Topological Entropy and ε-Entropy

for Damped Hyperbolic Equations

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Abstract. We study damped hyperbolic equations on the infinite line. We show that on the global attracting set $\mathcal{G}$ the ε-entropy (per unit length) exists in the topology of $W^{1,\infty}$. We also show that the topological entropy per unit length of $\mathcal{G}$ exists. These results are shown using two main techniques: Bounds in bounded domains in position space and for large momenta, and a novel submultiplicativity argument in $W^{1,\infty}$.

1. Introduction

This paper is an extension of our earlier papers [CE1, CE2] to mixed parabolic-hyperbolic equations in the infinite domain $\mathbf{R}$:

$$\eta^2\partial_t^2 u(x,t)+\partial_t u(x,t) \;=\; \partial_x^2 u(x,t)+U'\bigl(u(x,t)\bigr)\;, \tag{1.1}$$

where $U(s)=s^2/2-s^4/4$. The particular choice of the potential U is in fact not very important, but we will deal only with this one. This problem can be written as a system:

$$\begin{aligned}\partial_t u(x,t) \;&=\; v(x,t)\;,\\ \eta^2\partial_t v(x,t) \;&=\; -v(x,t)+\partial_x^2 u(x,t)+U'\bigl(u(x,t)\bigr)\;.\end{aligned} \tag{1.2}$$

The functions u will be real-valued, but the extension to vector-valued functions is easy and is left to the reader, since it only complicates notation.

The question we ask is about the nature of the attracting set for this problem, its complexity, and in particular its ε-entropy. We have developed this subject for parabolic problems in the two papers described above and we study now the complexity in this parabolic-hyperbolic setting. The difference with the parabolic case is the absence of regularization. In the parabolic case, the dispersion law is, written in Fourier space for the linearized equation

$$\partial_t\tilde u(k,t) \;=\; (1-k^2)\tilde u(k,t)\;, \tag{1.3}$$

when $U(s) = s^2/2 + \mathcal{O}(s^3)$ near $s = 0$. In the case we consider now, the problem is rather a system of the form

$$\begin{aligned} \partial_t \tilde{u}(k,t) &= \tilde{v}(k,t) , \\ \eta^2 \partial_t \tilde{v}(k,t) &= -\tilde{v}(k,t) + (1-k^2)\tilde{u}(k,t) . \end{aligned} \tag{1.4}$$

Thus, as is well known, (1.3) regularizes derivatives because $|k| \exp\bigl((1-k^2)t\bigr)$ is bounded in k when $t > 0$, while the real part of the eigenvalue of the system (1.4) is, for large $|k|$, only as negative as $-\mathcal{O}(\eta^{-2})$, and therefore the exponential is only bounded like $|k| \exp\bigl(-C\eta^{-2}t\bigr)$ for some $C > 0$. This diverges with $|k|$, but converges (non-uniformly) to 0 as $t \to \infty$.

One can ask whether this reduced form of regularization manifests itself in an increased complexity of either the attracting set, or some forward invariant set of bounded initial data. The conclusion of our paper is that *the complexity of the problem (1.4) is of the same order as that of (1.3).*

Since we work on the infinite line, we need local topologies. This will be achieved by choosing a cutoff function h:

$$h_\delta(x) = \frac{1}{(1+\delta^2 x^2)^2} .$$

We could take other functions with sufficiently strong polynomial decay, but the nice ideas of Mielke [M1, M2] using exponentially decaying cutoff functions do not seem to work here. We then consider local Sobolev norms of the form

$$\|(u,v)\|_{h_\delta,2}^2 = \int \mathrm{d}x\, h_\delta(x) \Bigl(u^2 + 2(u')^2 + (u'')^2 + \eta^2 \bigl(v^2 + (v')^2 \bigr) \Bigr)(x) ,$$

and then local spaces $\mathcal{H}^2_{\delta,\mathrm{loc}}$ with the norm

$$\|(u,v)\|_{\delta,\mathrm{loc},2} = \sup_{\xi \in \mathbf{R}} \|(u,v)\|_{h_{\delta,\xi},2} ,$$

where $h_{\delta,\xi}(x) = h_\delta(x-\xi)$. Note that this norm, and many others used in this paper, has *one more derivative in u than in v.* Such norms are typical when one writes equations such as (1.1) with two components as in (1.2).

We will show that every initial condition with finite $\|(u,v)\|_{\delta,\mathrm{loc},2}$ will end up after some finite time in a *bounded* set in this norm. We call this bounded set the attracting set $\mathcal{G}$. The attractor $\mathcal{A}$ is then defined as

$$\mathcal{A} = \bigcap_{t>0} \Phi^t(\mathcal{G}) ,$$

where $t \mapsto \Phi^t$ is the flow defined by (1.1). We will not only study the complexity of $\mathcal{A}$, but we can also make statements about functions which have "evolved for long enough," namely functions in $\mathcal{G}_T \equiv \bigcap_{T>t>0} \Phi^t(\mathcal{G})$ for some large T. Indeed, given some interval $[-L, L]$ in $\mathbf{R}$, with $L \gg 1/\varepsilon$, we show in Section 9.3 that $\mathcal{G}_T$ *when restricted to* $[-L, L]$ *in the variable* x can be covered by $N_L(\varepsilon) = C^{L \log 1/\varepsilon}$ balls of radius ε in $\mathcal{H}^2_{\delta,\mathrm{loc}}$. Our argument does not rely on compactness, but only on a comparison of the number of balls with radius ε relative to the number needed when the radius is 2ε:

$$N_L(\varepsilon) \leq N_{L+A/\varepsilon}(2\varepsilon)C^L ,$$

for some constants C and A (Proposition 8.4 and (9.20)). It is at this point that we use the invariance of $\mathcal{G}$, the fact that high-momentum parts of the solution are damped with an exponential rate of about η^{-2}, and that the low momentum parts are Fourier transforms of analytic functions, which can be finitely sampled by the Cartwright formula (8.10).

We then change topology to $W^{1,\infty}$ (functions in L^∞ whose derivatives are also in L^∞) and show that the results obtained for $\mathcal{H}^2_{\delta,\mathrm{loc}}$ give bounds in $W^{1,\infty}$. We introduce a new type of submultiplicativity bound in Section 9.2. Indeed, we show in Corollary 9.2 (which is an easy consequence of the Theorem 9.1) that if a bounded set of functions in $\mathcal{C}^2$ can be covered by N_1 balls of radius ε in $W^{1,\infty}_{I_1}$ and by N_2 on $W^{1,\infty}_{I_2}$, where I_1 and I_2 are disjoint intervals, then it can be covered by

$$C(\varepsilon)N_1N_2 \tag{1.5}$$

balls of radius ε in $W^{1,\infty}_{I_1\cup I_2}$. The point here is that C only depends on ε (and the bound on the functions) and that the balls on $I_1 \cup I_2$ have the *same* radius as the original balls. Indeed, if one allows a doubling of the radius, the corresponding inequality is trivial, but insufficient for taking the thermodynamic limit in the entropy. Thus, our bound is an essential tool for showing the existence of infinite volume limits in topologies where the "matching" of functions needs some care.

Once all these tools are in place, we can easily repeat the proofs of the existence of the topological entropy using the methods developed in [CE1] and [CE2].

The paper is structured as follows. In Section 2 we bound the flow in time, using localized versions of coercive functionals as introduced by Feireisl [F]. The main result is Theorem 2.6 and its corollary (2.22) and (2.23) which show that the solution to (1.2) is well behaved in $\mathcal{H}^3_{\delta,\mathrm{loc}}$. In Section 3 and Section 4 we study the linear part of (2.1) *localized in coordinate and momentum space*. We next study the decay of the high momentum part in Section 5 and Section 6. This allows, in Section 7 to study the time evolution of differences of two solutions of (2.1), in other words, we control now the *deformation* of balls (in the topology of $\mathcal{H}^2_{\delta,\mathrm{loc}}$). In Section 8 (Proposition 8.4 and (9.20)) we show how to cover the attracting set $\mathcal{G}$ with balls as explained in (1.5) above. In Section 9.2 we deal with the technically delicate submultiplicativity bound mentioned before. Finally, in Section 9.3 and Section 9.4 we show without effort the bound Theorem 9.5 on the ε-entropy (per unit length) and the Theorem 9.7 which shows the existence of the topological entropy per unit length.

2. Coercive functionals

In this section, we study some functionals which control the flow in time. The first part of this material is an adaptation from the work of Feireisl[F]. We consider here the problem (1.1) in the form

$$\begin{aligned} \dot{u} &= v\,, \\ \eta^2 \dot{v} &= -v + u'' + U'(u)\,, \end{aligned} \tag{2.1}$$

where we take $U(s) = \frac{s^2}{2} - \frac{s^4}{4}$, but many other choices are of course possible. To simplify things, we *assume throughout that* $0 < \eta < 1$, and in fact, in subsequent sections we will assume $\eta < \eta_0$ for some small η_0. We shall use throughout a localization function h_α which depends on a small parameter α, to be determined later on. The constant α will only depend on the coefficients of (2.1) (but not on $\eta < 1$).

We set

$$h_\alpha(x) = \frac{1}{(1+\alpha^2 x^2)^2}\,. \tag{2.2}$$

Note that $\alpha \int \mathrm{d}x\, h_\alpha$ is independent of α.

Remark. We will only use values of $0 < \alpha \le \frac{1}{2}$ and this will be tacitly assumed in all the estimates.

Using h_α, we introduce the norm

$$\|(u,v)\|^2_{h_\alpha,1} = \int \mathrm{d}x\, h_\alpha(x) \Big(\eta^2 v^2 + u^2 + (u')^2 \Big)(x)\,. \tag{2.3}$$

We also need a translation invariant topology on (u, v). Let $h_{\alpha,\xi}(x) = h_\alpha(x - \xi)$.

Definition 2.1. *We define the norm*

$$\|(u,v)\|_{\alpha,\mathrm{loc},1} = \sup_{\xi \in \mathbf{R}} \|(u,v)\|_{h_{\alpha,\xi},1}\,, \tag{2.4}$$

and the space $\mathcal{H}^1_{\alpha,\mathrm{loc}}$ *is defined by*

$$\mathcal{H}^1_{\alpha,\mathrm{loc}} = \Big\{ (u,v) \,\Big|\, \|(u,v)\|_{\alpha,\mathrm{loc},1} < \infty \Big\}\,. \tag{2.5}$$

The norm introduced above is not very convenient for estimates, and thus we introduce as in Feireisl[F] the quantity F_0 (which is not a norm) by

$$F_0(u,v) = \alpha \int \mathrm{d}x\, h_\alpha(x) \Big(\eta^2 v^2(x) + (u'(x))^2 + V(u(x)) + \eta^2 u(x) v(x) \Big)\,. \tag{2.6}$$

Here, V is chosen such that

$$U'(x) + 2V'(x) - \eta^2 x = 0\,, \tag{2.7}$$

with $V(0) = 0$. Note that $U(x) \to -\infty$ as $|x| \to \infty$ at a rate $\mathcal{O}(x^4)$, and therefore $V(x) \to +\infty$ at a rate $\mathcal{O}(x^4)$. In particular,

$$V(x) \ \geq \ x^2 \text{ for sufficiently large } |x| \ . \tag{2.8}$$

The following bound can be found in Feireisl[F].

Lemma 2.2. *There are constants $a_0 > 0$ and $b_0 > 0$ (independent of η for $0 < \eta < 1$) for which one has the inequality*

$$\partial_t F_0(u_t, v_t) \ \leq \ -a_0 F_0(u_t, v_t) + b_0 \ , \tag{2.9}$$

where $u_t(x) = u(x,t)$, $v_t(x) = v(x,t)$ is the solution of (2.1).

This bound can be used to bound $\|(u,v)\|_{h_\alpha,1}$. Recall that V diverges like $|x|^4$. Using the bound

$$\eta^2 |uv| \ \leq \ \frac{\eta^2 v^2}{2} + \frac{\eta^2 u^2}{2} \ ,$$

this implies

$$V(u) + \eta^2 uv \ \geq \ -\frac{\eta^2 v^2}{2} + u^2 - C_0 \ ,$$

for some constant C_0. Therefore, one has the inequality

$$\|(u,v)\|^2_{h_\alpha,1} \ \leq \ 2F_0(u,v) + C_1 \ .$$

Using (2.9) we conclude

Lemma 2.3. *There is a constant C_2 (independent of $0 < \eta < 1$) for which the following holds. Assume that $F_0(u_0, v_0) < \infty$. Then, for all $t > 0$ one has $\|(u_t, v_t)\|_{h_\alpha,1} < \infty$ and there is a $T = T(u_0, v_0)$ for which the solution (u_t, v_t) of (2.1) with initial data (u_0, v_0) satisfies for all $t > T$:*

$$\|(u_t, v_t)\|_{h_\alpha,1} \ \leq \ C_2 \ . \tag{2.10}$$

We can extend this result to the topology of $\mathcal{H}^1_{\alpha,\mathrm{loc}}$. Let $u_{0,\xi}(x) = u_0(x - \xi)$ and similarly for v_0.

Proposition 2.4. *There is a constant C_3 (independent of $0 < \eta < 1$) for which the following holds. Assume that $\sup_{\xi\in\mathbf{R}} F_0(u_{0,\xi}, v_{0,\xi}) < \infty$. Then there is a $T = T(u_0, v_0)$ for which the solution (u_t, v_t) of (2.1) with initial data (u_0, v_0) satisfies for all $t > T$:*

$$\|(u_t, v_t)\|_{\alpha,\mathrm{loc},1} \ \leq \ C_2 \ , \tag{2.11}$$

and

$$\|u_t\|_\infty \ \leq \ C_3 \ .$$

Proof. *Consider the quantities $F_{0,\xi}$ defined by replacing $h_\alpha(x)$ by its translate $h_\alpha(x+\xi)$ in Eq.(2.6). Then, $F_0(u_{t,\xi}, v_{t,\xi}) = F_{0,\xi}(u_t, v_t)$. Clearly, for every ξ we have*

$$\partial_t F_{0,\xi}(u_t, v_t) \le -a_0 F_{0,\xi}(u_t, v_t) + b_0 \,,$$

since (2.1) does not depend explicitly on x.

It follows from the above that if $\sup_\xi F(u_{0,\xi}, v_{0,\xi}) < \infty$ there is a finite time T after which

$$\|(u_t, v_t)\|_{\alpha,\mathrm{loc},1} \le C_2 \,. \tag{2.12}$$

This proves (2.11). To conclude that u is bounded we need the following easy

Lemma 2.5. *There is a constant $C_4 = C_4(\delta)$ such that*

$$\sup_{x\in[-1,1]} |f(x)| \le C_4 \|f\|_{h_\delta,1} \,. \tag{2.13}$$

Proof. From the explicit form of h_δ we conclude that

$$\int_{-1}^{1} \mathrm{d}x\, |f(x)|^2 \le (1+\delta^2)^2 \int \mathrm{d}x\, h_\delta |f(x)|^2 \,,$$

and similarly

$$\int_{-1}^{1} \mathrm{d}x\, |f'(x)|^2 \le (1+\delta^2)^2 \int \mathrm{d}x\, h_\delta |f'(x)|^2 \,.$$

The result follows from the standard Sobolev inequality. The proof of Lemma 2.5 is complete.

