda}(\hat N = n-|\bar\Lambda|)\ \phi_{\bar\Lambda}(\bar n) \tag{9.13}$$

with

$$\phi_{\bar\Lambda}(\bar n) = \frac{(zk)^{|\bar\Lambda|-\bar n}}{(n-\bar n)(n-\bar n-1)\times\ldots\times(n-|\bar\Lambda|+1)}. \tag{9.14}$$

Substitution of (9.13) into (9.11) gives

$$\frac{\mu^g(N=n)}{\mu(N=n)} = \frac{\sum_{\bar n=0}^{|\bar\Lambda|}\mu^g_{\bar\Lambda}(N=n)\ \phi_{\bar\Lambda}(\bar n)}{\sum_{\bar n=0}^{|\bar\Lambda|}\mu_{\bar\Lambda}(N=n)\ \phi_{\bar\Lambda}(\bar n)}. \tag{9.15}$$

From (9.14) it follows that there exists a $c=c(|\bar\Lambda|)>0$ such that for all $\Lambda\supset\bar\Lambda$

$$\sup_{0\le\bar n\le|\bar\Lambda|}|\phi_{\bar\Lambda}(\bar n)-1| \le \frac{c}{|\Lambda|}. \tag{9.16}$$

4. From (9.7), (9.15) and (9.16) the result in (9.6) follows. **QED**

Remark: It is clear from the above calculation that the assumption $\|f\|_\infty\le\frac14$ does not represent any loss of generality: in the generic case we get $4c\|f\|_\infty$ instead of c in the r.h.s. of (9.6). Moreover, the same estimate holds when the support $\bar\Lambda$ of f is replaced by any $\Lambda'\supset\bar\Lambda$: we get a different constant $c=c(\Lambda')$. It is easy to check that $c(\Lambda')\le\kappa|\Lambda'|$ for some $\kappa>0$.

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