Positive Kolmogorov-Sinai entropy for the Standard map

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Abstract

We prove that the Kolmogorov-Sinai entropy of the Chirikov-Standard map $T_{\lambda f} : (x, y) \mapsto (2x - y + \lambda f(x), x)$ with $f(x) = \sin(x)$ with respect to the invariant Lebesgue measure on the two-dimensional torus is bounded below by $\log(\lambda/2) - C(\lambda)$ with $C(\lambda) = \text{arcsinh}(1/\lambda) + \log(2/\sqrt{3})$. For $\lambda > \lambda_0 = (8/(6 - 3\sqrt{3}))^{1/2} = 3.1547...$, the entropy of $T_{\lambda \sin}$ is positive. This result is stable in Banach spaces of realanalytic symplectic maps: each Chirikov-Standard map with $\lambda > \lambda_0$ is contained in an open set of real-analytic, in general nonergodic area-preserving diffeomorphisms with positive entropy. Moreover, there is a C^0-dense set of realanalytic maps f in $C^0(\mathbf{T})$ for which the entropy of T_f is positive. The Lyapunov exponent estimates hold for a fixed cocycle uniformly for the entire group of measure preserving maps on the torus. This establishes new families of discrete ergodic one-dimensional Schrödinger operators $(Lu)_n = u_{n+1} + u_{n-1} + \lambda \cos(x_n) u_n$ with no absolutely continuous spectrum.

1 The entropy of the Standard map

The Chirikov-Taylor Standard map

$$T_{\lambda f} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2x - y + \lambda f(x) \\ x \end{pmatrix}$$

with $f(x) = \sin(x)$ and real parameter λ is a measure preserving diffeomorphisms on the two-dimensional torus $\mathbf{T}^2 = \mathbf{R}^2/(2\pi\mathbf{Z})^2$. It is probably the most famous example of a symplectic twist map. This map appeared in 1960 in the context of electron dynamics in microtrons (see [20]). It was first numerically studied by Taylor in 1968 and Chirikov in 1969 (these independent studies are unpublished but see [37, 19]) and also known under the name "kicked rotator" and describing ground states of the Frenkel-Kontorova model [79, 7]. It is often used to illustrate or motivate various more general mathematical theorems in smooth dynamical systems (i.e. [131]) the calculus of variations (i.e. [99, 98]) perturbation theory or renormalisation group techniques (for example to understand the break up of invariant tori [89, 135, 136]). While it is known that for $\lambda \neq 0$, the map $T_{\lambda f}$ is non-integrable, has positive topological entropy and horseshoes (i.e. [36, 4, 44]), the question, whether hyperbolicity can hold on a set of positive Lebesgue measure stayed open. Similarly, while many Lebesgue measure preserving diffeomorphism on the torus are known to be nonergodic with positive topological entropy (i.e. [140, 157]), it is not known whether positive metric entropy is dense in the C^∞ topology. The issue of the positivity of the Lyapunov exponents on some set of positive Lebesgue measure for Hamiltonian systems has been addressed at various places or reviews [105, 157, 19, 145, 88, 91, 52, 134, 137, 16, 24, 69, 32, 155, 90, 142]. According to [113, 32], the particular mathematical problem of positive entropy of the Chirikov Standard map

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was promoted in the early 80'ies by Sinai. The textbook [131] states on p. 144 a conjecture (H2) that the entropy of the Chirikov Standard map is positive for all $\lambda > 0$ and that the entropy grows to infinity for $\lambda \to \infty$.

Although numerical experiments show a very clear lower bound $\log(|\lambda|/2)$ for the entropy of the Chirikov Standard map (e.g. [37, 19, 115]), it was not known even for one single value of λ, whether the metric entropy with respect to the invariant Lebesgue measure can be positive. The following theorem solves this problem and the second part of the (H2) conjecture of Sinai. It confirms the numerical findings quantitatively.

Theorem 1.1 *The Kolmogorov-Sinai entropy $\mu(T_{\lambda \sin})$ of the Chirikov-Standard map $T_{\lambda \sin}$ with respect to the invariant Lebesgue measure is bounded below by $\log(|\lambda|/2) - C(\lambda)$ where $C(\lambda) = \text{arcsinh}(1/\lambda) + \log(2/\sqrt{3})$. The entropy is positive for $\lambda > \lambda_0 = 2\sqrt{2/(6-3\sqrt{3})} = 3.1547....$*

Numerical experiments tell us that with a more sophisticated proof one could probably get rid of the $C(\lambda)$-term in the case of the Standard map. We will see below that with a numerical evaluation of an integral, we can establish positive entropy for $\lambda > \lambda_2 = 2.84661$ and that with more effort, one could push down the theoretical threshold even further.

The result extends to more general Standard maps. One obtains for example explicit lower bounds $\log(|m\lambda|/2) - C(m\lambda)$ if $\lambda \sin(x)$ is replaced by $Ex + \lambda \sin(mx)$, where the integer $E \in \mathbf{Z}$ allows additionally to tune the homotopy type of the map.

This property of positive metric entropy is stable in the real analytic category and dense in the $C^0(\mathbf{T})$ topology:

Theorem 1.2 *There is in $C^0(\mathbf{T})$ a C^0-dense set of real-analytic Standard maps $f : \mathbf{T}^1 \mapsto \mathbf{R}$ for which $\mu(T_f) > 0$. Let f be in this dense set. In every Banach space of realanalytic maps, in which f is, there is an open neighborhood of realanalytic maps g, for which $\mu(T_g) > 0$ also.*

In other words, the so established sensitive dependence on initial conditions is persistent in the realanalytic category and can be obtained by realanalytic, $C^0(\mathbf{T})$-small perturbations of integrable maps. The stability of positive metric entropy with respect to realanalytic perturbations of the map makes Theorem (1.1) physically relevant. Other results in Hamiltonian dynamics that have both this stability and which deal with orbits forming a set of positive probability is the theory of Anosov maps (see i.e. [81, 152, 65]) (an example is $(x, y) \mapsto (4x + \lambda \sin(x) - y, x)$ for small λ) or KAM perturbation theory (see i.e. [105, 57, 67]) (which applies for example near the integrable SBKP map ([139, 12]) $(x, y) \mapsto (2x - y + 4\text{arg}(1 + \lambda \exp(-ix)), x)$). We establish so for the first time open sets of realanalytic Hamiltonian maps and flows for which there is quasiperiodic motion on a set of positive Lebesgue measure and simultaneously a conjugation to a Markov chain on a different set of positive Lebesgue measure. [1] These examples can be chosen in any homotopy type of Standard maps.

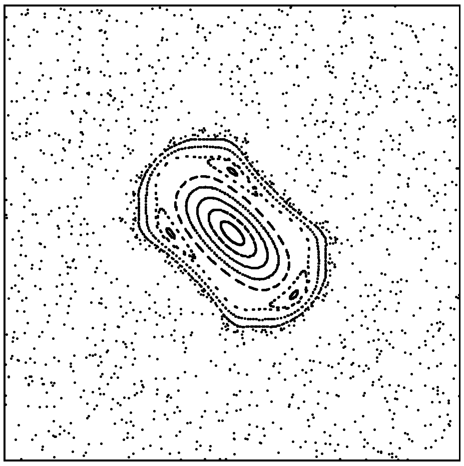
[1] We discarded in this statement a trivial way to achieve such mixed behavior on the union of two closed symplectic manifolds, where the map is Anosov on one component and integrable on the other. For smooth, not realanalytic examples in the literature that are unstable under perturbations of the map, see 13) in the discussion part.

The results are not restricted to maps on the two-dimensional torus. They could be extended to certain higher-dimensional symplectic maps $(x, y) \mapsto (Ex - y + \lambda f(x), x)$ on $\mathbf{T}^{2d}$, where E is a constant symmetric matrix in $GL(d, \mathbf{Z})$ and f is a vector valued, real-analytic function on the torus. Examples are Froeschle maps, where $Ex = 2x$, $f_i(x) = \lambda_i \sin(x_i) + \sin(\sum_j x_j)$ or classes of nearest neighbor coupled map lattices [63] on $\mathbf{T}^{2d}$, where $Ex = 2x$, $f_i(x) = \lambda \sin(x_i) + \epsilon(\sin(x_{i+1} - x_i) + \sin(x_i - x_{i-1}))$ and $x_{i+d} = x_i$. All averaged Lyapunov exponents are then nonzero for large enough λ.

As for any diffeomorphism T on a compact manifold M leaving invariant a smooth measure, the entropy is by the Pesin formula [92] equal to the sum of the positive integrated Lyapunov exponents. In the two-dimensional case, the entropy is $\lim_{n\to\infty} n^{-1} \int_M \log ||dT^n(x,y)||\ dxdy$. Since an estimate of the norm of the Jacobean of the Chirikov-Standard map $T_{\lambda \sin}$ shows that the integrated Lyapunov exponent is bounded above by $\int_{\mathbf{T}^2} \log ||dT_{\lambda \sin}(x,y)||\ dxdy < \log(\lambda/2) + c(\lambda)$ with $c(\lambda) = O(1/\lambda)$, Theorem 1.1 provides for large λ a rather accurate estimate for the actual values of the entropy of the Chirikov Standard map and shows that the measure of the Pesin region, the set on which the Lyapunov exponents is positive goes to 1, when $|\lambda| \to \infty$. In 9) of the discussion section of this article, explicit lower bounds on the area of the Pesin region are provided.

By Pesin theory [117, 94, 66, 118, 65], positive metric entropy (and in higher dimensions the nonvanishing of all Lyapunov exponents on a set of positive measure) implies the existence of invariant sets of positive Lebesgue measure on which T_λ is ergodic and on which some iterate of T_λ is mixing and actually measure-theoretically conjugated to a Bernoulli shift. Other consequences are the density of periodic orbits in the Pesin region and shadowing properties (see [64, 65]).

An obstacle for proving positive entropy estimates for the Standard map is Donskaya's observation that elliptic islands can exist for arbitrary large λ [19, 18, 86, 32] and which make it hard to find invariant cone bundles [148, 149] in the tangent bundle. In [32] it has been shown using renormalisation of coordinates near homoclinic tangencies [104] that for a residual set of large parameters λ one has Standard maps $T_{\lambda \sin}$ which are nonergodic.



Chirikov had formulated already concerns that the measure of the stable elliptic component might be close to 1 ([19] p. 333). It has been often asked whether an area-preserving monotone twist map has a dense set of elliptic islands in general (see e.g. [52, 112]). It is known that a Baire generic symplectic non-Anosov C^1-diffeomorphism has a dense set of elliptic periodic orbits [109]. Some research towards avoiding elliptic islands by smooth ergodic perturbations of the Chirikov Standard map has been done [132]. It is possible to get positive entropy by a smooth C^1 perturbation of the map [58]. Whether this is possible with C^∞ perturbations is not known [55].

The question whether a dense set in $\mathrm{Diff}^\infty_\mu(M)$ of measure preserving diffeomorphisms on a manifold M has positive metric entropy has been asked in [59].

It has been conjectured that there exists a set of parameters λ with full density at ∞ for which the Chirikov Standard map has no elliptic islands [16] (see also [32]).

Our proof of positive metric entropy depends on spectral and complex analytic techniques as well as the determinant theory of finite von Neumann algebras and avoids completely the ergodicity question. We obtain many non-ergodic realanalytic Standard maps with positive metric entropy. The picture above for example shows plots of some orbits of the Chirikov Standard map in the case $\lambda = 3.4$, where a linearly stable fixed point $(1/2, 1/2)$ coexists with a region with positive metric entropy.

There are Standard maps $T_{\lambda \sin}$ arbitrarily close to this map $T_{3.4 \sin}$ for which the Birkhoff normal form at $(1/2, 1/2)$ is such that the fixed point is surrounded by invariant KAM curves of positive Lebesgue measure and for which we still have ergodic components of positive Lebesgue measure. Together with [32], we know that there exist for large λ a open dense set of parameters for the Chirikov Standard map which lead to nonergodicity and positive Kolmogorov-Sinai entropy.

2 More general results on positive Lyapunov exponents

We formulate now more general theorems which imply the entropy results formulated above. Consider the matrix-valued map

$$(x,y) \in \mathbf{T}^2 \mapsto A_{E,\lambda f}(x,y) = \begin{pmatrix} E + \lambda f(x) & -1 \\ 1 & 0 \end{pmatrix} .$$

Together with a Lebesgue measure-preserving dynamical system $T : \mathbf{T}^2 \to \mathbf{T}^2$, it defines a cocycle $(x,y,n) \mapsto A^n_{E,\lambda f,T}(x,y) = A_{E,\lambda f}(T^{n-1}(x,y)) \circ \cdots \circ A_{E,\lambda f}(x,y)$ for which the integrated Lyapunov exponent

$$\mu(A_{E,\lambda f,T}) = \lim_{n \to \infty} \frac{1}{n} \int_{\mathbf{T}^2} \log ||A^n_{E,\lambda f,T}(x,y)|| \; dx \; dy$$

is defined. For general information on Lyapunov exponents, see for example [125, 84].

Denote by $\mathcal{X}$ the group of all homeomorphisms of the torus $\mathbf{T}^d$ which preserve the Lebesgue measure dx. Denote by $\mathcal{Y}$ the group of measure preserving automorphisms of the Lebesgue space $(\mathbf{T}^d, dx)$. We will mainly focus on the case $d = 2$.

Theorem 2.1 *For every $T \in \mathcal{Y}$, for all $E \in \mathbf{R}$ and all $\lambda \in \mathbf{R}$, one has $\mu(A_{E,\lambda \cos,T}) \geq \log(|\lambda|/2) - C(\lambda)$.*

It might come as a surprise to find that these estimates hold uniformly for all automorphisms T but this fact is actually responsible for a relatively simple proof.

Theorem 2.1 immediately implies Theorem 1.1.

Proof. Pesin's formula links the entropy of T_f with the Lyapunov exponent of dT_f and so with the Lyapunov exponent of the cocycle because $A_{E=2,df,T_f}$ agrees with the Jacobean dT_f of the map T_f.

Consider for $0 < \delta < 1$ the norm $||f||_\delta = \sup_{|z| \in [\delta, \delta^{-1}]} |f(z)|$ on the class A_δ of real-analytic periodic maps $x \mapsto f(e^{ix})$ on $\mathbf{T}^1 = \{|z| = 1\}$ which are analytic in the annulus $\{|z| \in (\delta, \delta^{-1})\ \}$.

Theorem 2.2 *Fix $\delta > 0$. Let $C(\lambda)$ be the function appearing in Theorem (2.1). There exist functions $\epsilon = \epsilon(\lambda) > 0, \beta = \beta(\lambda) > 0$ satisfying $\epsilon(\lambda) \to \infty, \beta(\lambda) \to 0$ for $\lambda \to \infty$, such that for all $f \in A_\delta$ with $||f||_\delta < \epsilon(\lambda)$ and $E \in \mathbf{R}$, one has $\mu(A_{E, \lambda \cos + f, T}) > \log(\lambda/2) - C(\lambda) - \beta(\lambda)$, uniformly for all $T \in \mathcal{Y}$.*

Theorem 2.2 implies Theorem 1.2:

Proof. We use the notation $\sin_m(x) = \sin(mx)$ and $\cos_m(x) = \cos(mx)$.

(i) $\mu(T_{\lambda \sin_m}) > \log|\lambda m/2| - C(m\lambda)$.
Proof. The Jacobean cocycle of $T = T_{\lambda \sin_m}$ is $A_{2, m\lambda \cos_m}$. The map $T_{\lambda \sin_m}$ on $\mathbf{T}^2 = (\mathbf{R}/(2\pi\mathbf{Z}))^2$ is a lift of a measure preserving map T_m which acts also on each of the m^2 tori $(\mathbf{R}/(\frac{2\pi}{m}\mathbf{Z}))^2$ so that $\mu(A_{2,(m\lambda)\cos_m, T}) = \mu(A_{2,(m\lambda)\cos, T_m})$. By Theorem 2.1, we have $\mu(A_{2, m\lambda \cos_m}) \geq \log|m\lambda/2| - C(m\lambda)$.

(ii) Fix $\lambda > 0$. For any trigonometric polynomial p and $\beta > 0$, we have $\mu(T_{\lambda \sin_m + p}) > \log|\lambda m/2| - C(m\lambda) - \beta$ if m is large enough.
Proof. This follows from Theorem 2.2 and (i).

(iii) There exists a dense set of realanalytic $f \in C^0(\mathbf{T})$ for which $\mu_{T_f} > 0$.
Proof. Given $h \in C^0(\mathbf{T})$ (which needs not to be realanalytic) and let $\epsilon > 0$. Using the Weierstrass approximation theorem, choose a trigonometric polynomial p with $|h - p|_{C^0(\mathbf{T})} \leq \epsilon/2$. Choose $0 < |\lambda| < \epsilon/2$. The trigonometric polynomials $f_m(x) = \lambda \sin(mx) + p(x)$ have the property that $|h - f_m|_{C^0(\mathbf{T})} \leq |h - p|_{C^0(\mathbf{T})} + |p - f_m|_{C^0(\mathbf{T})} \leq \epsilon/2 + \epsilon/2 = \epsilon$. By (i) and (ii), we know that $\mu(T_{f_m}) > 0$ for large m.

(iv) Assume $g \in A_\delta$. From Theorem 2.2, we know that $\mu(T_{f_m + g})$ is positive if m is so large that $\epsilon(\lambda m) > ||g||_{A_\delta}$.

3 A nonselfadjoint spectral problem

Consider the random operator

$$(L_w(x,y))u_n = u_{n+1} + u_{n-1} + \lambda(w^{-1}\exp(ix_n) + w\exp(-ix_n))u_n \ , \tag{1}$$

on $l^2(\mathbf{Z}, \mathbf{C})$, where $(x_n, y_n) = T^n(x, y)$ and $w \in \mathbf{C}$ is a complex parameter. For fixed $(x, y) \in \mathbf{T}^2$, this is a bounded operator on $l^2(\mathbf{Z}, \mathbf{C})$. The term 'random' is used because $(x, y) \mapsto L(x, y)$ is an operator valued random variable. Such operators are in a von Neumann algebra with finite trace $\mathrm{tr}(K) = \int_{\mathbf{T}^2} [K(x,y)]_{00}\, dxdy$ (see e.g. [107, 28, 21]).

Similar as in the case, when the complex parameter is the energy, the Lyapunov exponent $\mu(w)$ of the transfer cocycle $A_E(w)$ is a subharmonic function in w. As we will see below, the Thouless formula $\mu(w) = \log(\det(L_w)) = \mathrm{tr}(\log|L_w|)$ is still true for the in general nonselfadjoint operator L_w.

The problem is to compute $\det(L_w)$. The entropy of the Standard map is $\log\det L_{w=1}$.

The average value of the Lyapunov exponent $\mu(w)$ on the circle $|w| = 1$ can be estimated with the subharmonicity argument of Herman [56] or with a Jensen formula as used by Sorets-Spencer in [133]. In higher dimensions, an adaption [48] of [133] to higher-dimensional cocycles would apply.

A theorem of Lax [82] on continuous measure-preserving transformations allows to approximate the group $\mathcal{X}$ of measure preserving homeomorphisms by finite groups $\mathcal{Y}_k \subset \mathcal{Y}$ of measure preserving transformations. If one looks at the value of the Lyapunov exponent of the matrix-valued map $A_{E,\lambda\sin,T}$ on $\mathcal{Y}_k$, (where evaluating the Lyapunov exponent is a reliable finite dimensional integration), one can observe numerically that for every transformation T, one can find a transformation $\hat{T}$ such that $\mu(A_{E,\lambda\sin,T}) + \mu(A_{E,\lambda\sin,\hat{T}}) \geq 2\log(\lambda/2)$. This relation can be understood in terms of determinants.

By looking for a proof of this relation, we were motivated by Aubry duality which can be defined if T is in such a finite group $\mathcal{Y}_k$. Aubry duality is an involutive transformation $L \mapsto \hat{L}$, which preserves the density of states such that $\log(\det(L_\lambda)) = \log(\det((\lambda/2)\hat{L}_{(4/\lambda)})) = \log(\lambda/2) + \log\det(\hat{L}_{(4/\lambda)})$. In the Mathieu case $T(x,y) = (x+y,y)$, where $L_\lambda = \hat{L}_\lambda$, this leads with $\log(\det(\hat{L}_{4/\lambda})) \geq 0$ to the estimate $\log\det(L_\lambda) \geq \log(\lambda/2)$ which is expected to be true also if T is the Standard map. Here, we will show that there exists for each transformation T and corresponding operator $L = L_{T,\lambda}$ a different transformation S and operator $\hat{L} = L_{S,\lambda}$ such that $\log(\det(L\hat{L})) \geq 2\log(\lambda/2)$. Together with $\log(\det(\hat{L})) \leq \log(\lambda/2) + C(\lambda)$, we will obtain $\log(\det(L)) \geq \log(\lambda/2) - C(\lambda)$.

How is the operator $\hat{L}$ obtained? The group $R_\alpha : (x,y) \mapsto (x+\alpha,y)$ of translations on the torus acts by conjugation on the whole group $\mathcal{Y}$ of measure preserving transformations. The orbits of this action are $T_\alpha = R_\alpha T R_{-\alpha}$. The operator $\hat{L}$ will be obtained by changing T to one of the conjugates T_α. The value of the Lyapunov exponent $w = \exp(i\alpha) \mapsto \mu(A_{E,\lambda f,T_\alpha})$ extends to a subharmonic function from the circle $|w| = 1$ to the entire complex plane $\mathbf{C}$. If one considers the complex parameter w as a 'spectral parameter', the Riesz measure dk used to represent the Lyapunov exponent plays the role of the density of states. It is a measure in the complex plane which has in our case in general positive area in the complex plane. The measure dk has its support in an annular neighborhood of the unit circle (and an atom at 0).

Why does one use the artificial spectral parameter w and not simply complexity the variable x on the torus? The reason is that with respect to $z = re^{ix}$, there is no analyticity of $[z^n A^n(z,y)]_{ij}$.

(The last point is crucial. The motivation is trying to generalize results for maps which extend analytically to polydiscs like for $T(x,y) = (x+\alpha,y)$, where both Aubry duality [8] as well as Herman's subharmonicity [56] work directly. While the map $(te^{ix}, e^{iy}) \mapsto (te^{i(2x-y+\lambda\sin(x))}, e^{ix})$ (or any similar complexification attempt) is real-analytic in each of the variables t,x,y, it is definitely not analytic with respect to $z = te^{ix}$ and functions which needed to be subharmonic are not. Our first attack on the Standard map using plurisubharmonicity was done in the spring 1988, now more than eleven years ago. The approach was the observation that the Standard map family written in Hamiltonian form $(x,y) \mapsto (x+y+\lambda\sin(x), y+\lambda\sin(x))$ is induced on invariant tori of the single analytic map $U : (z,w,u,v) \mapsto (zwe^{z-u}, we^{z-u}, uve^{u-z}, ve^{u-z})$ on $\mathbf{C}^4$, and where nonanalyticity manifests itself that the U-invariant two-dimensional tori $S_\lambda = \{(z,w,u,v) \mid |z| = |u| = \lambda/2,\ |w| = |v| = 1\ z = \overline{u}, w = \overline{v}\}$ on which U induces T_λ, are not distinguished boundaries of polydiscs. Later attempts were to Vlasov-Toda deform the cocycle with the aim to estimate the

Lyapunov exponent on tori of isospectrally deformed operators (on which the Lyapunov exponent is constant) [71, 70] or to fix the dynamical system and to vary the cocycle [69]).

Because $w \mapsto g_n(w,x,y) = w^n[A_E^n(w,x,y)]_{11}$ is analytic, the $r = |w|$ dependence of the local Lyapunov exponent can be related with the average angular dependence of the argument: in polar coordinates $w = re^{i\phi}$, the Cauchy-Riemann differential equations

$$\frac{d}{d\phi}\frac{1}{r}\int_{\mathbf{T}^2} \arg(g_n(w,x,y))\ dxdy = \frac{d}{dr}\int_{\mathbf{T}^2} \log|g_n(w,x,y)|\ dxdy$$

are valid in any connected region of the w- resolvent set. In the complement, in the support of the w-spectrum, this formula can not be used and an integrated version, a Jensen formula replaces it. Besides the contribution of the radial change of the argument, there is an additional nonnegative subharmonic contribution to the Lyapunov exponent. For a pure complex analytic proof we hope to estimate both parts as well as possible.