□

Using this lemma, and observing that the $\|\cdot\|_{\alpha,\mathrm{loc},1}$ norm is translation invariant we conclude immediately from (2.12) that there is a constant C_3 for which

$$\sup_x |u(x,t)| \le C_3 \,, \tag{2.14}$$

for all $t > T$. The proof of Proposition 2.4 is complete.

□

We next deal with the slightly more complicated bounds on the *spatial derivatives* of u and v. Let $w = u'$ and let $z = v'$. They satisfy the equations

$$\begin{aligned} \dot{w}_t &= z_t \,, \\ \eta^2 \dot{z}_t &= -z_t + w_t'' + U''(u_t) w_t \,, \end{aligned} \tag{2.15}$$

where $U''(s) = 1 - 3s^2$, for the U we have taken above. We consider initial data (w_0, z_0) which will be bounded later and assume (in view of Proposition 2.4) that $\|(u_0, v_0)\|_{\alpha,\mathrm{loc},1} \le C_2$ which implies $\sup_x |u(x,t)| \le C_3$ for all $t > 0$.

We are going to bound the growth of (w, z) as a function of time. We introduce a positive constant μ (which we fix later) and set

$$F_1(w,z) \;=\; \alpha \int \mathrm{d}x\, h_\alpha(x) \Big(\eta^2 z^2(x) + (w'(x))^2 + \eta^2 \mu w(x) z(x) \Big) \,. \tag{2.16}$$

When no confusion is possible, we henceforth write $\int f$ for $\int \mathrm{d}x\, f(x)$. One finds for (w_t, z_t) satisfying (2.15):

$$\begin{aligned}
\tfrac{1}{2}\partial_t F_1(w_t,z_t) \;&=\; \alpha \int h_\alpha \Big(\eta^2 z_t \dot z_t + w_t' \dot w_t' + \frac{\eta^2\mu}{2} \dot w_t z_t + \frac{\eta^2\mu}{2} w_t \dot z_t \Big) \\
&=\; \alpha \int h_\alpha \Big(- z_t^2 + z_t w_t'' + U''(u_t) z_t w_t + w_t' z_t' + \frac{\eta^2 \mu}{2} z_t^2 \\
&\quad - \frac{\mu}{2} w_t z_t + \frac{\mu}{2} w_t w_t'' + \frac{\mu}{2} U''(u_t) w_t^2 \Big) \\
&=\; \alpha \int h_\alpha \Big(- z_t^2 (1 - \frac{\eta^2\mu}{2}) + U''(u_t) z_t w_t \\
&\quad - \frac{\mu}{2} w_t z_t - \frac{\mu}{2} (w_t')^2 + \frac{\mu}{2} U''(u_t) w_t^2 \Big) \\
&\quad - \alpha \int h_\alpha' \big(z_t w_t' + \frac{\mu}{2} w_t w_t' \big) \,.
\end{aligned} \tag{2.17}$$

Note now that by the definition (2.2) of h_α and the restriction $\alpha \le \frac{1}{2}$, we find that the quotient $|h_\alpha'/h_\alpha|$ is bounded by $2\alpha \le 1$. Using this, we get

$$|h_\alpha' z_t w_t'| \;\le\; \alpha h_\alpha \big(z_t^2 + (w_t')^2 \big) \,,$$

and

$$|\mu h_\alpha' w_t w_t'| \;\le\; \alpha \mu h_\alpha \big(w_t^2 + (w_t')^2 \big) \,.$$

We will also use the bound $|w_t z_t| \le \alpha^{-1} w_t^2 + \alpha z_t^2$. Finally note that there is a constant C_5 for which

$$\sup_{|s| \le C_3} |U''(s)| \;\le\; C_5 \,.$$

Combining these bounds with the last equality of (2.17), we get for times $t > 0$:

$$\begin{aligned}
\tfrac{1}{2}\partial_t F_1(w_t,z_t) \;\le\; & -\alpha \int h_\alpha z_t^2 \big(1 - \frac{\eta^2\mu}{2} - \alpha - \mu/2 - \alpha C_5 \big) \\
& - \alpha \int h_\alpha (w_t')^2 \big(\mu/2 - \alpha\mu/2 - \alpha \big) \\
& + \big(\alpha\mu + \mu/2 + \mu C_5/2 + \alpha^{-1} C_5 \big) \alpha \int h_\alpha w_t^2 \,.
\end{aligned} \tag{2.18}$$

It is clear that if we choose α and μ sufficiently small (but independent of η for small enough η), then we get, for some (large) constant C_6,

$$\tfrac{1}{2}\partial_t F_1(w_t, z_t) \;\leq\; \alpha \int h_\alpha \Big(-\frac{\mu}{4}\big(z_t^2 + (w_t')^2\big) + C_6 w_t^2 \Big)\,. \tag{2.19}$$

Note now that for small $\mu > 0$ and $\eta > 0$,

$$\eta^2 \mu w z \;\leq\; (1-\eta^2) z^2 + \frac{\eta^2 \mu}{4(1-\eta^2)} w^2\,,$$

which is equivalent to

$$-z^2 \;\leq\; -\eta^2 z^2 - \eta^2 \mu w z + \frac{\eta^2 \mu}{4(1-\eta^2)} w^2\,.$$

Therefore, (2.19) leads to

$$\tfrac{1}{2}\partial_t F_1(w_t, z_t) \;\leq\; -\frac{\alpha\mu}{4} \int h_\alpha \big(\eta^2 z_t^2 + (w_t')^2 + \eta^2 \mu w_t z_t\big) + C_7 \int h_\alpha w_t^2\,.$$

The last term is bounded by (2.11), and therefore we find:

$$\partial_t F_1(w_t, z_t) \;\leq\; -a_1 F_1(w_t, z_t) + b_1\,,$$

for some finite positive a_1 and b_1. Using again the methods leading to Proposition 2.4, we obtain

Theorem 2.6. *There are constants C_8, C_9 and C_{10} (independent of $\eta < 1$) for which the following holds. Assume $\sup_{\xi\in\mathbf{R}} F_0(u_{0,\xi}, v_{0,\xi}) < C_8$ and $\sup_{\xi\in\mathbf{R}} F_1(u_{0,\xi}, v_{0,\xi}) < C_8$. Then the solution (u_t, v_t) of (2.1) with initial data (u_0, v_0) satisfies for all $t > 0$:*

$$\|(u_t, v_t)\|_{\alpha,\mathrm{loc},2} \;\leq\; C_9\,, \tag{2.20}$$

and

$$\sup_{x\in\mathbf{R}} \big(|u(x,t)| + |u'(x,t)| + |v(x,t)|\big) \;\leq\; C_{10}\,. \tag{2.21}$$

Remark. The technique used above can be extended to show that any derivative of $u(x,t)$ and $v(x,t)$ is eventually bounded (if the potential U is sufficiently differentiable and the initial data are sufficiently regular). The details are left to the reader. We will in fact use bounds on the second derivative at some later point in the argument, *i.e.*, bounds of the form

$$\|(u_t, v_t)\|_{\alpha,\mathrm{loc},3} \;\leq\; C_9\,, \tag{2.22}$$

and

$$\sup_{x\in\mathbf{R}} \big(|u(x,t)| + |u'(x,t)| + |u''(x,t)| + |v(x,t)| + |v'(x,t)|\big) \;\leq\; C_{10}\,. \tag{2.23}$$

3. The linearized evolution

In this section, we study the linear part of the solution. By this we mean solutions of the equation

$$\begin{aligned} \dot{\mathfrak{u}} &= \mathfrak{v} , \\ \eta^2 \dot{\mathfrak{v}} &= -\mathfrak{v} + \mathfrak{u}'' . \end{aligned} \tag{3.1}$$

It will be useful to rewrite this system of equations as

$$\begin{pmatrix} \dot{\mathfrak{u}} \\ \dot{\mathfrak{v}} \end{pmatrix} = \mathcal{L} \begin{pmatrix} \mathfrak{u} \\ \mathfrak{v} \end{pmatrix} . \tag{3.2}$$

Next, we go through a second round of estimates, similar to the ones in Section 2, to see how fast $(\mathfrak{u}, \mathfrak{v})$ can grow. We use again the cutoff function

$$h_\delta(x) = \frac{1}{(1 + \delta^2 x^2)^2} ,$$

with a parameter δ different from α which will be fixed in Section 6. We consider the functional

$$H = \int h_\delta \big(\eta^2 \mathfrak{v}^2 + \mathfrak{u}^2 + (\mathfrak{u}')^2 \big) = \|(\mathfrak{u}, \mathfrak{v})\|_{h_\delta, 1} , \tag{3.3}$$

and proceed to bound it. One gets for the solution $(\mathfrak{u}_t, \mathfrak{v}_t)$ of (3.2) :

$$\begin{aligned} \tfrac{1}{2} \partial_t H(\mathfrak{u}_t, \mathfrak{v}_t) &= \int \mathrm{d}x \, h_\delta \big(\eta^2 \mathfrak{v}_t \dot{\mathfrak{v}}_t + \mathfrak{u}_t \dot{\mathfrak{u}}_t + \mathfrak{u}_t' \dot{\mathfrak{u}}_t' \big) \\ &= \int \mathrm{d}x \, h_\delta \big(-\mathfrak{v}_t^2 + \mathfrak{v}_t \mathfrak{u}_t'' + \mathfrak{v}_t \mathfrak{u}_t + \mathfrak{u}_t' \mathfrak{v}_t' \big) \\ &= \int \mathrm{d}x \, h_\delta \big(-\mathfrak{v}_t^2 + \mathfrak{v}_t \mathfrak{u}_t \big) - \int h_\delta' \mathfrak{v}_t \mathfrak{u}_t' . \end{aligned} \tag{3.4}$$

Observe that by construction,

$$|h_\delta' / h_\delta| \le 1 ,$$

when $\delta < \frac{1}{2}$ (which we assume throughout). We use now (since $\mathfrak{u}$ and $\mathfrak{v}$ are real):

$$|\mathfrak{v}_t| \big(|\mathfrak{u}_t| + |h_\delta' \mathfrak{u}_t' / h_\delta| \big) \le \tfrac{1}{2} \mathfrak{v}_t^2 + \mathfrak{u}_t^2 + (\mathfrak{u}_t')^2 .$$

We can use this inequality to bound the mixed terms in the last equality of (3.4), and compensate the term $\frac{1}{2}(\mathfrak{v}_t^2)$ with the negative term $-\mathfrak{v}_t^2$. Therefore, we have shown that there is a constant C_{11} *independent of* $\eta < 1$ for which

$$\partial_t H(\mathfrak{u}_t, \mathfrak{v}_t) \le C_{11} H(\mathfrak{u}_t, \mathfrak{v}_t) . \tag{3.5}$$

We next define a space in which both the functions and their derivatives are controlled. This is the space in which our final bounds will be spelled out. Using still the cutoff function

$$h_\delta(x) = \frac{1}{(1+\delta^2 x^2)^2} ,$$

we define the norm

$$\|(u,v)\|^2_{h_\delta,2} = \int h_\delta \Big(\eta^2 \big(v^2 + (v')^2\big) + u^2 + 2(u')^2 + (u'')^2 \Big) . \tag{3.6}$$

Definition 3.1. *The Hilbert space $\mathcal{H}^2_{h_\delta}$ is defined by*

$$\mathcal{H}^2_{h_\delta} = \Big\{ (u,v) \ \Big| \ \|(u,v)\|_{h_\delta,2} < \infty \Big\} . \tag{3.7}$$

We also need the translates of h_δ to define a translation invariant topology. Let $h_{\delta,\xi}(x) = h_\delta(x-\xi)$.

Definition 3.2. *The space $\mathcal{H}^2_{\delta,\mathrm{loc}}$ is defined by*

$$\mathcal{H}^2_{\delta,\mathrm{loc}} = \Big\{ (u,v) \ \Big| \ \sup_\xi \|(u,v)\|_{h_{\delta,\xi},2} < \infty \Big\} . \tag{3.8}$$

Remark. Note that

$$\|(u,v)\|^2_{h_\delta,2} = \|(u,v)\|^2_{h_\delta,1} + \|(u',v')\|^2_{h_\delta,1} .$$

Next, we observe that $\mathcal{L}$ and ∂_x commute. Therefore, the bounds on $\|(\mathfrak{u}_t,\mathfrak{v}_t)\|_{h_\delta,1}$ can be extended immediately to similar bounds on $\|(\mathfrak{u}_t,\mathfrak{v}_t)\|_{h_\delta,2}$ and we get from (3.5):

Theorem 3.3. *There is a constant C_{11} such that if $(\mathfrak{u}_t,\mathfrak{v}_t)$ solves (3.1) then*

$$\|(\mathfrak{u}_t,\mathfrak{v}_t)\|^2_{h_\delta,2} \le e^{C_{11}t} \|(\mathfrak{u}_0,\mathfrak{v}_0)\|^2_{h_\delta,2} . \tag{3.9}$$

4. Momentum localization

Let m be a bounded positive function of x which decays faster than any inverse power as $|x| \to \infty$. We define the convolution operator M_a by

$$\big(M_a f\big)(x) \;=\; \int \mathrm{d}y\, m\big(a(x-y)\big) f(y)\;. \tag{4.1}$$

Let again

$$h_\delta(x) \;=\; \frac{1}{(1+\delta^2 x^2)^2}\;. \tag{4.2}$$

Lemma 4.1. *There is a constant* C_{12} *such that if* $\delta > 0$ *and* $a > 0$, *then the operator* M_a *is bounded on* $\mathrm{L}^2(h_\delta\,\mathrm{d}x\,)$, *with norm bounded by*

$$\|M_a\|_\delta \;\le\; C_{12}\frac{1+\frac{\delta^2}{a^2}}{a}\;. \tag{4.3}$$

Proof. We will prove the result by bounding the operator $\hat{M}_a$ with integral kernel

$$h_\delta^{1/2}(x)\, m\big(a(x-y)\big)\, h_\delta^{-1/2}(y) \tag{4.4}$$

on $\mathrm{L}^2(\mathrm{d}x)$. Writing $1 = \chi(2|x| > |y|) + \chi(2|x| < |y|)$, with characteristic functions χ, and multiplying the kernel (4.4) with them, we induce a decomposition of this operator as a sum of two pieces.

