Algebraic Holography

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Abstract: A rigorous (and simple) proof is given that there is a one-to-one correspondence between causal anti-deSitter covariant quantum field theories on anti-deSitter space and causal conformally covariant quantum field theories on its conformal boundary. The correspondence is given by the explicit identification of observables localized in wedge regions in anti-deSitter space and observables localized in double-cone regions in its boundary. It takes vacuum states into vacuum states, and positive-energy representations into positive-energy representations.

1 Introduction and results

The conjectured correspondence (so-called "holography") [13, 18] between quantum field theories on $1{+}s$-dimensional anti-deSitter space-time $AdS_{1,s}$ (the "bulk space") and conformal quantum field theories on its conformal boundary $CM_{1,s-1}$ which is a compactification of Minkowski space $\mathbb{R}^{1,s-1}$, has recently raised enthusiastic interest. If anti-deSitter space is considered as an approximation to the space-time geometry near certain gravitational horizons (extremal black holes), then the correspondence lends support to the informal idea of reduction of degrees of freedom due to the thermodynamic properties of black holes [9, 17]. Thus, holography is expected to give an important clue for the understanding of quantum theory in strong gravitational fields and, ultimetely, of quantum gravity.

While the original conjecture [13] was based on "stringy" pictures, it was soon formulated [18] in terms of (Euclidean) conventional quantum field theory, and a specific relation between generating functionals was conjectured. These conjectures have since been exposed with success to many structural and group theoretical tests, yet a rigorous proof has not been given.

The problem is, of course, that the "holographic" transition from anti-deSitter space to its boundary and back, is by no means a point transformation, thus preventing a simple (pointwise) operator identification between bulk fields and boundary fields. In the present note, we show that in contrast, an identification between the *algebras generated by* the respective local bulk and boundary fields is indeed possible in a very transparent manner. These algebraic data are completely sufficient to reconstruct the respective theories.

We want to remind the reader of the point of view due to Haag and Kastler (see [7] for a standard textbook reference) which emphasizes that, while any choice of particular fields in a quantum field theory may be a matter of convenience without affecting the physical content of the theory (comparable to the choice of coordinates in geometry), the algebras they generate and their algebraic interrelations, notably causal commutativity, supply all the relevant physical information in an invariant manner. The interested reader will find in [1] a review of the (far from obvious, indeed) equivalence between quantum field theory in terms of fields and quantum field theory in terms of algebras, notably on the strategies available to extract physically relevant information, such as the particle spectrum, superselection charges, and scattering amplitudes, from the net of algebras without knowing the fields.

It is crucial in the algebraic approach, however, to keep track of the *localization* of the observables. Indeed, the physical interpretation of a theory is coded in the structure of a "causal net" of algebras [7] which means the specification of the sets of observables $B(X)$ which are localized in any given space-time region X.[1]

The assignment $X \mapsto B(X)$ is subject to the conditions of isotony (an observable localized in a region X is localized in any larger region $Y \supset X$, thus $B(Y) \supset B(X)$), causal commutativity (two observables localized at space-like distance commute with each other), and covariance (the Poincaré transform of an observable localized in X is localized in the transformed region gX; in the context at hand replace "Poincaré" by "anti-deSitter"). Each $B(X)$ should in fact be an algebra of operators (with the observables its selfadjoint elements), and to have sufficient control of limits and convergence in order to compute physical quantities of interest, it is convenient to let $B(X)$ be von Neumann algebras.[2]

For most purposes it is convenient to consider as typical compact regions the "double-cones", that is, intersections of a future directed and a past directed light-cone, and to think of point-like localization in terms of very small double-cones. On the other hand, certain aspects of the theory are better captured by "wedge" regions which extend to space-like infinity. A space-like wedge (for short: wedge) in Minkowski space is a region of the form $\{x : x_1 > |x_0|\}$, or any Poincaré transform thereof. The corresponding regions in anti-deSitter space turn out to be intersections of $AdS_{1,s}$ with suitable flat space wedges in the ambient space $\mathbb{R}^{2,s}$, see below.

It will become apparent in the sequel that to understand the issue of "holography", the algebraic framework proves to be most appropriate.

The basis for the holography conjectures is, of course, the coincidence between the anti-deSitter group $SO_0(2, s)$ and the conformal group $SO_0(2, s)$. ($SO_0(n, m)$ is the identity component of the group $SO(n, m)$, that is the proper orthochronous subgroup distinguished by the invariant condition that the determinants of the time-like $n \times n$ and of the space-like

[1] The assignment $X \mapsto B(X)$ is a "net" in the mathematical sense: a generalized sequence with a partially ordered index set (namely the set of regions X).

[2] A von Neumann algebra is an algebra of bounded operators on a Hilbert space which is closed in the weak topology of matrix elements. E.g., if ϕ is a hermitean field and $\phi(f)$ a field operator smeared over a region X containing the support of f, then operators like $\exp i\phi(f)$ belong to $B(X)$.

$m \times m$ sub-matrices are both positive.) The former group acts on $AdS_{1,s}$ (as a "deformation" from the flat space Poincaré group in 1+s dimensions, $SO_0(1,s) \ltimes \mathbb{R}^{1,s}$), and the latter group acts on the conformal boundary $CM_{1,s-1}$ of $AdS_{1,s}$ (as an extension of the Poincaré group in 1+($s-1$) dimensions, $SO_0(1,s-1) \ltimes \mathbb{R}^{1,s-1}$) by restriction of the former group action on the bulk. The representation theoretical aspect of this coincidence has been elaborated (in Euclidean metric) in [5].