Our paper contains more material than is actually necessary to prove the theorems. As the discussion part shows, many open questions are left and we think that more can be achieved through a more thorough analysis of the non-selfadjoint spectral problem to determine $w \mapsto \det(L_w)$ which is introduced here and which is at the heart of the matter. Methods of complex analysis, potential theory and spectral theory together could probably lead to sharper results. Any improvement for example in estimating the positive value of integrals like

$$\int_{|w|=1} \log(\det(L_w) - \log(\lambda/2) = -\int_{\{0<|z|<1\}} \log|E|\ dk(E) > 0\ ,$$

where dk is the 'w-density of states of $w \mapsto L_w$' would also improve on the lower bound on the Lyapunov exponent of the Standard map.

4 The main proposition and more heuristics

This section contains the formulation of the main proposition which will imply Theorem 2.1. It also includes some heuristic reasoning, which lead us to the proof of the proposition. Let $A_{E,\lambda\cos}(z,y) = \begin{pmatrix} E + \frac{\lambda}{2}(z+z^{-1}) & -1 \\ 1 & 0 \end{pmatrix}$.

Proposition 4.1 *For all $T \in \mathcal{Y}$, for all $E \in \mathbf{R}$ and for all $w \in \mathbf{C}$, $|w| = 1$, we have*

$$\mu(A_{w,E,\lambda\cos,T}) = \lim_{n\to\infty}\frac{1}{n}\int_{\mathbf{T}^2} \log||A^n_{E,\lambda f,T}(w^{-1}e^{ix},y)||\ dxdy \geq \log(\frac{|\lambda|}{2}) - C(\lambda)\ ,$$

where $C(\lambda) = \operatorname{arcsinh}(1/\lambda) + \log(2/\sqrt{3})$.

Theorem 2.1 follows from the stronger statement in Proposition 4.1.

In situations like the Mathieu case $T(x,y) = (x+y,y)$, where T extends to an analytic map on $\mathbf{D}\times\mathbf{T}$ with respect to $z = r\exp(ix)$, Herman's subharmonicity [56] gives $\mu(A_T) - \log(\lambda/2) \geq 0$. For a general map $T \in \mathcal{Y}$ which does no more commute with $R_\alpha : (x,y) \mapsto (x+\alpha,y)$ on $\mathbf{T}^2$, the function $w = \exp(i\alpha) \mapsto \mu(A_{E,\lambda\sin,T_\alpha})$ with $T_\alpha = R_\alpha T R_{-\alpha}$ is no more constant. In average, we have

$\int_{\mathbf{T}} \mu(A_{E,\lambda \sin,T_\alpha}) \, d\alpha \geq \log(\lambda/2)$. Furthermore $\mu(A_{E,\lambda \sin,T_\alpha})$ is bounded above by $\log(\lambda/2) + c(\lambda)$ with $c(\lambda) = O(1/\lambda)$. This means that for large λ, we have $\mu(A_{E,\lambda \sin,T_\alpha}) \geq \log(\lambda/2)$ for a large set of α's. (We used such a fact in a similar way in [69]).

However, we don't know the specific value of the upper-continuous function $\alpha \mapsto \mu(A_{E,\lambda \sin,T_\alpha})$ at the point $\alpha = 0$, which we are interested in. It could be zero a priori. Numerical computations however show that the minimum of $\alpha \mapsto \mu(A_{E,\lambda \sin,T_\alpha})$ is not farther away from the mean value than the maximum. This is the statement of Proposition 4.1. Why is this the case?

There is a heuristic explanation which uses the Jensen formula in a sector and which will assume $T \in \mathcal{X}$. This Jensen formula (see Section 13) is essentially an integrated Cauchy-Riemann differential equation and sharpens the subharmonicity tool. For now, it serves us only on a heuristic level. We also use the Lax approximation theorem (see Section 14) to explain the now following heuristics.

The Lyapunov exponent of the matrix-valued map $(x, y) \in \mathbf{T}^2 \mapsto B(e^{ix})$ with

$$B(z) = zA_{E,\lambda \cos}(z) = z \begin{pmatrix} E + \frac{\lambda}{2}(z + z^{-1}) & -1 \\ 1 & 0 \end{pmatrix}$$

is the same as the Lyapunov exponent of the cocycle $(x, y) \mapsto A_{E,\lambda \cos}(x)$ because $|z| = 1$.

Define for $(x, y) \in \mathbf{T}^2$ and $n \in \mathbf{N}$ the complex function

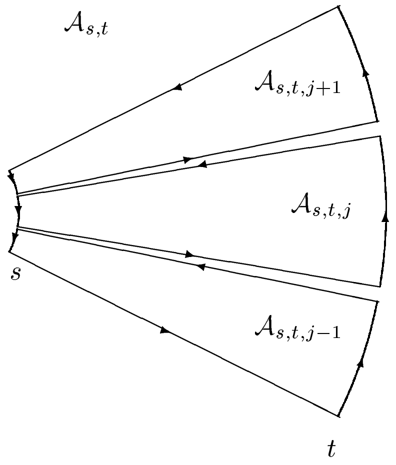
$$w \mapsto g_{n,T}(w^{-1} \exp(ix), y) = [B^n_{E,\lambda \cos,T}(w^{-1} \exp(ix), y)]_{11}$$

which is analytic in the complex plane.

Denote by $\mathrm{Arg}_{[z_1,z_2]}(g_{n,T})$ the argument change of the analytic function $g_{n,T}$ on the line from z_1 to z_2. Unlike $\arg(g_{n,T})$, the argument change $\mathrm{Arg}_{[z_1,z_2]}(g_{n,T})$ is a uniquely defined number.

Let T_k be a cyclic Lax cube exchange transformation approximating a homeomorphism $T \in \mathcal{X}$. The map T_k is extends from $\mathbf{T}^2$ to $\mathbf{D} \times \mathbf{T}$ by setting $T_k(r \exp(ix), y) = (r \exp(ix_1), y_1)$ with $T_k(x, y) = (x_1, y_1)$.

In each sector, we could also take the z variable instead of the w variable because $z \mapsto A^n(z, y)$ is analytic in each sector $\mathcal{A}_{s,t,j} = \{w \in \mathbf{C} \mid |w| \in [s, t], \arg(w) \in [2\pi j/k, 2\pi(j+1)/k) \}$.



Let $\Gamma^+(\alpha)$ be path connecting $se^{i\alpha}$ and $te^{i\alpha}$ on a radial line segment, Γ^- the reversed path. Let $\gamma_s^+(\alpha,\beta) = \{w, |w| = s, \arg(w) \in [\alpha,\beta]\}$ be a circular arc contained in the circle $\gamma_s = \{|w| = s\}$ and $\gamma_s^-(\alpha,\beta)$ the reversed path. The difference between the argument change of the function $g_{n,T}(w) = [B^n_{E,\lambda\sin,T}(w)]_{11}$ along the circles γ_s^- and γ_t^+ enclosing the annulus $\mathcal{A}_{s,t} = \{|w| \in [s,t]\}$ is the sum of the k angular argument changes $\mathrm{Arg}_{\gamma_t^+(2\pi j/k, 2\pi(j+1)/k)}(g_{n,T}) - \mathrm{Arg}_{\gamma_s^-(2\pi j/k, 2\pi(j+1)/k)}(g_{n,T})$ plus the sum of the k radial argument changes $\mathrm{Arg}_{\Gamma^+(2\pi j/k)}(g_{n,T}) - \mathrm{Arg}_{\Gamma^-(2\pi(j+1)/k)}(g_{n,T})$ with $j = 1,\dots,k$.

The total sum of the radial plus angular argument changes is nonnegative because it is the sum of the indices of the roots of $z \mapsto g_{n,T}$ inside the annulus $\mathcal{A}_{s,t}$, all of which are nonnegative.

The sum of the k radial argument changes is bounded above by a $O(1/\lambda)$-term, because a positive value contributes positively to the Lyapunov exponent estimate and the Lyapunov exponent is bounded above by $\log(\lambda/2) + c(\lambda)$ with $c(\lambda) = O(1/\lambda)$. There is no reason, why the sum of the k radial argument changes should differ much from its negative. Numerical experiments strongly confirmed that they don't. If this sum of the radial argument changes

$$\sum_{j=1}^{k} \mathrm{Arg}_{\Gamma^+(2\pi j/k)}(g_{n,T}) - \mathrm{Arg}_{\Gamma^-(2\pi(j+1)/k)}(g_{n,T}) \tag{2}$$

(which is averaged over $\mathbf{T}^2$ and bounded uniformly in k) would vanish, one would have the same estimates as in the subharmonic case and

$$n^{-1}\int_{\mathbf{T}^2} \log|g_{n,T}(x,y,w)|\; dxdy \geq n^{-1}\int_{\mathbf{T}^2} \log|g_{n,T}(x,y,0)|\; dxdy = \log(\lambda/2)$$

for all $|w| = 1$. This would imply $\mu(A_{E,\lambda\cos,T}) \geq \log(\lambda/2)$.

While the sum of the radial argument changes (2) does not vanish for a general $T \in \mathcal{Y}$, numerical data makes (2) appear to be small. Indeed, we will see that for every transformation $T \in \mathcal{Y}$, there is an other transformation $S_\alpha = R_\alpha T R_{-\alpha}$ such that the Lyapunov exponent of A_T plus the Lyapunov exponent of A_S beats $2\log(\lambda/2)$. This will imply that the fluctuations of

$$\alpha \mapsto \mu_n(\alpha) = n^{-1}\int_{\mathbf{T}^2} \log|[A^n(e^{i(\alpha+x)}, y)]_{11}|\; dxdy$$

around its mean $\int_{\mathbf{T}} \mu_n(\alpha)\; d\alpha \geq \log(\lambda/2)$ can not get too large.

We have not yet been able to prove such a statement using complex analytic methods only. We have to use the operator theoretical aspect of the situation also.

5 The spectrum of periodic difference operators

In this section we review some facts about not necessarily selfadjoint periodic Jacobi matrices and higher order difference operators. Unlike in the selfadjoint case, when the spectrum is confined to the real axes, the topology of the spectrum can then be more interesting. All we need to know for our purposes is that the spectrum is contained in a nowhere dense set.

While nonselfadjoint differential operators have been studied quite a bit by Russian mathematicians, nonselfadjoint higher order difference operators have appeared less frequently in the literature. But they do occur: these operators appear naturally when studying the stability of stationary solutions in coupled nonharmonic oscillators [144, 62]. For nonselfadjoint Schrödinger differential operators, see [45] and references.

Jacobi matrices. Three vectors $\underline{a}, \underline{b}, \underline{c} \in \mathbf{C}^p$ define periodic sequences $a_n = a_{n+p}, b_n = b_{n+p}, c_n = c_{n+p} \in l^\infty(\mathbf{Z}, \mathbf{C})$ which can be used to build a p-periodic Jacobi matrix

$$(Lu)_n = a_n u_{n+1} + c_{n-1} u_{n-1} + b_n u_n$$

which is a bounded operator on on $l^2(\mathbf{Z}, \mathbf{C})$. Denote by L_{per} the same operator L but acting on a different Hilbert space, namely the finite dimensional space of p-periodic sequences equipped with the norm $(\sum_{i=1}^p |u_i|^2)^{1/2}$. For $w \in \mathbf{C}$, (a parameter which has nothing to do with the complex parameter w in the operator (1)), this operator is given by the matrix

$$L_{per}(w) = \begin{pmatrix} b_1 & a_1 & 0 & \cdot & 0 & w^{-1}c_p \\ c_1 & b_2 & a_2 & \cdot & \cdot & 0 \\ 0 & c_2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & a_{p-2} & 0 \\ 0 & \cdot & \cdot & c_{p-2} & b_{p-1} & a_{p-1} \\ wa_p & 0 & \cdot & 0 & c_{p-1} & b_p \end{pmatrix}$$

and satisfies $L_{per} = L_{per}(1)$. Denote by $\Delta(z,w) = \det(L_{per}(w) - z)$ the characteristic polynomial of $L_{per}(w)$ and by $\Delta(z) = \Delta(z,1)$ the characteristic polynomial of L_{per}. Comparing coefficients in w and using $\Delta(z,1) = \Delta(z)$, one gets

$$\Delta(z,w) = \Delta(z) - aw - cw^{-1} - b$$

with $a = \prod_{j=1}^p a_j$ and $c = \prod_{j=1}^p c_j$ and $b = -a - c$. Denote by τ the shift $u_n \mapsto u_{n+1}$ on $l^2(\mathbf{Z}, \mathbf{C})$.

By the theorem of Cayley-Hamilton, we have $\Delta(L_{per}) = 0$. We deduce from this that the operator $\Delta(L, \tau^p)$ on $l^2(\mathbf{Z}, \mathbf{C})$ defined by the functional calculus (plug-in the operator L for z and the operator τ^p for w in $\Delta(z,w)$) satisfies

$$0 = \Delta(L_{per}) = \Delta(L, \tau^p) = \Delta(L) - a\tau^p - c\tau^{-p} - b \; .$$

This implies that the operator $K := \Delta(L) = a\tau^p + c(\tau^p)^* + b$ on $l^2(\mathbf{Z}, \mathbf{C})$ is a Laplacian with space independent entries. Conjugated by a Fourier transform $U : l^2(\mathbf{Z}, \mathbf{C}) \to L^2(\mathbf{T}, \mathbf{C})$, it is unitarily equivalent to the multiplication operator

$$(UKU^*)u(\theta) = (a\, e^{ip\theta} + b + c\, e^{-ip\theta})u(\theta) =: \lambda(\theta)u(\theta)$$

on $L^2(\mathbf{Z}, \mathbf{C})$ which has as the spectrum on an ellipse $\mathcal{E}$ possibly being degenerated to an interval. Since $\Delta(L) = K$, we know by the spectral theorem that $\sigma(\Delta(L)) = \Delta(\sigma(K))$. The spectrum of

L consists of p closed real curves. They are the zeros of $z \mapsto \Delta(z) - w$, when w is runs through the ellipse $\mathcal{E}$. The density of states of L is the push-forward measure $(\Delta^{-1} \circ \lambda)^* d\theta$ of the Lebesgue measure $d\theta$ on the circle $\mathbf{T} = \mathbf{R}/(2\pi\mathbf{Z})$ under the multi-valued map $\phi = \Delta^{-1} \circ \lambda : \mathbf{T} \mapsto \mathbf{C}$. (See [46] for a derivation in the case $a_n = c_n = 1$).

Remarks.
1) Since the spectrum of Jacobi matrices is simple, the curves $z_i(\theta)$ do not intersect transversely. However, by deforming along a one-parameter family of Jacobi matrices, curves could merge or separate. The discriminant $\{L \mid (\Delta_L)'(z) = 0 \text{ for some } z \in \sigma(L)\}$ is the set of Jacobi matrices, where the topology of the spectrum can change under small parameter variations by merging or separation of spectral arcs.

2) The Bloch variety of L is the set $BV(L) = \{(z, w) \in \mathbf{C}^* \times \mathbf{C}^* \mid \Delta(z) = aw + b + cw^{-1}\}$, where $\mathbf{C}^*$ is the Riemann sphere $\mathbf{C} \cup \{\infty\}$. It is the set of all (z, w) such that there exists a nontrivial solution of $Lu = zu$ satisfying $(\tau^p u)_n = u_{n+p} = wu_n$. The spectrum of L is the intersection of $BV(L)$ with $\{|w| = 1\}$.

3) A theorem of Polya (see [1]) implies that the projection of the spectrum of a periodic (not necessarily selfadjoint) Jacobi matrix onto any real line in the complex plane has Lebesgue measure ≤ 4.

4) In the case when a_j, b_j, c_j are real and $a_n = c_n$ and $a = \prod_j a_j = 1$, the Jacobi matrix L is real and selfadjoint and K is the free Laplacian $\tau + \tau^* - 2$ with spectrum $[-2, 2]$ so that $\sigma(L) = \{z \mid \Delta(z) \in [-2, 2]\}$. The spectrum is then a union of bands on the real line which can intersect only at the boundaries.

5) In the case when $a = \prod_j a_j = 0, c = \prod_j b_j = 0$, the spectrum is $\Delta^{-1}(0)$, the set of roots of the characteristic polynomial of L_{per} confirming that in this case, L and L_{per} have the same spectrum. (L is then a countable direct sum of operators L_{per}.)

Higher order difference operators. For a p-periodic operator $L = \sum_{i=-r}^{r} a_i \tau^i$, the spectrum of L_{per} is more complex. The spectral bands curves can cross and for selfadjoint matrices, the bands can overlap.

We consider the case $p = kN$, where the operator can be rewritten as a Jacobi operator $L = a\tau + c\tau^* + b$ on $l^2(\mathbf{Z}, \mathbf{C}^N)$, where a_j, b_j, c_j are $N \times N$ matrices satisfying $a_{j+k} = a_j, b_{j+k} = b_j, c_{j+k} = c_j$.

Lemma 5.1 (Spectrum of periodic matrix-valued Jacobi matrices) *The spectrum of the Jacobi operator L with $N \times N$-matrix valued coefficients is contained in the union of at most $p = kN$ closed real curves.*

Proof. Unlike in the case $N = 1$, the Bloch variety

$$\{(z, w) = \Delta(z, w) = \det(L_{per}(w) - z) = 0\} \subset \mathbf{C}^2$$

can in general no more given explicitly in the form $f(w) = g(z)$. The spectrum is still the intersection of this complex variety with $\{|w| = 1\}$, the parameter $|w| = 1$ labeling Bloch waves.

The intersection of the Bloch variety with $|w| = 1$ is a set in the complex plane which is contained in a union of at most $p = kN$ closed curves.

Let Det denote the determinant function on the algebra of complex $N \times N$ matrices. We write $\prod_{j=1}^{k} a_j = a_k \dots a_1$, also in the noncommutative case. With $a = \mathrm{Det} \prod_{j=1}^{k} a_j, c = \mathrm{Det} \prod_{j=1}^{k} c_j$, we have

$$\Delta(z,w) = \Delta(z) + aw^N + cw^{-N} + f(z,w) ,$$

where the mixed term $f(z,w)$ is a polynomial in w, w^{-1}, z which is of degree $< N$ in w, w^{-1} and of degree $< p$ in z.

Examples.
1) The case, which we actually need here is when the higher order difference operator is the product $L = L^{(1)} L^{(2)}$ of two Jacobi matrices $L^{(i)} = \tau_i + \tau_i^* + V^{(i)}$.

2) If a_n, b_n, c_n are diagonal, the operator L is the direct sum of N one-dimensional Jacobi matrices. There are therefore operators with $p = kN$ spectral curves.

6 Thouless formula for difference operators

In this section we review the Thouless formula for operators on the strip (Jacobi matrices with matrix-valued entries). We recall the proof in [80] because the class of operators differs a tiny bit from the operators considered in [80]: the off diagonal matrices are the identity to which some nilpotent part is added. An other change is that we do not assume selfadjointness of the operator. The proof of the Thouless formula however is essentially the same. We restrict ourself to a generality which is actually needed in our case. We also introduce here some notation which is used later.

The Thouless formula $\mu(A_E) = \int \log |E - E'| \, dk(E')$ relates the density of states dk of a random Jacobi operator $\tau + \tau^* + b$ with the Lyapunov exponent $\mu(A_E)$ of the transfer cocycle (see [23, 17]). This generalizes to operators $\tau + \tau^* + b$ on $l^2(\mathbf{Z}, \mathbf{C}^N)$, where $\tau u_n = u_{n+1}$ and $b(x)$ is a selfadjoint $N \times N$ matrix, if $\mu(A_E)$ is the average of the N largest Lyapunov exponent of the $2N \times 2N$-cocycle [80]. The Thouless formula extends also to the case of operators $a\tau + (a\tau)^* + b$, where $a(x,y)$ are unitary [74].
The next lemma provides a generalization to operators of the type $L = L^{(1)} L^{(2)}$ with Jacobi matrices $(L^{(k)}(x^{(k)}, y^{(k)})u)_n = u_{n+1} + u_{n-1} + V_n^{(k)}(x^{(k)}, y^{(k)}) u_n$, where the later are not necessarily selfadjoint operators on $l^2(\mathbf{Z}, \mathbf{C})$,

$$L(x^{(1)}, y^{(1)}, x^{(2)}, y^{(2)}) = \begin{pmatrix} \cdot & \cdot & \cdot & 1 & 0 & 0 & 0 \\ 1 & c_{n-2} & b_{n-1} & a_{n-1} & 1 & 0 & 0 \\ 0 & 1 & c_{n-1} & b_n & a_n & 1 & 0 \\ 0 & 0 & 1 & c_n & b_{n+1} & a_{n+1} & 1 \\ 0 & 0 & 0 & 1 & c_{n+1} & b_{n+2} & \cdot \\ 0 & 0 & 0 & 0 & 1 & \cdot & \cdot \end{pmatrix} , \tag{3}$$

where $a_n = a_n(\mathbf{x}) = V_n^{(1)}(x^{(1)}, y^{(1)}) + V_{n+1}^{(2)}(x^{(2)}, y^{(2)}), b_n = b_n(\mathbf{x}) = V_n^{(1)}(x^{(1)}, y^{(1)}) V_n^{(2)}(x^{(2)}, y^{(2)}) + 2, c_n = c_n(\mathbf{x}) = V_n^{(1)}(x^{(1)}, y^{(1)}) + V_{n-1}^{(2)}(x^{(2)}, y^{(2)})$. The potentials $V_n^{(j)}(x^{(j)}, y^{(j)})$ are obtained from $T_j \in \mathcal{Y}$ and functions $V^{(j)} : \mathbf{T}^2 \mapsto \mathbf{R}$ by $V_n^{(j)}(x^{(j)}, y^{(j)}) = V^{(j)}(T_j^n(x^{(j)}, y^{(j)}))$.

Each difference operator operator $L(\mathbf{x})$ on $l^2(\mathbf{Z},\mathbf{C})$ can be written as a Jacobi operator

$$\tilde{L} = \tilde{a}\tilde{\tau} + (\tilde{c}\tilde{\tau})^* + \tilde{b}, (\tilde{L}u)_{\tilde{n}} = \tilde{a}_{\tilde{n}} u_{\tilde{n}+1} + \tilde{c}_{\tilde{n}-2} u_{\tilde{n}-1} + \tilde{b}_{\tilde{n}} u_{\tilde{n}} \tag{4}$$

on $l^2(\mathbf{Z},\mathbf{C}^2)$ with $\tilde{\tau}u_{\tilde{n}} = u_{\tilde{n}+1}, \tilde{\tau}\tilde{c}(\mathbf{x}) = \tilde{c}(T_1^2(x^{(1)},y^{(1)}), T_2^2(x^{(2)},y^{(2)}))\tilde{\tau}$ and with matrix-valued entries

$$\tilde{b}_{\tilde{n}} = \begin{pmatrix} b_{n-1} & a_{n-1} \\ c_{n-1} & b_n \end{pmatrix}, \tilde{a}_{\tilde{n}} = \begin{pmatrix} 1 & 0 \\ a_n & 1 \end{pmatrix}, \tilde{c}_{\tilde{n}} = \begin{pmatrix} 1 & c_n \\ 0 & 1 \end{pmatrix},$$

where $n = 2\tilde{n}$. The transfer cocycle

$$A_E = \begin{pmatrix} (E - \tilde{b}_{\tilde{n}})\tilde{a}_{\tilde{n}-2}^{-1} & -\tilde{c}_{\tilde{n}-2} \\ \tilde{a}_{\tilde{n}-2}^{-1} & 0 \end{pmatrix} = \begin{pmatrix} E - b_{n-1} + a_{n-1}a_{n-2} & -a_{n-1} & -1 & -c_{n-2} \\ a_{n-2}(b_n - E) - c_{n-1} & E - b_n & 0 & -1 \\ 1 & 0 & 0 & 0 \\ -a_{n-2} & 1 & 0 & 0 \end{pmatrix} \tag{5}$$

satisfies

$$A_E \begin{pmatrix} \tilde{a}_{n-2}u_n \\ u_{n-1} \end{pmatrix} = \begin{pmatrix} \tilde{a}_n u_{n+1} \\ u_n \end{pmatrix}$$

if $(\tilde{L}(\mathbf{x})u)_n = Eu_n$ for $u \in l^2(\mathbf{Z},\mathbf{C}^2)$. The density of states of L defined as the functional $f \mapsto \int_{\mathbf{T}^4} [f(L(\mathbf{x})]_{00}\ d\mathbf{x}$ on $C(\mathbf{C})$ and the density of states of $\tilde{L}$ defined as the functional $f \mapsto \int_{\mathbf{T}^2} \mathrm{Trace}([f(\tilde{L}(\mathbf{x})]_{00})\ \mathbf{x}$ both exist and they are the same probability measure in $\mathbf{C}$, if Trace is the normalized trace on 2×2 matrices satisfying $\mathrm{Trace}(1) = 1$.