Fix $|x|$. We consider first the integration region $|y| < 2|x|$. In that region, we have a bound

$$0 \;<\; \frac{h_\delta^{1/2}(x)}{h_\delta^{1/2}(y)} \;\le\; 4\;.$$

Therefore, in this region, the integral kernel is bounded by $4|m(a(x-y))|$. Since m decreases like an arbitrary power we get for every $\ell > 0$ a bound

$$\int_{|y|<2|x|} \mathrm{d}y\, \frac{h_\delta^{1/2}(x)|m\big(a(x-y)\big)|}{h_\delta(y)^{1/2}}|f(y)| \;\le\; 4K_\ell \int_{|y|<2|x|} \mathrm{d}y\, \frac{1}{(1+a|x-y|)^\ell}|f(y)|\;.$$

When $\ell > 1$, then $a/(1+a|x|)^\ell$ is bounded in L^1 (uniformly in a) and therefore, Young's inequality shows that this piece of $\hat{M}_a f$ is bounded in $\mathrm{L}^2(\mathrm{d}x)$, with norm less that $C_{13}a^{-1}\|f\|_2$, and C_{13} independent of δ and a. In the region $|y| \ge 2|x|$, we use $|x-y| > |y|/2$ and $|x-y| > |x|$. Therefore, using $h_\delta(x) \le 1$, we find for $\ell > 3$,

$$\int_{|y|\ge 2|x|} \mathrm{d}y\, \frac{h_\delta^{1/2}(x)|m\big(a(x-y)\big)|}{h_\delta(y)^{1/2}}|f(y)| \;\le\; 2^{\frac{(\ell+2)}{2}} K_\ell \int_{|y|\ge 2|x|} \mathrm{d}y\, \frac{1+\delta^2 y^2}{(1+a|x|)^{\frac{\ell-2}{2}}(1+a|y|)^{\frac{\ell+2}{2}}}|f(y)|\;.$$

Note that

$$\frac{1+\delta^2 y^2}{(1+a|y|)^2} \;\le\; 1+\frac{\delta^2}{a^2}\;.$$

Using the Schwarz inequality yields a bound $(1+\delta^2/a^2)/a$ on the second piece of $\|M_a\|_\delta$. Combining the two pieces completes the proof of Lemma 4.1.

□

We need later the following variant of this result: Let $\vartheta=\vartheta(k)$ be a smooth characteristic function which equals 1 for $|k|\le 1$ and 0 for $|k|>2$. Let $q_a(x)=a\tilde{\vartheta}(ax)$, with $\tilde{\vartheta}$ the inverse Fourier transform of ϑ. For $a>0$, let $\mathcal{Q}_a$ be the convolution operator defined by

$$\big(\mathcal{Q}_a f\big)(x) \;=\; \int \mathrm{d}y\, q_a(x-y) f(y)\;. \tag{4.5}$$

This operator is a substitute for a projection onto momenta less than a. Setting $m(x)=\tilde{\vartheta}(x)$, we get from Lemma 4.1:

Corollary 4.2. *There is a constant C_{12} such that if $\delta>0$ and $a>0$ then the operator $\mathcal{Q}_a$ is bounded on $\mathrm{L}^2(h_\delta\,\mathrm{d}x)$, with norm bounded by*

$$\|\mathcal{Q}_a\|_\delta \;\le\; C_{12}\big(1+\frac{\delta^2}{a^2}\big)\;. \tag{4.6}$$

5. High momentum bounds

We consider again the function h_δ as defined in (4.2), and we study functions u for which $\int \mathrm{d}x\, h_\delta(x)\big(|u(x)|^2+|u'(x)|^2\big)<\infty$. Such functions have a Fourier transform $\tilde{u}$ in the sense of tempered distributions, and we define now

$$\mathcal{K}_a \;=\; \left\{ u \;\middle|\; \int \mathrm{d}x\, h_\delta(x)\big(|u(x)|^2+|u'(x)|^2\big)<\infty \text{ and supp } \tilde{u}\in \mathbf{R}\setminus(-a,a)\right\}\;.$$

Thus, apart from not being defined as a function, the Fourier transform $\tilde{u}$ of a $u\in\mathcal{K}_a$ has support at momenta larger than a. If $h_\delta(x)\equiv 1$ and $u'\in\mathrm{L}^2(\mathrm{d}x)$, then, obviously, for $u\in\mathcal{K}_a$, one has $\int |u|^2 \;\le\; a^{-2}\int |u'|^2$. The following proposition whose elegant proof was kindly provided by *H. Epstein*, shows that the cutoff function h_δ does only moderately change this property.

Proposition 5.1. *Assume that $a>0$ and $\delta>0$. There is a constant $\nu(a,\delta)<\infty$ such that for all $u\in\mathcal{K}_a$ one has the inequality*

$$\int \mathrm{d}x\, h_\delta(x)|u(x)|^2 \;\le\; \nu(a,\delta)\int \mathrm{d}x\, h_\delta(x)|u'(x)|^2\;. \tag{5.1}$$

There is a constant $C_{14} > 0$ *such that one can choose*

$$\nu(a,\delta) \;=\; C_{14}\frac{(1+\frac{\delta^2}{a^2})^2}{a^2} \;. \tag{5.2}$$

Remark. We will need the result only for $\delta < a$, so that we can use the simpler bound

$$\nu(a,\delta) \;\le\; \frac{C_{15}}{a^2} \;. \tag{5.3}$$

Proof. Let ϑ be a smooth characteristic function which equals 1 for $|k| \le \frac{1}{2}$ and 0 for $|k| > 1$. Let $u \in \mathcal{K}_a$. Since the distribution $\tilde{u}$ has support in the complement of the interval $(-a, a)$, and the Fourier transform $\widetilde{u'}$ of the derivative is $ik\tilde{u}(k)$, we see that

$$\tilde{u}(k) \;=\; \frac{1-\vartheta(k/a)}{ik}\;\widetilde{u'}(k) \;.$$

Define next

$$\tilde{m}(k) \;\equiv\; \frac{1-\vartheta(k)}{ik} \;.$$

The (inverse) Fourier transform, m, of $\tilde{m}$ decreases faster than any power of $|x|$ at infinity. If we let $m_a(x) = m(ax)$, then $\tilde{m}_a(k) = \tilde{m}(k/a)/a$. Thus, it follows with the notation of Section 4 that

$$u(x) \;=\; \big(M_a(u')\big)(x) \;.$$

By Lemma 4.1, we conclude that

$$\|u\|_\delta^2 \;=\; \int h_\delta |u|^2 \;=\; \int h_\delta |M_a(u')|^2 \;=\; \|M_a(u')\|_\delta^2 \;\le\; C_{12}^2\frac{(1+\frac{\delta^2}{a^2})^2}{a^2}\int h_\delta |u'|^2 \;,$$

and the claim (5.2) follows.

$\square$

6. The linear high frequency part

We begin by defining the projection onto high frequencies, on a space with weight $h_\delta(x) = (1+\delta^2 x^2)^{-2}$. We first recall the notion of projection onto low frequencies from Section 4. Denote by $\widetilde{\vartheta}$ a smooth characteristic function, equal to 1 for $|k| \le 1$ and vanishing for $|k| > 2$. We fix now a (large) cutoff scale k_* and we define as before

$$q_{k_*}(x) \;=\; k_* \vartheta(k_* x)\;,$$

and

$$\big(\mathcal{Q}_{k_*} f\big)(x) \;=\; \int \mathrm{d}y\, q_{k_*}(x-y) f(y)\;. \tag{6.1}$$

In Corollary 4.2, we showed that on $\mathrm{L}^2(h_\delta\,\mathrm{d}x)$, the operator $\mathcal{Q}_{k_*}$ is bounded by $C_{12}\big(1+\frac{\delta^2}{k_*^2}\big)$. Therefore, the projection onto high momenta

$$\mathcal{P}_{k_*} \;=\; 1 - \mathcal{Q}_{k_*}\;, \tag{6.2}$$

is also bounded on that space. Henceforth, we shall assume $\delta < k_*$, and thus we get immediately the bound

Lemma 6.1. *There is a constant C_{16} such that if $k_* > \delta > 0$, then the operator $\mathcal{P}_{k_*}$ satisfies*

$$\|\mathcal{P}_{k_*}\|_\delta \;\le\; C_{16}\;, \tag{6.3}$$

as a map on $\mathrm{L}^2(h_\delta\,\mathrm{d}x)$.

Lemma 6.2. *There is a constant C_{17} such that for $k_* > \delta > 0$ the operator $\mathcal{P}_{k_*} \oplus \mathcal{P}_{k_*}$ is bounded in norm by C_{17} as a map from $\mathcal{H}^2_{\delta,\mathrm{loc}}$ to itself.*

Proof. We have already checked in Lemma 6.1 that $\mathcal{P}_{k_*}$ is bounded on $\mathrm{L}^2(h_\delta\,\mathrm{d}x)$. Note that $\mathcal{P}_{k_*}$ is a convolution operator and so $\mathcal{P}_{k_*}$ and ∂_x commute, and the extension of the result to $\mathcal{H}^2_{\delta,\mathrm{loc}}$ (as defined in Definition 3.2) follows at once.

□

So far, we have argued that $\mathcal{P}_{k_*}$ is bounded. We will now use the high momentum bound of Section 5 with $a = k_*$, and $k_* \le \eta^{-1}$ to show that the semi-group generated by the free evolution (see below) is a (strong) contraction. In fact, we will show that the contraction rate is

$$\mathcal{O}(k_*^2) \text{ as long as } k_* < \eta^{-1}\;,$$
$$\mathcal{O}(\eta^{-2}) \text{ for any cutoff } k_* \ge \eta^{-1}\;.$$

This behavior is typical of the mixed parabolic-hyperbolic problems we consider here, since the linearized evolution, written in momentum space, has the generator

$$\begin{pmatrix} 0 & 1 \\ -\eta^{-2}k^2 & -\eta^{-2} \end{pmatrix}$$

with eigenvalues

$$\lambda_\pm = \frac{-1 \pm \sqrt{1-4k^2\eta^{-2}}}{2\eta^2} .$$

One can see from the expression for the eigenvalues that the real part never gets more negative than $-\mathcal{O}(\eta^{-2})$. We study now the properties of the operator $\mathcal{L}$ defined as in Eqs.(3.1) and (3.2) by

$$\mathcal{L}\begin{pmatrix} \mathfrak{u} \\ \mathfrak{v} \end{pmatrix} = \begin{pmatrix} \mathfrak{v} \\ \eta^{-2}\bigl(-\mathfrak{v}+\mathfrak{u}''\bigr) \end{pmatrix} .$$

We introduce parameters $\gamma > 0$, and $\delta > 0$ which will be fixed later and we consider the functional J:

$$J = J_{h_\delta,\gamma}(\mathfrak{u},\mathfrak{v}) = \|(\mathfrak{u},\mathfrak{v})\|^2_{h_\delta,2} + \eta^2\gamma \int \mathrm{d}x\, h_\delta(x)\bigl(\mathfrak{u}(x)\mathfrak{v}(x) + \mathfrak{u}'(x)\mathfrak{v}'(x)\bigr) , \qquad (6.4)$$

where the norm $\|(\mathfrak{u},\mathfrak{v})\|_{h_\delta,2}$ was defined in Eq.(3.6):

$$\|(\mathfrak{u},\mathfrak{v})\|^2_{h_\delta,2} = \int h_\delta\Bigl(\eta^2\bigl(\mathfrak{v}^2+(\mathfrak{v}')^2\bigr) + \mathfrak{u}^2 + 2(\mathfrak{u}')^2 + (\mathfrak{u}'')^2\Bigr) .$$

Consider the solution $(\mathfrak{u}_t,\mathfrak{v}_t)$ of (3.2): Then, writing J for $J_{h_\delta,\gamma}$, we find

$$\begin{aligned}
\tfrac{1}{2}\partial_t J(\mathfrak{u}_t,\mathfrak{v}_t) &= \int h_\delta\Bigl(\eta^2\bigl(\mathfrak{v}_t\dot{\mathfrak{v}}_t + \mathfrak{v}_t'\dot{\mathfrak{v}}_t'\bigr) + \mathfrak{u}_t\dot{\mathfrak{u}}_t + 2\mathfrak{u}_t'\dot{\mathfrak{u}}_t' + \mathfrak{u}_t''\dot{\mathfrak{u}}_t''\Bigr) \\
&\quad + \tfrac{1}{2}\eta^2\gamma\int h_\delta\bigl(\mathfrak{u}_t\dot{\mathfrak{v}}_t + \mathfrak{v}_t\dot{\mathfrak{u}}_t + \mathfrak{u}_t'\dot{\mathfrak{v}}_t' + \mathfrak{v}_t'\dot{\mathfrak{u}}_t'\bigr) \\
&= \int h_\delta\Bigl(-\mathfrak{v}_t^2 + \mathfrak{v}_t\mathfrak{u}_t'' - (\mathfrak{v}_t')^2 + \mathfrak{v}_t'\mathfrak{u}_t''' + \mathfrak{u}_t\mathfrak{v}_t + 2\mathfrak{u}_t'\mathfrak{v}_t' + \mathfrak{u}_t''\mathfrak{v}_t''\Bigr) \\
&\quad + \tfrac{1}{2}\gamma\int h_\delta\bigl(-\mathfrak{u}_t\mathfrak{v}_t + \mathfrak{u}_t\mathfrak{u}_t'' + \eta^2\mathfrak{v}_t^2 - \mathfrak{u}_t'\mathfrak{v}_t' + \mathfrak{u}_t'\mathfrak{u}_t''' + \eta^2(\mathfrak{v}_t')^2\bigr) \\
&= \int h_\delta\Bigl(-\bigl(\mathfrak{v}_t^2 + (\mathfrak{v}_t')^2\bigr)\bigl(1-\tfrac{1}{2}\eta^2\gamma\bigr) + \bigl(\mathfrak{u}_t\mathfrak{v}_t + \mathfrak{u}_t'\mathfrak{v}_t'\bigr)\bigl(1-\tfrac{1}{2}\gamma\bigr)\Bigr) \\
&\quad - \tfrac{1}{2}\gamma\int h_\delta\bigl((\mathfrak{u}_t')^2 + (\mathfrak{u}_t'')^2\bigr) - \int h_\delta'\bigl(\mathfrak{v}_t\mathfrak{u}_t' + \mathfrak{v}_t'\mathfrak{u}_t'' + \tfrac{1}{2}\gamma\bigl(\mathfrak{u}_t\mathfrak{u}_t' + \mathfrak{u}_t'\mathfrak{u}_t''\bigr)\bigr) .
\end{aligned} \qquad (6.5)$$

By construction, we have $|h_\delta'/h_\delta| \le C_{18}\delta$, and therefore the last integral in (6.5) can be bounded (in modulus) by

$$C_{18}\delta\int h_\delta\Bigl(\mathfrak{v}_t^2 + (\mathfrak{u}_t')^2 + (\mathfrak{v}_t')^2 + (\mathfrak{u}_t'')^2 + \tfrac{1}{2}\gamma\bigl(\mathfrak{u}_t^2 + 2(\mathfrak{u}_t')^2 + (\mathfrak{u}_t'')^2\bigr)\Bigr) .$$

Thus, we find

$$\begin{aligned}\tfrac{1}{2}\partial_t J(\mathfrak{u}_t,\mathfrak{v}_t) \;\le\; &-\int h_\delta\big(\mathfrak{v}_t^2+(\mathfrak{v}_t')^2\big)\big(1-\tfrac{1}{2}\eta^2\gamma - C_{18}\delta\big)\\ &-\tfrac{1}{2}\gamma\int h_\delta\big((\mathfrak{u}_t')^2+(\mathfrak{u}_t'')^2\big)\big(1-\frac{C_{18}\delta}{\gamma}-2C_{18}\delta\big)\\ &+\tfrac{1}{2}C_{18}\delta\gamma\int h_\delta \mathfrak{u}_t^2\\ &+\int h_\delta\big(\mathfrak{u}_t\mathfrak{v}_t+\mathfrak{u}_t'\mathfrak{v}_t'\big)\big(1-\tfrac{1}{2}\gamma\big)\;.\end{aligned} \tag{6.6}$$

Recall that $\eta > 0$ is given, and that we want to prove results for all $\eta < \eta_0$, where η_0 is our (only) small parameter.

