The Flux-Across-Surfaces Theorem for a Point Interaction Hamiltonian

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ABSTRACT. The flux-across-surfaces theorem establishes a fundamental relation in quantum scattering theory between the asymptotic outgoing state and a quantity which is directly measured in experiments. We prove it for a hamiltonian with a point interaction, using the explicit expression for the propagator. The proof requires only assuptions on the initial state and it covers also the case of zero-energy resonance. We also outline a different approach based on generalized eigenfunctions, in view of a possible extension of the result.

Dedicated to Sergio Albeverio

1. Introduction

In quantum scattering theory one is concerned with the derivation of the experimentally measurable quantities from the large time asymptotics of the quantum state. In particular it is reasonable to expect that the probability that a particle crosses the active surface Σ of a very far detector equals the probability that for large times the particle has a momentum in the cone $C(\Sigma) = \{\lambda\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} \in \Sigma, \lambda \geq 0\}$ generated by the surface Σ. The precise mathematical formulation of this flux-across-surfaces (FAS) conjecture is ([**1**], see also the discussion in [**4**])

$$\lim_{R\to\infty} \int_T^\infty dt \int_{\Sigma_R} \mathbf{j}^{\Psi_t} \cdot \mathbf{n}\, d\sigma = \int_{C(\Sigma)} |\widehat{\Psi}_{\rm out}(\mathbf{k})|^2\, d^3k \tag{1}$$

for any $T \in \mathbb{R}$, where $\mathbf{j}^{\Psi_t} := 2\mathrm{Im}(\Psi_t^* \nabla \Psi_t)$ is the density of probability current associated to $\Psi_t = e^{-iHt}\Psi_0$, $\Sigma_R = \{\mathbf{x} \in C(\Sigma) : |\mathbf{x}| = R\}$, $\Psi_{\rm out} = \Omega_+^{-1}\Psi_0$ is the asymptotic outgoing state and $\widehat{\Psi}$ denotes the Fourier transform of Ψ. We have chosen units in which $\hbar = 1$ and $m = \frac{1}{2}$.

The proof of (1) in the free case was given in [**2**], exploiting the explicit form of the free unitary group (see also [**3**] or [**4**] for a more direct proof). The interacting case has been studied in [**5**] for short range and in [**6**] for long range potentials respectively. The proof relies on the basic assumption that the asymptotic outgoing state $\Psi_{\rm out}$ has a Fourier transform with compact support not containing the

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origin. More recently, a different proof has been given in [**7**] for sufficiently smooth potentials, assuming the absence of zero-energy resonances and requiring Ψ_{out} in the Schwartz space $\mathcal{S}(\mathbb{R}^3)$.
All these results are obtained under assumptions that avoid the difficulty due to zero-energy resonances which produce a slower decay of the wave function for large times and then make problematic the convergence of the left hand side of (1).

Here we consider a specific model hamiltonian, i.e. hamiltonian with a point interaction, and we prove the FAS theorem using the explicit expression for the propagator, derived in [**9**].
This paper has, in a sense, a pedagogical purpose. It indicates the possibility to extend the FAS theorem to more general situations. In particular we show that (1) holds true in the case of point interaction even with a zero-energy resonance. The analysis of zero-energy resonances in the general case of potential scattering will be approached in a further work ([**10**]). We also stress that the result is proved assuming only some regularity on the initial state.

In order to formulate the result we denote by $H_{\alpha,\mathbf{y}}$ the Schrödinger operator in $L^2(\mathbb{R}^3)$ which corresponds to one point interaction placed at $\mathbf{y} \in \mathbb{R}^3$ whose strength is parametrized by $\alpha \in \mathbb{R}$.
It is well known that $H_{\alpha,\mathbf{y}}$ can be constructed as self-adjoint operator using the standard extension theory ([**8**]). Here we only recall that the continuous spectrum is purely absolutely continuous and $\sigma_{\mathrm{ac}}(H_{\alpha,\mathbf{y}}) = [0, +\infty)$. The point spectrum is empty if $\alpha \leq 0$ and $\sigma_{\mathrm{p}}(H_{\alpha,\mathbf{y}}) = \left\{ -(4\pi\alpha)^2 \right\}$ if $\alpha < 0$. For $\alpha = 0$ the hamiltonian exhibits a zero-energy resonance.
The scattering wave functions are

$$\psi_{\alpha,\mathbf{y}}(\mathbf{x},\mathbf{k}) = e^{i\mathbf{k}\cdot\mathbf{x}} + \frac{e^{i\mathbf{k}\cdot\mathbf{y}}}{(4\pi\alpha - ik)} \frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} \tag{2}$$