In terms of covariant nets of algebras of local observables ("local algebras"), it is thus sufficient to identify one suitable algebra in anti-deSitter space with another suitable algebra in the conformal boundary space, and then to let $SO_0(2,s)$ act to provide the remaining identifications. As any double-cone region in conformal space determines a subgroup of the conformal group $SO_0(2,s)$ which preserves this double-cone, it is natural to identify its algebra with the algebra of a region in anti-deSitter space which is preserved by the same subgroup of the anti-deSitter group $SO_0(2,s)$. It turns out that this region is a space-like wedge region which intersects the boundary in the given double-cone.

For a typical bulk observable localized in a wedge region, the reader is invited to think of a field operator for a Mandelstam string which stretches to space-like infinity. Its holographic localization on the boundary has finite size, but it becomes sharper and sharper as the string is "pulled to infinity". We shall see that one may be forced to take into consideration theories which possess only wedge-localized, but no compactly localized observables.

Our main algebraic result rests on the following geometric Lemma:[3]

Lemma: *Between the set of space-like wedge regions in anti-deSitter space, $pW \subset AdS_{1,s}$, and the set of double-cones in its conformal boundary space, $I \subset CM_{1,s-1}$, there is a canonical bijection $\alpha : pW \mapsto I = \alpha(pW)$ preserving inclusions and causal complements, and intertwining the actions of the anti-deSitter group $SO_0(2,s)$ and of the conformal group $SO_0(2,s)$*

$$\alpha(g(pW)) = \dot{g}(\alpha(pW)), \qquad \alpha^{-1}(\dot{g}(I)) = g(\alpha^{-1}(I))$$

where $\dot{g}$ is the restriction of the action of g to the boundary. The double-cone $I = \alpha(pW)$ associated with a wedge pW is the intersection of pW with the boundary.

Given the Lemma, the main algebraic result states that bulk observables localized in wedge regions are identified with boundary observables localized in double-cone regions:

Corollary 1: *The identification of local observables*

$$B(pW) := A(\alpha(pW)), \qquad A(I) := B(\alpha^{-1}(I))$$

[3] The symbol p stands for the projection from the hypersurface $H_{1,s} = \{x \in \mathbb{R}^{2,s} : x^2 = R^2\}$ onto its quotient $AdS_{1,s}$ by the identification of points x and $-x$. We denote double-cones in the boundary by the symbol I, *(i)* because we prefer to reserve the "standard" symbol O for double-cones in the bulk space, and *(ii)* because in 1+1 dimensions the "double-cones" on the boundary are in fact open intervals on the circle S^1. For further details, see Sect. 2.

gives rise to a 1:1 correspondence between isotonous causal conformally covariant nets of algebras $I \mapsto A(I)$ on $CM_{1,s-1}$ and isotonous causal anti-deSitter covariant nets of algebras $pW \mapsto B(pW)$ on $AdS_{1,s}$.

An observable localized in a double-cone O in anti-deSitter space is localized in any wedge containing O, hence the local algebra $B(O)$ should be contained in all $B(pW)$, $pW \supset O$. We shall define $B(O)$ as the intersection of all these wedge algebras. These intersections do no longer correspond to simple geometric regions in $CM_{1,s-1}$ (so points in the bulk have a complicated geometry in the boundary), as will be discussed in more detail in 1+1 dimensions below.

The following result also identifies states and representations of the corresponding theories:

Corollary 2: *Under the identification of Corollary 1, a vacuum state on the net A corresponds to a vacuum state on the net B. Positive-energy representations of the net A correspond to positive-energy representations of the net B. The modular group (in the sense of Tomita-Takesaki) of a wedge algebra $B(pW)$ in a vacuum state coincides with the modular group of the corresponding double-cone algebra $A(I)$ in the corresponding vacuum state, which is a one-parameter subgroup of $SO_0(2,s)$ preserving I. The modular conjugation of a wedge algebra $B(pW)$ in a vacuum state is a CPT-type transformation which maps pW onto its causal complement.*

Reference is here made to the modular theory of von Neumann algebras which states that every (normal and cyclic) state on a von Neumann algebra is a thermal equilibrium state with respect to a unique adapted "time" evolution (one-parameter group of automorphisms = modular group) of the algebra. In quantum field theories in Minkowski space, the modular groups have been computed for wedge algebras in the vacuum state [4] and were found to coincide with the boost subgroup of the Lorentz group which preserves the wedge. (In conformal theories, mapping wedges onto double-cones by suitable conformal transformations, the same result also applies to double-cones [8].) This result is an algebraic explanation of the Unruh effect according to which a uniformly accelerated observer attributes a temperature to the vacuum state, and provides also an explanation of Hawking radiation if the wedge region is replaced by the space-time region outside the horizon of a Schwarzschild black hole [16].

The modular theory also provides a "modular conjugation" which maps the algebra onto its commutant. In the quantum field theoretical application as before, the modular conjugation turns out to be a CPT-type reflection (CPT up to a rotation) along the "ridge" of the wedge which maps the wedge onto its causal complement, hence the algebras associated with complementary wedges not only commute, as required by causality, but either algebra is the *maximal* algebra commuting with the other one. This symmetry is known as essential Haag duality, or wedge duality [7].

The statement in Corollary 2 on the modular group thus implies that also in anti-deSitter space, a vacuum state of B in restriction to a wedge algebra $B(pW)$ is a thermal equilibrium state with respect to the associated one-parameter boost subgroup of the anti-deSitter group which preserves pW, i.e., the Unruh effect takes place for a uniformly accelerated observer. Furthermore, the CPT theorem holds for causal quantum field theories on anti-deSitter space.