We rewrite the last integral in (6.6) as

$$-\tfrac{1}{8}\eta^2\gamma^2\int h_\delta\big(\mathfrak{u}_t\mathfrak{v}_t+\mathfrak{u}_t'\mathfrak{v}_t'\big)+\int h_\delta\big(\mathfrak{u}_t\mathfrak{v}_t+\mathfrak{u}_t'\mathfrak{v}_t'\big)\big(1+\tfrac{1}{8}\eta^2\gamma^2-\tfrac{1}{2}\gamma\big)\;. \tag{6.7}$$

We introduce now the first restrictions on η and k_*: Fix

$$\eta_0 \;\le\; \frac{1}{\sqrt{40}}\;, \tag{6.8}$$

and

$$k_0 \;\ge\; \sqrt{40C_{15}}\;. \tag{6.9}$$

These bounds will be made more stringent below. We shall always require

$$0 < \eta < \eta_0\;,\quad \text{and}\quad \infty > k_* > k_0\;.$$

We next define

$$\gamma \;\equiv\; \min(\eta^{-2}, k_*^2/C_{15})/320\;, \tag{6.10}$$

and we choose the space-cutoff parameter δ as

$$\delta \;=\; \min\big(1/2, 1/(40C_{18})\big)\;. \tag{6.11}$$

Note that γ is essentially the inverse of the dispersion law as explained at the beginning of this section.

With the above requirements we find $\gamma > 2$ and

$$|1+\tfrac{1}{8}\eta^2\gamma^2-\tfrac{1}{2}\gamma| \;\le\; \gamma\;.$$

We polarize the *second* integral in (6.7) (but not the first) and bound it (in modulus) by

$$\begin{aligned}|1+\tfrac{1}{8}\eta^2\gamma^2-\tfrac{1}{2}\gamma|\int h_\delta(|\mathfrak{u}_t\mathfrak{v}_t|+|\mathfrak{u}_t'\mathfrak{v}_t'|) \;&\le\; \gamma\int h_\delta(|\mathfrak{u}_t\mathfrak{v}_t|+|\mathfrak{u}_t'\mathfrak{v}_t'|)\\ &\le\int h_\delta\Big(8\gamma^2\big(\mathfrak{u}_t^2+(\mathfrak{u}_t')^2\big)+\tfrac{1}{8}\big(\mathfrak{v}_t^2+(\mathfrak{v}_t')^2\big)\Big)\;.\end{aligned} \tag{6.12}$$

Combining (6.6) with the decomposition (6.7) and the bound (6.12), we find

$$\begin{aligned}
\tfrac{1}{2}\partial_t J(\mathfrak{u}_t,\mathfrak{v}_t) \;\le\; & -\int h_\delta\big(\mathfrak{v}_t^2+(\mathfrak{v}_t')^2\big)\big(1-\tfrac12\eta^2\gamma-C_{18}\delta-\tfrac18\big)\\
& -\tfrac12\gamma\int h_\delta\big(\mathfrak{u}_t^2+2(\mathfrak{u}_t')^2+(\mathfrak{u}_t'')^2\big)\big(1-\frac{C_{18}\delta}{\gamma}-2C_{18}\delta\big)\\
& +\int h_\delta\big(\mathfrak{u}_t^2+(\mathfrak{u}_t')^2\big)\big((8\gamma^2+\tfrac12 C_{18}\delta\gamma)+\tfrac12\gamma(1-\frac{C_{18}\delta}{\gamma}-2C_{18}\delta)\big)\\
& -\tfrac18\eta^2\gamma^2\int h_\delta\big(\mathfrak{u}_t\mathfrak{v}_t+\mathfrak{u}_t'\mathfrak{v}_t'\big)\;.
\end{aligned}\tag{6.13}$$

The bizarre decomposition of the terms involving $(\mathfrak{u}_t')^2$ will become clear soon. Note that by our choice of constants, (6.13) can be simplified to the slightly less good bound

$$\begin{aligned}
\tfrac{1}{2}\partial_t J(\mathfrak{u}_t,\mathfrak{v}_t) \;\le\; & -\tfrac12\int h_\delta\big(\mathfrak{v}_t^2+(\mathfrak{v}_t')^2\big)\\
& -\tfrac14\gamma\int h_\delta\big(\mathfrak{u}_t^2+2(\mathfrak{u}_t')^2+(\mathfrak{u}_t'')^2\big)\\
& +16\gamma^2\int h_\delta\big(\mathfrak{u}_t^2+(\mathfrak{u}_t')^2\big)\\
& -\tfrac18\eta^2\gamma^2\int h_\delta\big(\mathfrak{u}_t\mathfrak{v}_t+\mathfrak{u}_t'\mathfrak{v}_t'\big)\;.
\end{aligned}\tag{6.14}$$

We project onto high momenta, and exploit the contraction properties: *We assume from now on that* $\mathcal{Q}_{k_*}\mathfrak{u}_0=0$ *and* $\mathcal{Q}_{k_*}\mathfrak{v}_0=0$. Note that if this property holds at time zero, it holds for all times for the evolution defined by $\mathcal{L}$, because $\mathcal{L}$ commutes with $\mathcal{Q}_{k_*}\oplus\mathcal{Q}_{k_*}$. Using the bounds of Section 5, we have

$$\begin{aligned}
\int h_\delta \mathfrak{u}_t^2 \;&\le\; \nu\int h_\delta(\mathfrak{u}_t')^2\;,\\
\int h_\delta(\mathfrak{u}_t')^2 \;&\le\; \nu\int h_\delta(\mathfrak{u}_t'')^2\;,
\end{aligned}\tag{6.15}$$

where

$$\nu \;=\; C_{15}k_*^{-2}\;.$$

Thus, (6.14) can be improved to

$$\begin{aligned}
\tfrac{1}{2}\partial_t J \;\le\; & -\tfrac12\int h_\delta\big(\mathfrak{v}_t^2+(\mathfrak{v}_t')^2\big)\\
& -\tfrac14\gamma\int h_\delta\big(\mathfrak{u}_t^2+2(\mathfrak{u}_t')^2+(\mathfrak{u}_t'')^2\big)\\
& +16\nu\gamma^2\int h_\delta\big((\mathfrak{u}_t')^2+(\mathfrak{u}_t'')^2\big)\\
& -\tfrac18\eta^2\gamma^2\int h_\delta\big(\mathfrak{u}_t\mathfrak{v}_t+\mathfrak{u}_t'\mathfrak{v}_t'\big)\;.
\end{aligned}\tag{6.16}$$

This leads to a bound of the form

$$\begin{aligned}\tfrac{1}{2}\partial_t J \;\le\; & -\tfrac{1}{2}\eta^{-2}\int h_\delta \eta^2\big(\mathfrak{v}_t^2+(\mathfrak{v}_t')^2\big)\\ & -\tfrac{1}{8}\gamma\int h_\delta\big(\mathfrak{u}_t^2+2(\mathfrak{u}_t')^2+(\mathfrak{u}_t'')^2\big)\\ & -\tfrac{1}{8}\eta^2\gamma^2\int h_\delta\big(\mathfrak{u}_t\mathfrak{v}_t+\mathfrak{u}_t'\mathfrak{v}_t'\big)\ .\end{aligned}$$

Taking the least good bounds above, we finally get the decay of the high frequency part (since $\eta^{-2}\ge\gamma$):

$$\begin{aligned}\tfrac{1}{2}\partial_t J \;\equiv\; & \tfrac{1}{2}\partial_t\int h_\delta\Big(\eta^2\big(\mathfrak{v}_t^2+(\mathfrak{v}_t')^2\big)+\big(\mathfrak{u}_t^2+2(\mathfrak{u}_t')^2+(\mathfrak{u}_t'')^2\big)+\eta^2\gamma\big(\mathfrak{u}_t\mathfrak{v}_t+\mathfrak{u}_t'\mathfrak{v}_t'\big)\Big)\\ \le\; & -\tfrac{1}{8}\gamma\int h_\delta\Big(\eta^2\big(\mathfrak{v}_t^2+(\mathfrak{v}_t')^2\big)+\big(\mathfrak{u}_t^2+2(\mathfrak{u}_t')^2+(\mathfrak{u}_t'')^2\big)+\eta^2\gamma\big(\mathfrak{u}_t\mathfrak{v}_t+\mathfrak{u}_t'\mathfrak{v}_t'\big)\Big)\\ =\; & -\tfrac{1}{8}\gamma J \;=\; -\tfrac{1}{2560}\min(\eta^{-2},k_*^2/C_{15})J\ .\end{aligned}\tag{6.17}$$

Thus we have shown the

Proposition 6.3. *There is a (small)* $\eta_0>0$ *such that for all* $\eta<\eta_0$ *the following holds for the functional*

$$J_{h_\delta,\gamma}(\mathfrak{u}_t,\mathfrak{v}_t)\;=\;\int h_\delta(x)\Big(\eta^2\big(\mathfrak{v}_t^2+(\mathfrak{v}_t')^2\big)+\mathfrak{u}_t^2+2(\mathfrak{u}_t')^2+(\mathfrak{u}_t'')^2+\eta^2\gamma\big(\mathfrak{u}_t\mathfrak{v}_t+\mathfrak{u}_t'\mathfrak{v}_t')\big)\Big)(x)\ .$$

Let $(\mathfrak{u}_t,\mathfrak{v}_t)=e^{\mathcal{L}t}(\mathfrak{u}_0,\mathfrak{v}_0)$, *and assume* $(\mathfrak{u}_0,\mathfrak{v}_0)\in\mathcal{K}_{k_*}\oplus\mathcal{K}_{k_*}$. *Then*

$$J_{h_\delta,\gamma}(\mathfrak{u}_t,\mathfrak{v}_t)\;\le\;\exp\big(-\gamma t/80\big)\cdot J_{h_\delta,\gamma}(\mathfrak{u}_0,\mathfrak{v}_0)\ ,\tag{6.18}$$

where

$$\gamma\;=\;\min(\eta^{-2},k_*^2/C_{15})/320\ .\tag{6.19}$$

We come now back to the definition (6.18) of J, and compare it to the norm $\|\cdot\|_{h_\delta,2}$ defined in Eq.(3.6). These two quantities define equivalent topologies when considered on $\mathcal{K}_{k_*}\oplus\mathcal{K}_{k_*}$.

Lemma 6.4. *On* $\mathcal{K}_{k_*}\oplus\mathcal{K}_{k_*}$ *one has the bound*

$$\eta^2\gamma\left|\int h_\delta\big(\mathfrak{u}\mathfrak{v}+\mathfrak{u}'\mathfrak{v}'\big)\right|\;\le\;\int h_\delta\Big(\tfrac{1}{2}\big((\mathfrak{u}')^2+(\mathfrak{u}'')^2\big)+\tfrac{1}{8}\big(\mathfrak{v}^2+(\mathfrak{v}')^2\big)\Big)\ .$$

Remark. This lemma eliminates the somewhat arbitrary quantity γ from the topology, see Theorem 6.5 below.

Proof. This is a combination of earlier estimates. Indeed, we have already seen in Eq.(6.12) that the mixed terms in Eq.(6.7) can be bounded by

$$\eta^2\gamma\left|\int h_\delta\big(\mathfrak{u}\mathfrak{v}+\mathfrak{u}'\mathfrak{v}'\big)\right| \le \eta^2\int h_\delta\Big(8\gamma^2\big(\mathfrak{u}^2+(\mathfrak{u}')^2\big)+\tfrac{1}{8}\big(\mathfrak{v}^2+(\mathfrak{v}')^2\big)\Big) \equiv X\ .$$

Furthermore, by (6.15) and the choice of k_*, we get

$$X \le \eta^2 C_{15}k_*^{-2}\int h_\delta\Big(8\gamma^2\big((\mathfrak{u}')^2+(\mathfrak{u}'')^2\big)+\tfrac{1}{8}\big(\mathfrak{v}^2+(\mathfrak{v}')^2\big)\Big)\ .$$

Since we have also chosen $\gamma = \min(\eta^{-2}, k_*^2/C_{15})/320$, we get finally

$$X \le \int h_\delta\Big(\tfrac{1}{2}\big((\mathfrak{u}')^2+(\mathfrak{u}'')^2\big)+\tfrac{1}{8}\big(\mathfrak{v}^2+(\mathfrak{v}')^2\big)\Big)\ ,$$

which is what we asserted.

$\square$

Recall the definition (6.2) of the projection $\mathcal{P}_{k_*}$ onto momenta larger than k_*. From Lemma 6.4 and Proposition 6.3 we have immediately, with the notation of (3.6) and (6.4) in the topology of $\mathcal{H}^2_{\delta,\mathrm{loc}}$ (which does not depend on δ_0).

Theorem 6.5. *Assume η_0 and k_* satisfy (6.8) and (6.9), and assume $\delta \le 1/(40C_{18})$. For all η satisfying $0<\eta<\eta_0$ the following holds: If $\|(\mathfrak{u}_0,\mathfrak{v}_0)\|_{h_\delta,2}<\infty$ and $(\mathfrak{u}_t,\mathfrak{v}_t)=e^{\mathcal{L}t}(\mathfrak{u}_0,\mathfrak{v}_0)$ then one has the bounds*

$$J_{h_\delta,\gamma}(\mathcal{P}_{k_*}\mathfrak{u}_t,\mathcal{P}_{k_*}\mathfrak{v}_t)/2 \le \|(\mathcal{P}_{k_*}\mathfrak{u}_t,\mathcal{P}_{k_*}\mathfrak{v}_t)\|^2_{h_\delta,2} \le 2J_{h_\delta,\gamma}(\mathcal{P}_{k_*}\mathfrak{u}_t,\mathcal{P}_{k_*}\mathfrak{v}_t)\ , \tag{6.20}$$

and

$$\|(\mathcal{P}_{k_*}\mathfrak{u}_t,\mathcal{P}_{k_*}\mathfrak{v}_t)\|_{h_\delta,2} \le 4\exp(-\gamma t/80)\cdot\|(\mathcal{P}_{k_*}\mathfrak{u}_0,\mathcal{P}_{k_*}\mathfrak{v}_0)\|_{h_\delta,2}\ , \tag{6.21}$$

where $\gamma=\min(\eta^{-2},k_*^2C_{15})/320$.