Using the generalized eigenfunctions $\Phi_-(\mathbf{x},\mathbf{k}) = \psi_{\alpha,\mathbf{y}}(\mathbf{x},\mathbf{k})$ and $\Phi_+(\mathbf{x},\mathbf{k}) = \psi^*_{\alpha,\mathbf{y}}(\mathbf{x},-\mathbf{k})$ one can define two unitary maps $\mathcal{F}_\pm : \mathcal{H}_{ac}(H_{\alpha,\mathbf{y}}) \to L^2(\mathbb{R}^3)$ by

$$(\mathcal{F}_\pm f)(\mathbf{k}) = \operatorname*{s-lim}_{R\to+\infty} \frac{1}{(2\pi)^{3/2}} \int_{|\mathbf{x}|<R} \Phi^*_\pm(\mathbf{x},\mathbf{k}) f(\mathbf{x}) \; d^3x. \tag{3}$$

which spectralise the operator $H_{\alpha,\mathbf{y}}$ restricted to $\mathcal{H}_{ac}$, in the sense that $\mathcal{F}_\pm H_{\alpha,\mathbf{y}} \mathcal{F}_\pm^{-1}$ is a multiplication operator in $L^2(\mathbb{R}^3)$.

The well known relation with the wave operators $\Omega_\pm = \text{s-lim}_{t\to\pm\infty} e^{iH_{\alpha,\mathbf{y}}t} e^{-iH_0 t}$ (where $H_0 = -\Delta$) is expressed by the intertwining properties

$$\Omega_\pm^{-1} = \mathcal{F}^{-1}\mathcal{F}_\pm \qquad \text{and} \qquad \Omega_\pm = \mathcal{F}_\pm^{-1}\mathcal{F}. \tag{4}$$

With the above notation, our result is the following.

THEOREM 1. *Let us fix $\Psi_0 \in \mathcal{S}(\mathbb{R}^3) \cap \mathcal{H}_{ac}(H_{\alpha,\mathbf{y}})$. Then $\Psi_t := e^{-iH_{\alpha,y}t}\Psi_0$ is continuosly differentiable in $\mathbb{R}^3 \backslash \{0\}$ and relation (1) holds true, for every $T \in \mathbb{R}$.*

In Section 2 we shall give the details of the proof for the more interesting case $\alpha = 0$ and then we outline the procedure for the other cases. It will also be clear from the proof that the strong assumption $\Psi_0 \in \mathcal{S}(\mathbb{R}^3)$ has been considered only for the sake of simplicity; it can be replaced by the assumption that Ψ_0 is sufficiently many times differentiable and decays rapidly at infinity.

In Section 3 we briefly sketch a different proof based on the generalized eigenfunctions of $H_{\alpha,\mathbf{y}}$. In the case of one point interaction such method is less satisfactory than the method of the propagator, since it requires regularity of Ψ_{out}. Nevertheless it may be suitable for the extension of the result to general potential scattering in presence of zero-energy resonances.

2. Proof of the theorem

We shall denote by w the modulus of the vector $\mathbf{w} \in \mathbb{R}^3$ and with $\omega_{\mathbf{w}} := \frac{\mathbf{w}}{|\mathbf{w}|}$ the unit vector in the direction defined by $\mathbf{w} \neq \mathbf{0}$.

Without loss of generality we fix $\mathbf{y} = 0$ and, since the interaction is not trivial only in the s-wave ([**8**]), we choose a spherically symmetric initial state Ψ_0. Using the explicit propagator for $\alpha = 0$ ([**9**]) we have