We emphasize that the Hamiltonians $\frac{1}{R}M_{0,d}$ on $AdS_{1,s}$ and P^0 on $CM_{1,s-1}$ are *not* identified under the identification of the anti-deSitter group and the conformal group. Instead, $M_{0,d}$ is (in suitable coordinates) identified with the combination $\frac{1}{2}(P^0+K^0)$ of translations and special conformal transformations in the 0-direction of $CM_{1,s-1}$. This is different from the Euclidean picture [18] where the anti-deSitter Hamiltonian is identified with the dilatation subgroup of the conformal group. In Lorentzian metric, the dilatations correspond to a space-like "translation" subgroup of the anti-deSitter group. This must have been expected since the generator of dilatations does not have a one-sided spectrum as is required for the real-time Hamiltonian. The subgroup generated by $\frac{1}{2}(P^0+K^0)$ is well-known to be periodic and to satisfy the spectrum condition in every positive-energy representation. (Periodicity in bulk time of course implies a mass gap for the underlying bulk theory. This is not in conflict with the boundary theory being massless since the respective subgroups of time evolution cannot be identified.)

Different Hamiltonians give rise to different counting of degrees of freedom, since entropy is defined via the partition function. Thus, the "holographic" reduction of degrees of freedom [9, 17] can be viewed as a consequence of the choice of the Hamiltonian: The anti-deSitter Hamiltonian $M_{0,d} = \frac{1}{2}(P^0+K^0)$ has discrete spectrum and has a chance (at least in 1+1 dimensions) to yield a finite partition function. One the other hand, the partition function with respect to the boundary Hamiltonian P^0 exhibits the usual infrared divergence due to infinite volume and continuous spectrum.

A crucial aspect of the present analysis is the identification of compact regions in the boundary with wedge regions in the bulk. With a little hindsight, this aspect is indeed also present in the proposal for the identification of generating functionals [18]. While the latter is given in the Euclidean approach, it should refer in real time to a hyperbolic differential equation with initial values given in a double-cone on the boundary which determine its solution in a wedge region of bulk space.

We also show that in 1+1 dimensions there are sufficiently many observables localized in arbitrarily small *compact* regions in the bulk space to ensure that compactly localized observables generate the wedge algebras. This property is crucial if we want to think of local algebras as being generated by local fields:

Proposition: *Assume that the boundary theory A on S^1 is weakly additive (i.e., $A(I)$ is generated by $A(J_n)$ whenever the interval I is covered by a family of intervals J_n). If a wedge pW in $AdS_{1,1}$ is covered by a family of double-cones $O_n \subset pW$, then the algebra $B(pW)$ is generated by the observables localized in O_n:*

$$B(pW) = \bigvee_n B(O_n).$$

In order to establish this result, we explicitly determine the observables localized in a double-cone region on $AdS_{1,1}$. Their algebra $B(O)$ turns out to be non-trivial irrespective of the size of the double-cone O. It is the intersection of two interval algebras $A(I_i)$ on the boundary S^1 where the intersection of the two intervals I_i is a union of two disjoint intervals J_i. $B(O)$ contains therefore at least $A(J_1)$ and $A(J_2)$. In fact, it is even larger than that, containing also observables corresponding to a "charge transport" [7], that is, operators which annihilate a superselection charge in J_1 and create the same charge in J_2. The inclusion $A(I_1) \vee A(J_2) \subset B(O)$ therefore contains (complete) algebraic information about the superselection structure of the chiral boundary theory [10].

Shrinking the double-cone on $AdS_{1,1}$ shrinks the size of the intervals J_i without shrinking their distance, so points in 1+1-dimensional anti-deSitter space are related to pairs of points in conformal space. But we see that sharply localized bulk observables involve poorly localized boundary observables: the above charge transporters. This result provides an algebraic interpretation of the obstruction against a point transformation between bulk and boundary.

The issue of compactly localized observables in anti-deSitter space is more complicated in more than two dimensions, and deserves a separate careful analysis. Some preliminary results will be presented in Section 2.3. They show that if the bulk theory possesses observables localized in double-cones, then the corresponding boundary theory violates standard additivity properties, and conversely, if the boundary theory satisfies these additivity properties, then the observables of the corresponding bulk theory are always attached to infinity, as in topological (Chern-Simons) theories.

Let us point out that the conjectures in [13, 18] suggest a much more ambitious interpretation, namely that the correspondence pertains to bulk theories *involving quantum gravity*, while the anti-deSitter space and its boundary are understood in some asymptotic (semi-classical) sense. Indeed, the algebraic approach is no more able to describe proper quantum gravity as any other mathematically unambiguous framework up to now. Most arguments given in the literature in favour of the conjectures refer to gravity as perturbative gravity on a background space-time. Likewise, our present results concern the semi-classical version of the conjectures, treating gravity like any other quantum field theory as a theory of observables on a classical background geometry. In fact, the presence or absence of gravity in the bulk theory plays no particular role. This is only apparently in conflict with the original arguments for a holographic reduction of degrees of freedom of a bulk theory in the vicinity of a gravitational horizon [9, 17] in which gravity is essential. Namely, our statement can be interpreted in the sense that *once the anti-deSitter geometry is given* for whatever reason (e.g., the presence of a gravitational horizon), then it can support only the degrees of freedom of a boundary theory.

2 Identification of observables

We denote by $H_{1,s}$ the d=1+s-dimensional hypersurface defined through its embedding into ambient $\mathbb{R}^{2,s}$,

$$x_0^2 - x_1^2 - \ldots - x_s^2 + x_d^2 = R^2$$

with Lorentzian metric induced from the 2+s-dimensional metric

$$ds^2 = dx_0^2 - dx_1^2 - \ldots - dx_s^2 + dx_d^2.$$

Its group of isometries is the Lorentz group $O(2, s)$ of the ambient space in which the reflection $x \mapsto -x$ is central. Anti-deSitter space is the quotient manifold $AdS_{1,s} = H_{1,s}/\mathbb{Z}_2$ (with the same Lorentzian metric locally). We denote by p the projection $H_{1,s} \to AdS_{1,s}$.