7. The evolution of differences

In this section, we combine the results of Sections 3 and 6 into bounds on the evolution of the difference of solutions to (2.1). We will first treat the general case, and show a bound which diverges exponentially with time, and then we will treat the high frequency case where we have decay. We consider two initial conditions, and their respective evolutions under the semi-flow defined by (2.1). We call these functions (u_1, v_1) and (u_2, v_2), respectively. The evolution for the difference $(\mathfrak{u}, \mathfrak{v}) = (u_1 - u_2, v_1 - v_2)$ takes now the form

$$\begin{aligned} \dot{\mathfrak{u}} &= \mathfrak{v} \;, \\ \eta^2 \dot{\mathfrak{v}} &= -\mathfrak{v} + \mathfrak{u}'' + M(u_1, u_2)\mathfrak{u} \;, \end{aligned} \tag{7.1}$$

where $M(u_1, u_2)$ is defined by $M(u_1, u_2)(u_1 - u_2) = U'(u_1) - U'(u_2)$. It will be useful to rewrite this system of equations as

$$\begin{pmatrix} \dot{\mathfrak{u}}_t \\ \dot{\mathfrak{v}}_t \end{pmatrix} = \mathfrak{L} \begin{pmatrix} \mathfrak{u}_t \\ \mathfrak{v}_t \end{pmatrix} + \begin{pmatrix} 0 \\ M_{u_{1,t}, u_{2,t}} \mathfrak{u}_t \end{pmatrix} . \tag{7.2}$$

Note that $M(u_1, u_2)$ is really a space-time dependent coefficient of the linear problem (3.2). The important observation is now that on the attracting set, *i.e.*, for all sufficiently large t (depending on the initial conditions u_1, u_2, v_1, v_2) we have, by Theorem 2.6, a universal bound

$$\sup_{x \in \mathbf{R}} |M(u_{1,t}(x), u_{2,t}(x))| + \sup_{x \in \mathbf{R}} |\partial_x M(u_{1,t}(x), u_{2,t}(x))| \le M_* \,. \tag{7.3}$$

Since we already know bounds on the solution, we can write it as follows for $\mathfrak{u}_t(x) = u(x,t)$ and $\mathfrak{v}_t(x) = v(x,t)$:

$$\begin{pmatrix} \mathfrak{u}_t \\ \mathfrak{v}_t \end{pmatrix} = e^{t\mathfrak{L}} \begin{pmatrix} \mathfrak{u}_0 \\ \mathfrak{v}_0 \end{pmatrix} + \int_0^t \mathrm{d}s \, e^{(t-s)\mathfrak{L}} \begin{pmatrix} 0 \\ M_{u_{1,s}, u_{2,s}} \mathfrak{u}_s \end{pmatrix} . \tag{7.4}$$

Proposition 7.1. *Assume* $(u_{1,0}, v_{1,0})$ *and* $(u_{2,0}, v_{2,0})$ *are in* $\mathcal{G}$. *Let* $\mathfrak{u}_t = u_{1,t} - u_{2,t}$ *and let* $\mathfrak{v}_t = v_{1,t} - v_{2,t}$. *There are constants* C_{19} *and* C_{20} *such that for all* $t > 0$,

$$\|(\mathfrak{u}_t, \mathfrak{v}_t)\|_{h_\delta, 2} \le C_{19} e^{C_{20} t} \|(\mathfrak{u}_0, \mathfrak{v}_0)\|_{h_\delta, 2} \,. \tag{7.5}$$

Proof. We have already seen in (7.3) that $|M(u_{1,t}, u_{2,t})|$ and its derivative are bounded and then the result follows at once from the representation (7.4) and the bound of Theorem 3.3.

□

The handling of the high frequency part $\mathcal{P}_{k_*}(\mathfrak{u}_t, \mathfrak{v}_t)$ is similar. Instead of (7.2), we get

$$\begin{pmatrix} \partial_t \mathcal{P}_{k_*} \mathfrak{u}_t \\ \partial_t \mathcal{P}_{k_*} \mathfrak{v}_t \end{pmatrix} = \mathcal{L} \begin{pmatrix} \mathcal{P}_{k_*} \mathfrak{u}_t \\ \mathcal{P}_{k_*} \mathfrak{v}_t \end{pmatrix} + \begin{pmatrix} 0 \\ \mathcal{P}_{k_*} M_{u_{1,t}, u_{2,t}} \mathfrak{u}_t \end{pmatrix} . \tag{7.6}$$

The solution of this problem is

$$\begin{pmatrix} \mathcal{P}_{k_*} \mathfrak{u}_t \\ \mathcal{P}_{k_*} \mathfrak{v}_t \end{pmatrix} = e^{t\mathcal{L}} \begin{pmatrix} \mathcal{P}_{k_*} \mathfrak{u}_0 \\ \mathcal{P}_{k_*} \mathfrak{v}_0 \end{pmatrix} + \int_0^t \mathrm{d}s \, e^{(t-s)\mathcal{L}} \begin{pmatrix} 0 \\ \mathcal{P}_{k_*} M_{u_{1,s}, u_{2,s}} \mathfrak{u}_s \end{pmatrix} . \tag{7.7}$$

What is important here is that in *both* terms the operator $\mathcal{L}$ acts on functions with *high* frequencies.

Proposition 7.2. *Assume $(u_{1,0}, v_{1,0})$ and $(u_{2,0}, v_{2,0})$ are in $\mathcal{G}$. Let $\mathfrak{u}_t = u_{1,t} - u_{2,t}$ and let $\mathfrak{v}_t = v_{1,t} - v_{2,t}$. There are constants C_{21}, C_{22}, and C_{23} such that for all $t > 0$,*

$$\|(\mathcal{P}_{k_*} \mathfrak{u}_t, \mathcal{P}_{k_*} \mathfrak{v}_t)\|_{h_\delta,2} \le \left(C_{21} e^{-\gamma t/80} + C_{22} \frac{e^{C_{23} t}}{\gamma} \right) \|(\mathfrak{u}_0, \mathfrak{v}_0)\|_{h_\delta,2} , \tag{7.8}$$

where $\gamma = \min(\eta^{-2}, k_*^2/C_{15})/320$.

Remark. In fact, one can choose $C_{23} = C_{20}$.

Proof. We use again (7.3) to bound M and $\partial_x M$. Furthermore, $\mathcal{P}_{k_*}$ is bounded and then the result follows at once from the representation (7.7) and the bound (6.21) of Theorem 6.5 for the first term of (7.8) and additionally the bound (7.5) of Proposition 7.1 for the second.

□

8. Covering the attracting set

We define a new norm by

$$\|(u, v)\|_{\delta,L,2} = \sup_{\xi \in [-L,L]} \|(u, v)\|_{h_{\delta,\xi},2} , \tag{8.1}$$

where

$$h_{\delta,\xi}(x) = \frac{1}{(1 + \delta^2 (x - \xi)^2)^2} ,$$

and $\|(u, v)\|_{h_\delta,2}$ was defined in (3.6). This norm looks at a "window" of size $2L$ in $\mathcal{H}^2_{\delta,\mathrm{loc}}$. For $\varepsilon > 0$ we define $N_L(\varepsilon)$ as the minimum number of balls of radius ε (in the norm $\| \cdot \|_{\delta,L,2}$), needed to cover the attracting set $\mathcal{G}$.

Theorem 8.1. *There exist finite constants A, and C_{24} such that for all ε with $0 < \varepsilon < 1$ and all $L > A/\varepsilon$ one has*

$$N_{L-A/\varepsilon}(\varepsilon) \le C_{24}^L N_L(2\varepsilon) . \tag{8.2}$$

Proof. We denote $t \mapsto \Phi^t$ the flow defined by (2.1). Let $\mathcal{B}$ be a finite collection of balls of radius ε in the topology defined by $\|\cdot\|_{\delta,L,2}$ which cover $\mathcal{G}$.

We next define a natural *unit of time*, τ_*. We recall the definition (6.19) of γ: $\gamma = \min(\eta^{-2}, k_*^2/C_{15})/320$. We define

$$\tau_* = \frac{b}{\gamma} \log \gamma , \tag{8.3}$$

where the (small positive) constant b is chosen such that the factor in (7.8) is minimal and when γ is large (say, $\gamma > \gamma_0$), we get

$$C_{21} e^{-\gamma\tau_*/80} + C_{22} \frac{e^{C_{23}\tau_*}}{\gamma} \le \gamma^{-\varkappa} , \tag{8.4}$$

for some $\varkappa > 0$. We will use this bound in the sequel.

Since the flow Φ^t leaves $\mathcal{G}$ invariant, we see that

$$\Phi^{t+\tau}(\mathcal{G}) \subset \bigcup_{B\in\mathcal{B}} \Phi^\tau\big(B \cap \Phi^t(\mathcal{G})\big) = \bigcup_{B\in\mathcal{B}} \Phi^\tau\big(B\big) .$$

Consider now any of the B in $\mathcal{B}$. We are going to cover $\Phi^{\tau_*}(B)$ by balls of radius $\varepsilon/2$. Let φ_0 and ψ_0 be two elements of the ball B and assume furthermore φ_0 and ψ_0 are in the attracting set $\mathcal{G}$. This implies

$$\|\varphi_0 - \psi_0\|_{\delta,L,2} \le \varepsilon , \tag{8.5}$$

and, since φ_0 and ψ_0 are in the global attracting set $\mathcal{G}$, we also have

$$\|\varphi_0 - \psi_0\|_{h_\delta,2} \le C_{25} , \tag{8.6}$$

for some constant C_{25}. With τ_* as in (8.3), we let

$$\varphi = \Phi^{\tau_*}(\varphi_0) , \quad \psi = \Phi^{\tau_*}(\psi_0) .$$

We then rewrite $\varphi - \psi$ as

$$\varphi - \psi = \mathcal{P}_{k_*}(\varphi - \psi) + \mathcal{Q}_{k_*}(\varphi - \psi) , \tag{8.7}$$

where (the direct sums of) $\mathcal{P}_{k_*}$ and $\mathcal{Q}_{k_*}$ are the high- and low-momentum projections introduced earlier (in (6.1) and (6.2)). Our aim is to bound this difference in the norm $\|\cdot\|_{\delta,L-A/\varepsilon,2}$, where A is a large constant to be determined later.

We begin with $\mathcal{P}_{k_*}(\varphi-\psi)$. By our choice of τ_* in (8.3), we have, by (8.4) and Proposition 7.2,

$$\big\|\mathcal{P}_{k_*}(\varphi - \psi)\big\|_{\delta,L-A/\varepsilon,2} \le \big\|\mathcal{P}_{k_*}\big(\Phi^{\tau_*}(\varphi_0) - \Phi^{\tau_*}(\psi_0)\big)\big\|_{\delta,L,2} \le \gamma^{-\varkappa}\varepsilon .$$

We now fix $\eta_0 > 0$ so small and k_0 so large (and at least satisfying Eqs.(6.8) and (6.9)) such that for all $\eta < \eta_0$ and all $k_* > k_0$ one has

$$\gamma^{-\varkappa} = \big(\min(\eta^{-2}, k_*^2/C_{15})/320\big)^{-\varkappa} \le \frac{1}{8}\,, \tag{8.8}$$

and also $\gamma > \gamma_0$, see (8.4). Summarizing the bounds for this piece, we get

$$\|\mathcal{P}_{k_*}(\varphi - \psi)\|_{\delta,L-A/\varepsilon,2} \le \frac{\varepsilon}{8}\,. \tag{8.9}$$

We bound next $\mathcal{Q}_{k_*}(\varphi-\psi)$ by decomposing it into a finite sum plus a remainder. We will work with the two components of $\mathcal{Q}_{k_*}\varphi$ or $\mathcal{Q}_{k_*}\psi$ separately. Since the norm on the first component has 2 derivatives and the norm on the second only 1, we will deal only with the first case and leave the other case to the reader.

We will work with the notion of Bernstein class $\mathcal{B}_\sigma(K)$, defined by

$$\mathcal{B}_\sigma(K) = \left\{ h \;\middle|\; |h(x+iy)| \le K e^{\sigma|y|} \text{ for all } x, y \in \mathbf{R} \right\}.$$

If $h \in \mathcal{B}_\sigma(K)$, it can be represented by the Cartwright interpolation formula [KT, Eq. (191)] (or [B] for a proof) with $\sigma' = \pi/2$ and $\omega = \pi/4$) as

$$h(x) = \frac{\sin(2\sigma x)}{4} \sum_{j=-\infty}^{\infty} (-1)^j \frac{\sin(\frac{\sigma x}{2} - \frac{\pi j}{4})}{(\frac{\sigma x}{2} - \frac{\pi j}{4})^2} h(x_j)\,, \tag{8.10}$$

where the $x_j = \frac{j\pi}{2\sigma}$ are discrete sampling points. This class is useful in our context because of

Lemma 8.2. *There is a constant C_{26} such that if $u \in \mathrm{L}^\infty$, then*

$$\mathcal{Q}_{k_*} u \in \mathcal{B}_{2k_*}(C_{26} k_* \|u\|_\infty)\,. \tag{8.11}$$

Proof. This amounts to saying that a function with frequency support in $[-2k_*, 2k_*]$ is in the Bernstein class. This is almost obvious, except for the smooth cutoff. In fact, with the function ϑ as defined in Section 5, we consider

$$\int \mathrm{d}k\, e^{ik(x+iy)} \vartheta(k/k_*) = k_* \int \mathrm{d}\ell\, e^{ik_*\ell(x+iy)} \vartheta(\ell)\,, \tag{8.12}$$

which is in $\mathrm{L}^1(\,\mathrm{d}x\,)$ for any $y \in \mathbf{R}$. And the $\mathrm{L}^1(\,\mathrm{d}x\,)$ norm is bounded by

$$\mathcal{O}(1)\big(k_* + k_*^{-1}\big) e^{2k_*|y|}\,.$$

Therefore, the convolution operator defined by (8.12) maps u to $\mathcal{B}_{2k_*}(\mathcal{O}(k_*)\|u\|_\infty)$.

□

We next bound the functions appearing in (8.10) in our favorite topology:

Lemma 8.3. *Let $\sigma > 2$ and let f_j be defined by*

$$f_j(x) \equiv \frac{\sin\bigl(4(\frac{\sigma x}{2} - \frac{\pi j}{4})\bigr)\sin(\frac{\sigma x}{2} - \frac{\pi j}{4})}{4(\frac{\sigma x}{2} - \frac{\pi j}{4})^2} = (-1)^j \frac{\sin\bigl(2\sigma x\bigr)\sin(\frac{\sigma x}{2} - \frac{\pi j}{4})}{4(\frac{\sigma x}{2} - \frac{\pi j}{4})^2} .$$

There is a constant C_{27} independent of j and ξ, such that for all j and ξ one has:

$$\int \mathrm{d}x\, h_\delta(x-\xi)\Bigl(f_j^2(x) + 2\bigl(f_j'(x)\bigr)^2 + \bigl(f_j''(x)\bigr)^2\Bigr) \le \frac{\sigma^4 C_{27}}{1+(2\sigma\xi - \pi j)^4} . \tag{8.13}$$

Remark. The numerical coefficient C_{27} depends on δ, but δ has been fixed in Eq.(6.11): $\delta = 1/(40C_{18})$.

Proof. The function f_j can be bounded as

$$|f_j(x)| \le \left|\frac{\sin\bigl(4(\frac{\sigma x}{2} - \frac{\pi j}{4})\bigr)\sin(\frac{\sigma x}{2} - \frac{\pi j}{4})}{4(\frac{\sigma x}{2} - \frac{\pi j}{4})^2}\right| \le \frac{C_{28}}{1+(\frac{\sigma x}{2} - \frac{\pi j}{4})^2} ,$$

since the numerator vanishes simultaneously with the denominator (and to order 2). Similarly, the derivative is bounded by

$$\left|\partial_x^\ell\left(\frac{\sin\bigl(4(\frac{\sigma x}{2} - \frac{\pi j}{4})\sin(\frac{\sigma x}{2} - \frac{\pi j}{4})}{4(\frac{\sigma x}{2} - \frac{\pi j}{4})^2}\right)\right| \le \frac{C_{29}(\sigma/2)^\ell}{1+4(\frac{\sigma x}{2} - \frac{\pi j}{4})^2} , \quad \ell = 1, 2 , \tag{8.14}$$

since $\sigma > 2$ by assumption. It follows that

$$\int \mathrm{d}x\, h_{\delta,\xi}(x)|f_j(x)|^2 \le C_{30}\int \mathrm{d}x\, \frac{1}{(1+\delta^2(x-\xi)^2)^2} \cdot \frac{1}{(1+(2\sigma x - \pi j)^2)^2} .$$

Setting $\rho = \min(\delta, 2\sigma)$, we find that this is bounded by

$$\int \mathrm{d}x\, h_{\delta,\xi}(x)|f_j(x)|^2 \le \frac{C_{31}}{\rho}\frac{1}{\bigl(1+\rho^2(\xi - \frac{\pi j}{2\sigma})^2\bigr)^2} .$$

In view of (8.14) one gets a similar bound for the derivatives, and thus (8.13) follows.