Remarks.
1) For $L = L^{(1)}L^{(2)}$, where $L^{(i)}$ are Jacobi operators, there are other transfer matrices like the product of the individual 2×2 transfer matrices of the operators $L^{(i)}$. These are different matrix-valued functions in E and the Thouless formula would look different in this case.

2) Note that dk is normalized to be a probability measure. In some of the literature like for example in [80], the usual Trace for $N \times N$ matrices is taken which implies that the total mass of the density of states dk for operators L on the strip becomes N.

Lemma 6.1 (Thouless formula for operators on the strip) *Let dk be the density of states of the operator (4). Then*

$$\int_{\mathbf{C}} \log|E - E'|\ dk(E') = \mu(\wedge^2 A_E)/2\ ,$$

where $\mu(\wedge^2 A_E)/2$ is the arithmetic mean of the two largest Lyapunov exponents of the 4×4 transfer cocycle A_E of the operator (4).

Proof. Small changes are needed to the known proofs [23, 17, 80] for the same statements for selfadjoint operators L, where $a_n = c_n = 1, n \in \mathbf{Z}$. Write $\mathbf{x} = (x^{(1)}, y^{(1)}, x^{(2)}, y^{(2)})$ for a point on $\mathbf{T}^4$.

(i) It is enough to prove the statement for periodic, (nonergodic) transformations $T \times S$.
Proof. Every $T \times S \in \mathcal{Y} \times \mathcal{Y}$ can be approximated by periodic transformations $T_n \times S_n$ using Rohlin's lemma [53] (which implies that there exist measurable sets Y_n whose Lebesgue measure on $\mathbf{T}^4$ goes to 1 for $n \in \infty$ so that $(T \times S)^k(\mathbf{x}) = (T_n \times S_n)^k(\mathbf{x})$ for $k = 0, \ldots n-1$).

The Thouless formula is an identity for subharmonic functions $f_{T\times S} = g_{T\times S}$, parameterized by $T\times S \in \mathcal{Y}\times\mathcal{Y}$. Let $dk(f)$ denote the Riesz measure of a subharmonic function f it satisfies $dk(f) = \Delta f$ in the sense of distributions, where Δ is the Laplacian. It follows from the Avron-Simon lemma and the dominated convergence theorem that if $T_n\times S_n \to T\times S$ in the uniform topology for measure preserving transformations, then $dk(f_{T_n\times S_n}) \to dk(f_{T\times S})$ weakly. Also $dk(g_{T_n\times S_n}) \to dk(g_{T\times S})$ weakly holds because $dk(g_T)$ is a weak limit of measures $dk(g_T^{(k)}) = \int_{\mathbf{T}^4} \sum_j \delta(E_j^{(k)}(\mathbf{x}))\, d\mathbf{x}$, where $E_j^{(k)}(\mathbf{x})$ are the roots of $E \mapsto [A_{T,E}^k(\mathbf{x})]_{11}$ and $T \mapsto dk(g_T^{(k)})$ is continuous from $\mathcal{Y}$ with the uniform topology to the space of measures on $\mathbf{C}$ with the weak topology.
By assumption, we have $dk(f_{T_n\times S_n}) = dk(g_{T_n\times S_n})$. Therefore $dk(f_{T\times S}) = dk(g_{T\times S})$ which is $\Delta f_{T\times S} = \Delta g_{T\times S}$. From this, $f_{T\times S} = g_{T\times S}$ follows by Weyl's lemma [121].

(ii) Next, it is sufficient to assume that E is outside the spectrum of a periodic operator L (which is now contained in a finite union of one-dimensional curves (Lemma (5.1))) as well as that E is outside the spectrum of $L^{(0)}$, where $L^{(0)}$ is the Jacobi matrix on $l(\mathbf{Z}^+, \mathbf{C})$ with zero Dirichlet boundary condition at $n = 0$. The spectrum of $L^{(0)}$ is a finite set of points.
Proof. This is the Craig-Simon subharmonicity argument [22]. Taking balls $B_r(E)$ around a point E in the spectrum, the result holds for the over $B_r(E)$ averaged Lyapunov exponent. Passing to the limit $r \to 0$ is possible because of subharmonicity.

(iii) If $\lambda_j(E) = \prod_{k=1}^n (E - \lambda_{jk}), j = 1, \dots, N$ is an eigenvalue of the $N\times N$ matrix $[A_E^n]_{11}$, (where the $2N\times 2N$ matrix A_E is written as a 2×2 matrix of $N\times N$ matrices), then each root λ_{jk} of $E \mapsto \lambda_j(E)$ gives rise to an eigenvector $u^{(jk)}$ of a truncation $L^{(n)}$ of L, where a_k, b_k, c_k are put to zero for $k < 0$ as well as for $k > n$. The measures $dk_n = (2n)^{-1} \sum_{j,k} \delta(\lambda_{jk})$ converge for $n \to \infty$ weakly to the density of states dk which is supported on finitely many curves.
This is the Thouless-Avron-Simon lemma [23] which holds in general for random finite-difference operators which do not need to be selfadjoint.

(iv) The fact that

$$\int_{\mathbf{C}} \log|E - E'|\, dk_n(\mathbf{x}, E') = \sum_{j,k} \log|E - \lambda_{jk}(\mathbf{x})| = \log|\mathrm{Det}([A_E^n]_{11}(\mathbf{x}))|$$

converges to $\mu(\wedge^2 A_E(\mathbf{x}))/2$ for almost all $\mathbf{x} \in \mathbf{T}^4$ in the limit $n \to \infty$ is seen in the same way as in the Appendix of [80]: If $m^\pm(\mathbf{x}) = a(\mathbf{x})u^\pm((T\times S)\mathbf{x})u^\pm(\mathbf{x})^{-1}$ are the matrix-valued Titchmarsh-Weyl functions, where $Lu_n^\pm(\mathbf{x}) = Eu_n^\pm(\mathbf{x})$ and $\{u_n^\pm(\mathbf{x})\}_{n\in\pm\mathbf{Z}^+} \in l^2(\pm\mathbf{Z}^+, M(N, \mathbf{C}))$, then

$$(\mp 1/2) \int_{\mathbf{T}^4} \log|\mathrm{Det}(m^\pm(\mathbf{x}))|\, d\mathbf{x}$$

is the arithmetic mean $\mu(\wedge^2 A_E(\mathbf{x})/2$ of the two largest Lyapunov exponents of A_E. Furthermore, the maximal Lyapunov exponent averaged over $\mathbf{T}^4$ is

$$-\int_{\mathbf{T}^4} \log||m^+(\mathbf{x})||\, d\mathbf{x} = \int_{\mathbf{T}^4} \log||m^-(\mathbf{x})||\, d\mathbf{x}\ .$$

Because E is not an eigenvalue of $L(\mathbf{x})$, there are two $N\times N$-matrices U, V such that the functions $m^\pm, m^{(0)} = au^{(0)}(T\times S)(u^{(0)})^{-1}$ satisfy $m^{(0)} = m^- U + m^+ V$. Here $u^{(0)}(\mathbf{x})$ is an other solution of $L(\mathbf{x})u = Eu$ satisfying a Dirichlet boundary condition at 0. The matrix U is invertible because else there would exists $a \neq 0$ with $Ua = 0$ and $m^{(0)}a = m^+ Va$ would be in $l^2(\mathbf{Z}^+, \mathbf{C}^2)$ and contradict the fact that E was also chosen away from the spectrum of $L^{(0)}$. (In the selfadjoint case, this would already have been taken care of by choosing $\mathrm{Im}(E) \neq 0$.) Writing

$$m_n^{(0)} = (1 + m_n^+ V U^{-1} (m_n^-)^{-1}) m_n^- U$$

and using $m_n^{(0)}(\mathbf{x}) = [A_E^n]_{11}(\mathbf{x})$, one concludes

$$\begin{aligned}
\log\det(L) &= \int_{\mathbf{C}} \log|E - E'| \; dk(E') \\
&= \lim_{n\to\infty} \int_{\mathbf{T}^4} \int_{\mathbf{C}} \log|E - E'| \; dk_n(\mathbf{x}, E') \\
&= \lim_{n\to\infty} \int_{\mathbf{T}^4} \log \mathrm{Det}([A_E^n(\mathbf{x})]_{11}) \; d\mathbf{x} \\
&= \frac{1}{2} \lim_{n\to\infty} \int_{\mathbf{T}^4} \log|\mathrm{Det}(m^{(0)}(\mathbf{x})_n)| \; d\mathbf{x} \\
&= \frac{1}{2} \int_{\mathbf{T}^4} \log|\mathrm{Det}(m^-(\mathbf{x}))| \; d\mathbf{x} = \mu(\wedge^2 A_E)/2 \; .
\end{aligned}$$

7 Subharmonic estimate of a determinant

This section is central in the proof of Theorem 2.1. We estimate

$$\int_{\mathbf{T}} \int_{|w|=|\exp(2\pi i\beta)|=1} \log\det(L_{T,E} L_{S_\alpha,w}) \; d\beta \; d\alpha$$

from below, where S_α is a one-parameter family of maps parameterized by the torus $\mathbf{T}$. Together with an upper bound on $\det(L_{S,w})$ established in the next section, it will lead to the lower bound on $\log\det(L_{T,E})$ which we are looking for. The calculation of the determinant needs an estimate for the sum of the two largest Lyapunov exponents of a 4×4 matrix cocycle. This is the largest Lyapunov exponent of a 6×6 matrix cocycle.

The somehow tedious calculation of the 36 matrix entries of this 6x6 cocycle is quite straightforward and possible to do by hand. (The fact that many matrix entries are zero simplifies the task). The input for doing the computation on a symbolic manipulation software package is included in Appendix A.

Let

$$(L^{(j)}_{T_j,E_j,w_j}(x^{(j)}, y^{(j)}) u)_n = u_{n+1} + u_{n-1} + (w_j^{-1} z_n^{(j)} + w_j (z_n^{(j)})^{-1}) u_n + E_j u_n, j = 1, 2$$

be two random Jacobi matrices parameterized by the complex numbers E_j and w_j, where $z_n^{(j)} = \exp(i x_n^{(j)})$ with $T_j^n(x_0^{(j)}, y_0^{(j)}) = (x_n^{(j)}, y_n^{(j)})$. We write $w_2 = w$ and chose $w_1 = 1$. To simplify further the notation we write $T = T_1, S = T_2$ write $s_n = z_n^{(1)}, z_n = z_n^{(2)}$ as well as $L = L_{T,E_1} = L_{T,E_1} = L_{T_1,E_1,w_1=1}, K = L_{S,E_2,w} = L_{T_2,E_2,w_2=w}$.

Lemma 7.1 *For fixed $T \in \mathcal{Y}$ and for all $E_i \in \mathbf{R}$, there exists $S \in \mathcal{Y}$ such that*

$$\int_{|w|=|\exp(2\pi i\beta)|=1} \log\det(L_{T,E_1} L_{S,E_2,w})\, d\beta \geq 2\log(\lambda/2) \ .$$

Proof. The product $L_{T,E_1} L_{S,E_2,w}$ of the two Jacobi matrices

$$\begin{aligned} L &= L_{T,E_1} = g(s_n + s_n^{-1}) + \tau + \tau^* + E_1 \\ K_w &= L_{S,E_2,w} = g(w^{-1}z_n + wz_n^{-1}) + \tau + \tau^* + E_2 \end{aligned}$$

with $\tau u_n = u_{n+1}$, diagonal operators $u_n \mapsto s_n u_n$, $u_n \mapsto z_n u_n$, $g = \lambda/2$ is an operator (3) which is rewritten as a matrix-valued Jacobi operator (4).

This operator LK_w is a random operator over the dynamical system $T \times S$ on the four-dimensional torus $\mathbf{T}^4$ with the invariant Lebesgue measure.

The transfer cocycle (5) of LK_w is (since we assume $E = 0$),

$$A(w) = \begin{pmatrix} -b_{n-1} + a_{n-1}a_{n-2} & -a_{n-1} & -1 & -c_{n-2} \\ a_{n-2}b_n - c_{n-1} & -b_n & 0 & -1 \\ 1 & 0 & 0 & 0 \\ -a_{n-2} & 1 & 0 & 0 \end{pmatrix} ,$$

with

$$\begin{aligned} a_n &= g(s_n + s_n^{-1} + w^{-1}z_{n+1} + wz_{n+1}^{-1}) + E_1 + E_2 \\ b_n &= g^2(s_n + s_n^{-1})(w^{-1}z_n + wz_n^{-1}) + 2 + E_2\ g(s_n + s_n^{-1}) + E_1\ g(w^{-1}z_n + wz_n^{-1}) + E_1E_2 \\ c_n &= g(s_n + s_n^{-1} + w^{-1}z_{n-1} + wz_{n-1}^{-1}) + E_1 + E_2 \ , \end{aligned}$$

where $g = \lambda/2$. Let $B(w) = \wedge^2 A(w)$ be the representation of the 4×4 matrix cocycle on the 6 dimensional vector space $\wedge^2\mathbf{C}^4 \equiv \mathbf{C}^6$, which is the exterior product of $\mathbf{C}^4$ with itself.

In physical terminology, $B(w) = \wedge^2 A(w)$ is the representation of $A(w)$ in the 'Fermionic two-particle space'. The maximal Lyapunov exponent of $B(w)$ is the sum of the two largest Lyapunov exponents of $A(w)$ (see e.g. [125]).

While the Lyapunov exponent does not depend on the chosen basis in $\wedge^2\mathbf{C}^4$, we make the specific choice

$$f_1 = e_1 \wedge e_2, f_2 = e_1 \wedge e_3, f_3 = e_1 \wedge e_4, f_4 = e_2 \wedge e_3, f_5 = e_2 \wedge e_4, f_6 = e_3 \wedge e_4 \ ,$$

where the $\{e_j\}_{j=1}^4$ is the standard basis in $\mathbf{C}^4$. We have $B_{ij} = (f_i, Bf_j)$, where $B(u \wedge v) = Au \wedge Av$ and the scalar product is $(u_1 \wedge v_1, u_2 \wedge v_2) = \det\begin{pmatrix} (u_1,u_2) & (u_1,v_2) \\ (v_1,u_2) & (v_1,v_2) \end{pmatrix}$. In the basis $\{f_j\}_{j=1}^6$, one has

$$B = \begin{pmatrix} A_{11}A_{22} - A_{12}A_{21} & A_{11}A_{23} - A_{13}A_{21} & A_{11}A_{24} - A_{14}A_{21} & A_{12}A_{23} - A_{13}A_{22} & A_{12}A_{24} - A_{14}A_{22} & A_{13}A_{24} - A_{14}A_{23} \\ A_{32}A_{11} - A_{12}A_{31} & A_{11}A_{33} - A_{13}A_{31} & A_{11}A_{34} - A_{14}A_{31} & A_{12}A_{33} - A_{13}A_{32} & A_{12}A_{34} - A_{14}A_{32} & A_{13}A_{34} - A_{14}A_{33} \\ A_{11}A_{42} - A_{12}A_{41} & A_{11}A_{43} - A_{13}A_{41} & A_{11}A_{44} - A_{14}A_{41} & A_{12}A_{43} - A_{13}A_{42} & A_{12}A_{44} - A_{14}A_{42} & A_{13}A_{44} - A_{14}A_{43} \\ A_{21}A_{32} - A_{22}A_{31} & A_{21}A_{33} - A_{23}A_{31} & A_{21}A_{34} - A_{24}A_{31} & A_{22}A_{33} - A_{23}A_{32} & A_{22}A_{34} - A_{24}A_{32} & A_{23}A_{34} - A_{24}A_{33} \\ A_{42}A_{21} - A_{22}A_{41} & A_{21}A_{43} - A_{23}A_{41} & A_{21}A_{44} - A_{24}A_{41} & A_{22}A_{43} - A_{23}A_{42} & A_{22}A_{44} - A_{24}A_{42} & A_{23}A_{44} - A_{24}A_{43} \\ A_{31}A_{42} - A_{32}A_{41} & A_{31}A_{43} - A_{33}A_{41} & A_{31}A_{44} - A_{34}A_{41} & A_{32}A_{43} - A_{33}A_{42} & A_{32}A_{44} - A_{34}A_{42} & A_{33}A_{44} - A_{34}A_{43} \end{pmatrix}$$

To make the estimates possible, we will now change the coordinate system in $\mathbf{C}^6$. Without this, we would have to multiply with a too high power of w to make the entries analytic in w and the estimates would not work. [2]

Let $D(w) = \mathrm{Diag}(w, 1, 1, 1, 1, 1)$. The Lyapunov exponent of

$$G(w) = w^2 D(w) B(w) D^{-1}(w) s_n s_{n-1} \tag{6}$$

is for $|w| = 1$ the same as the Lyapunov exponent of $B(w)$.

The matrix-valued map $w \mapsto G(w)$ is analytic: each matrix entry is a polynomial in w of degree ≤ 6. We don't write down the entries here.

¿From Formula (6) one sees that $\mu(G(w))$ is $2 \log |w| + \mu(B(w))$ and so $\mu(G(w)) = \mu(B(w))$ if $|w| = 1$. The Lyapunov exponent $\mu(G(w))$ is a subharmonic function in w.

Let $R : \mathbf{T}^2 \mapsto \mathbf{T}^2$ be an arbitrary measure preserving transformation and let $Y = Y_R = \{(x, y, R(x, y))\} \subset \mathbf{T}^4$ be the graph in $\mathbf{T}^4$ defined by R. The set Y_R is equipped with the measure m coming from the projection $\pi_1 : (x_1, y_1, x_2, y_2) \mapsto (x_1, y_1) \in \mathbf{T}^2$: i.e. $m(A)$ is the Lebesgue measure of $\pi_1(A)$.

As shown in Appendix B, for every $w \in \mathbf{C} \setminus \{0\}$, the arithmetic mean of the two largest Lyapunov exponents of the 4×4 transfer cocycle of $L((x^{(1)}, y^{(1)}) K_w(x^{(2)}, y^{(2)})$ which is

$$\log \det(L(x^{(1)}, y^{(1)}) K_w(x^{(2)}, y^{(2)}))$$

exists for m-almost all $(x^{(1)}, y^{(1)}, x^{(2)}, y^{(2)})$. The integral over Y_R with measure m is the same as the integral over $\mathbf{T}^4$ with the Lebesgue measure and therefore does not depend on R.

It follows that $\mu(B(w))$ and so $\mu(G(w))$ can be obtained as an integral over Y_R of pointwise Lyapunov exponents $\mu(B(w))(x^{(1)}, y^{(1)}, x^{(2)}, y^{(2)})$ rsp. $\mu(G(w))(x^{(1)}, y^{(1)}, x^{(2)}, y^{(2)})$ and that the integral does not depend on R.

With $g = \lambda/2$, we have

$$G(0) = \begin{pmatrix} [G(0)]_{11} & 0 & g^3 s_{n-1}(g + E_1 s_n + g s_n^2) z_{n-3} z_{n-1} z_n & 0 & 0 & 0 \\ g s_{n-1} s_n z_n & 0 & 0 & 0 & 0 & 0 \\ -(g(g + E_1 s_{n-1} + g s_{n-1}^2) s_n z_{n-1}) & 0 & -(g^2 s_{n-1} s_n z_{n-3} z_{n-2}) & 0 & 0 & 0 \\ g s_{n-1}(g + E_1 s_n + g s_n^2) z_n & 0 & 0 & 0 & 0 & 0 \\ -(g s_{n-1} s_n z_{n-2}) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with $[G(0)]_{11} = g^2(g(g + E_1 s_n + g s_n^2) z_{n-1} + g s_{n-1}^2 (g + E_1 s_n + g s_n^2) z_{n-1} + s_{n-1}(-(s_n z_{n-2}) + E_1 g z_{n-1} + E_1^2 s_n z_{n-1} + E_1 g s_n^2 z_{n-1})) z_n$. Note that all negative powers of z_n, z_k, s_n, s_k etc. have disappeared.

[2] In this part, it was essential for us to have a computer equipped with an symbolic manipulation program assisting us in the search of a proof. Without it, we would probably have been discouraged to continue at this point.

By subharmonicity, the mean of the Lyapunov exponent of $G(w)$ over the circle $|w|=1$ is bounded below by the Lyapunov exponent of $G(0)$. In the rest of the proof, we will estimate this value from below, using a second complex parameter z, using again subharmonicity.

Let P be the projection from $\mathbf{C}^6$ onto the two-dimensional complex vector space

$$\{(u_1, 0, u_3, 0, 0, 0) \mid u_1, u_3 \in \mathbf{C}\} .$$

The cocycle $G(0)$ has the same maximal Lyapunov exponent as the cocycle $F = PG(0)P$. We have with $\epsilon = g^{-1} = 2/\lambda$,

$$F = g^4 \begin{pmatrix} F_{11} & s_{n-1}(1 + E_1 \epsilon s_n + s_n^2) z_{n-3} z_{n-1} z_n \\ -(\epsilon^2 (1 + E_1 \epsilon s_{n-1} + s_{n-1}^2) s_n z_{n-1}) & -(\epsilon^2 s_{n-1} s_n z_{n-3} z_{n-1}) \end{pmatrix}$$

where $F_{11} = ((1+s_{n-1}^2)(1+s_n^2) z_{n-1} + E_1 \epsilon (s_{n-1} + s_n + s_{n-1}^2 s_n + s_{n-1} s_n^2) z_{n-1} + \epsilon^2 s_{n-1} s_n (-z_{n-2} + E_1^2 z_{n-1})) z_n$.

Until now, no relation between z_n and s_n is specified and

$$\int_{|w|=|\exp(2\pi i \beta)|=1} \log \det(L_{T,E_1} L_{S,E_2,w}) \, d\beta \geq \mu(F) ,$$

(where $\mu(F)$ is the maximal Lyapunov exponent of the 2×2 cocycle F), holds for any pair of dynamical systems T, S leading for any $(x_0^{(1)}, y_0^{(1)}, x_0^{(2)}, y_0^{(2)})$ to orbits s_n, z_n and for all R linking the initial conditions $(x_0^{(1)}, y_0^{(1)}) = R(x_0^{(2)}, y_0^{(2)})$.

We determine now S and R by requiring $z z_n = s_n$, where $|z|=1$ is a second and new complex parameter. This corresponds to restricting the initial conditions $(x_0^{(1)}, y_0^{(1)}, x_0^{(2)}, y_0^{(2)})$ of the orbits to Y_R, where $R(x,y) = R_\alpha(x,y) = (x+\alpha, y)$ on $\mathbf{T}^2$ and to put $S = R_\alpha T R_{-\alpha}$ if $z = r\exp(i\alpha)$.

The 6×6 matrix cocycle $G(0)$ as well as the 2×2 matrix cocycle $F = F(z)$ depend now on the parameter z. Each matrix entry of $F(z)(x,y)$ is a polynomial in z.

$F(z)$ is an analytically parameterized 2x2 cocycle over the dynamical system $(\mathbf{T}^2, T, dxdy)$: every $(x,y) \in \mathbf{T}^2$ determines

$$(x_n^{(1)}, y_n^{(1)}, x_n^{(2)}, y_n^{(2)}) = (T^n(x,y), S^n(R_\alpha(x,y))) = (T^n(x,y), R_\alpha T^n(x,y))$$

on $\mathbf{T}^4$ and for all $(x,y) \in \mathbf{T}^2$ the matrix entries of $z \mapsto F^n(z,x,y)$ are analytic. As a consequence, $z \mapsto \log ||F^n(z,x,y)||$ is subharmonic.