Two open regions in anti-deSitter space are called "causally disjoint" if none of their points can be connected by a time-like geodesic. The largest open region causally disjoint from a given region is called the causal complement. In a causal quantum field theory on the quotient space $AdS_{1,s}$, observables and hence algebras associated with causally disjoint regions commute with each other.

The reader should be worried about this definition, since causal independence of observables should be linked to causal connectedness by time-like *curves* rather than geodesics. But on anti-deSitter space, any two points can be connected by a time-like curve, so they are indeed causally connected, and the requirement that causally *disconnected* observables commute is empty. Yet, as our Corollary 1 shows, if the boundary theory is causal, then the associated bulk theory is indeed causal in the present (geodesic) sense. We refer also to [2] where it is shown that vacuum expectation values of commutators of observables with causally disjoint localization have to vanish whenever the vacuum state has reasonable properties (invariance and thermodynamic passivity), but without any a priori assumptions on causal commutation relations (neither in bulk nor on the boundary).

Thus in the theories on anti-deSitter space we consider in this paper, observables localized in causally disjoint but causally connected regions commute; see [2] for a discussion of the ensuing physical constraints on the nature of interactions on anti-deSitter space.

The causal structure of $AdS_{1,s}$ is determined by its metric modulo conformal transformations which preserve angles and geodesics. As a causal manifold, $AdS_{1,s}$ has a boundary (the "asymptotic directions" of geodesics). The boundary inherits the causal structure of the bulk space $AdS_{1,s}$, and the anti-deSitter group $SO_0(2, s)$ acts on this space. It is well known that this boundary is a compactification $CM_{1,s-1} = (S^1 \times S^{s-1})/\mathbb{Z}_2$ of Minkowski space $\mathbb{R}^{1,s-1}$, and $SO_0(2, s)$ acts on it like the conformal group.

The notions of causal disjoint and causal complements on $CM_{1,s-1}$ coincide, up to conformal transformations, with those on Minkowski space [11]. In d =1+1 dimensions, $s = 1$, the conformal space is S^1, and the causal complement of an interval I is $I^c = S^1 \backslash \overline{I}$.

Both anti-deSitter space and its causal boundary have a "global time-arrow", that is, the distinction between the future and past light-cone in the tangent spaces (which are ordinary Minkowski spaces) at each point x can be globally chosen continuous in x (and consistent with the reflection $x \mapsto -x$). The time orientation on the bulk space induces the time orientation on the boundary. The time arrow is crucial in order to distinguish representations of positive energy.

2.1 Proof of the Lemma

Any ordered pair of light-like vectors (e, f) in the ambient space $\mathbb{R}^{2,s}$ such that $e \cdot f < 0$ defines an open subspace of the hypersurface $H_{1,s}$ given by

$$W(e, f) = \{x \in \mathbb{R}^{2,s} : x^2 = R^2, e \cdot x > 0, f \cdot x > 0\}.$$

This space has two connected components. Namely, the tangent vector at each point $x \in W(e, f)$ under the boost in the e-f-plane, $\delta_{e,f}\, x = (f \cdot x)e - (e \cdot x)f$, is either a future or a past directed time-like vector, since $(\delta_{e,f}\, x)^2 = -2(e \cdot f)(e \cdot x)(f \cdot x) > 0$. We denote by $W_+(e, f)$ and $W_-(e, f)$ the connected components of $W(e, f)$ in which $\delta_{e,f}\, x$ is future and past directed, respectively. By this definition, $W_+(f, e) = W_-(e, f)$, and $W_+(-e, -f) = -W_+(e, f)$.

The wedge regions in the hypersurface $H_{1,s}$ are the regions $W_\pm(e, f)$ as specified. The wedge regions in anti-deSitter space are $pW_\pm(e, f)$. One has $pW_+(e, f) = pW_-(f, e) = pW_+(-e, -f)$, and $pW_+(e, f)$ and $pW_-(e, f)$ are each other's causal complements. For an illustration in 1+1 dimensions, cf. Figure 1.