$$\begin{aligned}
\Psi_t(\mathbf{x}) &= \int_{\mathbb{R}^3} \frac{e^{i\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}}}{(4\pi i t)^{3/2}} \Psi_0(y)\, d^3y + \frac{2it}{x} \int_{\mathbb{R}^3} \frac{\Psi_0(y)}{y} \frac{e^{i\frac{(x+y)^2}{4t}}}{(4\pi i t)^{3/2}}\, d^3y \\
&= \frac{e^{i\frac{x^2}{4t}}}{(2it)^{3/2}} \left[\frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \Psi_0(y) \left(e^{-i\frac{\mathbf{x}\cdot\mathbf{y}}{2t}} + \frac{2it}{x} \frac{e^{i\frac{x}{2t}y}}{y} \right) d^3y \right] + R(\mathbf{x},t) \\
&= \frac{e^{i\frac{x^2}{4t}}}{(2it)^{3/2}} \widehat{\Psi}_{\text{out}}\left(\frac{\mathbf{x}}{2t}\right) + R(\mathbf{x},t) \equiv P(\mathbf{x},t) + R(\mathbf{x},t) \qquad (5)
\end{aligned}$$

where we used (4) and we have denoted

$$\begin{aligned}
R(\mathbf{x},t) &= \frac{e^{i\frac{x^2}{4t}}}{(4\pi i t)^{3/2}} \int_{\mathbb{R}^3} e^{-i\frac{\mathbf{x}\cdot\mathbf{y}}{2t}} \left(e^{i\frac{y^2}{2t}} - 1 \right) \Psi_0(y)\, d^3y + \\
&+ \frac{e^{i\frac{x^2}{4t}}}{(2it)^{1/2} x} \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{i\frac{xy}{2t}} \left(e^{i\frac{y^2}{2t}} - 1 \right) \frac{\Psi_0(y)}{y}\, d^3y \\
&=: R_1(\mathbf{x},t) + R_2(\mathbf{x},t) \qquad (6)
\end{aligned}$$

An explicit computation gives

$$\nabla P(\mathbf{x},t) = \frac{i}{2}\frac{\mathbf{x}}{t} P(\mathbf{x},t) + \frac{e^{i\frac{x^2}{4t}}}{(2it)^{3/2}} \nabla_{\mathbf{x}} \widehat{\Psi}_{\text{out}}\left(\frac{\mathbf{x}}{2t}\right) \equiv \frac{i}{2}\frac{\mathbf{x}}{t} P(\mathbf{x},t) + \mathbf{Q}(\mathbf{x},t) \qquad (7)$$

Taking (7) into account one obtains

$$\mathbf{j}^{\Psi_t}(\mathbf{x},t) := 2\text{Im}\,(\Psi_t^*(\mathbf{x},t)\nabla\Psi_t(\mathbf{x},t)) = \frac{\mathbf{x}}{t}|P(\mathbf{x},t)|^2 + \mathbf{N}(\mathbf{x},t)$$

where

$$\mathbf{N} = 2\text{Im}\,(P^*\mathbf{Q} + P^*\nabla R + R^*\nabla P + R^*\nabla R) \qquad (8)$$

A change of the integration variable yields

$$\begin{aligned}
\lim_{R\to\infty} \int_T^\infty dt \int_{\Sigma_R} |P(\mathbf{x},t)|^2 \frac{\mathbf{x}}{t} \cdot \mathbf{n}\, d\sigma &= \lim_{R\to\infty} \int_T^\infty dt \int_{\Sigma_R} \frac{1}{(2t)^3} \left| \widehat{\Psi}_0\left(\frac{\mathbf{x}}{2t}\right) \right|^2 \frac{\mathbf{x}}{t} \cdot \mathbf{n}\, d\sigma \\
&= \int_{C(\Sigma)} |\widehat{\Psi}_{\text{out}}(\mathbf{k})|^2\, d^3k.
\end{aligned}$$

We are then reduced to prove that

$$\lim_{R\to\infty} \int_T^\infty dt \int_{\Sigma_R} \mathbf{N}(\mathbf{x},t) \cdot \mathbf{n}\, d\sigma = 0 \qquad (9)$$

It is convenient to introduce the following definition.

DEFINITION 1. We say that $F : \mathbb{R}^3 \times \mathbb{R} \to C^r$ $(r \in N)$ satisfies the hypothesis $\mathcal{O}(n)$ (for $n \in N$), and we write $F = \mathcal{O}(n)$, if there exist $T_0 > 0$, $R_0 > 0$ such that

$$\sup_{x \geq R_0, t \geq T_0} \left(\frac{x}{t}\right)^q |F(\mathbf{x},t)| \leq C$$

for each $q \leq n$. If $F = \mathcal{O}(n)$ for every $n \in N$ we will write $F = \mathcal{O}(\infty)$.