The map $z \mapsto \mu(F(z))$ is subharmonic. Therefore, the average of the Lyapunov exponent $\mu(F(z))$ of $F(z)$ over $|z|=1$ is bounded below by the Lyapunov exponent of $F(0) = g^4 \begin{pmatrix} z_n z_{n-1} & 0 \\ 0 & 0 \end{pmatrix}$ which is $\log(g^4)$. This implies that there exists $|z|=1$ for which $\mu(F(z)) \geq \log(g^4)$.

This means that there exists $|w|=1, |z|=1$ such that half of the largest Lyapunov exponent of the 6×6 cocycle $B(w,z) = \wedge^2 A(w,z)$ (where $A(w,z)$ the 4×4 cocycle $A(w)$, newly parameterized by z) is bounded below by $(\log |g^4|)/2 = \log |g^2|$.

Because the sequence $zz_n = s_n$ $|z| = |\exp(i\alpha)| = 1$ belong to orbits of a transformation $T_\alpha = R_\alpha T R_{-\alpha}$, we find for fixed $T \in \mathcal{Y}$ always $S = T_\alpha \in \mathcal{Y}$ such that

$$\int_{|w|=|\exp(2\pi i\beta)|=1} \log\det(L_T L_S(w))\, d\beta \geq \log(g^2) = 2\log(\lambda/2) \; .$$

We have used in the last step the Thouless formula.

Why does this proof work? Isn't it just repeating twice Herman's subharmonicity [56] argument? Kind of, but there is a twist. A closely related calculation does not lead to the result: the map

$$(z,w) \mapsto h(z,w) = \log(zw\det(L_{T,z,E_1} L_{S,wE_2})) = \log(z\det(L_{T,z,E_1})) + \log(w\det(L_{S,w,E_2}))$$

is plurisubharmonic in (z,w). When integrating over the torus $\{|w| = 1, |z| = 1\}$, we get again a value bounded below by $h(0,0) = 2\log(\lambda/2)$. But now, nothing is gained because it just means that we can conjugate T and S by rotations to get the lower bound - which is clear by looking at each of the two determinants separately. In the proof, the setup is slightly different. It is just because the calculation in the proof is so close to a calculation which does not lead to the result, that one can easily miss it.

8 An upper bound for the Lyapunov exponent

In this section we determine the constant $C(\lambda)$ in Theorem (1.1) and formulate a Proposition which outlines, how one can improve the constant with more effort. The reason to stick to the constant $C(\lambda)$ is that it allows explicit expressions. We also show in this section that $C(\lambda)$ can be replaced by $c(\lambda) = O(\lambda^{-1})$.

Because $||A^n|| \leq ||A||^n$ for any matrix A, the number

$$m(E,\lambda) := \int \log||A_{E,\lambda\cos}(x,y)||\; dxdy$$

is an upper bound for the Lyapunov exponent of the Standard map, rsp. the Lyapunov exponent of the cocycle $A_E(w)$ for $|w| = 1$. Define $C(E,\lambda) = m(E,\lambda) - \log(\lambda/2)$ and $C(\lambda) = C(0,\lambda)$.

Lemma 8.1 *$C(E,\lambda) = \log(2/\sqrt{3}) + O(1/\lambda)$ for large λ and $C(0,\lambda) > 0$ for $\lambda > \lambda_0 = 2\sqrt{2/(6-3\sqrt{3})} = 3.1547....$*

Proof. With the norm $|||A||| = \max_{i=1,2}|Ae_i|$, where e_i are the basis vectors, there is an explicit expression for an upper bound of the integral $m(E,\lambda)$. Following [60], define

$$M(E,\lambda) = |E + i + \sqrt{(E+i)^2 - \lambda^2}|/\sqrt{3}$$

where the square-root takes the solution with positive imaginary part.

For matrices $A = \begin{pmatrix} c & 1 \\ -1 & 0 \end{pmatrix}$ one has $|||A||| = |\sqrt{1+c^2}| \geq (\sqrt{3}/2)||A||$. Furthermore, with the analytic function $f(z) = z^2(1 + (E - \lambda(z+z^{-1})/2)^2) = \lambda/2(z-a_-)(z-a_+)$ where $|a_-| < 1$, one has using the Jensen formula

$$\begin{aligned}
\int_{\mathbf{T}} \log |||A_{E,\lambda\cos}(x)||| \; dx &= \frac{1}{2}\int_{\mathbf{T}} \log|1+(E-\lambda\cos(x))^2| \; dx \\
&= \frac{1}{2}\int_{\{|z|=|\exp(ix)|=1\}} \log|f(z)| \; dx \\
&= \frac{1}{2}(\log(f(0)) - \log(|a_-|)) \\
&= \log(\frac{\sqrt{3}}{2}M(E,\lambda)) \; .
\end{aligned}$$

Therefore $C(E,\lambda) = \log(M(E,\lambda)) - \log(\lambda/2) = \log(2/\sqrt{3}) + \log(|(E+i)/\lambda + \sqrt{(E+i)^2/\lambda^2-1}|)$ and $C(0,\lambda) = C(\lambda) = \log(1/\lambda + \sqrt{1/\lambda^2+1}) + \log(2/\sqrt{3}) = \mathrm{arcsinh}(1/\lambda) + \log(2/\sqrt{3})$. For $\lambda = 3.1547.. = 2\sqrt{2/(6-3\sqrt{3})}$, one has $C(\lambda) = 0$.
The function $\lambda \mapsto C(\lambda)$, having a positive derivative, is strictly monotone.

Lemma 8.2 *We have*

$$\int_{\mathbf{T}^2} \log ||dT_{E,\lambda\cos}(x,y)|| \; dxdy \leq c(E,\lambda) := \int_{\mathbf{T}} \log\sqrt{2+(E+\lambda\cos(x))} \; dx = O(\lambda^{-1}) \; .$$

Proof. We use Hadamard's determinant theorem, which tells that for a finite matrix L, one has $\log\det(L) \leq \sum_{j=1}^n ||a_j||_2$, where a_j are the rows of the matrix $L = (a_1,\cdots,a_n)$. (The relation follows from the fact that if $A = (a_1,\dots,a_n)$, then $\det(A)/\prod_{j=1}^n ||a_j||_2 = \det(B)$, where $B = (a_1/||a_1||,\dots,a_n/||a_n||)$. The determinant theorem is equivalent to $\det(B) \leq 1$, a relation which can easily be seen geometrically (i.e. [9, 29]).)

In the random case, where the a_j are vector valued random variables on $\mathbf{T}^2$ with Lebesgue measure, and $L^{(n)}$ is a finite dimensional $n\times n$ approximation of L and where $n^{-1}\log\det(L^{(n)})$ converges to to the Fuglede-Kadison determinant $\log\det(L)$, we obtain from Hadamard's determinant inequality and Birkhoff's ergodic theorem

$$\log\det(L) \leq \int_{\mathbf{T}^2} \log||a(x,y)|| \; dxdy \; .$$

In the case of random Jacobi operators $L = \tau + \tau^* + b(x) = \tau + \tau^* + \lambda\cos(x) + E$, where $a(x,y) = (\dots,0,0,1,b(x),1,0,\dots)$, this gives

$$\begin{aligned}
\log\det(L) \leq \int_{\mathbf{T}} \log\sqrt{2+(E+\lambda\cos(x))^2} \; dx &= \log(\lambda/2) + \int_{\mathbf{T}} \log\sqrt{8/\lambda^2 + 4(E/\lambda+\cos(x))^2} \; dx \; , \\
&= \log(\lambda/2) + O(\lambda^{-1}) \; .
\end{aligned}$$

Improved smaller constants $C_n(\lambda) < C_{n-1}(\lambda)$ with $C_1(\lambda) = C(\lambda)$ could be obtained by estimating

$$n^{-1}\int_{\mathbf{T}^2} \log||A^n(x,y)|| \; dxdy$$

from above, where n is small. But already for $n = 3$, explicit bounds become more difficult due to the transcendental nature of the integrals. For $n = 2$, one would have to estimate the integral

$$\int_{\mathbf{T}^2} \log || \begin{pmatrix} E - \lambda\cos(y) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} E - \lambda\cos(x) & -1 \\ 1 & 0 \end{pmatrix} || \; dxdy$$

from above. With a numerical evaluation of this integral, we get an upper bound for the Lyapunov exponent which shows that for $\lambda_2 > 2.84661...$, the entropy of the Chirikov-Standard map is positive.

An other improvement of the lower bound of the entropy of the Standard map could be obtained by estimating better the bound $\int_{|w|=1} \mu(A_{E,\lambda\cos}) \geq \log(\lambda/2)$:

Proposition 8.3 *Let $C_n(\lambda) = n^{-1} \int_{\mathbf{T}^2} \log ||A^n_{2,\lambda\cos}(x,y)|| \; dxdy$. Let λ_n be the solution of $C_n(\lambda) = 2\log(\lambda/2)$. Then, for $\lambda > \lambda_n$, the Kolmogorov-Sinai entropy of the Standard map $T_{\lambda\sin}$ is positive.*

See Section 11. The Lyapunov exponent is bounded above by $C_n(\lambda)$ and $\int_{|w|=1} \log\det(L_T K_{S,w}) \geq 2\log(\lambda/2)$ so that for Kolmogorov-Sinai entropy $\mu(\lambda)$ satisfies $\mu(\lambda) + C_n(\lambda) \geq 2\log(\lambda/2)$.

Proposition (4.1) was obtained with $n = 1$, where the integral $C_1(\lambda)$ can be estimated explicitly. The values of $C_n(\lambda)$ could be estimated computer assisted. We expect that like this, one should reach λ_n close to 2 for large n. Maybe one could even get $\lambda_n \to 0$ for $n \to \infty$, if also the lower bound of the determinant $\int_{|w|=1} \log \det L_T K_{S,w}$ could be improved further with the Jensen formula.

9 Uniform hyperbolicity: a result of Ruelle

In this section we look at a class of uniformly hyperbolic cocycles and apply a result of Ruelle [124] to conclude that the lower bound on the Lyapunov exponent depends in our case smoothly on parameters. Furthermore, the result implies that the neighborhood around such a cocycle, for which one has positive Lyapunov exponents, becomes large if the Lyapunov exponent is large.

Consider a $(d+1) \times (d+1)$- matrix-valued map

$$(x,y) \mapsto A(x,y) = \begin{pmatrix} \lambda/2 + a(x,y) & b(x,y) \\ c(x,y) & d(x,y) \end{pmatrix} ,$$

where $a(x,y) \in \mathbf{C}$, $d(x,y)$ is a $d \times d$ matrix and $b(x,y)$ and $c(x,y)^T$ are vectors in $\mathbf{C}^d$. All entries are assumed to be bounded and measurable functions. Together with $T \in \mathcal{Y}$, it defines a cocycle $(x,y) \mapsto A^n(x,y)$. We look at cocycles in a neighborhood of $\begin{pmatrix} \lambda/2 & 0 \\ 0 & 0 \end{pmatrix}$, which has one Lyapunov exponent $\log(\lambda/2)$ and $(d-1)$ Lyapunov exponents $-\infty$.

The maximal Lyapunov exponent in such a uniformly hyperbolic situation (which generalizes the logarithm of the Perron-Frobenius eigenvalue of a positive matrix) depends smoothly on parameters [124]. We formulate this in a special case:

Lemma 9.1 (Ruelle) *The maximal Lyapunov exponent $A \mapsto \mu(A)$ of is Fréchet differentiable and realanalytic in a neighborhood of $A_0 = \begin{pmatrix} \lambda/2 & 0 \\ 0 & 0 \end{pmatrix} \in L^\infty(\mathbf{T}^2, M(d+1,\mathbf{C}))$, where $M(d+1,\mathbf{C})$ is* *the vector space of all complex $(d+1)\times(d+1)$ matrices. If $w \mapsto A_w$ be a complex parameterization of the cocycle such that for all $\{|w| \leq r\}$, the inequalities*

$$|a_w(x,y)|, ||b_w(x,y)||, ||c_w(x,y)||, ||d_w(x,y)|| \leq \epsilon$$

hold. Then, for fixed λ and small enough $|\epsilon|$, or for fixed $\epsilon > 0$ and large enough λ, the Lyapunov exponent $w \in \{|w| \leq r\} \mapsto \mu(A(w))$ is a harmonic function in w.

Proof. If $|a(x,y)|, ||b(x,y)||, ||c(x,y)||, ||d(x,y)|| \leq \epsilon$ and ϵ is small enough, the cocycle maps the constant cone bundle

$$(x,y) \mapsto C(x,y) = C = \{c \cdot (1,\eta) \mid ||\eta|| \leq 1, c \in \mathbf{C}\ \}$$

strictly into itself.

(One could write the complex matrix cocycle as a real matrix cocycle on $\mathbf{R}^{2(d+1)}$ for which the cone

$$C = \{c \cdot (1,\eta) \mid ||\eta|| \leq 1, \eta \in \mathbf{R}^{2d+1}, c \in \mathbf{R}\ \}$$

in $\mathbf{R}^{2d+2}$ is strictly invariant. This is equivalent to the fact that one can find a basis and $\delta > 0$ such that $|[A(x)]_{ij}| \geq \delta$. By ([124] 4.7), one can deal directly with the complex case.)

[124] is written in the case of continuous cone bundles and homeomorphisms, but remark 4.9 in [124] extends this to the measure theoretic case.

This can now be applied to the 6×6 cocycle $G(w,z)$ from Section (7) parameterized by w and z. From Proposition 4.8 in [124] we obtain especially:

Corollary 9.2 *For small r, the Lyapunov exponent $(w,z) \mapsto \mu(G(w,z))$ is pluriharmonic (harmonic both in w and z) on $\{(w,z) \mid 0 \leq |w| < r, 0 < |z| < r\ \}$.*

Corollary 9.3 *Given $R > 0$ and $\epsilon > 0$. Let $B_R(A)$ be the ball of radius R around A. For large enough λ, the ball $B_R(G(w,z))$ is contained in the open subset of uniformly hyperbolic cocycles and $|\mu(B) - \mu(A)| \leq \epsilon$ for all $B \in B_R(A)$.*

Proof. The Fréchet derivatives of the function $A \mapsto \mu(A)$ can be computed (see formula (3.6) in [124]) and satisfy $D^n_A \mu(A) \leq C^n/\lambda^n$, where C is a constant independent of λ.

Actually, one only needs the first derivative: Bloch's theorem which says that if a function f is analytic in a disc B_R of radius R, then $f(B_R)$ contains a disc of radius $|Df(a)|Rl$, where l is a constant not depending on anything (see [3] p. 39).

10 Determinants of random difference operators

In this section we look at determinants for classes of random operators. We will see that for a class of random operators, which do not need to be invertible, the determinant is nonnegative and satisfies $\det(LK) = \det(L)\det(K)$. This product formula of Fuglede and Kadison [38] for operators in a type II_1 von Neumann algebra usually assumes the operators L and K to be invertible. In the case of random, finite order difference operators, we can approximate the operators by periodic operators, which have the spectrum on a union of one-dimensional curves. By changing the energy we can approximate the operators by invertible operators for which the formula holds. The formula holds then for the random difference operators also.

A measure dk in the complex plane is called the density of states of a (not necessarily selfadjoint) random bounded operator $(x,y) \in \mathbf{T}^2 \mapsto L(x,y) = \sum_{j=-p}^{p} a^{(j)}(x,y)\tau^j$ on $l^2(\mathbf{Z}, \mathbf{C}^N)$ with $N \times N$ matrices $a^{(j)}$ and $\tau u_n = u_{n+1}$, if for every polynomial f, one has

$$\int_{\mathbf{T}^2} \mathrm{Trace}([f(L(x,y))]_{00})\; dx\; dy = \int_{\mathbf{C}} f(E)\; dk(E)$$

where Trace is the usual trace for $N \times N$ matrices normalized in such a way that Trace(1) = 1.

For operators L with a density of states dk one has a trace

$$\mathrm{tr}(L) = \int E\; dk(E) = \int \mathrm{Trace}[L(x,y)]_{00}\; dx\; dy\ .$$

If the Thouless formula holds and the Lyapunov exponents of the transfer cocycles are all finite, then the measure dk has finite logarithmic energy and the potential $E \mapsto \mathrm{tr}(\log|L-E|) = \int_{\mathbf{C}} \log|E-E'|\; dk(E')$ is defined and finite for all E.

In general, if L has a density of states dk, a determinant $\det(L)$ of L can be defined by

$$\log\det(L) = \mathrm{tr}(\log(|L|))\ .$$

If the value $\int_{\mathbf{C}} \log|E|\; dk(E)$ of dk at the point $E=0$ should be $-\infty$, one would put $\det|L| = 0$.

(The extension of the determinant to noninvertible operators, which we look at, is what Fuglede and Kadison call an 'analytic extension of the determinant' to singular operators. This is in contrast to the 'algebraic extension' for which the determinant is put always to zero if the operator is noninvertible).

Given $T, S \in \mathcal{Y}$ and $a^{(j)}, b^{(j)} \in L^\infty(\mathbf{T}^2, dxdy)$. Two not necessarily selfadjoint elements $L = \sum_j a^{(j)}(x,y)\tau^j, K = \sum_j b^{(j)}(x,y)\sigma^j$ in the crossed products define random operators

$$(L(x,y)u)_n = \sum_{j=-p}^{p} a_n^{(j)}(x,y)u_{n+j},\ \ (K(x,y)u)_n = \sum_{j=-p}^{p} b_n^{(j)}(x,y)u_{n+j}$$

with $a_n^{(j)}(x,y) = a^{(j)}(T^n(x,y)), b_n^{(j)}(x,y) = b^{(j)}(S^n(x,y))$.

Lemma 10.1 *The operators L, K, LK have all density of states. The Thouless formula holds for L, K and LK and $\det(LK) = \det(L)\det(K)$.*

Proof. (i) The operators L, K can be embedded in a single von Neumann algebra without changing the determinant.
Proof. L is in the crossed product $\mathcal{Z}_T$ of $L^\infty(\mathbf{T}^2, dxdy)$ with the $\mathbf{Z}$-action $\alpha_T : f \mapsto f(T)$ and K is in the crossed product $\mathcal{Z}_S$ of $L^\infty(\mathbf{T}^2, dxdy)$ constructed with the $\mathbf{Z}$-action $\alpha_S : f \mapsto f(S)$. The operator LK is in the crossed product $\mathcal{Z}_{T\times S}$ of $L^\infty(\mathbf{T}^2 \times \mathbf{T}^2, dxdy \times dxdy)$, constructed with the $\mathbf{Z}$-action $\alpha_{T\times S}$, where $T \times S(x_1,y_1,x_2,y_2) = (T(x_1,y_1), S(x_2,y_2))$ on $\mathbf{T}^2 \times \mathbf{T}^2$. One can embed

the two crossed products $\mathcal{Z}_T$ and $\mathcal{Z}_S$ in $\mathcal{Z}_{T\times S}$. The density of states of LK exists in the same way as it exists for L and K. Moreover, the density of states of L (rsp. K) in $\mathcal{Z}_{T\times S}$ is the same as the density of states of L in $\mathcal{Z}_T$ (rsp. in $\mathcal{Z}_S$).

(ii) For the verification of the Thouless formula, see Section 15).

(iii) We can assume without loss of generality that L and K are invertible.
Proof. By ergodic decomposition (i.e. [26]), one can assume T, S to be ergodic.
Approximate $T, S \in \mathcal{Y}$ by periodic transformations T_k using Rholin's lemma [53]: the measure of $\{(x,y) \in \mathbf{T}^2,\ T(x,y) \neq T_k(x,y)\}$ is then smaller than $1/k$ and $T_k^k = Id$. By ergodic decomposition, the determinant is in such a case an integral of determinants of type II_1 factors, in which case L_k, K_k as well as the product $L_k K_k$ are periodic operators which have their spectrum contained in a finite union of real smooth curves in the complex plane. In this case, there are arbitrarily small complex numbers E_k such that $L_k - E_k, K_k - E_k$ are invertible allowing to apply the determinant formula. Now, $E \mapsto \det(L_k - E), E \mapsto \det(K_k - E)$ as well as $E \mapsto \det((L_k - E)(K_k - E))$ are continuous. $L_k - E_k, K_k - E_k$ are invertible and $\det(L_k - E_k) \to \det(L)$, $\det(K_k - E_k) \to \det(K)$, and $\det((L_k - E_k)(K_k - E_k)) \to \det((L - E)(K - E))$ for $n \to \infty$.

(iv) The product formula.
Proof. We can now invoke the determinant theory for finite type von Neumann algebras [38] (see [27] chapter I) which tells that $\det(LK) = \det(L)\det(K)$ for invertible operators L, K in a type II_1 von Neumann algebra.

Corollary 10.2 *If $L^{(i)} = a^{(i)}\tau + (c^{(i)}\tau)^* + b^{(i)}$ are random Jacobi matrices over dynamical systems $T^{(i)} : \mathbf{T}^2 \mapsto \mathbf{T}^2$, then*

$$\det(L^{(1)}L^{(2)}) = \det(L^{(1)})\,\det(L^{(2)})\ .$$

Proof. Apply Lemma (10.1).

11 Proof of the key proposition

This section contains the proof of Proposition (4.1) which implies Theorem (1.1).

Let $z = \exp(ix)$. The cocycle $A_{w,E}(x,y) = A_E(w^{-1}z, y)$ with $A_E(z,y) = \begin{pmatrix} E - \frac{\lambda}{2}(z + z^{-1}) & -1 \\ 1 & 0 \end{pmatrix}$ is the transfer cocycle for the (in general no more selfadjoint) operator

$$(L_{T,E,w}(x,y)u)_n = u_{n+1} + u_{n-1} + \lambda(w^{-1}\exp(ix_n) + w\exp(-ix_n))u_n - Eu_n$$

with $T^n(x,y) = (x_n, y_n)$. The map $w \mapsto \mu(A_{w,E})$ is subharmonic and for $|w| = 1$, the Thouless formula

$$\log(\det(L_{T,w,E})) = \mu(A_{w,E})$$

holds (see Proposition (15.1)).

There is an upper bound on the Lyapunov exponent of $\mu(A_{S,w,E})$:

$$\log(\det(L_{S,w,E})) = \mu(A_{w,E}) \le \log(|E + i + \sqrt{(E+i)^2 - \lambda^2}|/\sqrt{3}) = \log(\lambda/2) + C(E,\lambda)\ ,$$

where $C(E,\lambda)$ is known by Lemma (8.1). In the case $E=0$, it simplifies to $C(\lambda)=C(0,\lambda)=\text{arcsinh}(1/\lambda)+\log(2/\sqrt{3})$ and we have

$$\log(\det(L_{S,E=0,w})) \leq C(\lambda) \; . \tag{7}$$

Consider now the function $w \mapsto \log(\det(L_{T,E_1=E} L_{S,w,E_2=0}))$, where $T,S \in \mathcal{Y}$. The operator $L_{T,E} L_{S,w}$ is a higher order difference operator. It is parameterized by the energy parameter E in the first operator $L_{T,E}$ and the auxiliary complex parameter w for the second operator $L_{S,w}$.

We have seen in Lemma (6.1) that the Thouless formula on the strip generalizes to this nonselfadjoint case. It assures that $\log(\det(L_{T,E}L_{S,w}))=\mu_2(A(w))$, where $\mu_2(w)=\mu(\wedge^2 A(w))/2$ is the arithmetic mean of the two largest Lyapunov exponents of the transfer cocycle $A(w)=A_{E,S,w}$ of $L_{T,E}L_{S,w}$.

We have seen in Lemma (7.1) that there exists S such that $\int_{|w=\exp(2\pi i\beta)|=1} \mu_2(A(w)) \; d\beta \geq 2\log(\lambda/2)$. There exists therefore $|w|=1$ such that

$$\log(\det(L_{T,E}L_{S,w})) \geq 2\log(\lambda/2) \; . \tag{8}$$

Using (7), (8) and Corollary (10.2) we have

$$\begin{aligned} \log(\det(L_{T,E})) &= \log(\det(L_{T,E}L_{S,w})) - \log(\det(L_{S,w})) \\ &\geq 2\log(\lambda/2) - (\log(\lambda/2) + C(\lambda)) = \log(\lambda/2) - C(\lambda) \; . \end{aligned}$$

12 Proof of stability

The proof of Proposition (4.1) suggests that whatever Lyapunov exponent estimates for an analytic cocycle can be proven for irrational rotations $x \mapsto x+\alpha$ as in [56, 133], they can up to some correction $C(\lambda)$ also be proven for this cocycle, where the underlying system is an arbitrary measure preserving transformation $T \in \mathcal{Y}$. For example, for any realanalytic nonconstant f, the Lyapunov exponent of $A_{T,\lambda f}$ should be positive for large enough λ. This still remains to be done.