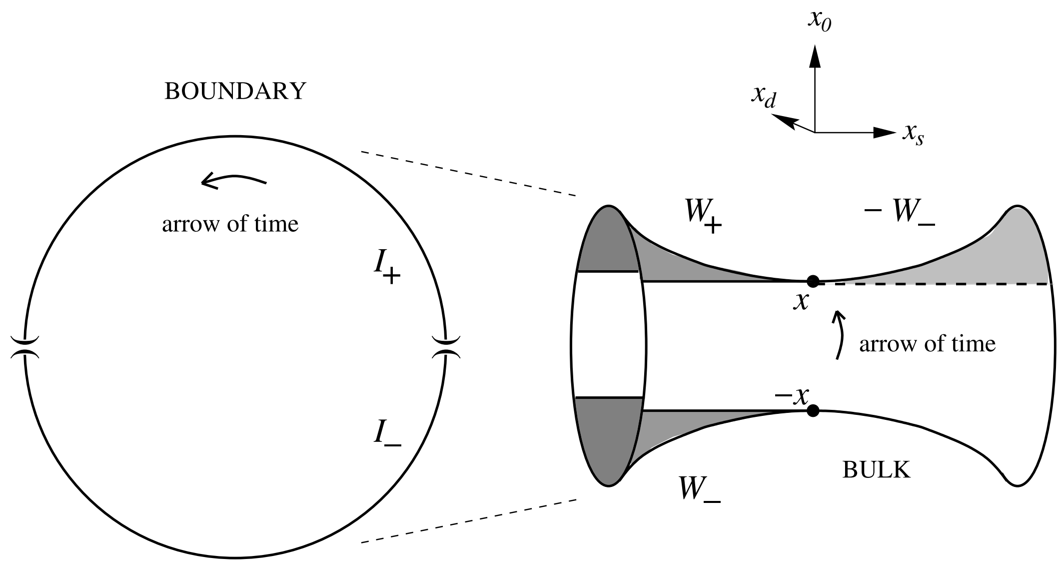

Figure 1. *Wedge regions $W_+(e, f)$ and $W_-(e, f)$ in 1+1 dimensions, and their intersections with the boundary. The light-like vectors e and f are tangent to $-W_-$ in its apex x. In anti-deSitter space, W_- is identified with $-W_-$, and pW_+ and pW_- are causal complements of each other.*

We claim that the projected wedges $pW_\pm(e,f)$ intersect the boundary of $AdS_{1,s}$ in regions $I_\pm(e,f)$ which are double-cones of Minkowski space $\mathbb{R}^{1,s-1}$ or images thereof under some conformal transformation. Note that any two double-cones in $\mathbb{R}^{1,s-1}$ are connected by a conformal transformation, and among their conformal transforms are also the past and future light-cones and space-like wedges in $\mathbb{R}^{1,s-1}$.

We claim also that the causal complement $pW_-(e,f)$ of the wedge $pW_+(e,f)$ intersects the boundary in the causal complement $I_-(e,f) = I_+(e,f)^c$ of $I_+(e,f)$.

It would be sufficient to compute the intersections of any single pair of wedges $W_\pm(e,f)$ with the boundary and see that it is a pair of causally complementary conformal double-cones in $CM_{1,s-1}$, since the claim then follows for any other pair of wedges by covariance. For illustrative reason we shall compute two such examples.

We fix the "arrow of time" by declaring the tangent vector of the rotation in the 0-d-plane, $\delta_t\, x = (-x_d, 0, \dots, 0, x_0)$, to be future directed.

In stereographic coordinates $(y_0, \vec{y}, x_-)$ of the hypersurface $x^2 = R^2$, where $x_- = x_d - x_s$ and $(y_0, \vec{y}) = (x_0, \vec{x})/x_-$, $\vec{x} = (x_1, \dots, x_{s-1})$, the boundary is given by $|x_-| = \infty$. Thus, in the limit of infinite x_- one obtains a chart $y_\mu = (y_0, \vec{y})$ of $CM_{1,s-1}$. The induced conformal structure is that of Minkowski space, $dy^2 = f(y)^2(dy_0^2 - d\vec{y}^2)$.

Our first example is the one underlying Figure 1: we choose $e_\mu = (0, \dots, 0, 1, 1)$ and $f_\mu = (0, \dots, 0, 1, -1)$. The conditions for $x \in W(e,f)$ read $x_d - x_s > 0$ and $x_d + x_s < 0$, implying $x_d^2 - x_s^2 < 0$ and hence $x_0^2 - \vec{x}^2 > R^2$. The tangent vector $\delta_{e,f}\, x$ has d-component $\delta_{e,f}\, x_d = -2x_s > 0$. Hence, it is future directed if $x_0 > 0$, and past directed if $x_0 < 0$:

$$W_+(e,f) = \{x : x^2 = R^2, x_s < -|x_d|, x_0 > 0\}.$$

After dividing $(x_0, \vec{x})$ by $x_- = x_d - x_s \nearrow \infty$, we obtain the boundary region

$$I_+(e,f) = \{y = (y_0, \vec{y}) : y_0^2 - \vec{y}^2 > 0, y_0 > 0\},$$

that is, the future light-cone in Minkowski space; similarly, $I_-(e,f)$ is the past light-cone, and $I_\pm(e,f)$ are each other's causal complements in $CM_{1,s-1}$.

Next, we choose $e_\mu = (1, 1, 0, \dots, 0)$ and $f_\mu = (-1, 1, 0, \dots, 0)$. The conditions for $x \in W(e,f)$ read $x_1 < -|x_0|$, implying $x_0^2 - \vec{x}^2 < 0$ and hence $x_d^2 - x_s^2 > R^2 > 0$. The tangent vector $\delta_{e,f}\, x$ has 0-component $\delta_{e,f}\, x_d = -2x_1 > 0$. Hence, it is future directed if $x_d < 0$ hence $x_d - x_s < 0$, and past directed if $x_d > 0$ hence $x_d - x_s > 0$:

$$W_+(e,f) = \{x : x^2 = R^2, x_1 < -|x_0|, x_- < 0\}.$$

After dividing $(x_0, \vec{x})$ by $x_- = x_d - x_s \searrow -\infty$, we obtain the boundary region

$$I_+(e,f) = \{y = (y_0, \vec{y}) : y_1 > |y_0|\},$$

that is, a space-like wedge region in Minkowski space; similarly, $I_-(e,f)$ is the opposite wedge $y_1 < -|y_0|$, which is again the causal complement of $I_+(e,f)$ in $CM_{1,s-1}$.

Both light-cones and wedge regions in Minkowski space are well known to be conformal transforms of double-cones, and hence they are double-cones on $CM_{1,s-1}$. The pairs of regions computed above are indeed causally complementary pairs.