It is easy to check that for every $f \in \mathcal{S}(\mathbb{R}^3)$ we have

$$F(\mathbf{x},t) := \int_{\mathbb{R}^3} e^{-i\frac{\mathbf{x}\cdot\mathbf{y}}{2t}} f(\mathbf{y})\ d^3y = \mathcal{O}(\infty). \tag{10}$$

Likewise, by iterated integration by parts, one can prove that for every $f \in \mathcal{S}(\mathbb{R}^3)$

$$F_0(\mathbf{x},t) := \int_{\mathbb{R}^3} e^{i\frac{x}{2t}y}\ f(\mathbf{y})\ d^3y = \mathcal{O}(2) \tag{11}$$

$$F_{-1}(\mathbf{x},t) := \int_{\mathbb{R}^3} e^{i\frac{x}{2t}y} \frac{f(\mathbf{y})}{y}\ \ d^3y = \mathcal{O}(1) \tag{12}$$

Using (10), (11) and (12) we can write

$$\begin{aligned} P(\mathbf{x},t) &= \frac{e^{i\frac{x^2}{4t}}}{(2it)^{3/2}}\widehat{\Psi}_0\left(\frac{\mathbf{x}}{2t}\right) + \frac{e^{i\frac{x^2}{4t}}}{(2it)^{1/2}x}\frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3}\Psi_0(y)\ \frac{e^{i\frac{x}{2t}y}}{y}\ d^3y \\ &= \frac{\mathcal{O}(\infty)}{t^{3/2}} + \frac{\mathcal{O}(1)}{xt^{1/2}}. \end{aligned} \tag{13}$$

By a direct computation we obtain

$$\begin{aligned} \mathbf{Q}(\mathbf{x},t) &= \frac{e^{i\frac{x^2}{4t}}}{(2it)^{3/2}}\frac{1}{(2\pi)^{3/2}}\left\{-\frac{i}{2t}\int_{\mathbb{R}^3} e^{-i\frac{\mathbf{x}\cdot\mathbf{y}}{2t}}\mathbf{y}\Psi_0(y)\ d^3y\ + \right. \\ &- \left. i\frac{2t}{x^2}\omega_{\mathbf{x}}\int_{\mathbb{R}^3} e^{i\frac{x}{2t}y}\ \frac{\Psi_0(y)}{y}\ d^3y - \frac{1}{x}\omega_{\mathbf{x}}\int_{\mathbb{R}^3} e^{i\frac{x}{2t}y}\ \Psi_0(y)\ d^3y\right\} \\ &= \frac{1}{t^{5/2}}\mathcal{O}(\infty) + \frac{1}{x^2t^{3/2}}\mathcal{O}(1) + \frac{1}{xt^{3/2}}\mathcal{O}(2). \end{aligned} \tag{14}$$

where we have used (10) and (12) and the inequality

$$\left|e^{iw} - 1\right| \leq |w|\,. \tag{15}$$

Now we analyse $R_1(\mathbf{x},t)$ in (6) following the line of [**4**]. For any $q \in N$ one has

$$\left(\frac{x}{2t}\right)^q e^{-i\frac{\mathbf{x}\cdot\mathbf{y}}{2t}} = i^q\,(\omega_{\mathbf{x}}\cdot\nabla_y)^q\,e^{-i\frac{\mathbf{x}\cdot\mathbf{y}}{2t}}$$

and, by iterated integration by parts, we obtain

$$\begin{aligned} \left(\frac{x}{t}\right)^q |R_1(\mathbf{x},t)| &= \frac{1}{(4\pi it)^{3/2}}\left|\int_{\mathbb{R}^3}\left[(\omega_{\mathbf{x}}\cdot\nabla_y)^q\,e^{-i\frac{\mathbf{x}\cdot\mathbf{y}}{2t}}\right]\ \left(e^{i\frac{y^2}{2t}} - 1\right)\Psi_0(y)\ d^3y\right| \\ &\leq \frac{1}{(4\pi it)^{3/2}}\left|\int_{\mathbb{R}^3}(\omega_{\mathbf{x}}\cdot\nabla_y)^q\ \left(\left(e^{i\frac{y^2}{2t}} - 1\right)\Psi_0(y)\right)\ d^3y\right| \leq \frac{C_q}{t^{5/2}} \end{aligned}$$