We give here a proof of the stability result Theorem 2.2 of the positive Lyapunov exponent.

Proof. Let $G(z,w)$ be the cocycle (6) over the dynamical system $(\mathbf{T}^2, T, dxdy)$, where $s_n = zz_n$. The matrix entries of $G(z,w)$ are rational in z and analytic in w. (We have seen that for $w=0$, $G(0)=G(z,0)$ is analytic in z.)

The function

$$(w,z) \mapsto \phi(w,z) = \mu(G(z,w))$$

is plurisubharmonic and $\phi(0,0) = 2\log|\lambda/2|$ by Proposition (4.1).

For sufficiently small r, the function $(w,z) \mapsto \phi(w,z)$ is pluriharmonic on $\{0 \leq |w| \leq r, 0 < |z| \leq 1\}$ by Corollary (9.2), and the maximal Lyapunov exponent $\mu(B(w,z))$ of $B(w,z) = \wedge^2 A(w,z)$ on $\{|z|=1, |w|=r\}$ depends smoothly on an external parameters. Therefore, the integral

$$\int_{|w|=r,|z|=1} \mu(G(w,z))$$

depends smoothly on parameters.

Especially,

$$f \mapsto \int_{|z|=1} \int_{|w|=r} \log(\det(w^2 L_{T,E,z,\lambda \sin +f} L_{T,w,\lambda \sin +f}))$$

is smooth and has a value $\geq 2\log(\lambda/2)$ for $f = 0$.

We proceed now to the proof of the density statement. Let g denote a periodic function in a Banach space A_δ of realanalytic functions with norm $|| \cdot ||_\delta$.

It follows from Corollary (9.3) that there exists a function $\beta_1(\lambda) \to 0$ such that for any $\tilde{F}$ in a ball B_R around the 2x2 cocycle $F(0)$, an estimate $\mu(\tilde{F}) \geq (2\log|\lambda/2| - \beta_1(\lambda))$ holds uniformly for all $T \in \mathcal{Y}$. (The topology on cocycles is the product topology, with A_δ topology for each matrix entry.)
We know therefore that for any $\epsilon > 0$ there exists $|z| = 1, |w| = 1$, such that for all $||g||_\delta \leq \epsilon$, then

$$\log(\det(L_{T,\lambda \sin +g} L_{T_z,\lambda \sin +g}(w))) \geq (2\log|\lambda/2| - \beta_1(\lambda)) \qquad (9)$$

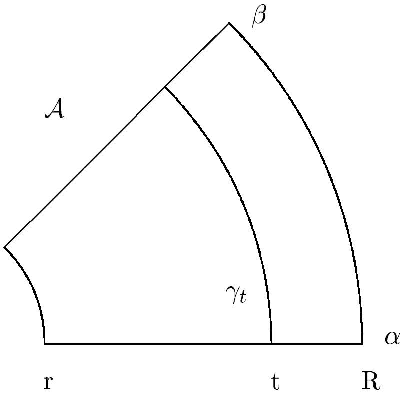
for all λ, because the cocycle $A_{T,\lambda \sin +g}$ is in a ball B_R of realanalytic 2x2-cocycles centered at $F(0)$ if $||g||_\delta$ is small.

By chosing for example $\beta(\lambda) = \beta_1(\lambda) + 1/\lambda$, this relation (9) even holds for $||g||_\delta \leq \epsilon(\lambda)$ and for some functions $\epsilon(\lambda)$ satisfying $\epsilon(\lambda) \to \infty$ for $\lambda \to \infty$.

Note that we only proved the analytic dependence of a lower bound for the Lyapunov exponent and not the analytic dependence of the Lyapunov exponent itself.

13 The Jensen formula in a sector

In this section we prove an integrated form of the Cauchy-Riemann differential equations. Unlike the classical Jensen formula which holds in an annulus (for a short proof see [43]), we formulate it in a sector. This allows to apply it in piecewise analytic situations. While we don't use this formula explicitly, we used it to explain some heuristics in the introduction. We furthermore think it could be useful for further improvements on the Lyapunov estimates.



Consider a sector $\mathcal{A} = \{z \in \mathbf{C} \mid |z| \in [r,R], \arg(z) \in [\alpha,\beta]\ \}$ and denote by $\gamma_t = \{|z| = t\} \cap \mathcal{A}$ a circular arc in $\mathcal{A}$. Let g be a bounded continuous, complex-valued function on $\mathcal{A}$ which is analytic and nonzero on $\mathcal{A} \setminus \mathcal{S}$, where $\mathcal{S}$ is a finite set of points in $\mathcal{A}$. Define for $t \in [r,R]$

$$\mathrm{Arg}_{\gamma_t}(g) = \mathrm{Re}\frac{1}{2\pi i}\int_{\gamma_t} \frac{g'}{g}(z)\, dz\ .$$

(It is defined except possibly at finitely many t, where $\gamma_t \cap \mathcal{S}$ is nonempty.)

Lemma 13.1 (Jensen formula in a sector) *If g is analytic in $\mathcal{A}$ and nonzero in $\mathcal{A} \setminus \mathcal{S}$, where $\mathcal{S}$ is a finite set of points, then*

$$\int_\alpha^\beta \log |g(Re^{2\pi i x})| \, dx = \int_\alpha^\beta \log |g(re^{2\pi i x})| \, dx + \int_r^R \frac{dt}{t} \mathrm{Arg}_{\gamma_t}(g) \, dt \; .$$

Proof. (i) Assume first that g is analytic in $\mathcal{A}$ and that it has no roots in $\mathcal{A}$. Clearly,

$$\int_\alpha^\beta \log |g(Re^{2\pi i x})| \, dx = \int_\alpha^\beta \log |g(re^{2\pi i x})| \, dx + \int_{\mathcal{A}} \frac{\partial}{\partial t} \log |g(te^{2\pi i x})| \, dt \, dx \; .$$

Because g is analytic, the Cauchy-Riemann equations $\frac{\partial}{\partial \overline{z}} g(z) = 0$ hold. With $z = t\exp(2\pi i x), g'(z) = (\partial/\partial z) g(z)$, we obtain

$$\frac{\partial}{\partial t} \log |g(te^{2\pi i x})| = \mathrm{Re} \left(\frac{g'(te^{2\pi i x})}{g(te^{2\pi i x})} e^{2\pi i x} \right) = \mathrm{Re} \left(\frac{g'(z)}{g(z)} \, \frac{z}{t} \right) \; .$$

We have therefore, using $2\pi i dx = dz/z$,

$$\begin{aligned} \int_\alpha^\beta \log |g(Re^{2\pi i x})| \, dx - \int_\alpha^\beta \log |g(re^{2\pi i x})| \, dx \quad &= \quad \mathrm{Re} \frac{1}{2\pi i} \int_r^R \int_{\gamma_t} \frac{g'(z)}{g(z)} \, \frac{dz}{t} \, dt \\ &= \quad \int_r^R \mathrm{Arg}_{\gamma_t}(g) \, \frac{dt}{t} \; . \end{aligned}$$

(ii) In general, we partition $\mathcal{A} = [r, R] \times [\alpha, \beta]$ into n^2 small sectors $\mathcal{A}_{jk} = [r_j, r_{j+1}] \times [\alpha_k, \alpha_{k+1}]$ and apply the formula to small sectors $\mathcal{A}_{jk}$ which do not intersect S. The assumption on $\mathcal{S}$ and g assures that for all the integrals, the contribution of the sectors which intersect $\mathcal{S}$ can be neglected in the limit $n \to \infty$.

Remarks.
1) The Jensen formula in the annulus [133, 10] follows: if $\mathcal{A}$ is the annulus and if g is analytic in $\{\arg(z) \in (\alpha_i, \alpha_{i+1}), \; |z| \in (r, R) \; \}$ and if $g(z) = \prod_{j=1}^m (z - a_j) h(z)$ such that $h(z) \neq 0$ everywhere on $\mathcal{A}$, then

$$\int_0^{2\pi} \log |g(Re^{2\pi i x})| \, dx - \int_0^{2\pi} \log |g(re^{2\pi i x})| \, dx = \int_r^R \frac{dt}{t} \mathrm{Arg}_{\gamma_t}(h) \, dt + \sum_{j=1}^m \log(\frac{R}{|a_j|}) \; .$$

2) The Jensen formula for subharmonic functions in the annulus $\mathcal{A}$ follows: for subharmonic functions $z \mapsto \log |g(z)| = \int \log |z - z'| \, dk(z')$ in the disc $\{|z| < R\}$, one has

$$\int_0^{2\pi} \log |g(Re^{2\pi i x})| \, dx = \int_0^{2\pi} \log |g(re^{2\pi i x})| \, dx + \int_{\mathcal{A}_{r,R}} \log(\frac{R}{|z|}) \, dk(z') \; .$$

3) Lemma (13.1) holds for sets S which are bigger than just a finite point set. For example, S can be a subset of a finite union of smooth curves.

14 Toral homeomorphisms and Lax approximation

In this section we review an approximation result for homeomorphisms on the torus. It allows to approximate the group of measure preserving homeomorphisms by a finite group of piecewise linear transformations. Investigating the Lyapunov map $T \mapsto \mu(A_{T,\lambda \sin})$ on finite groups $\mathcal{Y}_n$ of measure preserving transformations was important for us because the evaluation of the Lyapunov exponent is a finite dimensional integration and quite reliable.

On the group $\mathcal{Y}$ of all measurable, invertible transformations on the torus $\mathbf{T}^d$ which preserve the Lebesgue measure, one has the metric

$$\rho(T_1, T_2) = |\text{dist}(T_1(x), T_2(x))|_\infty ,$$

where dist is the geodesic distance on the flat torus and where $|\cdot|_\infty$ is the L^∞-norm.

A cube exchange transformation on $\mathbf{T}^d$ is a periodic, piecewise affine measure-preserving transformation which permutes rigidly all the cubes $\prod_{i=1}^d [k_i/n, (k_i+1)/n]$, where $k_i \in \{0, \dots, n-1\}$. Every point in $\mathbf{T}^d$ is periodic. Such a transformation is determined by a permutation of the set $\{1, \dots, n\}^d$. If the permutation is cyclic, the exchange transformation is called cyclic. A theorem of Lax [82] states that every $T \in \mathcal{X}$ can be approximated in the metric ρ by cube exchange transformations. Actually, T_k can be chosen to be cyclic [2].

Lemma 14.1 (Lax approximation) *For every $\epsilon > 0$ and every homeomorphism $T \in \mathcal{X}$, there exists a cyclic cube exchange transformation T_k which satisfies $\rho(T_k, T) \leq \epsilon$.*

Lax's 'book proof' of this result uses Hall's marriage theorem in graph theory (for a 'book proof' of the later theorem, see [1]).

Remarks.
1) Periodic approximations of symplectic maps work surprisingly well for relatively small n (see [120]). On the Pesin region this can be explained in part by the shadowing property [65]. The approximation by cyclic transformations make however long time stability questions look different [51].

2) It has been measured that certain Lax approximations T_n of the Standard map have longest orbits of size $\geq \delta n^2$ with $\delta > 0$ [158].

3) If $T \in \mathcal{Y}_k$, the density of states $dk(x,y)$ of a periodic operator $L(x,y)$ is contained in a finite union of curves. The density of states of L is the measure $\int_0^{1/k} \int_0^{1/k} dk(x,y)\ dxdy$ which has positive area in the complex plane. For a cyclic T_k, the Lyapunov exponent is computed as

$$\mu(A_{T_k}) = k^{-2} k^2 \int_0^{1/k} \int_0^{1/k} \log \lambda_{\max}(A^{k^2}(x,y))\ dxdy ,$$

where $\lambda_{max}(A)$ is the maximal eigenvalue of a matrix A. The factor k^2 comes because we integrate over a region with Lebesgue measure k^{-2}.

15 Thouless formula for nonselfadjoint operators

In this section we look at the Thouless formula for random operators which can have spectrum with positive area in the complex plane. Nonselfadjoint operators appear naturally, when a problem is parameterized by an external complex parameter like for operators (1). The proof of the Thouless formula which we are looking at in this section relies on the fact that for a general parameterization, there is an abstract Thouless formula which relies on the Riesz decomposition theorem in potential theory.

Consider an analytically parameterized, not necessarily selfadjoint, random Jacobi matrix

$$(L_z(x)u)_n = u_{n+1} + u_{n-1} + f_z(x_n)u_n \tag{10}$$

with a complex potential f_z parameterized by a complex parameter z. Examples are the w-parameterization

$$f_w(x) = (w^{-1}\exp(ix) + w\exp(-ix))/2 \tag{11}$$

for $\mathbf{C} \setminus \{0\}$ and for fixed $w \neq 0$, the energy parameterization

$$E \mapsto f_{E,w}(w,x) = f_w(x) - E\ . \tag{12}$$

Proposition 15.1 (Thouless formula for nonselfadjoint operators) *In the energy parameterization case (12), the Riesz measure of the Lyapunov exponent $\mu(A_{T,E})$ of the transfer cocycle $A_{T,E}$ is the density of states of the operator (1).*

Proof. For any general analytic parameterization $z \mapsto L(z)$, the Lyapunov exponent $\mu(A_z)$ of $L(z)$ is a a subharmonic function of z. The Riesz decomposition theorem (see i.e. [54, 121]) implies the abstract Thouless formula $\mu(z) = \int \log|z - z'|\ dk(z') + h(z)$, where $z \mapsto h(z)$ is harmonic.

The name 'abstract Thouless formula' is from [56].

If $z \mapsto \mu(z)$ is harmonic for large $|z|$ and grows like $\log|az|$ for $|z| \mapsto \infty$ then $\mu(z) = \log(a) + \int \log|z - z'|\ dk(z')$.

The Riesz measure dk of the subharmonic function $z \mapsto \mu(z)$ satisfies $dk = \Delta\mu$ in the sense of distributions, where Δ is the Laplacian on distributions. The measure dk coincides in the periodic case with the density of states because in this case the Lyapunov exponent is zero exactly on the spectrum which implies that the density of states is the equilibrium measure on the spectrum.

Consider the Riesz measures $dk_n(z) = \Delta\mu_n(z)$ of the subharmonic functions

$$z \mapsto \mu_n(z) = n^{-1} \int_{\mathbf{T}^2} \log|(A^n_{z,T}(x,y))_{11}|\ dx\ dy$$

and let $dk(z) = \Delta\mu(z)$ be the Riesz measure of the Lyapunov exponent $z \mapsto \mu(A_{z,T})$. Because $\mu_n(A_{z,T})(x,y)$ converges by the multiplicative ergodic theorem to $\mu(A_{z,T})(x,y)$ for almost all (x,y), the convergence holds also in the sense of distributions. Then also the Riesz measures $dk_n = \Delta\mu_n$ converge to $\Delta\mu$ as distributions for $n \to \infty$. But then, because smooth functions are dense in all continuous functions, $dk_n = \Delta\mu_n$ converges weakly as measures to $dk = \Delta\mu$.

In the energy parameterization case (12), the density of states of L is defined as the measure dk in the parameter plane $\mathbf{C}$ satisfying $\int_{\mathbf{T}^2} [g(L(x,y))]_{00}\ dxdy = \int g(E')\ dk(E')$ for every continuous function g.

For E outside the spectrum of L, the sequence $\int_{\mathbf{C}}(E'-E)^{-1}\, dk_n(x,y,E')$ converges for $n \to \infty$ uniformly to $\int_{\mathbf{C}}(E'-E)^{-1}\, dk(x,y,E')$. Because

$$\mathrm{tr}((L^{(n)}(x,y)-E)^{-1}) = \int_{\mathbf{C}} (E'-E)^{-1}\, dk_n(x,y,E') ,$$

where the Riesz measure $dk_n(x,y) = \Delta\mu_n(x,y)$ is a finite point measure located on the point spectrum of $L^{(n)}(x,y)$ defined in an earlier section, the formula holds also after integration over $\mathbf{T}^2$ and dk_n is the density of states of the random operator $(x,y) \mapsto L^{(n)}(x,y)$.

The functions $g_j(E) = (E-z_j)^{-1}$, where $\{z_j\}_{j\in\mathbf{N}}$ is a dense set in the complement of the spectrum of L, span a dense set in $C(\mathbf{C})$. For such functions, we have

$$\int_{\mathbf{T}^2} [g_j(L^{(n)}(x,y))]_{00}\; dxdy \mapsto \int_{\mathbf{T}^2} [g_j(L(x,y))]_{00}\; dxdy$$

by the Avron-Simon lemma.

We know therefore $\int_{\mathbf{T}^2}[g(L(x,y))]_{00}\; dxdy = \int g(E')\; dk(E')$ for all $g \in C(\mathbf{C})$ so that dk is the density of states.

Examples.
1) If $w \mapsto B(w) = w^l A(w)$ is analytic in $\mathbf{C}$, where $A(w)$ is the transfer cocycle of L_w, then the Lyapunov exponent of B satisfies

$$\mu(B(w)) = C + \int_{\mathbf{C}} \log|w-w'|\; dk(w')$$

where dk has total mass $(l+1)$ and where C is a constant.
2) For periodic operators, where the support of the Riesz measure is on spectral bands, the Lyapunov exponent of $w \mapsto A(w)$ is $\mu(w) = \mathrm{Re}[i \arccos(\mathrm{Tr}(A^n)(w)/2)]$. With the parameterization (11), we have

$$\mu(A(w)) = -\log|w| + \int_{\mathbf{C}} \log|w-w'|\; dk(w') + \log|\lambda/2|$$

because $w \mapsto \mu(A(w))$ grows like $\log|w\lambda/2|$ for $|w| \to \infty$ and $dk(\mathbf{C}) = 2$. The Lyapunov exponent of $B(w) = wA(w)$ which is harmonic near 0 satisfies then

$$\mu(B(w)) = \int_{\mathbf{C}} \log|w-w'|\; dk(w') + \log|\lambda/2| \; . \tag{13}$$

We expect the function $w \mapsto \mu(B(w))$, which is harmonic near $w=0$ to extend often to a harmonic function in the entire complex plane.

Remark. By the Hadamard three circle theorem [10], the function $\mu_{max}(r) = \max_{|w|=r}|\mu(w)|$ is a increasing function in r which is convex in $\log(r)$. Aiming to get rid of the $C(E,\lambda)$ term in Theorem (1.1), it would be good to know $M = \{|w| = 1 \mid \mu(w) = \mu_{max}(|w|)\ \}$ which is contained in the subset of the unit circle on which the potential is $\geq u(0)$. In the case of the Standard map, a candidate for a maximum on $|w| = 1$ is $w = 1$ because the pointwise Lyapunov exponent $\mu(x,y,w)$ satisfies $\mu(x,y,w) = \mu(-x,-y,\overline{w})$ so that $\mu(w) = \mu(\overline{w})$. See 18) in the discussion section.

16 Log-Holder continuity and capacity

In this section we consider a potential theoretical property of the Riesz measure of the Lyapunov exponent. This applies to the Riesz measure of the parameterization $w \mapsto L_w$ or, for fixed w to the energy parmeterisation $E \mapsto L_w - E$, where the Riesz measure dk is the density of states.

As in the selfadjoint case, the fact that the Lyapunov exponent is nonnegative implies mild regularity for the measure dk. Log-Holder continuity is a property of measures with finite potential theoretical energy. It implies that such a measure is absolutely continuous with respect to the zero-dimensional Hausdorff measure [35].

Lemma 16.1 (Log-Holder continuity) *Let $w \mapsto L_w$ be an analytic parameterization of a random Jacobi operator. The Riesz measure dk of the nonnegative subharmonic function $w \mapsto \mu(A_T(w))$ is log-Holder continuous: there exists a constant C such that*

$$dk(B) \leq C \cdot (\log(1/|B|))^{-1}$$

for every ball B with diameter $|B|$. The constant C only depends on L and $|E|$.

Proof. The proof is the same as in the real case [22, 17]: assume dk is contained in a ball of radius c and assume that B is any ball of radius $r < 1$ around $z \in \mathbf{C}$. With $A = \{z' \mid |z' - z| \geq 1 \}$, we have

$$0 \leq \int_A \log|z - z'| \, dk(z') \leq \log(1 + c + |z|) \; .$$

Therefore, using in the first inequality of the following identities also that $|z - z'| \leq 1$ on A^c and $\int_{\mathbf{C}} \log|z - z'| \, dk(z') \geq 0$, we have

$$0 \geq \int_{A^c} \log|z - z'| \, dk(z') = \int_{\mathbf{C}} \log|z - z'| dk(z') - \int_A \log|z - z'| \, dk(z') \geq 0 - \log(1 + c + |z|) \; .$$

For $z_1 \neq z$ with $|z_1 - z| = r$, and because $|z - z'| < |z - z_1|$ for $z' \in A^c$

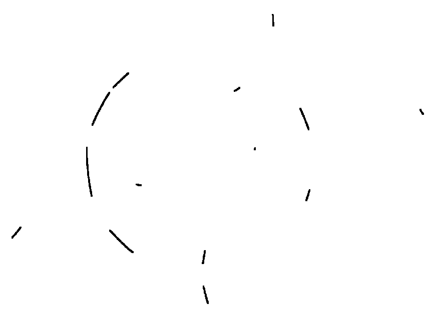
$$\log|z - z_1| \, dk(B) \geq \int_{A^c} \log|z - z'| \, dk(z') \geq -\log(1 + c + |z|)$$

so that $0 \leq \log(1 + c|z|) + \log|z - z_1| \, dk(B)$ which is the claim with $C = \log(1 + c|z|)$.

Example. Consider the case, when $T = T_k$ is a Lax cube exchange transformation on the torus and where the operator

$$L_w(x, y) = \tau + \tau^* + \lambda/2(w^{-1} z_j + w z_j^{-1})$$

with $z_j = z_j(x, y) = \exp(ix_j), (x_j, y_j) = T_k^j(x, y)$ is a periodic, nonselfadjoint Jacobi matrix.



We are interested in the potential theory of the w-spectrum

$$\sigma(L(x,y)) = \{w \mid \mathrm{tr}(A_w^n(x,y)) \in [-2,2]\ \} ,$$

which supports the 'w-density of states' $dk(x,y)$ of $L(x,y)$. In the picture to the left, we see an example of the 'w-spectrum' $\sigma(L(x,y))$ for $\lambda = 2.1$ where $k = 7$. One can see six bands located on the unit circle and eight bands away from the unit circle.

The 'w-density of states' of L is then $dk = \int_{\mathbf{T}^2} dk(x,y)$ and its support, the 'w-spectrum' of the random operator L is the union $\bigcup_{x,y\in\mathbf{T}^2} \sigma(L(x,y))$ and has in general positive area in the complex plane.

The potential theoretical energy

$$I(dk(x,y)) = -\int_{\mathbf{C}}\int_{\mathbf{C}} \log|z-z'|\ dk(x,y)(z')\ dk(x,y)(z)$$

is $I(dk(x,y)) = \log(\lambda/2)\ dk(x,y)(\mathbf{C}) = 2\log(\lambda/2)$ because $dk(x,y)(\mathbf{C}) = 2$ and the Thouless formula assures that the Lyapunov exponent $\mu(x,y)(z)$ is related to the potential $u(x,y)(z) = \int_{\mathbf{C}} \log|z-z'|\ dk(x,y)(z')$ of $dk(x,y)$ by $\mu(x,y)(z) = \log(\lambda/2)+u(x,y)(z)-\log(w)$ and the Lyapunov exponent is zero on the spectrum $\sigma(L(x,y))$. The measure $dk(x,y)/2$ is the equilibrium measure on the set $\sigma(L(x,y))$. It has the potential theoretical energy $\log(\lambda/2)/2$ and the capacity $C(\sigma(L(x,y)))$ is $\sqrt{\frac{2}{\lambda}}$.