We now consider the map $\alpha : pW_+(e,f) \mapsto I_+(e,f)$. Since the action of the conformal group on the boundary is induced by the action of the anti-deSitter group on the bulk, we see that $W_\pm(ge,gf) = g(W_\pm(e,f))$ and $I_\pm(ge,gf) = \dot{g}(I_\pm(e,f))$, hence α intertwines the actions of the anti-deSitter and the conformal group. It is clear that α preserves inclusions, and we have seen that it preserves causal complements for one, and hence for all wedges. Since $SO_0(2,s)$ acts transitively on the set of double-cones of $CM_{1,s-1}$, the map α is surjective. Finally, since $pW_+(e,f)$ and $I_+(e,f)$ have the same stabilizer subgroup of $SO_0(2,s)$, it is also injective.

This completes the proof of the Lemma. □

2.2 Proof of the Corollaries

We identify wedge algebras on $AdS_{1,s}$ and double-cone algebras on $CM_{1,s-1}$ by

$$B(pW_\pm(e,f)) = A(I_\pm(e,f)),$$

that is, $B(pW) = A(\alpha(pW))$. The Lemma implies that if A is given as an isotonous, causal and conformally covariant net of algebras on $CM_{1,s-1}$, then $B(pW)$ defined by this identification constitute an isotonous, causal and anti-deSitter covariant net of algebras on $AdS_{1,s}$, and vice versa. Namely, the identification is just a relabelling of the index set of the net which preserves inclusions and causal complements and intertwines the action of $SO_0(2,s)$. Thus we have established Corollary 1. □

As for Corollary 2, we note that, as the algebras are identified, states and representations of the nets A and B are also identified.

Since the identification intertwines the action of the anti-dSitter group and of the conformal group, an anti-deSitter invariant state on B corresponds to a conformally invariant state on A. The generator of time translations in the anti-deSitter group corresponds to the generator $\frac{1}{2}(P^0 + K^0)$ in the conformal group which is known to be positive if and only if P^0 is positive (note that K^0 is conformally conjugate to P^0). Hence the conditions for positivity of the respective generators of time-translations are equivalent.

By the identification of states and algebras, also the modular groups are identified. The modular group and modular conjugation for double-cone algebras in a vacuum state of conformally covariant quantum field theories are conformally conjugate to the modular group and modular conjugation of a Minkowski space wedge algebra, which in turn are given by the Lorentz boosts in the wedge direction and the reflection along the ridge of the wedge [4, 8]. It follows that the modular group for a wedge algebra on anti-deSitter space is given by the corresponding subgroup of the anti-deSitter group which preserves the wedge (for a wedge $pW_+(e,f)$, this is the subgroup of boosts in the e-f-plane), and the modular conjugation is a CPT transformation which maps pW_+ onto pW_-.

These remarks suffice to complete the proof of Corollary 2. □

Let us mention that the correspondence given in Corollary 1 holds also for "weakly local" nets both on the bulk and on the boundary. In a weakly local net, the vacuum expectation value of the commutator of two causally disjoint observables vanishes, but not necessarily the commutator itself. Weak locality for quantum field theories on anti-deSitter space follows [2] from very conservative assumptions on the vacuum state without any commutation relations assumed. Thus, also the boundary theory will always be weakly local.

2.3 Double-cones algebras on anti-deSitter space

Let us first note that as the ridge of a wedge is shifted into the interior of the wedge, the double-cone on the boundary shrinks. Thus, sharply localized boundary observables correspond to bulk observables at space-like infinity. We now show that sharply localized bulk observables do not correspond to a simple geometry on the boundary, but must be determined algebraically.

An observable localized in a double-cone O of anti-deSitter space must be contained in every wedge algebra $B(pW)$ such that $O \subset pW$. The algebra $B(O)$ is thus at most the intersection of all $B(pW)$ such that $O \subset pW$. We may define it as this intersection, thereby ensuring isotony, causal commutativity and covariance for the net of double-cone algebras in an obvious manner.

Double-cone algebras on anti-deSitter space are thus delicate intersections of algebras of double-cones and their conformal images on the boundary, and might turn out trivial. In 1+1 dimensions, the geometry is particularly simple since a double-cone is an intersection of only two wedges. We show that the corresponding intersection of algebras is non-trivial, and shall turn to $d > 1+1$ below.

Let us write (in 1+1 dimensions) the relation

$$B(O) = B(pW_1) \cap B(pW_2) \equiv A(I_1) \cap A(I_2) \qquad \text{whenever} \qquad O = pW_1 \cap pW_2,$$

where pW_i are any pair of wedge regions in $AdS_{1,1}$ and $I_i = \alpha(pW_i)$ their intersections with the boundary, that is, open intervals on S^1.

The intersection $pW_1 \cap pW_2$ might not be a double-cone. It might be empty, or it might be another wedge region. Before discussing the above relation as a definition for the double-cone algebra $B(O)$ if $O = pW_1 \cap pW_2$ is a double-cone, we shall first convince ourselves that it is consistent also in these other cases.

If pW_1 contains pW_2, or vice versa, then O equals the larger wedge, and the relation holds by isotony. If pW_1 and pW_2 are disjoint, then the intersections with the boundary are also disjoint, and $B(\emptyset) = A(I_1) \cap A(I_2)$ is trivial if the boundary net A on S^1 is irreducible (that is, disjoint intervals have no nontrivial observables in common).

Next, it might happen that pW_1 and pW_2 have a nontrivial intersection without the apex of one wedge lying inside the other wedge. In this case, the intersection is again a wedge, say pW_3. Namely, any wedge in $AdS_{1,1}$ is of the form $pW_+(e, f)$ where e and f are a future and a past directed light-like tangent vector in the apex x (the unique point in $AdS_{1,1}$ solving $e\cdot x = f\cdot x = 0$). The condition $e\cdot f < 0$ implies that both tangent vectors point in the same (positive or negative) 1-direction. The wedge itself is the surface between the two light-rays emanating from x in the directions $-e$ and $-f$ (cf. Figure 1). The present situation arises if the future directed light-ray of pW_1 intersects the past directed light-ray of pW_2 (or vice versa) in a point x_3 without the other pair of light-rays intersecting each other. The intersection of the two wedges is the surface between the two intersecting light-rays travelling on from the point x_3, which is another wedge region pW_3 with apex x_3. It follows that the intersection of the intersections I_i of pW_i with the boundary equals the intersection I_3 of pW_3 with the boundary. Hence consistency of the above relation is guaranteed by $A(I_1) \cap A(I_2) = A(I_3)$ where I_1 and I_2 are two intervals on S^1 whose intersection I_3 is again an interval.