□

Consider the element $(u, v) \in \mathcal{G}$. We know that $\|(u, v)\|_{\delta,\mathrm{loc},2} \le C_9$. For the first component, u, this means

$$\sup_{\xi\in\mathbf{R}} \int \mathrm{d}x\, h_\delta(x-\xi)\Bigl(|u(x)|^2 + 2|u'(x)|^2 + |u''(x)|^2\Bigr) \le C_9^2 .$$

From this, we conclude using the Sobolev inequality in the form of Lemma 2.5 that $\|u\|_\infty \le C_{32}$ for some finite C_{32}. By Lemma 8.2 we then get that $\|\mathcal{Q}_{k_*}u\|_\infty \le C_{26}k_*C_{32}$ and furthermore, $\mathcal{Q}_{k_*}u \in \mathcal{B}_{2k_*}(C_{26}k_*C_{32})$. Thus, we can apply the Cartwright formula to $h = \mathcal{Q}_{k_*}u$, with $\sigma = 2k_*$.

Throughout, Lk_* has to be sufficiently large. We define

$$\mathcal{S}_L(h) \;=\; \frac{\sin(4k_*x)}{4} \sum_{|j|\le 2Lk_*} (-1)^j \frac{\sin(k_*x - \frac{\pi j}{4})}{(k_*x - \frac{\pi j}{4})^2} h(x_j) \;, \tag{8.15}$$

where $x_j = \frac{j\pi}{4k_*}$ are the discrete sampling points. We decompose

$$\mathcal{Q}_{k_*}u \;=\; \Big(h - \mathcal{S}_L(h)\Big) + \mathcal{S}_L\big(h\big) \;. \tag{8.16}$$

The first term in (8.16) will be small because $h - \mathcal{S}_L(h)$ is the remainder of the converging sum in (8.10), and for the second one we will use a covering argument.

We first show that $X_L \equiv h - \mathcal{S}_L(h)$ is small when L is large. The difference can be written as

$$\big(h - \mathcal{S}_L(h)\big)(x) \;=\; \sum_{|j|>2Lk_*} (-1)^j \frac{\sin(4k_*x)\sin(k_*x - \frac{\pi j}{4})}{4(k_*x - \frac{\pi j}{4})^2} h(\frac{j\pi}{4k_*}) \;.$$

Using (8.13), we get as a bound for X_L *when* $|\xi| \le L$:

$$\begin{aligned}\Big(\int \mathrm{d}x\, h_\delta(x-\xi)\big(|X_L(x)|^2 + 2|X_L'(x)|^2 + |X_L''(x)|^2\big)\Big)^{1/2} \\ \le\; C_{27} \sum_{|j|\ge 2Lk_*} \frac{1}{1 + (k_*\xi - \frac{\pi j}{4})^2} \;\le\; \frac{C_{33}}{1 + \big|L - |\xi|\big|} \;.\end{aligned}$$

This argument can be repeated for the second component. Since in the definition (8.1) of $\|\cdot\|_{\delta,L-A/\varepsilon,2}$ we have $|\xi| \le L - A/\varepsilon$, we find a bound on the exterior part of $\mathcal{Q}_{k_*}(\varphi - \psi)$:

$$\Big\|\mathcal{Q}_{k_*}(\varphi-\psi) - \mathcal{S}_L\big(\mathcal{Q}_{k_*}(\varphi-\psi)\big)\Big\|_{\delta,L-A/\varepsilon,2} \;\le\; \frac{C_{34}\varepsilon}{A} \sup_{\xi\in\mathbf{R}} \|\varphi-\psi\|_{h_{\delta,\xi},2} \;\le\; \frac{C_{34}\varepsilon}{A} C_{25} \;, \tag{8.17}$$

using (8.6). Clearly, if A is sufficiently large (but independent of ε and L), we get the bound

$$\Big\|\mathcal{Q}_{k_*}(\varphi - \psi) - \mathcal{S}_L\big(\mathcal{Q}_{k_*}(\varphi - \psi)\big)\Big\|_{\delta,L-A/\varepsilon,2} \;\le\; \frac{\varepsilon}{8} \;. \tag{8.18}$$

We finally deal with the central part, namely $\mathcal{S}_L(\mathcal{Q}_{k_*}(\varphi - \psi))$. This is described in

Proposition 8.4. *There is a constant C_{35} such that the following holds. Let B be a ball of radius ε in the topology defined by $\|\cdot\|_{\delta,L,2}$. Then the set $\mathcal{S}_L(B \cap \mathcal{G})$ can be covered by no more than*

$$C_{35}^L$$

balls of radius $\varepsilon/8$.

Proof. Since $\varphi, \psi \in \mathcal{G}$, Lemma 8.2 implies $\mathcal{Q}_{k_*}(\varphi-\psi) \in \mathcal{B}_{2k_*}(X)$, where

$$X \;=\; \mathrm{diam}_{\mathrm{L}^\infty}(\mathcal{G}) C_{26} k_* \;.$$

Moreover, from Corollary 4.2 we deduce

$$\|\mathcal{Q}_{k_*}(\varphi-\psi)\|_{\delta,L,2} \;\le\; C_{36}\varepsilon \;.$$

Using the Sobolev inequality from Lemma 2.5, this implies

$$\sup_{x\in[-L,L]} \big|\big(\mathcal{Q}_{k_*}(\varphi-\psi)\big)(x)\big| \;\le\; C_{37}\varepsilon \;.$$

We use next the bounds

$$\big|\big(\mathcal{Q}_{k_*}(\varphi-\psi)\big)(x_j)\big| \;\le\; C_{37}\varepsilon \;,$$

for $|j| < 2Lk_*$. We let n be a large integer which will be fixed at the end of the proof. The set of values of each of the 2 components of $\big(\mathcal{Q}_{k_*}(\varphi-\psi)\big)(x_j)$ can be covered by $8nC_{37}$ balls of radius $\varepsilon/(4n)$, for each of the $2(2Lk_*)+1$ possible values of j. We bound now in detail the sum in $\mathcal{S}_L\big(\mathcal{Q}_{k_*}(\varphi-\psi)\big)$ as defined in (8.15).

We fix one of the $(8nC_{37})^{4(2Lk_*)+1}$ grid points for the components of $\big(\mathcal{Q}_{k_*}(\varphi-\psi)\big)(x_j)$. For each component, we get a set of $2(2Lk_*)+1$ numbers q_ℓ, with $|\ell| \le 2Lk_*$. We pick numbers r_ℓ satisfying $|r_\ell - q_\ell| < \varepsilon/(4n)$ for all ℓ and we want to show that the function

$$\Delta(x) \;=\; \frac{\sin(4k_*x)}{4} \sum_{|j|\le 2Lk_*} (-1)^j \frac{\sin(k_*x - \frac{\pi j}{4})}{(k_*x - \frac{\pi j}{4})^2}(r_j - q_j)$$

has a $\|\cdot\|_{\delta,L,2}$ norm less than $\varepsilon/8$. This will clearly suffice to show Proposition 8.4. By Lemma 8.3, we get

$$\|\Delta\|_{\delta,L,2} \;\le\; C_{38} \sup_{|\xi|\le L} \sum_{|j|\le 2Lk_*} \frac{1}{1+(4k_*\xi - \pi j)^2}\frac{\varepsilon}{4n} \;\le\; C_{39}\frac{\varepsilon}{n} \;.$$

We choose $n = 8C_{39}$, and we see that, all in all, one needs $(8nC_{37})^{4(2Lk_*)+1} \le C_{35}^L$ balls of radius $\varepsilon/8$ to cover $\mathcal{S}_L(B\cap\mathcal{G})$. (Note that C_{38} and C_{39} depend on k_*. In fact they are bounded by $\mathcal{O}(k_*^4)$.)

$\square$

Proof of Theorem 8.1. We combine now the various estimates to prove (8.2). Let B be one of the $N_L(\varepsilon)$ balls of radius ε needed to cover $\mathcal{G}$ and let $f \in B\cap\mathcal{G}$. All we need to show is that the set of all $g \in B\cap\mathcal{G}$ can be covered by C_{24}^L balls of radius $\varepsilon/2$ in the topology of the norm $\|\cdot\|_{\delta,L-A/\varepsilon,2}$. We decompose $\varphi-\psi$ according to (8.7) and then $\mathcal{Q}_{k_*}(\varphi-\psi)$ according to (8.16), so that we have three terms. The first is bounded by $\varepsilon/8$ using (8.9) and the second is bounded by (8.18). Since $\mathcal{S}_L(B\cap\mathcal{G})$ can be covered by C_{35}^L balls of radius $\varepsilon/8$ in the norm $\|\cdot\|_{\delta,L,2}$ it can also be covered by the same number of balls in the norm $\|\cdot\|_{\delta,L-A/\varepsilon,2}$. Thus the sum of the three contributions can be covered by C_{35}^L balls of radius $3\varepsilon/8<\varepsilon/2$. The proof of Theorem 8.1 is complete.

□

9. The ε-entropy and the topological entropy

9.1. Introduction

In this section, we exploit the results obtained so far to show that the ε-entropy and the topological entropy per unit length can be defined for the Eq.(2.1). The reasoning here is very close to the one used in [CE2], and so there is no need to repeat it here. What needs however some special attention is the choice of topology for which the entropy per unit length can be defined. We basically need a topology which has a *submultiplicativity property* which we define below. The most simple example of such a topology was used in [CE2], namely L^∞. The property which we used there is that if a set S of functions is defined on the union of 2 adjacent intervals, say $I_1\cup I_2$, then the following is true: If S restricted to I_1 can be covered by N_{I_1} balls of radius ε in $\mathrm{L}^\infty(I_1)$, and $S|_{I_2}$ can be covered by N_{I_2} balls in $\mathrm{L}^\infty(I_2)$, then $S|_{I_1\cup I_2}$ can be covered by $N_{I_1}\cdot N_{I_2}$ balls in $\mathrm{L}^\infty(I_1\cup I_2)$ (all of radius ε). In L^∞, this property is obvious: Let $B_{1,i}$, with $i=1,\dots,N_{I_1}$ be the balls covering $S|_{I_1}$ and $B_{2,j}$, with $j=1,\dots,N_{I_2}$ those covering $S|_{I_2}$. Then one can just take the set $S_{i,j}$ of functions

$$S_{i,j}=\left\{f\;\middle|\;f|_{I_1}\in B_{1,i}\text{ and }f|_{I_2}\in B_{j,2}\right\},$$

and this *is* a ball of radius ε in $\mathrm{L}^\infty(I_1\cup I_2)$.

The difficulty with topologies which are finer than L^∞ is that we have to patch the functions on I_1 and I_2 together in such a way that the patched function is an element of a ball in the topology on $I_1\cup I_2$. We do not know how to do this in the topologies used in the earlier sections, and therefore we go to a new topology in which the submultiplicativity property holds in the sense that there is a constant $C=C(\varepsilon)$ *independent of* I_1 *and* I_2 such that the functions on the union of I_1 and I_2 can be covered by

$$N_{I_1}(\varepsilon)\cdot N_{I_2}(\varepsilon)\cdot C(\varepsilon) \tag{9.1}$$

balls of radius ε. It is well known from the literature on statistical mechanics (see *e.g.*, Ruelle [R]) and easy to see that this weaker form of submultiplicativity suffices to prove the existence of limits (of the logarithms) per unit length.

The topology we will use is $W^{1,\infty}$, defined by

$$\|f\|_{W^{1,\infty}} \equiv \max\bigl(\sup_{x\in\mathbf{R}}|f(x)|,\ \sup_{y\in\mathbf{R}}|f'(y)|\bigr)\ . \tag{9.2}$$

This is a "good" topology for our problem, because we can control the evolution of functions in $W^{1,\infty}$. However, it is obvious that the submultiplicativity property is not immediate, since the matching of functions has to be continuous and once differentiable.

9.2. Submultiplicativity in $W^{1,\infty}$

We develop here the estimates leading to Eq.(9.1) for balls in $W^{1,\infty}$. Our main result will be Corollary 9.2. We let $R > 5$ be a large constant which will be determined in Eq.(9.8) below.

Notation. Throughout, we will use the notation

$$|g|_I = \sup_{x\in I}|g(x)|\ .$$

We let $W_I^{1,\infty}$ be the space of continuously differentiable functions $g : I \to \mathbf{R}$, equipped with the norm

$$\|g\|_I = \max\bigl(|g|_I, |g'|_I\bigr)\ .$$

(Thus, comparing with (9.2) we have $\|g\|_{W^{1,\infty}} = \|g\|_{\mathbf{R}}$.) Assume $g_{\mathrm{L}} \in W^{1,\infty}_{[-R,0]}$ and $g_{\mathrm{R}} \in W^{1,\infty}_{[0,R]}$ and let

$$\begin{aligned} E_{\varepsilon,G,g_{\mathrm{L}},g_{\mathrm{R}}} = \bigl\{u \in \mathcal{C}^2([-R,R]) \ :\ & |u''|_{[-R,R]} \le G\ , \\ & \|u-g_{\mathrm{L}}\|_{[-R,0]} \le \varepsilon\ , \|u-g_{\mathrm{R}}\|_{[0,R]} \le \varepsilon\bigr\}\ . \end{aligned}$$

Theorem 9.1. *There are a K (depending only on ε and G), and functions $g_1, \dots, g_N \in W^{1,\infty}_{[-R,R]}$ satisfying*

$$g_i(-R) = g_{\mathrm{L}}(-R)\ ,\quad g_i(R) = g_{\mathrm{R}}(R)\ , \tag{9.3}$$

for $i = 1, \dots, N$, such that the following holds: For every $u \in E_{\varepsilon,G,g_{\mathrm{L}},g_{\mathrm{R}}}$, there is a $j = j(u) \in \{1, \dots, N\}$ such that

$$\|u-g_j\|_{[-R,R]} \le \varepsilon\ .$$

Definition. We say a set $\{g_1, \dots, g_K\}$ of functions $g_i \in W^{1,\infty}$ *ε-covers a set $\mathcal{F}$ of $W^{1,\infty}$ functions on I* if for every $g \in \mathcal{F}$ there is a $k \in \{1, \dots, K\}$ for which

$$|g-g_k|_I \le \varepsilon\ ,\quad \text{and}\quad |g'-g_k'|_I \le \varepsilon\ .$$

Corollary 9.2. *Assume that a collection $\mathcal{F}$ of $\mathcal{C}^2$ functions is given on $[-L,L']$ and assume that each $f\in\mathcal{F}$ satisfies $|f|_{[-L,L']}\le\alpha$, $|f'|_{[-L,L']}\le\beta$, and $|f''|_{[-L,L']}\le\gamma$. There are constants R, $\varepsilon_0>0$ and a family of constants K_ε (depending only on α, β, and γ) such that the following holds for any $L,L'>R$ and any $\varepsilon\le\varepsilon_0$: If $\mathcal{F}|_{[-L,0]}$ and $\mathcal{F}|_{[0,L']}$ can be ε-covered by S, (resp. S') functions in $W^{1,\infty}_{[-L,0]}$ and $W^{1,\infty}_{[0,L']}$ respectively, then $\mathcal{F}|_{[-L,L']}$ can be ε-covered by no more than $S\cdot S'\cdot K_\varepsilon$ functions in $W^{1,\infty}_{[-L,L']}$.*

Proof of Theorem 9.1. We will first find finite constants a, b, c (>1) with the following property: Fix $g_{\rm L}$ and $g_{\rm R}$ and assume $E\equiv E_{\varepsilon,G,g_{\rm L},g_{\rm R}}\ne\emptyset$ (that is, there is a connecting function in an ε-neighborhood of $g_{\rm L}$ and $g_{\rm R}$). We claim one can construct a $W^{2,\infty}$ function g for which the following inequalities hold:

$$\|g-g_{\rm L}\|_{[-R,0]}\;\le\;a\varepsilon\;,\quad \|g-g_{\rm R}\|_{[0,R]}\;\le\;b\varepsilon\;,\quad |g''|_{[-R,R]}\;\le\;c+G\;. \tag{9.4}$$

Furthermore, g will satisfy

$$g(-R)\;=\;g_{\rm L}(-R)\;,\quad g(R)\;=\;g_{\rm R}(R)\;. \tag{9.5}$$

In other words, this is in principle a good approximation, which in addition matches *exactly* at the boundary, but the bound has deteriorated to $a\varepsilon$ and $b\varepsilon$ and a and b might be larger than 1. The point of Theorem 9.1 and Corollary 9.2 is that a (and b) can be pushed down to 1 by increasing the number of connecting functions to a number of functions which *does not* depend on $g_{\rm L}$ and $g_{\rm R}$.