17 Harmonic continuation of some singular integrals

In the case of the Standard map, the function $\mu(w) = \log(w\ \det(L_w))$ with L_w given by (1) is for small $|w|$ the real part of an analytic function. By the Cauchy-Riemann differential equations, the $\arg(w)$-dependence of the function $w \mapsto \exp(\mu(w))$ is related to the change of the harmonic conjugate along the radial direction. The Jensen formula extends this to the w-spectrum of L_w, where μ is no more analytic. There are cases (and we expect it to be the case in the Standard map), in which the harmonic conjugate can be extended as a harmonic function on a neighborhood of the unit disc. Estimating this harmonic conjugate with complex analytic methods could lead to possibly better lower bounds of the Lyapunov exponent. Alternatively, one could look at the argument change (which differs from the harmonic conjugation) and use the Jensen formula directly to estimate the oscillation of the Lyapunov exponent $w \mapsto \mu(A(w))$ for w the unit circle. In this section we formulate the problem of computing the harmonic continuation rsp. the argument change along radial lines, when the w-spectrum is located on the unit circle. In the next section, we will see that assuming dk to be supported on $\mathbf{T}$ is no big loss of generality for estimation purposes because projecting the spectrum onto the unit circle does not change much the potential (which is the Lyapunov exponent.)

A continuous, real-valued periodic function $x \mapsto \phi(x)$ on the circle defines a complex-valued function $x \mapsto a(x) = \exp(i(x + \phi(x)))$ and a measure $dk = h(x)dx$ on the circle $\mathbf{T}$ by pushing forward the Lebesgue measure $dk = a^* dx$. Denote by $a_n = \int \exp(-inx)\, h(x)dx$ the Fourier coefficients of this measure. The function $f(z) = \int_{\mathbf{T}} dk(x')/(z - \exp(ix')) = \int_{\mathbf{T}} dx/(z - a(x))$ defines two analytic functions $g_\pm(z)$, where g_+ is analytic on $|z| < 1$ and $g_-(z)$ is analytic in $|z| > 1$. These functions are defined by their Taylor expansions at $z_+ = 0$ and $z_- = \infty$:

$$\begin{aligned} h_+(z) &= -\int_{\mathbf{T}} a^{-1}(x) \frac{1}{1 - z/a(x)}\, dx = -\sum_{n=0}^{\infty} z^n a_{-(n+1)} , \\ h_-(z) &= \int_{\mathbf{T}} z^{-1} \frac{1}{1 - a(x)/z}\, dx = \sum_{n=0}^{\infty} z^{-(n+1)} a_n = \sum_{n=-1}^{-\infty} z^n a_{-(n+1)} \ . \end{aligned}$$

The function $h = h_- - h_+ = \sum_{n \in \mathbf{Z}} z^n a_{-(n+1)}$ is like this represented by the boundary value of two analytic functions $h_\pm$. One has for example that if h is in L^p, then $h_\pm$ are in the Hardy space H^p. The function h is realanalytic if and only if h is analytic in some strip around the unit circle. In this case, we can write $f(z) = h_+(z) = h(z) + h_-(z)$ for $\{1 - \epsilon < |z| \leq 1\}$ and $f(z) = h_-(z) = -h(z) + h_+(z)$ for $\{1 < |z| < 1 + \epsilon\}$. The functions $h_\pm$ can then be analytically continued to a larger domain. If dk extends to an entire function, then both $h_\pm$ extend to entire functions.

The integral $H(z) = \int_{\mathbf{T}} \log(z - a(x))\, dx$ with $H(0) = 0$ can be expanded in the same way at $z = 0$ and ∞ and

$$H = H_- - H_+ = \sum_{n \in \mathbf{Z}} z^{n+1} a_{-(n+1)}/(n+1) = \sum_{n \in \mathbf{Z}} z^n a_{-n}/n \ .$$

The imaginary part of H on $|z| = 1$ is the average argument change from 0 to ∞ of the function $a \mapsto z - a(x)$ along the line going through $z = \exp(i\alpha)$.

Near $z_- := 0$, the function $\mathrm{Im}(H_-(z))$ is the argument change of $\exp(H_-(z))$ from 0 to z. Near $z_+^{-1} := 0$, the function $\mathrm{Im}(H_+(z))$ is the argument change of $\exp(H_+(z))$ from ∞ to z.

Because of the Jensen formula (13.1), we are interested in the argument change $\alpha(z)$ from 0 to z along a straight line of the function $\int_{\mathbf{T}} \log(z - a(x))\, dx$. For each point $a(x)$, the argument change $\alpha(z)$ from 0 to z is a well defined angle. For $|z| < 1$, we have $\alpha(z) = \mathrm{Im}(H_-(z))$. For $|z| > 1$ we can write $\alpha(z) = \mathrm{Im}(H_+(z)) + \beta(z)$, where $\beta(z) = \beta(z/|z|)$ is the argument change along the straight line from 0 to ∞ which passes through z. Now $\beta(z) = \mathrm{Im}(H(z/|z|))$ so that

$$\begin{aligned} \alpha(z) &= \mathrm{Im}(H_+(z)) + H(z/|z|)) \ , \ |z| \geq 1 \\ \alpha(z) &= \mathrm{Im}(H_-(z)) \ , \ |z| \leq 1 \ . \end{aligned}$$

18 Lipshitz approximation of the Lyapunov exponent

In the theory of Schrödinger operators one uses the fact that the Lyapunov exponent $\mu(A_E)$ of the transfer cocycle A_E of L is the Hilbert transform of the integrated density of states (or rotation number [123, 25]) $\rho(E) = dk(-\infty, E)$, where dk is the density of states of L. For the spectral problem $w \mapsto L_w$, where the w-spectrum is contained in an annulus $\mathcal{A}_r = \{|z| \in (r, r^{-1})\ \}, r < 1$ containing the unit circle, one can ask for the regularity of the map $\alpha \mapsto \mu(L(\exp(i\alpha))$. The situation for the unit circle in the last section is relevant to this problem. The idea is to project the measure dk onto the unit circle with the map $\pi(z) = \arg(z)$. While this does not change the Lyapunov exponent much, one can use the classical properties for the new Lyapunov exponent which is the potential of a measure on the unit circle. A Lipshitz continuity of the 'integrated density of states' $\alpha \mapsto dk(\{z \mid \arg(z) \in (0,\alpha)\})$ would imply that $\alpha \mapsto \mu(\alpha) = \mu(L(\exp(i\alpha)))$ would be close to a Lipshitz continuous map.

Let dk be the Riesz measure of the Lyapunov exponent $\mu(w)$ of $w \mapsto L_w$ so that $\mu(w) = \int_{\mathbf{C}} \log|z-z'| \, dk(z')$. Let $\pi^* dk$ be the measure on the unit circle obtained as the push-forward of dk under the projection $\pi(r\exp(i\phi)) = \exp(i\phi)$.

The fact that the Hilbert transform preserves Lipshitz continuity can be derived from the Jensen formula in a sector (13.1) (rsp. the Cauchy-Riemann differential equations). Let H be the function as in the last section with the measure $\pi^* dk$ on the unit circle. The statement is that if the radial argument change $\mathrm{Im}(H(z))$ of the function $z \mapsto g(z) = \exp(H(z))$ is Lipshitz continuous on the unit circle and the Lipshitz constant is C, then the function $\mathrm{Re}(H(z)) = \int \log|z-z'| \, \pi^* dk(z')$ is Lipshitz continuous on the unit circle with Lipshitz constant C.

Lemma 18.1 *If dk is supported on $\mathcal{A}_r = \{|z| \in (r, r^{-1})\}$ and dk is the Riesz measure of a Lyapunov exponent, then*

$$\int_{\mathbf{C}} \log|z-z'| \, dk(z') \geq \int_{\mathbf{C}} \log|z-z'| \, \pi^*(dk)(z') - d(r) \; .$$

where $d(r) \to 0$ for $r \to 1$.

Proof. Given a point z_0 with $|z_0| = 1$. The set $Y(z_0, r) = \{w \in \mathcal{A}_r \mid |w - z_0| > |\pi(w) - z_0| \}$ is an open set. We have $Y(z_0, r) \subset Y(z_0, r')$ for $r < r'$. Write

$$\int_{\mathbf{C}} \log|z-z'| \, dk(z') = \int_{Y(z_0,r)} \log|z-z'| \, dk(z') + \int_{\mathcal{A}_r - Y(z_0,r)} \log|z-z'| \, dk(z') \; .$$

We have $\int_{\mathcal{A}_r - Y(z_0,r)} \log|z-z'| \, dk(z') \geq \int_{\mathcal{A}_r - Y(z_0,r)} \log|z-z'| \, \pi^* dk(z')$ and

$$\int_{Y(z_0,r)} \log|z-z'| \, dk(z') \geq \int_{Y(z_0,r)} \log|z-z'| \, \pi^* dk(z') - d(r) \; ,$$

where $d(r) \to 0$.

The punch line is that a numerical bound on the Lipshitz constant of $\alpha \mapsto \int_0^\alpha \pi^*(dk)(\beta) \; d\beta = dk(\{z \mid \arg(z) \in (0,\alpha)\})$, where dk is the Riesz measure of the Lyapunov exponent $w \mapsto \mu(A_{E,\lambda f,T}(w))$ would allow to estimate the Lyapunov exponent of the transfer cocycle $A_{E,\lambda f,T}$ of the operator (1). See also 22) in the discussion part.

19 An estimate for harmonic maps

We have numerical evidence that in the case of the Standard map the Lyapunov exponent $\mu(w)$ is the sum of a positive subharmonic function and a harmonic function obtained by harmonic continuation of $\mathrm{Re} \int \log(w - w') \, dk(w') + \log|w|$ from a neighborhood of $\{w = 0\}$. If this should turn out to be true, estimates for harmonic maps could be used to give alternative and maybe better estimates for the Lyapunov exponent.

For a general harmonic function h, one can estimate $\inf_{|z|=r} h(z)$ in terms of $C(r) = \sup_{|z|=r} h(z)$. We present two ways. The first way uses Harnack's inequality. A second approach uses the distortion theorem for univalent functions.

Lemma 19.1 (Lower bound for harmonic maps) *Let h be a harmonic map in a disc $\mathbf{D}(r') = \{|z| < r'\}$ with $r' > r, r' > 1$, such that $h(0) = 0$ and $h(z) < C(r)$ for all $z \in \mathbf{D}(r)$. Then a)*

$$h(z) \geq -C(r)\frac{2}{r-1} \; .$$

for $|z| = 1$ and b)

$$h(z) \geq -C(r)(1 + \frac{(1+r^{-1})}{(1-r^{-1})^3})$$

for $|z| = 1$.

Proof. a) $v(z) = C(r) - h(z)$ is nonnegative in the disc $\mathbf{D}(r) = \{|z| < r\}$ and $v(0) = C(r)$. Harnack's inequality gives

$$C(r) - h(z) = v(z) \leq C(r)\frac{r+1}{r-1}$$

so that $h(z) - C(r) \geq -C(r)\frac{r+1}{r-1}$ and $h(z) \geq C(r)(1 - \frac{r+1}{r-1}) = -C(r)\frac{2}{r-1}$.

b) The function $C(r) - h(z)$ is positive in $\mathbf{D}(r)$. Let $H(z)$ be the analytic function in $\mathbf{D}(r)$ which has h as its real part and which vanishes at 0. By the Noshiro-Warschawski criterium [33], the analytic function

$$F(z) = \int_0^z (C(r) - H(z)) \, dz$$

is univalent in $\mathbf{D}(r)$ because it has a derivative which has a positive real part in $\mathbf{D}(r)$. The function $\tilde{F}(z) = F(rz)/(rC(r))$ is in the class S of functions which are analytic and univalent in the unit disc and satisfy $\tilde{F}'(0) = 1$. By the distortion theorem for maps in the class S, we have

$$|\tilde{F}'(z)| \leq \frac{(1+R)}{(1-R)^3}$$

for $|z| = R < 1$. Assuming $r > 1$ and applying this for $R = r^{-1}$, we get for $|z| = R$

$$|\tilde{F}'(z)| = |\frac{F'(rz)}{C(r)}| = |1 - \frac{H(rz)}{C(r)}| \leq \frac{(1+r^{-1})}{(1-r^{-1})^3} \; .$$

In other words, if $|z| = 1$, then

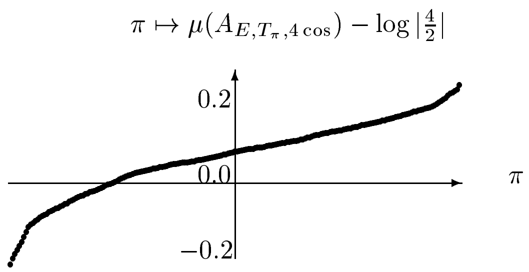
$$|1 - \frac{H(z)}{C(r)}| \leq \frac{(1+r^{-1})}{(1-r^{-1})^3}$$

which leads to $|H(z)| \leq C(r)(1 + \frac{(1+r^{-1})}{(1-r^{-1})^3})$ and implies the claim.

20 Remarks and questions

In this section we put the results in the context of some open questions in the subject and add some remarks.

1) **Measure preserving maps**.



All cyclic permutations T_π of $\{1,\cdots,9\}$. They are ordered such that the excess to $\log|\lambda/2|$, shown here for $\lambda = 4$, increases.

We have shown our result for a general automorphisms T of the Lebesgue space $(\mathbf{T}^2, dxdy)$. The only thing which matters is that the random variables x_n where $T^n(x,y) = (x_n, y_n)$ have as their law the Lebesgue measure. One could enlarge the class of transformations even so it is not much gain of generality [87]. The conjectured lower bound $\log(\lambda/2)$ can be expected to hold in general only for homeomorphisms $T \in \mathcal{X}$. The reason is that for some piecewise periodic measure preserving transformations $T_\pi : (x,y) \mapsto (x + \pi(x)/n \mod 1, y)$ with a permutation π on $\{1, 2, \ldots, n\}$, the value of $\mu(A_{T,\lambda\cos})$ can be slightly smaller than $\log(\lambda/2)$. The Figure to the left shows the distribution of the random variable $\pi \mapsto \mu(A_{T_\pi,\lambda\cos})$ on the group $\mathcal{Y}_9 \subset \mathcal{Y}$ of all permutations of the 9 annuli $\{(j/9, (j+1)/9] \times \mathbf{T} \subset \mathbf{T}^2\}_{j=1}^{9}$ in the case $\lambda = 4.0$.

We did not find $T \in \mathcal{Y}$, for which $\mu(A_{T,\lambda\cos}) = 0$ for some $\lambda > 2$. It would be interesting to know $\inf_{T\in\mathcal{Y}}\mu(A_{T,\lambda\cos})$ for given $\lambda > 0$ and whether the infimum is attained. By upper continuity of the Lyapunov exponent the infimum taken on the discrete set $\bigcup_n \mathcal{Y}_n \subset \mathcal{Y}$ is the same as the infimum taken on $\mathcal{X}$.

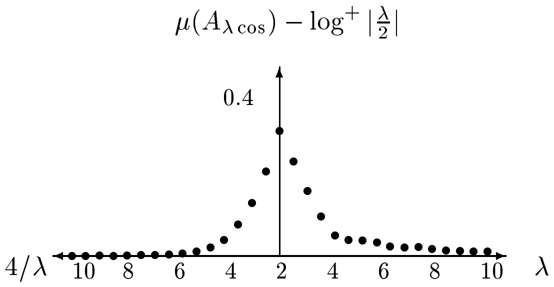
2) **Hamiltonian flows**.

Like for a large class of monotone twist maps, any Standard map is the time-one map of a Hamiltonian differential equation with time-dependent periodic Hamiltonians [106]. This is also true in higher dimensional cases [11, 49]. Since by Theorem (1.2), positive Kolmogorov-Sinai entropy is stable under perturbations of the map, there are many real-analytic, time-dependent periodic potentials V for which the time-dependent Hamiltonian system $\ddot{x} = V(t,x)$ has positive Kolmogorov-Sinai entropy. To see this, we note that the potential $V(t,x) = \delta_{t\in\mathbf{Z}}\lambda\sin_m(x)$ (which is a distribution in t) has as a Poincaré map the Standard map $T_{\lambda\sin_m}$. The smoothed-out potentials $V_\epsilon(t,x) = \phi_\epsilon(t) * V(t,x)$, where ϕ_ϵ is a mollifier function, has a Poincaré map which is realanalytic in (x,y) and which is arbitrarily close to $T_{\lambda\sin_m}$ in a Banach space of realanalytic maps. For fixed λ, the Poincaré map has now positive metric entropy for small ϵ and large m because the stability result as stated in Theorem (1.1) extends to general realanalytic perturbations of the map (and not only to perturbations of the function f).
There might be an alternative direct way to prove positive metric entropy for such Hamiltonian flows using results for continuous Schrödinger operators [133]. We don't see however yet how this should be done: one difficulty with such a direct approach is that the positivity of the Lyapunov exponent of the transfer cocycle of the continuous Schrödinger operator is not assured for all energy parameters in the spectral problem.
It would be nice to see subharmonic entropy estimates apply to Hamiltonian flows on three dimen-

sional energy surfaces of systems $\ddot{x} = V(x)$, where $V(x)$ is a potential on $\mathbf{T}^2$ or to geodesic flows on the torus $\mathbf{T}^2$ or to realanalytic, strictly convex Birkhoff billiards for which we have no example, for which positive Kolmogorov-Sinai entropy has been proven.
Even more off-limit seem problems in celestial mechanics like the Störmer problem of particles trapped in the van Allen belts of the Earth's magnetic dipole field [14] (for which Pesin theory in the form [66] can be shown to apply) or particular Newtonian three body problems.

3) **Smaller** λ.
Estimating the entropy in the case of small λ is still open for the Chirikov Standard map. Experiments indicate that $\mu(\lambda) > 0$ for all $\lambda > 0$.

Numerically, we find $\mu(\lambda) \geq \log(\lambda/2)$ for all λ and as a "rule of thumb" that the averaged Lyapunov exponent $\mu(\lambda)$ satisfies almost but not exactly the Aubry duality $\mu(\lambda) \sim \log(\lambda/2) + \mu(4/\lambda)$.



How close are we to Aubry duality?
For each λ we averaged 70x70=4900 different orbits of length 10^6 on the phase space.

The duality holds in the case of the dynamical system T_0 (the almost Mathieu case). It does not hold for $T_{\lambda \sin}$ as the Figure to the left shows for $\lambda \in [4/10, 2]$ and $\lambda \in [2, 10]$. Indeed, by computing moments of the density of states of L using a random walk expansion [73], one can check that the discrete random Schrödinger operator $L_{T,\lambda \cos} = \Delta + \lambda \cos(x)$ over the dynamical system T_λ does not have the same density of states as the naive "dual operator" $\tilde{L} = (\lambda/2)\Delta + 2\cos(x) = (\lambda/2)L_{T,(4/\lambda)\cos}$. (It is not excluded however that changing T in $\tilde{L}$ could restore a generalized duality.)

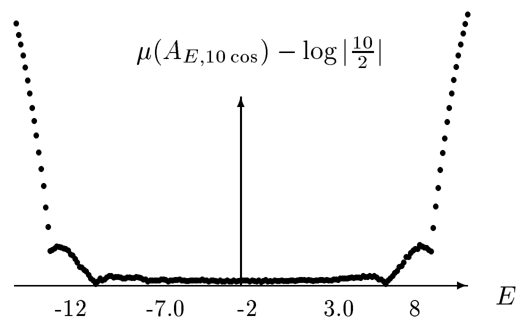
As in the almost Mathieu case, there is a two-dimensional magnetic operator involved. The magnetic field at a plaquette $P_{n,m}$ is $\exp(i(x_n - x_{n-1})$ independent of m. In the Standard map case, the magnetic fields depend on space and are correlated. See 25) for more on the Aubry duality.

4) **Smooth cocycles**.
One could extend the proof of Proposition (4.1) to show that cocycles $A = A_{E_1,\lambda_1 \cos} \circ \cdots \circ A_{E_m,\lambda_m \cos}$ with $E_j \in \mathbf{R}$ and λ_j large, have Lyapunov exponent $\mu(A) \geq \min_j(\log(|\lambda_j|/2) - C(\lambda_j))$. Therefore, large classes of random maps have positive Lyapunov exponents. The results of our paper are restricted to realanalytic cocycles and we do not reach results like in [154, 156] which are more subtle and depend on the dynamical system, e.g. Diophantine conditions for irrational rotations. By approximating a C^r-cocycle by piecewise analytic cocycles, estimates like the ones established here might be glued together for sufficiently large r. If this should turn out to be true, it would be interesting to know, at which threshold r, smooth perturbations of cocycles can destroy positive Lyapunov exponents. It is known that positive Lyapunov exponents of $SL(2, \mathbf{R})$ cocycles can be destroyed away from Anosov systems by measurable perturbations of the cocycle [68], by continuous perturbations [39] or using the unproven "Last theorem of Mané" [95], by C^1-perturbations of a non-Anosov map.

5) **Equilibrium measures**.

Theorem 1.1 gives via the variational principle (see e.g. [143]) lower bounds on the topological entropy for the Chirikov Standard map. These bounds are comparable with previous known lower bounds [32, 75]. The estimates in [75] are obtained using a Gronwall argument which estimates the spectral radius of the operator L^{-1} (where L is defined by a hyperbolic invariant measure of the Chirikov map) and which are far from optimal (the bounds in [75] are less good than the estimates in [32] but are proven when f is a nonconstant Morse function and not only in the case of the Standard map where $f = \sin$.) Note that the cited known lower bounds on the topological entropy do not become obsolete because they only require that f is smooth and Morse, while the bound on the metric entropy established here require f to be real analytic. The topological entropy estimates in the smooth case might be beaten however, if one could push the subharmonicity results to smooth cocycles (see the previous Point 4)). The equilibrium measure is a T-invariant measure on $\mathbf{T}^2$ which maximizes the entropy and has as the metric entropy the topological entropy. In the case of the Standard map, it is not known whether there is an absolutely continuous equilibrium measure.

6) **Absolutely continuous spectrum**.



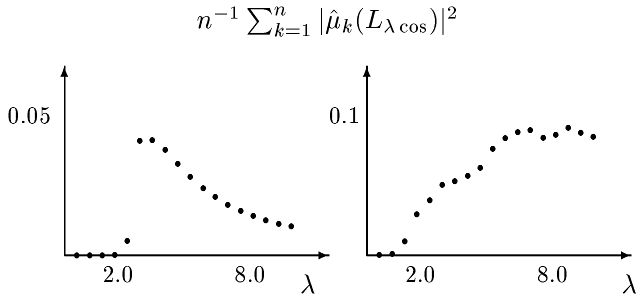
The Lyapunov exponent of the Chirikov Hamiltonian. The graph shows a nonnegative excess to $\log|\lambda/2|$ for all E.

Our theorems have applications in the theory of discrete random Schrödinger operators [23, 17, 116]. (The attribute 'random' is used in that theory often in the same way as 'random variable' is used in probability theory. Indeed, random operators are classes of operator-valued random variables. There is in general no randomness assumed in the potential. For example, if T is a translation on the torus, the random operators are quasiperiodic). Using Pastur-Kotani theory, Theorem 2.1 enlarges the class of (ergodic) random discrete one-dimensional random Schrödinger operators L, for which there is no absolutely continuous spectrum.

Theorem 2.1 implies with [111] that for a Baire generic ergodic $T \in \mathcal{X}$, the corresponding ergodic operators have no absolutely continuous spectrum for large λ. [3] For non-ergodic T, the absolutely continuous spectrum is only absent for a large set of $x \in \mathbf{T}^d$ because the Lyapunov exponent can be zero on a set of positive measure. Indeed, for invariant measures on KAM tori, the corresponding ergodic almost periodic Schrödinger operator can have some absolutely continuous spectrum for small λ [78]. [4] The size of the spectrum, where the Lyapunov exponent vanishes can be estimated if the orbits do not stay too long at the same x value for a long time [138].

7) **Discrete Spectrum**.
The question of localization of the operators $L_{\lambda\cos}$ [134] stays open. The conjugation of the map to a Bernoulli shift on some positive Lebesgue measure suggests that eigenvalues should exist for similar reasons as in the case of independent, identically distributed potentials.