where the last inequality follows from an explicit differentiation and from (15). We conclude that

$$R_1(\mathbf{x},t) = \frac{1}{t^{5/2}}\mathcal{O}(\infty) \tag{16}$$

An analogous computation gives

$$\begin{aligned}
\nabla R_1(\mathbf{x},t) &= \frac{\mathbf{x}}{t} R_1(\mathbf{x},t) + \frac{e^{i\frac{x^2}{4t}}}{(4\pi i t)^{3/2}} \frac{-i}{2t} \int_{\mathbb{R}^3} e^{-i\frac{x}{2t}y} \left(e^{i\frac{y^2}{2t}} - 1 \right) \mathbf{y} \Psi_0(y)\, d^3y \\
&= \frac{\mathbf{x}}{t^{7/2}} \mathcal{O}(\infty) + \frac{1}{t^{7/2}} \mathcal{O}(\infty). \qquad (17)
\end{aligned}$$

We analyse now $R_2(\mathbf{x},t)$. Using the inequality (15) one easily gets

$$|R_2(\mathbf{x},t)| \leq \frac{C}{xt^{3/2}}. \qquad (18)$$

Moreover, using the radial simmetry of Ψ_0, one has (for $q = 1, 2$)

$$\begin{aligned}
i^q \left(\frac{x}{2t} \right)^q R_2(\mathbf{x},t) &= \frac{C}{xt^{1/2}} \int_{\mathbb{R}^3} \left(\frac{d^q}{dy^q} e^{i\frac{x}{2t}y} \right) \left(e^{i\frac{y^2}{2t}} - 1 \right) \frac{\Psi(y)}{y}\, d^3y \\
&= \frac{C}{xt^{1/2}} \int_0^{+\infty} \left(\frac{d^q}{dy^q} e^{i\frac{x}{2t}y} \right) \left(e^{i\frac{y^2}{2t}} - 1 \right) \Psi(y) y\, dy.
\end{aligned}$$

Integrating by parts one obtains

$$\left| \left(\frac{x}{2t} \right)^q R_2(\mathbf{x},t) \right| \leq \frac{C}{xt^{1/2}} \int_0^{+\infty} \left| \frac{d^q}{dy^q} \left(\left(e^{i\frac{y^2}{2t}} - 1 \right) \Psi(y) y \right) \right| dy \leq \frac{C}{xt^{3/2}} \qquad (19)$$

By inequalities (18) and (19) one has

$$R_2(\mathbf{x},t) = \frac{1}{xt^{3/2}} \mathcal{O}(2) \qquad (20)$$

A direct computation yields

$$\begin{aligned}
\nabla R_2(\mathbf{x},t) &= \frac{\mathbf{x}}{t} R_2(\mathbf{x},t) + \frac{e^{i\frac{x^2}{4t}}}{(2it)^{1/2}} \frac{1}{(2\pi)^{3/2}} \left\{ -\frac{\omega_{\mathbf{x}}}{x^2} \int_{\mathbb{R}^3} e^{i\frac{x}{2t}y} \left(e^{i\frac{y^2}{2t}} - 1 \right) \frac{\Psi(y)}{y}\, d^3y \right. \\
&+ \left. \frac{1}{x} \frac{i}{2t} \int_{\mathbb{R}^3} e^{i\frac{x}{2t}y} \left(e^{i\frac{y^2}{2t}} - 1 \right) \mathbf{y} \Psi_0(y)\, d^3y \right\}.
\end{aligned}$$

and, by an argument analogous to the previous one, we obtain

$$\nabla R_2(\mathbf{x},t) = \frac{1}{t^{5/2}} \mathcal{O}(2) + \frac{1}{x^2 t^{3/2}} \mathcal{O}(1) + \frac{1}{xt^{5/2}} \mathcal{O}(2). \qquad (21)$$