Fix an arbitrary function $u_0\in E$. We construct a function g which interpolates between $g_{\rm L}$ and $g_{\rm R}$, using u_0 as a bridge. Let ψ be a $\mathcal{C}^\infty$ function, $0\le\psi(x)\le1$ satisfying $\psi(x)=0$ for $x<R-3$ and $\psi(x)=1$ for $x\ge R$. We define g by

$$g(x)\;=\;u_0(x)-\psi(x)\cdot\bigl(u_0(R)-g_{\rm R}(R)\bigr)-\psi(-x)\cdot\bigl(u_0(-R)-g_{\rm L}(-R)\bigr)\;. \tag{9.6}$$

This function is clearly continuously differentiable since u_0 is continuously differentiable. Let $I=[0,R]$. From (9.6) we find for $x\in I$:

$$g(x)-g_{\rm R}(x)\;=\;\bigl(u_0(x)-g_{\rm R}(x)\bigr)-\psi(x)\cdot\bigl(u_0(R)-g_{\rm R}(R)\bigr)\;,$$

and therefore,

$$\begin{aligned}|g-g_{\rm R}|_I\;&\le\;|u_0-g_{\rm R}|_I+|u_0-g_{\rm R}|_I\cdot|\psi|_I\;\le\;2\varepsilon\;,\\ |g'-g'_{\rm R}|_I\;&\le\;|u'_0-g'_{\rm R}|_I+|u_0-g_{\rm R}|_I\cdot|\psi'|_I\;\le\;\varepsilon\bigl(1+|\psi'|_I\bigr)\;.\end{aligned}$$

The negative x are handled in the same way. Finally, the last inequality of Eq.(9.4) follows at once from Eq.(9.6). We note that by the construction in Eq.(9.6), the boundary condition (9.5) is fulfilled.

Definition. We denote by $F_{\varepsilon,A,B,C}$ the set of $\mathcal{C}^2$ functions defined by

$$F_{\varepsilon,A,B,C} \;=\; \{f \;:\; |f(\pm R)| \le \varepsilon \,,\, |f|_{[-R,R]} \le A \,,\, |f'|_{[-R,R]} \le B \,,\, |f''|_{[-R,R]} \le C\} \,.$$

We shall need later the sets

$$F^0_{\delta,A,B,C,\xi} \;=\; \{f \;:\; |f(0)| \le \delta \,,\;\; |f|_{[0,\xi]} \le A \,,\, |f'|_{[0,\xi]} \le B \,,\, |f''|_{[0,\xi]} \le C\} \,.$$

Let $u \in E_{\varepsilon,G,g_{\rm L},g_{\rm R}}$ and define g as in Eq.(9.6). If we let $f = u - g$, then by Eq.(9.4), and the construction of g, we see that $f \in F_{\varepsilon,A,B,C}$, with $A = B = (a+b)\varepsilon$ and $C = c + G$.

We will now use the following bound on $F_{\varepsilon,A,B,C}$.

Proposition 9.3. *Fix $0 < \varepsilon \le A$, $B \ge 0$, and $C \ge 1$. There is a* **finite** *set H of $W^{1,\infty}$ functions $H = \{h_1, \dots, h_N\}$, which ε-covers $F_{\varepsilon,A,B,C}$ on $[-R,R]$ and which furthermore satisfies*

$$h_j(\pm R) \;=\; 0 \,, \tag{9.7}$$

for $j = 1, \dots, N$.

Using Proposition 9.3 we can complete the proof of Theorem 9.1. Given $g_{\rm L}$ and $g_{\rm R}$ as above, we construct first a g as in Eq.(9.6). When $u \in E$, then $f = u - g$ is in $F_{\varepsilon,A,B,C}$ by the bounds Eq.(9.4) and the equality (9.5). Thus, by Proposition 9.3 the f are ε-covered by the N functions $\{h_1, \dots, h_N\}$. Define now $u_i = h_i + g$, $i = 1, \dots, N$, and then the set $E_{\varepsilon,G,g_{\rm L},g_{\rm R}}$ of functions u is ε-covered by the u_i, since

$$u - u_i \;=\; (u - g) - (u_i - g) \;=\; f - h_i \,,$$

and we have just stated that the f are ε-covered by a finite number of u_i. Furthermore, the h_i vanish at the boundary of $[-R,R]$. Thus, we have interpolated between $g_{\rm L}$ and $g_{\rm R}$, with N functions in $W^{1,\infty}$ which ε-cover the original set. The proof of Theorem 9.1 is complete.

□

The corollary then follows at once since the factor N does not depend on the choice of $g_{\rm L}$ and $g_{\rm R}$ (except that the bound is too pessimistic in case $E_{\varepsilon,G,g_{\rm L},g_{\rm R}}$ happens to be empty).

Remark. The difficulty in proving Proposition 9.3 lies in the fact that the h_j vanish at the endpoints while the functions f in $F_{\varepsilon,A,B,C}$ may be as large as ε near the boundary, $|f(\pm R)| = \varepsilon$, so there is no space near $\pm R$ with which just to construct an open cover.

The main ingredient to the proof of Proposition 9.3 is the following local lemma. Before we formulate it, we assume, without loss of generality, that $C > 1$. Since we are interested in small ε, we shall also assume $\varepsilon < 1$.

We introduce two fundamental scales ξ and τ in our analysis:

$$\xi \;=\; \frac{\varepsilon}{10C} \,, \quad \text{and} \quad \tau \;=\; \frac{\varepsilon}{10} \,.$$

We will first consider a (small) interval J whose left endpoint is the origin.

Definition of R. We can now fix R by setting it to

$$41 \geq R \geq 40 , \quad R = m_* \xi , \tag{9.8}$$

where m_* is an integer. This choice is only good for $\varepsilon \leq 10C$ and we leave the trivial modifications for arbitrary ε to the reader.

Lemma 9.4. *Let $J = [0, \xi]$. There is a finite set of linear functions of the form $g_j(x) = j\tau x$, which ε-covers $F^0_{\delta,A,B,C,\xi}$, for every $\delta \in [0, \varepsilon]$. One has in fact better bounds: there is for every $f \in F^0_{\delta,A,B,C,\xi}$ a j with $|j| \leq \frac{B}{\tau} + 2$ for which*

$$|f - g_j|_J \leq \max(\delta, \varepsilon^2 \nu) , \quad \text{and} \quad |f' - g'_j|_J \leq \varepsilon \frac{3}{10} ,$$

and furthermore, at the right endpoint, one has

$$|f(\xi) - g_j(\xi)| \leq \max\bigl(\delta - \mu\varepsilon^2, \varepsilon^2 \nu\bigr) , \tag{9.9}$$

where

$$\nu = \frac{1}{40C} , \quad \mu = \frac{1}{200C} . \tag{9.10}$$

Proof of Lemma 9.4. This is just a construction of the "right" j, followed by some verifications. Note first that if $f \in F^0_{\delta,A,B,C,\xi}$, then we have

$$\begin{aligned} f(x) &= f(0) + xf'(0) + x^2 v(x) , \\ f'(x) &= f'(0) + xw(x) , \end{aligned}$$

with $|v|_{[0,\xi]} \leq C/2$ and $|w|_{[0,\xi]} \leq C$. We will pursue the proof for the case when $f(0) \geq 0$, the other case is handled by symmetry. We define

$$j = \left[\frac{f'(0)}{\tau} + 2 \right] = \frac{f'(0)}{\tau} + 1 + \rho , \tag{9.11}$$

with $\rho \in (0, 1]$. Here $[x] = \inf_{n \in \mathbf{Z}, n \geq x} n$ is the integer part of x. Now set $g(x) = cx$, with $c = j\tau$:

$$c = f'(0) + \tau + \tau\rho .$$

Clearly, g equals one of the g_j of Lemma 9.4 if we take the finite set of j to contain $|j| \leq \frac{B}{\tau} + 2$. Next, we estimate the quality of the approximation. First we have

$$f'(x) - g'(x) = f'(x) - c = f'(0) + xw(x) - f'(0) - \tau - \tau\rho .$$

This leads, for $x \in [0, \xi]$, to

$$\begin{aligned} f'(x) - c &\le \quad C\xi - \tau \le C\xi \le \frac{\varepsilon}{10} , \\ f'(x) - c &\ge -C\xi - 2\tau \ge -\frac{\varepsilon}{10} - \frac{2\varepsilon}{10} . \end{aligned}$$

We conclude that

$$|f' - g_j'|_J \le \frac{3}{10}\varepsilon .$$

We consider next $f(x) - cx$. We find

$$f(x) - cx = f(0) + xf'(0) + x^2 v(x) - xf'(0) - \tau x - \rho\tau x .$$

This leads to the bounds

$$f(x) - cx \le f(0) + \frac{C}{2}x^2 - \tau x \le \delta + \frac{C}{2}x^2 - \tau x = \delta - \frac{\varepsilon}{10}x(1 - x\frac{C}{2} \cdot \frac{10}{\varepsilon}) , \tag{9.12}$$

$$f(x) - cx \ge f(0) - \frac{C}{2}x^2 - 2\tau x \ge -\frac{C}{2}x^2 - 2\tau x . \tag{9.13}$$

Since we consider only $x \in [0, \xi]$, we find that $1 - x\frac{10C}{2\varepsilon} \ge 1 - \frac{\varepsilon}{10C}\frac{10C}{2\varepsilon} = \frac{1}{2}$ and therefore

$$f(x) - cx \le \delta . \tag{9.14}$$

Recall that we deal with the case $f(0) \ge 0$. Thus, we also get for $x \in [0, \xi]$,

$$f(x) - cx \ge -\frac{C}{2}\frac{\varepsilon^2}{10^2C^2} - \frac{2\varepsilon^2}{100C} = -\frac{\varepsilon^2}{200C} - \frac{4\varepsilon^2}{200C} = -\frac{\varepsilon^2}{40C} \equiv -\varepsilon^2\nu . \tag{9.15}$$

Thus, we conclude that $|f - g_j|_J \le \varepsilon$, provided $\varepsilon \le 40C$. (This un-intuitive bound comes from having chosen $\xi = \varepsilon/(10C)$ which is unreasonable when $C \ll 1$.)

We next show that the bound on $f(x) - cx$ is tighter than what we got so far when $x = \xi$. Indeed, we get in this case from Eq.(9.12),

$$f(\xi) - c\xi \le \delta - \frac{1}{2}\frac{\varepsilon^2}{100C} \equiv \delta - \mu\varepsilon^2 . \tag{9.16}$$

The assertion Eq.(9.9) follows by combining Eq.(9.16) with Eq.(9.15).

It remains to see that the set of possible j is finite. Considering Eq.(9.11) and the fact that $f \in F^0_{\delta,A,B,C,\xi}$ we see that j can take at most $2(B/\tau + 2) + 1$ possible values. The proof of Lemma 9.4 is complete.

$\square$

Proof of Proposition 9.3. This proof is a repeated application of Lemma 9.4. We retain the assumptions and notations from that proof. Let $\eta = \xi\tau = \frac{\varepsilon^2}{100C}$. We consider the grid (in the (x, y)-plane):

$$\left\{ \left(m\xi, n\eta\right) \ : \ m = -m_*, -m_* + 1, \ldots, m_* \ ; \ n = -n_*, \ldots, n_* \right\} ,$$

where

$$m_* = R/\xi \ , \quad n_* = [A/\eta] + 1 \ ,$$

recalling that R/ξ is an integer. In other words, we cover the range of possible arguments (in $[-R, R]$) and values (in $[-A, A]$) of $f \in F_{\varepsilon,A,B,C}$ by a fine grid. Consider now the set of all continuous, piecewise linear functions $h(x)$, connecting linearly successive lattice points $(m\xi, n\eta)$ with $((m + 1)\xi, n'\eta)$, with $-m_* \le m < m_*$, $|n| \le n_*$ and $|n'| \le n_*$. Furthermore, we require that $h(-R) = h(R) = 0$. There are a finite number of such functions, namely at most $(2n_* + 1)^{2m_*-1}$.

Note that η has been chosen in such a way that the slopes of the straight pieces of h are integer multiples of τ. We show next that every $f \in F_{\varepsilon,A,B,C}$ is, together with its derivative, ε-close to one of the h.