[3] The space $\mathcal{X}$ becomes a complete metric space with the metric $\rho(T,S) + \rho(T^{-1}, S^{-1})$.
[4] The results in [78] have not been written down in a final form since both of us got engaged in other projects.



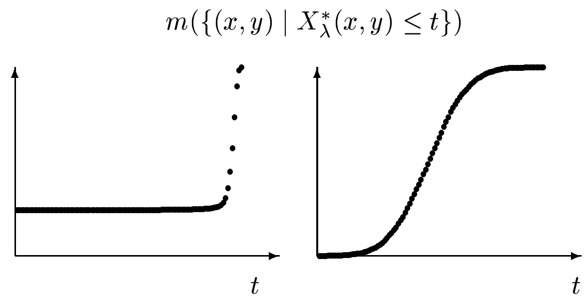
Wiener test for localisation for $\lambda \in [0, 10]$. To the left, the Mathieu case (n=100'000), to the right, the Chiricov case (n=10'000 averaged over 50x50=2500 orbits).

In the Figure to the left, numerical results are shown for the Fourier coefficients $\hat{\mu}_n = (\phi, \exp(-inL)\phi) = \int_{\mathbf{T}} e^{-in\theta}\, d\mu_\phi(\theta)$ of the spectral measure μ_ϕ where $\phi = (\dots, 0, 1, 0, \dots) \in l^2(\mathbf{Z}, \mathbf{C})$. We see the case of the Standard map operator and Mathieu operator. With a Wiener criterion for the existence of discrete spectrum, (where we compute a fast discrete quantum evolution using a related operator which has the same type of spectral measures [76]), we find numerical evidence for some point spectrum for $L(x, y)$ for a set of $(x, y) \in \mathbf{T}^2$ of positive measure, if λ is large.

In the picture to the left, we confirm numerically the theoretically established localization transition for the Mathieu operator at $\lambda = 2$ [60, 61]. In the picture to the right we see an indication for some point spectrum in the Chirikov case for $\lambda > 0$.

Note that in the same way as for the Anderson operator, the operator $L_T(x, y)$ has for an Anosov T purely singular continuous spectrum for a Baire generic set of (x, y). This can be derived from general principles [130]. Also, for Baire generic $(T, (x, y)) \in \mathcal{X} \times \mathbf{T}^2$ and $\lambda > \lambda_0$, one knows that $L_{T,\lambda\cos}(x, y)$ has purely singular continuous spectrum. This means that $L(x, y)$ on $l^2(\mathbf{Z}, \mathbf{C})$ has Baire generically no bound states and no extended states.

8) **The distribution of the Lyapunov exponents**.



The distribution functions of the random variable X_λ^* for $\lambda = 2, 10$. For $\lambda = 2$, $m(X_\lambda = 0) > 0$ (KAM).

For every λ, the pointwise Lyapunov exponent $(x, y) \mapsto \mu_\lambda(x, y)$ is a random variable on $\mathbf{T}^2$ which is T-invariant and has a mean $\geq \log(|\lambda|/2) - C(\lambda)$. The numerical experiments indicate that the random variables $X_\lambda(x, y) = \mu_\lambda(x, y)$ have the property that the normalizations $X_\lambda^* = (X_\lambda - \mathrm{E}[X_\lambda])/\sqrt{\mathrm{Var}[X_\lambda]}$ converge in distribution for $\lambda \to \infty$. The numerical results indicate that such a central limit theorem might indeed hold. While the tools used here for estimating $\mathrm{E}[X_\lambda]$ do not give information about the distribution, it is reasonable that the randomness in the map assured by Pesin theory makes the Lyapunov exponents in the limit $\lambda \to \infty$ behave like the Lyapunov exponents of Markovian cocycle, for which central limit theorems are known [5, 13].

9) **The size of the Pesin region**.

Because the averaged Lyapunov exponent of the Standard map can not be bigger than $\log(\lambda/2)+C(2,\lambda)$, we know that the size of the Pesin region, the set where the pointwise Lyapunov exponent of the Standard map is positive, has Lebesgue measure which is bounded below by

$$\frac{(\log(\lambda/2)-C(\lambda))}{(\log(\lambda/2)+C(2,\lambda))}=\frac{(\log(\lambda/2)-\operatorname{arcsinh}(1/\lambda)-\log(2/\sqrt{3}))}{(\log(\lambda/2)+C(2,\lambda)}\ . \tag{14}$$

For example, our result implies that for $\lambda=5.42$, the Pesin region already covers more than half of the phase space. (In comparison, it was measured numerically that for $\lambda=5.0$, the size of the elliptic regions is less than 2 percent of the phase space [19]). It follows from formula (14) that the set, where the Lyapunov exponent is zero has measure which is smaller than $O(\log(4/3)/\log(\lambda))$ for $\lambda\to\infty$. However, $C(\lambda)$ could be replaced by $c(\lambda)=m(2,\lambda)=\int_{\mathbf{T}^2}\log\|dT_{\lambda\cos}(x,y)\|\ dxdy=O(1/\lambda)$ for which we don't have an explicit formula. This shows that the complement of the Pesin region is of the order $O(1/(\lambda\log(\lambda)))$.

Empirically, the set with stable behavior has been measured in [19]. Empirical formulas have to be taken with a grain of salt for large λ because the size of the elliptic islands is usually so small that say for $\lambda=100$ already, a computer can hardly resolve individual islands. We know that from formula (14) that the total measure must then be less then $3/1000$. The size of the elliptic islands of a periodic orbit with period p is expected to be of the order M^{-3}, where $M=\sup_{j<p}\|dT^j(x)\|$ [16]. There are examples in Hamiltonian dynamics where it is possible to prove that elliptic regions cover a substantial part of the phase space [108]. For other nonergodic Hamiltonian flows, see [30]. Open is the Carleson problem (see point 27) below), which is the question whether there are λ for which $T_{\lambda\sin}$ is ergodic.

10) **Vlasov-Toda deformation of Twist maps**.

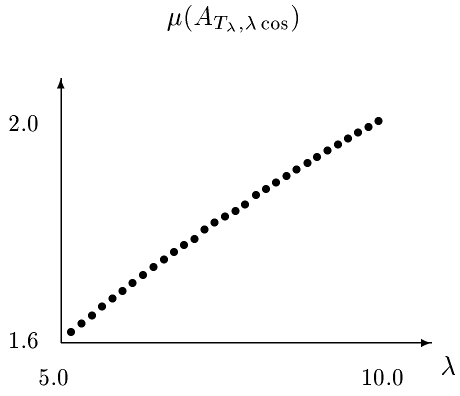
Other cocycles with the same bounds on the Lyapunov exponent can be obtained with isospectral Vlasov-Toda deformations of the random Jacobi operators $L=\Delta+V=a\tau+(a\tau)^*+b$ [71]. Assume we deform L and so the transfer cocycle with the first Toda flow $\dot a=a(b(T)-b)$, $\dot b=2a^2-2a^2(T^{-1})$, where

$$\begin{aligned}
a(x,y) &= h_{12}(x,y)=\partial_x\partial_y h(x,y)\\
b(x,y) &= h_{11}(x,y)+h_{22}(T^{-1}(x,y))=\partial_x\partial_x h(x,y)+\partial_y\partial_y h(T^{-1}(x,y))\ .
\end{aligned}$$

It is not clear whether the deformed operator $L_t=a_t\tau+(a_t\tau)^*+b_t$ is the Hessian of a critical point defining a twist map. The question is whether a generating function h can be derived from the deformed a,b such that h generates a twist map T satisfying discrete time Euler-Lagrange equations $h_2(T^{-1})+h_1=0$. Because the deformation keeps the Lyapunov exponents of the cocycle constant, the lower bound on the entropy would hold also for the deformed twist maps. In any case, even if the 'isospectral deformation of the twist maps' should not exist in general, Toda deformation gives infinite dimensional manifolds of cocycles over a fixed $T\in\mathcal{Y}$ obtained from the cocycle $A_{T,\lambda\sin}$ of the Standard map for which one has the same Lyapunov exponent.

11) **Continuity of the entropy**.

The map $\lambda\mapsto\mu(T_{\lambda\sin})$ is upper-semicontinuous. We don't know whether it is continuous. On the other hand, the topological entropy depends continuously on λ because it is always a continuous function on C^∞ diffeomorphisms on two-dimensional manifolds [110].



The topological entropy of surface diffeomorphisms changes only through homoclinic bifurcations [151]. Motivated from results on unimodal maps [31] and consistent with experiments, it is reasonable to ask whether the topological entropy or the metric entropy with respect to the invariant Lebesgue measure depends monotonically on λ. The Figure to the left shows the numerically computed Kolmogorov-Sinai entropy for 30 values of λ in the interval $[5.0, 10.0]$ (Lyapunov exponents averaged over 40×40 orbits each of length 10^6.)

The dependence of the entropy on λ even appears to be smooth. Oscillations indeed become small for larger λ because the established lower bound is realanalytic in λ and the effective value of the entropy is in a small corridor $\log(\lambda/2)+[-c(\lambda), c(\lambda)]$ which becomes increasingly narrow for $\lambda \to \infty$.

12) **Nonuniform hyperbolicity and homoclinic tangencies**.
Assume λ is such that T_λ has positive Lyapunov exponents on a set of positive Lebesgue measure. The Pesin region of positive Lebesgue measure is contained in the closure of transverse homoclinic points or in the closure of hyperbolic periodic orbits with transverse homoclinic intersections (see [65]). We don't expect however in the case of the Standard map that for some λ, we have uniform hyperbolicity on a set Y of positive Lebesgue measure. This is supported by the fact that for large λ, there exists a dense set of parameter values λ for which homoclinic bifurcations [114] happen in the Standard map family [32]. Uniform hyperbolicity on $\mathbf{T}^2$ is excluded because the Chirikov Standard map can not be Anosov on the whole torus $\mathbf{T}^2$: it would be topologically conjugated to a hyperbolic automorphism on the torus [96] and by homotopy to $(x, y) \mapsto (2x - y, x)$ which is not hyperbolic. Equivalent to uniform hyperbolicity on a set $Y \subset \mathbf{T}^2$ is the property that there exists an interval I containing $E = 0$ such that I is disjoint from in the spectrum of $L(x, y)$ for almost all $(x, y) \in Y \subset \mathbf{T}^2$. This is equivalent to the fact that the random operator L associated to an invariant ergodic set of positive Lebesgue measure is invertible in the corresponding crossed product algebra. It is an open question whether hyperbolic sets always have measure zero or one in the Chirikov Standard map case. This property is Baire generic in the class of measure preserving C^1-diffeomorphisms [95] and probably always holds for realanalytic diffeomorphisms on the torus. Having excluded uniform hyperbolicity almost everywhere, the positive entropy result could provide a new proof that homoclinic tangencies and consequently elliptic islands occur in the Chirikov Standard map for a dense set of parameters in $[\lambda_0, \infty)$.

13) **Coexistence**.
Theorem 1.2 establishes the existence of explicit real analytic maps on the torus, where coexistence of elliptic islands and positive metric entropy holds. Previously known were piecewise smooth maps [146, 147] or C^∞ maps [119]. Whether true coexistence (in the sense that the Pesin region as well as its complement are dense [137]) can hold on open sets of the phase space for some Standard maps is not known. According to [137], this coexistence problem was posed already before 1969 by Sinai. It was measured numerically in [141] that the closure of some orbits might have fractional box counting dimension. In higher dimensions, Mather conjectured [100] that generically one should have transitivity in the sense that there exists orbits which are dense in the phase space. No mathematical results about these questions are yet available. It has also been conjectured that transitive components in the complement of quasiperiodic sets are trellises [34], closed sets obtained by closing the unstable manifold of some hyperbolic periodic point. For the Standard map,

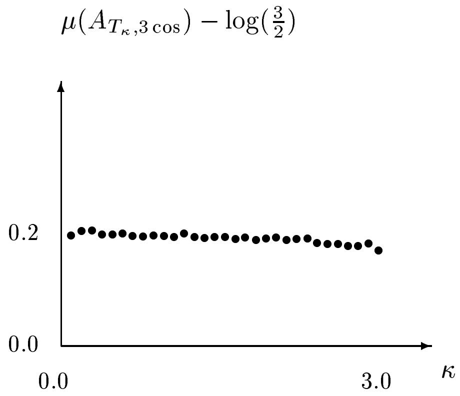
where we know now the existence of ergodic sets of positive measure, one can ask about the nature of these ergodic sets (i.e. the Hausdorff dimension of the closure). Careful measurements done in [101] indicate that some ergodic components are obtained as the closure of some aperiodic orbits.

14) **The complement of the Pesin region**.
An interesting question is what is left on the complement of the union of the almost periodic KAM set and the Pesin region. Is the dynamics weakly mixing but not mixing on a set of positive measure? While Pesin theory together with a genericity result for shift invariant measures [77] provide many invariant measures which have this property, they are in general supported on sets of zero Lebesgue measure. This observation by [6] can be strengthened by estimating the Hausdorff dimension of the measure. It was first observed in [18] that there are invariant measures μ_λ whose Hausdorff dimension goes to 2 for $\lambda \to \infty$. This observation is based on Young's formula [153] for the Hausdorff dimension $H(\mu)$ of μ:

$$H(\mu) = 2h_\mu(T)/\lambda(\mu) \ ,$$

where $\lambda(\mu)$ is the Lyapunov exponent integrated with respect to the measure μ and where $h_\mu(T)$ is the metric entropy of T with respect to the measure μ.
While periodic orbits are dense on the Pesin region [64] and on the closure of KAM orbits (in the sense that for x in the KAM set, there exists periodic $y_n \to x$), the question whether periodic orbits are dense in $\mathbf{T}^2$ is open and related to the problem to analyze the dynamics on the complement of the union of the Pesin and KAM regions.

15) **A variational problem**.



For fixed λ, is there (a realanalytic) $T \in \mathcal{X}$ (rsp. $\mathcal{Y}$) for which the Lyapunov exponent of the fixed matrix valued map $A_{\lambda \cos}$ is maximal? If T is the identity, we have $\mu(A_{\lambda \cos, T}) \geq \log|\lambda/2|$. There are many maps $R_\alpha T R_{-\alpha}$ with $R_\alpha(x,y) = (x+\alpha, y)$ for which the Lyapunov exponent is bigger than $\log(\lambda/2)$. If we fix the cocycle A_λ, how does the Lyapunov exponent depend on T_κ, when we deform T_κ from the Mathieu case T_0 to the Standard map case T_λ? The graph to the left shows a numerical computation in the case $\lambda = 3.0$. The averaged Lyapunov exponent seems to decrease slightly when κ is increasing.

The fact that the Lyapunov exponent is close to the one in the Mathieu case $\kappa = 0$ is compatible with the empirical fact of being close to Aubry duality (see 3) 25)).
It would be good to know the Taylor expansion of

$$\kappa \mapsto \int_{\mathbf{T}^2} \log([A^n_{E,\lambda,T_\kappa}(w)(x,y)]_{11}) \ dxdy$$

at $\kappa = 0$, for $|w| < 1$ which exists and converges for large λ. This requires to understand the dynamics for complex κ, where (x_n, y_n) become complex too.

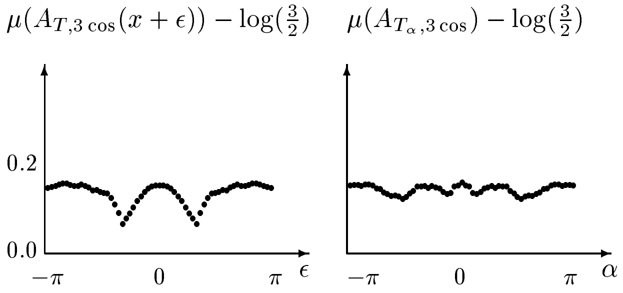
16) **Thermodynamic limit**.
For classes of symplectic coupled map lattices like finite dimensional versions of [63], we could obtain lower bounds for the entropy which are independent of the dimension of the map. In the thermodynamic limit, these maps define homeomorphisms T on the compact metric space $(\mathbf{T}^2)^{\mathbf{Z}}$ which preserve the product measure. While one has no more an Oseledec theorem in this infinite dimensional situation (the results in [126, 93] do not apply), the averaged maximal Lyapunov exponent $\lim_{n\to\infty} n^{-1} \int_{(\mathbf{T}^2)^{\mathbf{Z}}} \log ||dT^n(\mathbf{x}|| \; d\mathbf{x}$ is defined and positive for large λ and small ϵ. The cocycle $A = dT$ is now a bounded linear operator on $l^2(\mathbf{Z}, \mathbf{C}^2)$. By Vesantini's theorem [121], the map $w \mapsto n^{-1} \log(\rho(A^n(w))$ is subharmonic if $\rho(A)$ denotes the spectral radius of A. The estimates can be done directly for the operator-valued cocycle. It is not clear however whether the metric entropy of the homeomorphism T with invariant product measure is the limit of the metric entropies in the finite-dimensional situations.

17) **Response formulas**.
The subharmonic estimates could apply for classes of dissipative Standard maps on the cylinder. We expect the averaged Lyapunov exponent to depend analytically on the map in open sets of realanalytic maps because our results imply that smooth observables see a pretty uniform hyperbolic behavior of the dynamics. Indeed, by taking periodic Rohlin approximations to the map T, we get

$$n^{-1} \int_{\mathbf{T}^2} \log ||dT^n_{\lambda \sin}|| \; dxdy \geq \log(\lambda/2) - C_n(\lambda) \; ,$$

where $C_n(\lambda) = C(\lambda) + (\log(\lambda/2) + C(\lambda))/n$. The hyperbolicity appears to be uniform when observing the dynamics with smooth observables.
Could it be possible even that Ruelle's response formula [128] holds for some Standard maps with positive Lyapunov exponents? This formula says that for an infinitesimal perturbation $T + \delta T$ of an Anosov map T, the natural T-invariant measure ρ_T will change to $\rho_{T+\delta T} = \rho_T + \delta\rho_T$ with $\delta\rho_T(\Phi) = -\sum_{n=0}^{\infty} \rho_T((\Phi \circ T^n) \cdot \mathrm{div}(X))$, where $X = \delta T \circ T^{-1}$ is the vector field associated to the change of the map and where $\Phi \in C(\mathbf{T}^d)$ is an observable. The response formula would be expected to hold only for real analytic perturbations of T in $\mathcal{X}$ and for a subset of smooth observables Φ which do nowhere vanish on the torus. One has to see this question in the more general context of the 'chaotic hypothesis' [42].

18) **Symmetry**.
The symmetry $T(-x,-y) = -T(x,y), DT(x,y) = DT(-x,-y)$ which holds in the case of the Chirikov Standard map seems to make the Lyapunov exponent extremal with respect to some parameter perturbations. This is supported by some numerical experiments. It is not known whether the Lyapunov exponent is continuous as a function of w for example when $w \mapsto L_w$ is the random operator (1), so that symmetry does not imply a local extremum.



Computing the Lyapunov exponent in two cases with lack of symmetry. To the right, we use $T_\alpha(x,y) = (2x + 3\sin(x) - y + \alpha, x) \in \mathcal{X}$. The estimate known for $\epsilon = \alpha = 0$ extend.

In the Figure to the left, we see two experiments. The left graph shows the Lyapunov exponent of the deformed cocycle $\phi \mapsto A(x+\phi)$, where $A(x)$ is the Jacobean cocycle of the Standard map. This is equivalent to a deformation of the map T by conjugation $T_\phi = R_\phi^{-1} \circ TR_\phi$. In the right graph, the map T is moved along a path $\alpha \mapsto T_\alpha = R_\alpha \circ T$ in $\mathcal{X}$, where $R_\alpha(x,y) = (x+\alpha, y)$. For $\phi = \alpha = 0$, we have the Chirikov Standard map case with $\lambda = 3$. In the graph to the left, Herman subharmonicity argument shows that $\int_{\mathbf{T}} \mu(A_{T,3\cos}(\cdot + \alpha))\, d\alpha - \log(3/2) > 0$. Nevertheless, in both experiments, we never see a Lyapunov exponent below $\log(\lambda/2)$.

19) **Diffusion and Sinai's (H1) conjecture**.
A discrete Legendre transform brings the Standard map into the Hamiltonian form $T : (x,y) \mapsto (x+y+f(x), y+f(x))$ which is a map on the cylinder, the cotangent bundle $T^*\mathbf{T} = \mathbf{T} \times \mathbf{R}$ of the circle. Let $A \subset \mathbf{T}^2$ be a measurable T-invariant set of positive Lebesgue measure. If $(x_j, y_j) = T^j(x,y)$ is an orbit in the universal cover $\mathbf{R}^{2d}$, then $X_j = y_j - y_{j-1} = f(x_j) = \psi(x_j, y_j)$ are random variables on the probability space Ω equipped with the normalized Lebesgue measure. They have the mean $\mathrm{E}[X_j] = \int_{\mathbf{T}^2} f(x_j)\, dxdy = 0$ and the variance $\mathrm{Var}[X_j] = \int_{\mathbf{T}^2} \psi(x,y)^2\, dxdy$. Interesting is the growth rate of $S_n^2 = (\sum_{j=0}^{n-1} X_j)^2 = (y_n - y_0)^2$. Using translational invariance $\mathrm{E}[X_j X_l] = \mathrm{E}[X_{j+1}, X_{l+1}]$, the variance of S_n is

$$\mathrm{Var}[S_n] = \mathrm{E}[(\sum_{j=0}^{n-1} X_j)^2] = n\mathrm{Var}[X] + \sum_{j=1}^{n-1}(n-j)\mathrm{E}[X_0 X_j] = n\mathrm{Var}[X] + \sum_{j=1}^{n-1}(n-j)\hat{\mu}_\psi(j)\ ,$$

where μ_ψ is the spectral measure of $\psi \in L^2(\Omega), \psi(x,y) = f(x)$ with respect to the unitary Koopman operator $g \mapsto g(\tilde{T})$, where $\tilde{T}$ is the map T induced on Ω. The Fourier transform of μ_ψ is $\hat{\mu}_\psi(j) = \mathrm{E}[X_0 X_j] = \mathrm{Cov}[X_0, X_j]$, a correlation function. Let β be the infimum over all real numbers for which $\limsup_{n\to\infty} n^{-\beta} \sum_{j=1}^{n-1}(n-j)\hat{\mu}_\psi(j)$ is finite. If $\beta = 1$, then S_n behaves like a random walk (a case, where the X_j are independent) and $D = \int_{\mathbf{T}^2} V'(x)^2\, dx + \limsup_{n\to\infty} n^{-1} \sum_{j=1}^{n}(n-j)\hat{\mu}_\psi(j)$ is the diffusion constant.
It is a conjecture of Sinai (H1) on page 144 in [131] that there exists a set Ω of positive Lebesgue measure for which $\beta = 1$ if λ is larger then λ_{crit}, where the last homotopically nontrivial KAM torus disappears. If $\hat{\mu}_\psi(n)$ would decay fast enough, then $D = \int_{\mathbf{T}^2} V'(x)^2\, dx + \sum_{j=0}^{\infty} \hat{\mu}_\psi(j)$. First numerical experiments were done by Chirikov and Hizanidis [122]. Numerically, the Standard is reported to show such Brownian diffusive behavior for large λ [86, 83]. The fact of having positive Lyapunov exponents on a set of positive measure makes it plausible that the random variables X_0, X_n get decorrelated for $n \to \infty$. When computing numerically the first few hundred Fourier coefficients for smaller g (like $g = 5$) and checking with the Wiener theorem, we got the impression that μ_f still has some atoms (which prevents decorrelation) if $\Omega = \mathbf{T}^2$. Indeed, the presence of elliptic islands could be responsible for an almost periodic component in the Fourier transform of μ_f. In numerical experiments, one can get rid of this discrete part of the spectrum by adding stochastic noise [86].

In any case, the (H1) conjecture of Sinai would be settled for $\lambda > \lambda_0$ if one could show a fast enough decay of correlation of the spectral measure of $\psi(x,y) = f(x)$ on a mixing component of the Pesin set Ω. No exponential decay is necessary. It is enough to establish a power law decay of correlation $\hat{\mu}_\psi(j) = O(j^{-2})$. This finite differentiability condition for the spectral measure μ_ψ is reasonable since the dynamics on Ω is conjugated to a Markov chain. We even expect many spectral measures to be realanalytic leading to exponential decay of correlations.