In order to prove (9) we first observe that

$$\begin{aligned}
&P^*(\mathbf{x},t)\mathbf{Q}(\mathbf{x},t) = \\
&= \mathbf{A}_0(\mathbf{x},t) + \frac{\mathcal{O}(\infty)}{t^4} + \frac{\mathcal{O}(\infty)\mathcal{O}(1)}{x^2 t^3} + \\
&+ \frac{\mathcal{O}(\infty)\mathcal{O}(2)}{xt^3} + \frac{\mathcal{O}(1)\mathcal{O}(\infty)}{xt^3} + \frac{\mathcal{O}(1)\mathcal{O}(2)}{x^2 t^2}.
\end{aligned}$$

where

$$\mathbf{A}_0(\mathbf{x},t) := \frac{1}{(2\pi)^3} \frac{\mathbf{x}}{2tx^4} \left| \int_{\mathbb{R}^3} e^{i\frac{x}{2t}y}\, \frac{\Psi_0(y)}{y}\, d^3y \right|^2. \qquad (22)$$

Since we are interested in the imaginary part of (22) we can neglet the term $\mathbf{A}_0(\mathbf{x},t)$. For the other terms in (22) a direct application of the dominated convergence theorem yields

$$\lim_{R\to\infty} \int_T^{+\infty} \int_{\Sigma_R} |\mathrm{Im}\,(P(\mathbf{x},t)\mathbf{Q}(\mathbf{x},t))|\, R^2 d\Omega dt = 0.$$

Using the estimates (13), (14), (16), (17), (20) and (21) we similarly prove that

$$\lim_{R\to\infty}\int_T^{+\infty}\int_{\Sigma_R}\left|\mathrm{Im}\,(P^*\nabla R+R^*\nabla P+R^*\nabla R)\right|R^2 d\Omega dt=0,$$

proving the claim for an appropriate $T>0$. Using the invariance by finite time translations like in [**7**] we obtain the thesis for every $T\in\mathbb{R}$, in the case $\alpha=0$.

Now we sketch the proof in the case $\alpha\neq 0$. For $\alpha>0$, using the explicit form of the propagator, one obtains

$$\begin{aligned}\Psi_t(\mathbf{x}) &= \frac{e^{i\frac{x^2}{4t}}}{(4\pi i t)^{3/2}}\int_{\mathbb{R}^3}\Psi_0(y)\left(e^{-i\frac{\mathbf{x}\cdot\mathbf{y}}{2t}}+\frac{1}{4\pi\alpha-i\frac{x}{2t}}\frac{e^{i\frac{x}{2t}y}}{y}\right)d^3y+\sum_{j=1}^{3}R_j(\mathbf{x},t)\\ &= \frac{e^{i\frac{x^2}{4t}}}{(2it)^{3/2}}\widehat{\Psi}_{\mathrm{out}}\left(\frac{\mathbf{x}}{2t}\right)+\sum_{j=1}^{3}R_j(\mathbf{x},t)\end{aligned}$$

where R_1 and R_2 are given by (6) and

$$R_3(\mathbf{x},t):=\frac{-2\alpha e^{i\frac{x^2}{4t}}}{(4\pi i t)^{1/2}x}\int_{\mathbb{R}^3}d^3y\ e^{i\frac{x}{2t}y}\frac{\Psi_0(y)}{y}\int_0^{+\infty}du\ e^{-4\pi\alpha u}\left(e^{\frac{i}{4t}(u^2+y^2+2uy)}-1\right)e^{\frac{i}{2t}ux}.$$

All the estimates proved for $\alpha=0$ hold true in the present case. Moreover, concerning the leading term it is easy to see that

$$\left|\frac{e^{i\frac{x^2}{4t}}}{(2it)^{3/2}}\widehat{\Psi}_{\mathrm{out}}\left(\frac{\mathbf{x}}{2t}\right)\right|\leq\frac{C}{t^{3/2}}. \tag{23}$$

We stress that the estimate (23) holds true only for $\alpha\neq 0$. Using the radial symmetry of Ψ_0 and applying Fubini's theorem we can write

$$\begin{aligned}\widetilde{R}_3(\mathbf{x},t) &:= \int_{\mathbb{R}^3}d^3y\ e^{i\frac{x}{2t}y}\frac{\Psi_0(y)}{y}\int_0^{+\infty}du\ e^{-4\pi\alpha u}\left(e^{\frac{i}{4t}(u^2+y^2+2uy)}-1\right)e^{\frac{i}{4t}2ux}\\ &= 4\pi\int_0^{+\infty}du\int_0^{+\infty}dy\ e^{i\frac{x}{2t}(y+u)}y\Psi_0(y)e^{-4\pi\alpha u}\left(e^{\frac{i}{4t}(u+y)^2}-1\right)\\ &= 4\pi\int_0^{+\infty}du\int_u^{+\infty}dw\ e^{i\frac{x}{2t}w}(w-u)\Psi_0(w-u)e^{-4\pi\alpha u}\left(e^{\frac{i}{4t}w^2}-1\right)\end{aligned}$$