We begin by constructing the piecewise linear approximation h. We start at the point $x = -R$, $y = 0$, and shift the origin to this point by defining:

$$f_0(x) = f(x + R) \ .$$

Then f_0 is in $F^0_{\varepsilon,A,B,C,2R} \supset F^0_{\delta,A,B,C,\xi}$, with $\delta = \varepsilon$, and by Lemma 9.4, f_0 is approximated by one of the linear functions, say $n_0\tau x$, with $n_0 = [2 + f_0'(0)/\tau]$ on the interval $[0, \xi]$ (when $f(0) > 0$). Note that we also have (when ε is small) $|f_0(\xi) - n_0\tau\xi| \le \varepsilon - \mu\varepsilon^2$. We define

$$h(x) = n_0\tau(x + R) \ , \text{ for } x \in [-R, -R + \xi] \ .$$

Next shift the origin of the (x, y)-plane to $(-R + \xi, n_0\eta) = \bigl(-R + \xi, h(\xi)\bigr)$, and define

$$f_1(x) = f(x + R - \xi) - h(x + R - \xi) = f(x + R - \xi) - n_0\eta \ .$$

The definition of the first segment of h and the bounds on f_0 show that

$$f_1 \in F^0_{\varepsilon-\mu\varepsilon^2, A+|n_0|\eta, B, C, 2R-\xi} \ .$$

We now apply Lemma 9.4 to f_1. Note that f_1 is not in $F^0_{\delta,A,B,C,\xi}$ but in a space with a worse bound on the absolute value. However, the value of A does *not* enter the construction of the proof of Lemma 9.4 and hence is irrelevant for our inductive construction of h. Applying Lemma 9.4 to f_1, we find the second linear piece of the function h, and get a piecewise linear, continuous approximation of f on $[-R, -R + 2\xi]$. The final point of the approximation by h is now $(-R + 2\xi, n_1\eta)$, and we construct f_2 by translating the origin to that point. Assuming that $\varepsilon - 2\mu\varepsilon^2 > \nu\varepsilon^2$, we conclude that

$$f_2 \in F^0_{\varepsilon-2\mu\varepsilon^2, A+|n_0+n_1|\eta, B, C, 2R-2\xi} \ .$$

Note that the construction can not "drift away" in the y-direction, since we assumed from the outset that $|f|_{[-R,R]} \leq A$, and hence the y-translates never exceed A by more than ε (since h *is* an approximation to f). We continue the construction in the same way as before, until $x = 0$ is reached. At this point we have achieved the following: The original function is approximated by the piecewise linear function h on $J = [-R, 0]$ with the bound

$$|f - h|_J \leq \varepsilon\ , \quad |f' - h'|_J \leq \varepsilon\ .$$

Furthermore, at the point $x = \xi$ the approximation is really "good:" Consider the definition (9.8) of R. The number of steps from $-R$ to 0 is $m_* \geq 40 \cdot \frac{1}{\xi} - 1 = \frac{40\cdot 10C}{\varepsilon} - 1$ and in each step we gain a constant $\mu\varepsilon^2$, as long as $\delta > \varepsilon^2\nu$. Therefore,

$$|f(0) - h(0)| \leq \max\bigl(\varepsilon^2\nu, \varepsilon - m_*\varepsilon^2\mu\bigr) = \varepsilon^2\nu\ , \tag{9.17}$$

where the last equality follows from

$$\varepsilon^2\mu m_* \geq \varepsilon^2\mu(m_* - 1) \geq \varepsilon^2\mu\bigl(\frac{400C}{\varepsilon} - 2\bigr) = \varepsilon^2\frac{1}{200C}\cdot\bigl(\frac{400C}{\varepsilon} - 2\bigr) \geq 2\varepsilon - \frac{2\varepsilon^2}{200C} \geq \varepsilon\ , \tag{9.18}$$

when $\varepsilon \leq 100C$.

We repeat the same construction from the right endpoint, (with $m_* - 1$ steps, which is also covered by (9.18)) obtaining the piecewise linear function h on the set $J = [\xi, R]$, and again a bound, using (9.10):

$$|f(\xi) - h(\xi)| \leq \max\bigl(\varepsilon^2\nu, \varepsilon - (m_* - 1)\varepsilon^2\mu\bigr) \leq \varepsilon^2\nu\ . \tag{9.19}$$

We complete the definition of h by connecting $\bigl(0, h(0)\bigr)$ linearly with $\bigl(\xi, h(\xi)\bigr)$. Note that it is necessarily a line segment connecting two of the grid points and so h is one of the functions we counted earlier. We need to verify the bounds on $J = [0, \xi]$. It is here that the Eqs.(9.17) and (9.19) are relevant. We write

$$f(x) = f(0)\cdot(1 - \frac{x}{\xi}) + f(\xi)\frac{x}{\xi} + r(x)\ ,$$

and then by the bounds on the second derivative of f we get $|r|_J \leq C\xi^2/8$, and $|r'|_J \leq C\xi/2$. Since

$$h(x) = h(0)\cdot(1 - \frac{x}{\xi}) + h(\xi)\frac{x}{\xi}\ ,$$

we find for $\varepsilon \leq 800C/21$,

$$\begin{aligned}|f - h|_J &\leq \varepsilon^2\nu + \frac{C}{8}\frac{\varepsilon^2}{C^2 10^2} = \varepsilon(\frac{\varepsilon}{40C} + \frac{\varepsilon}{8\cdot 10^2 C}) \leq \varepsilon\ ,\\ |f' - h'|_J &\leq \frac{2}{\xi}\varepsilon^2\nu + \frac{C\xi}{2} = \frac{2\cdot 10C}{\varepsilon}\cdot\varepsilon^2\cdot\frac{1}{40C} + \frac{C\varepsilon}{2\cdot 10C} = \frac{11}{20}\varepsilon \leq \varepsilon\ .\end{aligned}$$

Thus, we have shown the required bound on all of $[-R, R]$. The piecewise linear, continuous function obtained in this way will be called $h_f(x)$. It is clearly one of the functions we constructed. It approximates f and f' on all of $[-R, R]$. We have thus found a finite family of piecewise linear functions which ε-covers $F_{\varepsilon,A,B,C}$. The proof of Proposition 9.3 is complete.

$\square$

9.3. The ε-entropy of Kolmogorov and Tikhomirov

We proceed as in [CE1], but with a change of topology as explained above. We have defined in Section 8 the minimum number $N_L(\varepsilon)$ of balls in the norm $\|\cdot\|_{\delta,L,2}$ needed to cover the attracting set. We also showed in Theorem 8.1 that

$$N_{L-A/\varepsilon}(\varepsilon) \leq C_{24}^L N_L(2\varepsilon) , \tag{9.20}$$

with some constants A and C_{24} depending only on the coefficients of the problem (2.1). If we iterate Eq.(9.20) m times, we get

$$N_L(\varepsilon) \leq C_{24}^{L+A/\varepsilon} C_{24}^{L+2A/\varepsilon} \cdots C_{24}^{L+mA/\varepsilon} N_{L+A/\varepsilon+2A/\varepsilon+\cdots+mA/\varepsilon}(2^m\varepsilon) . \tag{9.21}$$

In (2.20) we have shown that there is a constant C_9' which bounds the radius of $\mathcal{G}$ in $\mathcal{H}_{\delta,\text{loc},2}$. (The bound C_9 in (2.20) was for $\mathcal{H}_{\alpha,\text{loc},2}$.) Therefore, *one* ball of radius C_9' suffices to cover $\mathcal{G}|_{[-L,L]}$. Choosing $m = m(\varepsilon)$ in such a way that $2^m\varepsilon > C_9'$, we conclude that $\mathcal{G}|_{[-L,L]}$ can be covered by a *finite* number of balls in $\mathcal{H}_{\delta,\text{loc},2}$.[1]

Remark. This argument does not use compactness of $\mathcal{G}_{[-L,L]}$ and does not prove it either. It is here that our method differs from that of Feireisl[F]. He shows that the intersection of the $\Phi^T(\mathcal{G}|_{[-L,L]})$ is compact, whereas our approach shows that $\mathcal{G}|_{[-L,L]}$ itself can be covered by a finite number of balls.

We define similarly for any interval I, the minimal number $M_I(\varepsilon)$ of balls needed to cover $\mathcal{G}$ in the topology $\|\cdot\|_{W_I^{1,\infty}}$. By the Sobolev inequality from Lemma 2.5, we see that

$$\|u\|_{W^{1,\infty}_{[-L,L]}} \leq C_4 \|u\|_{\delta,L,2} .$$

Therefore, if $\mathcal{G}$ can be covered by $N_L(\varepsilon/C_4)$ balls of radius ε/C_4 in the norm $\|\cdot\|_{\delta,L,2}$, it can obviously be covered by the same number of balls of radius ε in the norm $\|\cdot\|_{W^{1,\infty}_{[-L,L]}}$. Thus we have

$$M_{[-L,L]}(\varepsilon) \leq N_L(\varepsilon/C_4) . \tag{9.22}$$

We now apply Corollary 9.2. We first note that by Eqs.(2.22) and (2.23) it is adequate to consider functions with *bounded* second derivative. (In fact this is the only place where these higher derivatives are needed.) Thus, we can apply Corollary 9.2 and we conclude that for two intervals I_1 and I_2, one has

$$M_{I_1\cup I_2}(\varepsilon) \leq M_{I_1}(\varepsilon) M_{I_2}(\varepsilon) K_\varepsilon . \tag{9.23}$$

Thus, we have established submultiplicativity (in I) and finiteness of $M_I(\varepsilon)$. Furthermore, from the construction of m in (9.21) with $2^m\varepsilon > C_9'$, we find by choosing the minimal such m:

$$M_I(\varepsilon) \leq C_{24}^{C_{40}\log(\varepsilon^{-1})|I| + \log(\varepsilon^{-1})\varepsilon^{-1} C_{41} A} .$$

[1] The argument used here is more elegant than the one used in [CE1]. We thank Y. Colin de Verdière for suggesting it.

Using this bound and (9.23), we get convergence and a bound on the ε-entropy $H_\varepsilon(\mathcal{G})$:

Theorem 9.5. *The ε-entropy per unit length of $\mathcal{G}$ in $W^{1,\infty}$ exists and is bounded by*

$$H_\varepsilon(\mathcal{G}) \;=\; \lim_{L\to\infty} \frac{1}{L} \log\bigl(M_{[-L,L]}(\varepsilon)\bigr) \;\le\; C_{42} \log(1/\varepsilon)\;. \tag{9.24}$$

9.4. Existence of the topological entropy per unit length

This material is taken from [CE2], and we introduce it without proofs, just to show what follows from the bounds of the previous sections.

For any $\varepsilon > 0$ and any interval I in $\mathbf{R}$, we define $\mathcal{W}_I^\varepsilon$ as the set of all finite covers of $\mathcal{G}$ by open sets in $W_I^{1,\infty}$ of diameter at most ε. Note that by the argument of Section 9.3, such a finite cover exists. Note also that elements of $W_I^{1,\infty}$ are pairs of functions (u, v) and that the topology is $W_I^{1,\infty}$ on the u-component and $\mathrm{L}^\infty(I)$ of the v-component.

Let $\tau > 0$ be a fixed time step, and let $T = n\tau$ with $n \in \mathbf{Z}$.

Definitions. Let $\mathcal{U} \in \mathcal{W}_I^\varepsilon$. We say that two elements A_1 and A_2 in $\mathcal{G}$ are $(\mathcal{U}, T)$-*separated* if there is at least one $k \in \{0, \dots, n\}$ for which the points $\Phi^{k\tau}(A_1)$ and $\Phi^{k\tau}(A_2)$ do not belong to the same atom of $\mathcal{U}$. We define

$$N_{T,\tau}(\mathcal{U})$$

to be the largest number of elements which are pairwise $(\mathcal{U}, T)$-separated (and considered with time-step τ.) Note that this number is finite since it is at most $(\mathrm{Card}\,\mathcal{U})^{2T/\tau}$. Finally, we define

$$N_{I,T,\tau,\varepsilon} = \inf_{\mathcal{U}\in\mathcal{W}_I^\varepsilon} N_{T,\tau}(\mathcal{U})\;.$$

Lemma 9.6. *(Lemma 2.1. of [CE2]). Let I_1 and I_2 be two disjoint intervals (perhaps with common boundary) and let $I = I_1 \cup I_2$. The functions $N_{I,T,\tau,\varepsilon}$ satisfy the following bounds: There is a constant $C = C(\varepsilon)$ such that:*
i) $N_{I,T,\tau,\varepsilon}$ is non-increasing in ε.
ii) $N_{I,T_1+T_2,\tau,\varepsilon} \le N_{I,T_1,\tau,\varepsilon}\, N_{I,T_2,\tau,\varepsilon}$.
iii) $N_{I_1\cup I_2,T,\tau,\varepsilon} \le C\, N_{I_1,T,\tau,\varepsilon}\, N_{I_2,T,\tau,\varepsilon}$.

Remark. It is important here that $C(\varepsilon)$ does *not* depend on the lengths of I_1 and I_2.

Proof. The properties i) and ii) are shown exactly as in [CE2]. However, the proof and the statement of iii) are now modified since we consider the topology of $W^{1,\infty}$.

In order to prove iii), we consider $\mathcal{U}_1 \in \mathcal{W}_{I_1}^\varepsilon$ and $\mathcal{U}_2 \in \mathcal{W}_{I_2}^\varepsilon$. Since we are using the $W^{1,\infty}$ norm we have

$$N_{T,\tau}(\mathcal{U}_1 \cap \mathcal{U}_2) \;\le\; N_{T,\tau}(\mathcal{U}_1)\, N_{T,\tau}(\mathcal{U}_2)\;.$$

We also have easily

$$\mathcal{W}^{\varepsilon}_{I_1} \cap \mathcal{W}^{\varepsilon}_{I_2} \subset \mathcal{W}^{\varepsilon}_{I_1 \cup I_2} \,.$$

The claim iii) now follows easily.

□

Remark. Henceforth, we shall work with domains which are intervals $I_L = [-L, L]$.

Theorem 9.7. *The following limit exists*

$$h \,=\, \lim_{\varepsilon \to 0} \lim_{L \to \infty} \frac{1}{L} \lim_{T \to \infty} \frac{1}{T} \log N_{I_L, T, \tau, \varepsilon} \,. \tag{9.25}$$

Moreover, h does not depend on τ. It is called the topological entropy per unit volume of the system.

Proof. The proof is given in [CE2].

Remark. It also follows from Section 9.3 that h is bounded.

9.5. Sampling

The results we describe in this section are, on the surface, the same as those obtained in [CE2]. This means that by *discrete* sampling of the signal in a space-time region

$$[-L - A \log(1/\varepsilon), L + A \log(1/\varepsilon)] \times [0, \tau_* \log(1/\varepsilon)] \,, \tag{9.26}$$

the function observed can be determined to precision ε *everywhere* on the interval $[-L, L]$ at time $\tau_* \log(1/\varepsilon)$.

In the current context this result can be worked out in detail in the following sense: Assume that two solutions u_1 and u_2 and their first and second space derivatives (as well as $\partial_t u_1$ and $\partial_t u_2$ and their first derivatives) coincide to within ε in the region (9.26) on a space-time grid with mesh $\mathcal{O}(1/k_*) \times \mathcal{O}(\tau_*)$. Then one can conclude that

$$\begin{aligned} &\|u_1(\tau_* \log(1/\varepsilon), \cdot) - u_2(\tau_* \log(1/\varepsilon), \cdot)\|_{W^{1,\infty}_{[-L,L]}} \\ &+\|\partial_t u_1(\tau_* \log(1/\varepsilon), \cdot) - \partial_t u_2(\tau_* \log(1/\varepsilon), \cdot)\|_{\mathrm{L}^{\infty}_{[-L,L]}} \,\le\, C_{43} \varepsilon \,, \end{aligned}$$

for some universal constant C_{43}. This result allows, in principle, to reconstruct the K_2-entropy.

In our view, the result sketched above is somewhat unsatisfactory, and its clarification needs further work. Namely, we would like to be able to make positive statements based on sampling *only* function values, and not their derivatives, in particular, not the second derivative. (They are needed to bound the difference in $W^{1,\infty}$.) Indeed, a quick inspection of properties of the Bernstein class shows that we have no reasonable bound on $\mathcal{S}_L(\mathcal{Q}_{k_*} f) - \mathcal{S}_L(f)$ in $W^{1,\infty}$ if we only have information about the function and not about its derivatives.

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