20) **Dissipative Standard maps**.
The Lyapunov exponent of dissipative Standard maps numerically often satisfy the lower bound of the conservative case. The Lyapunov exponent can drop however to zero. In order that the proof carries over to the dissipative case, it appears however, that the random variables x_j need a smooth distribution μ sufficiently close to the uniform distribution. Results for Henon maps lead to the expectation that for many parameters, there exist invariant SRB measures for dissipative Standard maps like

$$T_b : (x,y) \mapsto (x + y + \lambda \sin(x), b(y + \lambda \sin(x))) ,$$

with $b < 1$. Results like in [15, 103] support this. Viana conjectured that in general a map with nonzero Lyapunov exponents almost everywhere in the phase space has an SRB measure [142]. If this conjecture is true, it indicates that for most values of b, an SRB measure should exist.
The estimates of Lyapunov exponents using spectral methods could even carry over to the case $b = 0$, where on each invariant set $y = \alpha$ we get the one-dimensional Arnold family $x \mapsto x + \alpha + \lambda \sin(x)$. If such a circle map has a smooth invariant measure sufficiently close to the Lebesgue measure, the Lyapunov exponent with respect to this measure should be $\geq \log(\lambda/2) - C(\lambda)$ for large λ.

21) **More general stability of positive metric entropy?**
The stability, we have established holds for estimates of the entropy which are done using subharmonicity rsp. the Jensen formula. This result provokes the question, whether, in general, positive metric entropy is an open property in the realanalytic category: is it true that for any realanalytic, measure preserving diffeomorphism T on the torus with positive metric entropy, there exists a Banach space of realanalytic measure-preserving maps such that an open neighborhood of T has positive metric entropy?
A question related to this stability problem is whether the Riesz measure dk of the w-parameterization of the operator L for a general twist map has the property that the potential $\int \log |w - w'| \, dk(w')$ does not fluctuate too much around its mean on the unit circle $\{|w| = 1\}$. This could be used to establish the stability if the entropy is comparable to upper bounds of the entropy like in the Standard map case.

22) **The Herman spectrum**.
One can also look at different analytic parameterizations of the cocycle. The Herman spectrum of a cocycle A is the set of complex numbers z, such that $R(z)A$ is not uniformly hyperbolic, where $R(z) = \begin{pmatrix} \frac{z+z^{-1}}{2} & \frac{z-z^{-1}}{2i} \\ -\frac{z-z^{-1}}{2i} & \frac{z+z^{-1}}{2} \end{pmatrix} \in SL(2,\mathbf{C})$ which has the property that $R(e^{i\alpha}) = R(\alpha) \in SO(2,\mathbf{R})$.
It is a subset of $\{|z| = 1\}$ because for $|z| \neq 1$, one can find a strict coinvariant cone field in $\mathbf{C}P^1$. There is a measure μ supported by the Herman spectrum and a harmonic function g such that the Lyapunov exponent $\mu(z) = \mu(zR(z)A)$ satisfies

$$\mu(z) = \int \log |z - z'| \, d\mu(z) + g(z) .$$

This abstract Thouless formula [56] follows from Riesz theorem and μ is the Riesz measure of the subharmonic function $\mu(z)$. The proof of Theorem 2.1 shows $\log(\mu(\beta)) > \log(\cos(\beta)\lambda/2) - O(1/\lambda)$ for all β. The Lyapunov exponent is realanalytic and positive outside the Herman spectrum. At such points, the directional derivative with respect to variations of the angle β in $z = re^{i\beta}$ can be computed using a formula of Ruelle [124] for the Fréchet derivative of Lyapunov exponents on the open set of uniformly hyperbolic cocycles. One gets $\frac{d}{d\beta}\mu(\beta) = \int_{\mathbf{T}^2} \cot(\omega(x,y))\, dm(x)$, where $\omega(x)$ is the angle between the stable and unstable directions m^+, m^- at the point (x,y) (see [72]). This formula shows that the Lyapunov exponent can change a lot if the stable and unstable manifolds are close. In the case $A(x) = \begin{pmatrix} c & b(x) \\ 0 & c^{-1} \end{pmatrix}$ with constant $|c| \neq 1$, one obtains

$$\frac{d}{d\beta}\mu(A(\beta)) = \int_{\mathbf{T}^2} \cot(\omega(x,y))\, dxdy = \frac{c}{1-c^2} \int b\, dxdy \; .$$

If A is the cocycle of a Standard map in Hamiltonian form like in 19), there is a spectral gap containing $\beta = -\pi/4$ in the Herman spectrum, because $R(-\pi/4)dT_\lambda = \begin{pmatrix} 2^{-1/2} & 0 \\ 2^{-1/2} + 2^{1/2}\lambda \cdot \cos(x) & 2^{1/2} \end{pmatrix}$. We compute $\frac{d}{d\beta}A(-\pi/4) = 1$. The point $z = e^{i0}$ is in the Herman spectrum because it follows from [97, 127] (see also [72]) that $z = 1$ is outside the Herman spectrum if and only if the map $T_{\lambda f}$ is Anosov. The rotation number of Ruelle [127] defined for lifts of A into the universal cover of $\mathrm{SL}(2, \mathbf{R})$ plays the role of the integrated density of states in the case of the Schroedinger spectrum. It is uniquely defined up to a multiple of 2π if one fixes $\rho(A(0)) = 0$. The rotation number is constant on an interval I if and only if I is not in the Herman spectrum [72]. Furthermore, $\rho(\beta) = \int_0^\beta \, d\mu(\alpha)$, showing that μ is an 'integrated density of states'. From the stable and unstable direction fields, one can construct (nonselfadjoint) random Jacoby operators $L(z)$ having those direction fields $m^\pm(z)$ as Titchmarsh-Weyl functions (with energy $E = 0$). They are analytic in z outside the Herman spectrum. Call a subset of the unit circle a part of the 'absolutely continuous spectrum' of $L(z)$ if there, $m^+(z) = \overline{m^-(z)}$. This absolutely continuous spectrum is a subset of the Herman spectrum. An adaption of Kotani theory [23] shows that the 'absolutely continuous spectrum' is the essential closure of the set, where the Lyapunov exponent is zero and that in the ergodic case, the existence of some absolutely continuous spectrum implies that T is 'deterministic' (see [40, 85, 80]). We know therefore for the Standard map that there is no absolutely continuous Herman spectrum for almost all (x,y) in the Pesin region if λ is large. We call z an 'eigenvalue' of $L(z)(x,y)$ if $L(z)(x,y)$ has an eigenvalue $E = 0$. The set of these 'eigenvalues' forms a 'discrete spectrum' of $L(z)(x,y)$ which is a subset of the Herman spectrum and which we expect to be nonempty for Lebesgue almost all (x,y) in the Pesin region.

24) **Quasiconformality**.
The map

$$z \mapsto G_n(z) = \int_{\mathbf{T}^1} \log |z^n A_E^n(z,y)| \, dy$$

is not conformal for small $|z|$ in general. For realanalytic T near a linear automorphism $(x,y) \mapsto (x+y, y)$ it is quasiconformal for all n. Question: is there an open set of realanalytic, measure-preserving maps T for which G_n are quasiconformal? If yes, it would be important to estimate the complex dilation $\kappa(z) = \overline{\partial} G_n(z)/\partial G_n(z)$ and to estimate the Lyapunov exponent in terms of κ. For $\kappa = 0$, one has conformality and the subharmonic estimates of Herman apply.

25) **The Aubry duality transform**.
The Aubry duality transform is an involutive map on a class of operators. It provides in the

Mathieu case an elegant way to estimate the Lyapunov exponent. The transform can be defined in more general situations: Let T_k be a cyclic interval exchange transformation on $\mathbf{T}$ and let $L_{\lambda \cos, T_k}$ be the corresponding Schrödinger operator. The density of states of the random operator L is the same as the density of states of the operator

$$(Lu)_n(\theta) = u_{n+1}(\theta) + u_{n-1}(\theta) + \lambda \cos(T^n\theta) u_n(\theta)$$

on the Hilbert space $H = L^2(\mathbf{T} \times \mathbf{Z})$. There is a piecewise smooth potential V such that $\cos(T^n(\theta)) = V(\theta + n\alpha)$, where $\alpha = 1/k$. Let $V(\theta) = \sum_n V_n \exp(in\theta)$ be its Fourier series. The duality transform (see [50])

$$(Uu)_m(\eta) = \sum_{n \in \mathbf{Z}} \int_{\mathbf{T}} e^{-(\eta + 2\pi m\alpha)n} e^{-im\theta} u_n(\theta) \, d\theta$$

satisfies $\tau U = (\lambda/2) U \sigma$, where $\tau u_n(\theta) = u_{n+1}(\theta)$ and $\sigma u_n(\theta) = \exp(i(\theta + n\alpha)) u_n(\theta)$. The operator L can be written as $L = \tau + \tau^* + \lambda \sum_n V_n \sigma^n$ so that $U^* L U = \sigma + \sigma^* + \lambda/2 \sum_n V_n(\tau^n + (\tau^n)^*)$. We are interested in the density of states dk_ϵ of $L_\epsilon = (2/\lambda) U^* L U = \epsilon(\sigma + \sigma^*) + \sum_n v_n(\tau^n + \tau_n^*)$ and $f(\epsilon) = \log(\det(L_\epsilon)) = \int \log|E| \, dk_\epsilon(E)$ which satisfies $f(0) = 0$. Because $\log(\det(L_{\lambda \cos, T_k})) \geq \log(\lambda/2) + f(2/\lambda)$, we want to estimate $f(\epsilon)$ from below for small ϵ. If $T_k(x) = x + 1/k$, where $f(\epsilon)$ is the Lyapunov exponent of a symplectic transfer cocycle, we have $\int \log|E| \, dk_\epsilon(E) \geq 0$. In general, the Thouless formula just becomes $\log(\det(L_\epsilon)) = \lim_{N \to \infty} \sum_{k=1}^{2N} \int_{\mathbf{T}} \log(\lambda_j(x, \epsilon)) \, dx$, where $\lambda_j(\epsilon, x)$ are the eigenvalues of the truncated $N \times N$ matrix $L_{\epsilon,N}(x)$. While $f(0) = 0$ and a perturbation lemma of Lidskii ([129]) assures that $|\lambda_j(\epsilon, x) - \lambda_j(\epsilon, 0)| \leq \epsilon$, this does not allow us to estimate $f(\epsilon)$ from below. The perturbation problem to estimate $f(\epsilon)$ seems still difficult and it is not clear, whether the Aubry transform, which transformed the perturbation problem $V \mapsto V + \epsilon\Delta$ into a perturbation $\Delta \mapsto \Delta + \epsilon V$ has made things simpler.

26) **Lower bound for topological entropy?**
The topological entropy of $T \in \mathcal{X}$ on $\mathbf{T}^2$ is bounded below by $\log(\mathrm{sp}(T_*))$, where $\mathrm{sp}(T_*)$ is the spectral radius of $T_* : H_*(\mathbf{T}^2) \mapsto H_*(\mathbf{T}^2)$ [102]. Can the metric entropy of a (smooth) $T \in \mathcal{X}$ with respect to the invariant Lebesgue measure become smaller than $\mathrm{sp}(T_*)$?

27) **Conjectures on the Standard map**.
The following of conjectures about Standard maps have still to be settled.

I) In [19], the entropy of the Standard map $T_{\lambda \sin}$ was measured $\geq \log(\lambda/2)$. Chirikov formulated the possibility that the elliptic islands might cover arbitrary large regions of the phase space.

> **I [72]:** The Kolmogorov-Sinai entropy of the Chirikov Standard map is $\geq \log(\lambda/2)$.

II) [134] introduced the problem of determining the spectrum of random operators $(L(x,y)u)_n = u_{n+1} - 2u_n + u_{n-1} + \lambda \cos(x_n) u_n$, where $T^n(x,y) = (x_n(x,y), y_n(x,y))$ is the orbit starting at (x,y).

> **II [134]:** The operator $L(x,y)$ has some point spectrum for a set (x,y) of positive Lebesgue measure if λ is large enough.

III) [16] asked whether there exists a set of parameters λ with full density at ∞ for which the Chirikov Standard map has no elliptic islands. We know that we have for large λ an open dense set of parameters with no ergodicity and positive entropy.

III [16]: There are parameters λ with full density at ∞ for which the Chirikov Standard map $T_{\lambda \sin}$ is ergodic.

Heuristic arguments in [47] predict that the Lebesgue measure of the set of parameters $\lambda \in [r, r+1]$ which lead to nonergodic Standard maps $T_{\lambda \sin}$ is of the order $O(1/r)$.

IV) The following problem is related to III):

IV [132]: For large λ, the Standard map has (in some Baire topology of maps) a neighborhood in which a residual set of f's gives ergodic maps T_f.

V) In the textbook [131] p. 144, conjecture (H2) contained as a second part the statement that the entropy grows to infinity for $\lambda \to \infty$. While we solved this part of the problem here, the first part of the conjecture is still open:

V [131]: the entropy of the Chirikov Standard map is positive for all $\lambda > 0$.

One should be able to prove that for any real-analytic non-constant periodic f, there exists $\lambda_0 > 0$ such that the Standard map $T_{\lambda f}$ has positive Kolmogorov-Sinai entropy for all $\lambda > \lambda_0$ and that for any real-analytic, non-constant, periodic map $f : \mathbf{T}^d \to \mathbf{R}^N$ and every symmetric, constant matrix $E \in GL(N, \mathbf{Z})$, all the Lyapunov exponents of the symplectic map $T_{Ex+\lambda f}$ are nonzero on a set of positive Lebesgue measure for large enough $|\lambda|$. All these properties are expected to be stable with respect to realanalytic perturbations of the map $T_{Ex+\lambda f}$, and hold for a fixed cocycle for all $T \in \mathcal{Y}$.

Appendix A: Computation of $A(w), B(w), G(w), G(0)$.

The calculation of the transfer cocycle $A(w)$ as well as its representation $B(w)$ in the 2 particle Fermionic space is straightforward but unpleasant to do by hand. We go through the calculation with a computer even so help from a machine is not necessary in principle. In order to be as independent of the algebraic manipulation tool as possible (here we use Mathematica [150]), we use only ASCII symbols (i.e. no indices or Greek symbols, even so the software would allow this). So, we write $an = a_n, am = a_{n-1}, ak = a_{n-2}, al = a_{n-3}, g = \lambda/2$.

The code for this computation is available online at

http://www.ma.utexas.edu/ ˜knill/Twist/Mathematica/appendix.m

We first compute the 4×4 transfer matrix $A(E, w)$ as a 2×2 matrix of 2×2 matrices. (The variable EE is used because E is already reserved for $e = 2.718....$) $I2 = \{\{1,0\},\{0,1\}\}$ is the 2×2 identity matrix.

```
Bn= {{bm, am},{cm,bn}};
Ak= {{ 1 , 0},{ak, 1}};
Ck= {{ 1 ,ck},{0 , 1}};
N2= {{ 0 , 0},{0 , 0}};
I2= {{ 1 , 0},{0 , 1}};
AA={{(EE*I2-Bn).Inverse[Ak], -Ck},
    {Inverse[Ak]           ,  N2}}
```

We rewrite this as a 4×4 matrix. (The actual input really starts now).

```
A={{EE-bm+am*ak ,         -am ,   -1,   -ck},
   {-cm-ak*(EE-bn)    ,EE-bn ,0    ,   -1 },
   {1                 ,     0 ,0    ,    0 },
   {-ak               ,     1 ,0    ,    0 }};
```

We plug in the coefficients. We need the transfer matrix only in the case $EE = 0$. The entries are meromorphic functions in the variable w and analytic in $E1 = E_1, E2 = E_2$. We use again the notation $sn = s_n, zn = z_n$ etc. With $L^{(i)} = \tau + \tau^* + V^{(i)} + E_j$, with $V_n^{(1)} = g(w^{-1}z_n + wz_n^{-1})$, $V_n^{(2)} = g(s_n + s_n^{-1})$, $\tau u_n = u_{n+1}, \tau^* u_n = u_{n-1}$, we get the entries of the transfer cocycle of $L_{T,E_1} L_{S,w,E_2}$:

```
EE=0;
ak = g*( sk + sk^(-1) + w^(-1)*zm + w*zm^(-1) ) + E1+E2;
am = g*( sm + sm^(-1) + w^(-1)*zn + w*zn^(-1) ) +E1+E2;
bn = g^2*(sn + sn^(-1) )*( w^(-1)*zn + w*zn^(-1) )+2.0+E1*E2+
      E2*g*( sn + sn^(-1) ) + E1*g*( w^(-1) zn + w*zn^(-1) );
bm = g^2*(sm + sm^(-1) )*( w^(-1)*zm + w*zm^(-1) )+2.0+E1*E2+
      E2*g*(sm + sm^(-1) ) + E1*g*( w^(-1) zm + w*zm^(-1) );
ck = g*( sk + sk^(-1) + w^(-1)*zl + w*zl^(-1) ) + E1+E2;
cm = g*( sm + sm^(-1) + w^(-1)*zk + w*zk^(-1) ) + E1+E2;
cn = g*( sn + sn^(-1) + w^(-1)*zm + w*zm^(-1) ) +E1+E2;
```

(With $A[[1,1]], \dots, A[[4,4]]$ one can access at this point the entries of A).

The variable A represents a 4×4 transfer matrix $A(E_1, w)$ which has as entries rational functions in the variable w. We represent now this matrix A as a matrix B in $\wedge^2 \mathbf{C}^4 = \mathbf{C}^6$.

```
b[m_,n_]:=Module[{ u={{1,2},{1,3},{1,4},{2,3},{2,4},{3,4}}  },
  {i,j}=u[[m]]; {k,l}=u[[n]];
  A[[i,k]]*A[[j,l]]-A[[i,l]]*A[[j,k]]];
B=Table[b[m,n],{m,6},{n,6}];
```

(With $B[[1,1]], \dots, B[[6,6]]$ one can access at this point the (pretty messy) entries of B).

To compute the polynomial matrix entries at $w = 0$, we conjugate with a diagonal matrix D (which is written as DD because again the symbol D is already reserved for the differential operator) and multiply with w^2 and sn, sm, which both does not affect the Lyapunov exponent when $|s_n| = 1, |w| = 1$. We get now the matrix $G(w)$ and after setting $w = 0$, we obtain the matrix $G(0)$.

```
Clear[w];
DD={{w,0,0,0,0,0},
    {0,1,0,0,0,0},
    {0,0,1,0,0,0},
    {0,0,0,1,0,0},
    {0,0,0,0,1,0},
    {0,0,0,0,0,1}};
G=Expand[sn*sm*w^2*DD.B.Inverse[DD]];
w=0;
G0=Simplify[Chop[G]]
```

As expected, $\mathrm{G0} = G(0)$ does not depend on E_2 (which we have put zero anyway in the proof in section (7)). We cut now away the nilpotent part (the additional Chop is used to eliminate zero terms like 10^{-16} which are zero but which were not recognized to be so).

```
F= Chop[{{G0[[1,1]],G0[[1,3]]},{G0[[3,1]],G0[[3,3]]}}]
```

Now, we introduce a new variable z which links z_n with s_n.

```
Clear[z];
sm=z*zm; sn=z*zn; sl=z*zl;sk=z*zk;
```

We are interested in what is left when one puts $z = 0$:

```
z=0; F
```

The cocycle $F(0)$ has the Lyapunov exponent $\log(g^4) = 2\log(\lambda/2)$ and the function $z \mapsto \mu(F(z))$ is subharmonic.

Appendix B: determinant of a product of operators

This appendix was added later to answer a question from early readers of the May 27) preprint.

Linking the initial conditions of the orbits restricts the dynamics to a diagonal like set Y_R of initial conditions in the product system $T \times S : \mathbf{T}^2 \to \mathbf{T}^2$ which is not $T \times S$ invariant.

Lemma 20.1 *Let $T, S \in \mathcal{Y}$. Let $R \in \mathcal{Y}$ be an additional measure preserving transformation. Let L be a random Jacobi operator over $(\mathbf{T}^2, T, dxdy)$ and let K be a random Jacobi operator over $(\mathbf{T}^2, S, dxdy)$. Let $Y = Y_R \subset \mathbf{T}^4$ be the graph $Y_R = \{((x,y), R(x,y)) \mid (x,y) \in \mathbf{T}^2\} \subset \mathbf{T}^4$. Let m be the measure on Y which is the pull back of the projection $\pi_1 : \mathbf{T}^4 \mapsto \mathbf{T}^2, \pi_1(x_1, y_1, x_2, y_2) = (x_1, y_1)$.*

The arithmetic mean $\mu_2(x,y)$ of the two largest Lyapunov exponents of the 4×4 transfer cocycle of LK exists for m-almost all $(x,y,R(x,y)) \in Y$. The map $(x,y) \mapsto \mu_2(x,y)$ is measureable.

The integral of $\mu_2(x,y)$ over Y_R with measure m is the same as the integral over $\mathbf{T}^4$ with the Lebesgue measure and therefore does not depend on R.

Proof. For Lebesgue almost all $(x,y) \in \mathbf{T}^2$, there is an ergodic T-invariant measure $m_{(x,y)}$ for which (x,y) is generic (see e.g. [26] or [41].) This means that for all $f \in C(\mathbf{T}^2)$, the Cesaro averages $n^{-1} \sum_{k=0}^{n-1} f(T^k(x,y)) \mapsto \int_{\mathbf{T}^2} f(x',y')\, dm_{x,y}(x',y'))$.

Because L becomes so an ergodic operator over the dynamical system $(\mathbf{T}^2, T, m_{(x,y)})$, we can so define $\log\det L(x,y)$ for almost every $(x,y) \in \mathbf{T}^2$ and it is equal to the pointwise Lyapunov exponent $\mu(x,y)$ of the transfer cocycle of L.

Let K is the second additional random operator over $S \in \mathcal{Y}$. For almost all $(x,y) \in \mathbf{T}^2$, and almost all $(u,v) \in \mathbf{T}^2$, one has from the Fuglede-Kadison theory

$$\log\det(L(x,y)K(u,v)) = \log\det L(x,y) + \log\det K(u,v) \; .$$

(The operator $L(x,y)K(u,v)$ is a higher order random difference operator determined by (x_n, y_n, u_n, v_n), where $(x_n, y_n) = T^n(x,y)$ and $(u_n, v_n) = S^n(u,v)$. Lebesgue almost all points $(x,y,u,v) \in \mathbf{T}^4$ are generic with respect to $T \times S$.)

Especially, if $R : \mathbf{T}^2 \mapsto \mathbf{T}^2$ is a measure preserving map then, for almost all (x,y), the relation

$$\log\det(L(x,y)K(R(x,y))) = \log\det L(x,y) + \log\det K(R(x,y)) \tag{15}$$

holds.
Its value is equal to the arithmetic mean $\mu_2(x,y)$ of the two largest Lyapunov exponents of the 4×4 transfer cocycle of $L(x,y)K(R(x,y))$.

Therefore, using (15), the definition of m and the measure preserving property of R, we have

$$\begin{aligned}
\int_Y \log\det(L(x_1,y_1)K(x_2,y_2))\, dm &= \int_{\mathbf{T}^2} \log\det(L(x,y)K(R(x,y)))\, dxdy \\
&= \int_{\mathbf{T}^2} \log\det(L(x,y))\, dxdy + \int_{\mathbf{T}^2} \log\det(K(R(x,y)))\, dxdy \\
&= \int_{\mathbf{T}^2} \log\det(L(x,y))\, dxdy + \int_{\mathbf{T}^2} \log\det(K(x,y))\, dxdy \\
&= \int_{\mathbf{T}^4} \log\det(L)(x_1,y_1))\, dx_1 dy_1 + \log\det(K(x_2,y_2))\, dx_2 dy_2 \\
&= \int_{\mathbf{T}^4} \log\det(L(x_1,y_1)K(x_2,y_2))\, dx_1 dy_1 dx_2 dy_2 \; .
\end{aligned}$$

The right hand side is independent of R.

We have shown with this lemma that the value of the averaged Lyapunov exponent can be obtained by integrating the pointwise Lyapunov exponents over a diagonal Y_R without affecting the value of the Lyapunov exponent and that the value does not depend on R.

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