where we used the change of integration variable $w=y+u$. The integration domain is

$$D=\left\{(u,w)\in\mathbb{R}_+^2 : u\in[0,+\infty), w>u\right\}=\left\{(u,w)\in\mathbb{R}_+^2 : w\in[0,+\infty), u<w\right\}$$

so

$$\begin{aligned}\widetilde{R}_3(x,t) &= 4\pi\int_0^{+\infty}dw\int_0^{w}du\ e^{i\frac{x}{2t}w}\left(e^{\frac{i}{4t}w^2}-1\right)(w-u)\Psi_0(w-u)e^{-4\pi\alpha u}\\ &= 4\pi\int_0^{+\infty}dw\ \ e^{i\frac{x}{2t}w}\left(e^{\frac{i}{4t}w^2}-1\right)e^{-4\pi\alpha w}\int_0^{w}ds\ s\Psi_0(s)e^{4\pi\alpha s}.\end{aligned}$$

Using the fact that the function

$$\varphi(w):=e^{-4\pi\alpha w}\int_0^{w}ds\ s\Psi_0(s)e^{4\pi\alpha s}$$

satisfies

$$\lim_{w\to+\infty} w^n \varphi(w) = 0 \qquad (n \in N) \tag{24}$$

we can integrate by parts showing that $\widetilde{R}_3(x,t) = \frac{1}{t}\mathcal{O}(2)$. Then we conclude that

$$R_3(\mathbf{x},t) = \frac{\mathcal{O}(2)}{xt^{3/2}}. \tag{25}$$

Following the same line we obtain the corrisponding estimate for $\nabla R_3(\mathbf{x},t)$

$$\nabla R_3(\mathbf{x},t) = \frac{\mathcal{O}(2)}{t^{5/2}} + \frac{\mathcal{O}(2)}{x^2 t^{3/2}} + \frac{\mathcal{O}(2)}{xt^{5/2}}. \tag{26}$$

The previous estimates (23), (25) and (26) allow us to use dominated convergence and then to obtain the result in the case $\alpha > 0$.

Finally, in the case $\alpha < 0$ the propagator can be written in the form

$$\begin{aligned}\Psi_t(\mathbf{x}) &= \left(e^{-iH_0 t}\Psi_0\right)(\mathbf{x}) - e^{i(4\pi\alpha)^2 t}\Psi_\alpha(\mathbf{x})\int_{\mathbb{R}^3}\Psi^*_\alpha(y)\Psi_0(y)\, d^3y + \\ &+ \int_{\mathbb{R}^3}\frac{\Psi_0(y)}{y}\frac{e^{i\frac{(x+y)^2}{4t}}}{(2it)^{1/2}x}d^3y - 4\pi\alpha\frac{2it}{x}\int_{\mathbb{R}^3} d^3y\, \frac{\Psi_0(y)}{y}\int_0^{+\infty} du\; e^{4\pi\alpha u}\frac{e^{i\frac{(u+x+y)^2}{4t}}}{(4\pi i t)^{3/2}}\end{aligned}$$

where Ψ_α is the eigenfunction relative to the eigenvalue $\lambda_\alpha = -(4\pi\alpha)^2$. The second term is identically zero due to the assumption $\Psi_0 \in \mathcal{H}_{\mathrm{ac}}(H_{\alpha,\mathbf{y}}) = \mathcal{H}_{\mathrm{p}}(H_{\alpha,\mathbf{y}})^{\perp}$ and all the remaining terms can be treated exactly as in the case $\alpha > 0$, so we omit the details.