Now we come to the case that $pW_1 \cap pW_2$ is a double-cone O in the proper sense. This is the case if the closure of the causal complement of pW_1 is contained in pW_2. It follows that the closure of the causal complement of I_1 is contained in I_2, hence the intersection of I_1 and I_2 is the union of two disjoint intervals J_1 and J_2. The latter are the two light-like geodesic “shadows”, cast by O onto the boundary.

Thus, the observables localized in a double-cone in anti-deSitter space $AdS_{1,1}$ are given by the intersection of two interval algebras $A(I_i)$ on the boundary for intervals I_i with disconnected intersections (or equivalently, by duality, the joint commutant of two interval algebras for disjoint intervals). Such algebras have received much attention in the literature [15, 19, 10], notably within the context of superselection sectors. Namely, if $I_1\cap I_2 = J_1\cup J_2$ consists of two disjoint intervals, then the intersection of algebras $A(I_1)\cap A(I_2)$ is larger than the algebra $A(J_1) \vee A(J_2)$. The excess can be attributed to the existence of superselection sectors [10], the extra operators being intertwiners which transport a superselection charge from one of the intervals J_i to the other.

We conclude that (certain) compactly localized observables on anti-deSitter space correspond to strongly delocalized observables (charge transporters) on the boundary. Yet there is no obstruction to have conventional point-like localized Wightman fields in both theories.

In order to present a simple example with point-like fields we choose suitable coordinates $x_\mu = R\cdot(\cos t, \cos x, \sin t)/\sin x$ such that the bulk is the strip $(t, x) \in \mathbb{R}\times(0, \pi)$ with points $(t, x) \sim (t + \pi, \pi - x)$ identified, while the boundary are the points $(0, u)$, $u \in \mathbb{R} \bmod 2\pi$. The metric is a multiple of $dt^2 - dx^2$, thus the light rays emanating from the bulk point (t, x) hit the boundary at the points $u_\pm = t \pm x \bmod 2\pi$. We see that, as the double-cone

O shrinks to a point (t,x) in bulk, the two intervals J_i on the boundary also shrink to points (namely $u_\pm$) while their distance remains finite.

Now consider the abelian current field $j(u)$ on the boundary. The corresponding fields in bulk are the conserved vector current j^μ with components $j^0(t,x) = j(t+x) + j(t-x)$, $j^1(t,x) = -j(t+x) + j(t-x)$, and the fields $\phi_\alpha(t,x) = \exp i\alpha \int_{t-x}^{t+x} j(u)du$ (suitably regularized, of course), $\alpha \in \mathbb{R}$. Since the charge operator $\int_{S^1} j(u)du$ is a number q in each irreducible representation, $\phi_\alpha(t,x)$ may also be represented as $q \exp -i\alpha \int_{t+x}^{t-x+2\pi} j(u)du$ and hence is localized in both complementary intervals $[t-x, t+x]$ and $[t+x-2\pi, t-x]$ which overlap in the points $t-x$ and $t+x$.

For the more expert reader, we mention that our identification of double-cone algebras in bulk with two-interval algebras on the boundary also shows how the notorious difficulty to compute the modular group for two-interval algebras [15] is related to the difficulty to compute the modular group of double-cone algebras in massive theories. (We discuss below that, as the size of the double-cones shrinks and the massive anti-deSitter theory approaches a conformal flat space theory, the modular group can again be computed.)

We now prove the Proposition of Sect. 1. It asserts that the algebras $B(O_n)$ generate $B(pW)$ whenever a family of double-cones $O_n \subset pW$ covers the wedge $pW \subset AdS_{1,1}$.

Each $B(O_n)$ is of the form $A(I_{n1}) \cap A(I_{n2})$ where $I_{n1} \subset I = \alpha(pW)$ and $I_{n1} \cap I_{n2} = J_{n1} \cup J_{n2}$ is a union of two disjoint intervals. By definition, the assertion is equivalent to

$$A(I) = \bigvee_n A(I_{n1}) \cap A(I_{n2}),$$

where the inclusion "$\supset$" holds since each $A(I_{n1})$ is contained in $A(I)$. On the other hand, the algebras on the right hand side are larger than $A(J_{n1}) \vee A(J_{n2})$. If O_n cover the wedge pW, then the intervals J_{n1} and J_{n2}, as n runs, cover the interval $I = \alpha(pW)$. So the claim follows from weak additivity of the boundary theory. □

In $d \geq 2+1$ dimensions, the situation is drastically different. For simplicity, let $d = 2+1$, thus double-cones on the boundary are diamonds in two-dimensional Minkowski space. Consider a double-cone I_1 and a space-like translate I_2 thereof such that their intersection is a third double-cone I. If pW_i are the corresponding wedge regions in $AdS_{1,2}$, then – and this marks the difference to $1+1$ dimensions – pW is strictly smaller than the intersection $pW_1 \cap pW_2$. In fact, the latter contains some bulk double-cone O which belongs to the causal complement of pW. Thus, the identification of observables (Corollary 1) implies that $B(O)$ is contained in $A(I_1) \cap A(I_2) \cap A(I)'$.

Now, in the commonly studied models of 1+1-dimensional conformal quantum field theory based on Wightman fields (with the localization of operators determined in terms of smearing functions), $A(I_1) \cap A(I_2)$ is equal to $A(I)$, and the latter has no intersection with its own commutant. Thus, $B(O)$ must be trivial. Conversely, if there are double-cone localized bulk observables (e.g., if the bulk theory is itself described by a Wightman field [6]), then either $A(I_1) \cap A(I_2)$ must be strictly larger than $A(I)$, or $A(I)$ must have a nontrivial center. The second alternative can be ruled out without loss of generality by passing to an irreducible representation. The first alternative is – under the mild assumption of essential duality – equivalent to the violation of the additivity property $A(J) = A(J_1) \vee A(J_2)$ whenever J_1 and J_2 are two double-cones (diamonds) which cover the space-like basis of the double-cone J. But this additivity property of the boundary theory holds whenever its dynamics is generated by a Hamiltonian which is an integral over a local energy density, because the observables in a neighborhood of the space-like basis of J algebraically determine all observables within J. Its violation (existence of compactly localized bulk observables) therefore excludes the existence of a stress-energy tensor in the boundary theory.

These issues certainly deserve a more detailed and caraful analysis. For the moment, we conclude that the holographic correspondence necessarily relates, in more than 1+1 dimensions, boundary theories with a stress-energy tensor to bulk theories without compactly localized observables (topological theories), in agreement with a remark on Chern-Simons theories in [18], and, conversely, bulk theories with point-like fields to boundary theories which violate additivity as above.

3 Speculations

It is an interesting side-aspect of the last remark in the previous section that one may exploit the holographic correspondence, in both directions, as a device to construct quantum field theories of unconventional type from, say, Wightman theories. It might therefore provide a constructive scheme giving access to new classes of quantum field theoretical models.

If one is interested in quantum field theories on Minkowski rather than anti-deSitter space, one may consider the flat space limit in which the curvature radius R of anti-deSitter space tends to infinity, or equivalently consider a region of anti-deSitter space which is much smaller than the curvature radius. The regime $|x| << R$ asymptotically becomes flat Minkowski space, and the anti-deSitter group contracts to the Poincaré group. Thus, one obtains a Minkowski space theory on $\mathbb{R}^{1,s}$ from a conformal theory on $CM_{1,s-1}$ through a scaling limit [3] of the associated theory on $AdS_{1,s}$.

For $d = 1+1$, this can be done quite explicitly. The double-cones algebras $B(O)$ are certain extensions of the algebras $A(J_1) \vee A(J_2)$, as discussed before. Now in the flat regime the intervals J_i become small of order $|O|/R$. Thus for a substantial portion of Minkowski space, the relevant intervals J_i are all contained in a suitable but fixed pair of non-overlapping intervals K_i. Let us now assume that the conformal net has the split property (an algebraic property valid in any chiral quantum field theory for which Tr exp $-\beta L_0$ exists), which ensures that states can be independently prepared on causally disjoint regions with a finite distance [1]. Then $A(K_1) \vee A(K_2)$ is unitarily isomorphic to $A(K_1) \otimes A(K_2)$, and the isomorphism is inherited by all its subalgebras $A(J_1) \vee A(J_2) \simeq A(J_1) \otimes A(J_2)$. Under this isomorphism, the larger algebra $B(O)$ is identified [10] with the standard construction [12] of 1+1-dimensional conformal Minkowski space observables from a given chiral conformal net (which corresponds to the diagonal modular invariant and is sometimes quoted as the Longo-Rehren net): $B(O) \simeq B_{\rm LR}(J_1 \times J_2)$ if O corresponds to $I_1 \cap I_2 = J_1 \cup J_2 \subset K_1 \cup K_2$. The unitary isomorphism of algebras, however, does not take the vacuum state on B to the vacuum state on the LR net.

Thus, the flat space limit of the anti-deSitter space theory in 1+1 dimensions associated with a given chiral conformal theory, is given by the LR net associated with that same chiral theory. Note that the LR net has 1+1-dimensional conformal symmetry, but of course the anti-deSitter net is not conformally invariant due to the presence of the curvature scale R.

It would be interesting to get an analogous understanding of the flat space limit of the anti-deSitter space theory in higher dimensions in terms of the associated conformal theory.

One might speculate whether one can "iterate holography", and use the flat space limit of the bulk theory on $AdS_{1,s}$ as a boundary input for a new bulk theory on $AdS_{1,s+1}$. Here, however, a warning is in order. Namely, the limiting flat space theory on Minkowski space $\mathbb{R}^{1,s}$ will, like the LR net, in general not be extendible to the conformal compactification $CM_{1,s}$ but rather to a covering thereof. One should therefore endeavour to extend the present analysis to theories on covering spaces both of anti-deSitter space and of its boundary.

There is an independent and physically motivated reason to study quantum field theories on a covering of anti-deSitter space. Namely, it has been observed (see above, [2]) that the local commutativity for causally disjoint but not causally disconnected observables leads to severe constraints on the possible interactions on anti-deSitter space proper. These constraints will disappear on the universal covering space.

This "anti-deSitter causality paradox" parallels very much the old "conformal causality paradox" that causal commutativity on $CM_{1,s}$ proper excludes most conformal theories of interest; it was solved [14] by the recognition that conformal fields naturally live on a covering space. Holography tells us that both problems are the two sides of the same coin.

Extending the present analysis to covering spaces seems a dubious task for $d = 1 + 1$ since the boundary of the covering of two-dimensional anti-deSitter space has two connected components. In higher diemsnions, however, we do not expect any serious obstacles.

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