Remark 1. In the proof of Theorem 1 for $\alpha = 0$ we have estimated the *absolute value* of the terms in parenthesis in (8), except for $\mathbf{A}_0$ (see (22)), which is real and then it doesn't contribute to $\mathbf{N}$.
This is a crucial point since, in general, one has

$$\lim_{R\to\infty}\int_T^{\infty} dt \int_{\Sigma_R} |\mathbf{A}_0 \cdot \mathbf{n}|\; d\sigma = +\infty \tag{27}$$

(choose e.g. $\Psi_0(y) = e^{-y}$).
The divergent limit (27) is a consequence of the slower decay of the wave function in presence of a zero-energy resonance. This means that, in such case, one cannot hope to prove the theorem by simply estimating the absolute value of $\mathbf{N}$, unless one assumes the pseudo-orthogonality condition $\Psi_0 \in \mathcal{W}$ (see Remark 2). On the other hand this difficulty doesn't arise for $\alpha \neq 0$. In fact the term $\mathbf{A}_0$ is now replaced by

$$\mathbf{A}_\alpha(\mathbf{x},t) = \frac{1}{t^3}\left|\int_{\mathbb{R}^3} e^{i\frac{x}{t}y}\,\frac{\Psi_0(y)}{y}\, d^3y\right|^2 \frac{1}{\alpha + i\frac{x}{t}}\left(\frac{d}{dx}\frac{1}{\alpha - i\frac{x}{t}}\right)\omega_{\mathbf{x}} \tag{28}$$

which is easily estimated taking the absolute value.

3. The method of generalized eigenfunctions

In this section we outline a proof of the FAS conjecture based on the generalized eigenfunctions of $H_{\alpha,\mathbf{y}}$ and we compare it with the result obtained in the previous section. This tecnique was previosly used in [**7**]. Here we generalize the method in order to allow for the presence of zero-energy resonances.

THEOREM 2. *Let us fix $\Psi_{\rm out} \in \mathcal{S}(\mathbb{R}^3)$. Then $\Psi_t := e^{-iH_{\alpha,\mathbf{y}}t}\Omega_+\Psi_{\rm out}$ is continuosly differentiable in $\mathbb{R}^3\backslash\{0\}$ and (1) holds true for every $T \in \mathbb{R}$.*

Proof. Let be $\Psi_0 = \Omega_+\Psi_{\rm out}$. Using the properties of $\mathcal{F}_+$ and (4) we obtain

$$\begin{aligned}\Psi_t(\mathbf{x}) &= \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3} e^{-ik^2t}\widehat{\Psi}_{\rm out}(\mathbf{k})\Phi_+(\mathbf{x},\mathbf{k})\,d^3k \\ &= \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3} e^{-ik^2t}\widehat{\Psi}_{\rm out}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}\,d^3k + \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3} e^{-ik^2t}\frac{\widehat{\Psi}_{\rm out}(\mathbf{k})}{4\pi\alpha+ik}\frac{e^{-ikx}}{x}d^3k \\ &=: a(\mathbf{x},t)+b(\mathbf{x},t)\,.\end{aligned}$$

The density current is

$$\mathbf{j}^{\Psi_t} = \mathrm{Im}(a^*\nabla a + a^*\nabla b + b^*\nabla a + b^*\nabla a). \tag{29}$$

The first term $\mathbf{j}_0 = \mathrm{Im}(a^*\nabla a)$ corresponds to the free evolution of $\Psi_{\rm out}$, so using the free flux-across-surfaces theorem [**2**] one has

$$\lim_{R\to\infty}\int_T^{+\infty} dt\int_{\Sigma_R} \mathbf{j}_0(\mathbf{x},t)\cdot\mathbf{n}\,d\sigma = \int_{C(\Sigma)} |\widehat{\Psi}_{\rm out}(\mathbf{k})|^2\,d^3k\,.$$

It remains to show that

$$\lim_{R\to\infty}\int_T^{+\infty} dt\int_{\Sigma_R} |\mathbf{j}_1(\mathbf{x},t)\cdot\mathbf{n}|\,d\sigma = 0 \tag{30}$$

where $\mathbf{j}_1 := \mathrm{Im}(a^*\nabla b + b^*\nabla a + b^*\nabla b)$.
In order to prove (30) we need estimates on a, b and their gradients. In the notation of the previous section (see Definition 1), one has

$$a(\mathbf{x},t) = \frac{\mathcal{O}(\infty)}{t^{3/2}} \qquad \text{and} \qquad \nabla a(\mathbf{x},t) = \frac{\mathcal{O}(\infty)}{t^{3/2}}. \tag{31}$$

Concerning b,∇b we distinguish between the cases $\alpha \neq 0$ and $\alpha = 0$.
Case I. In the case $\alpha \neq 0$ we use a stationary phase tecnique, following [**7**]. Posing

$$\chi(k) \equiv \frac{k^2t+kx}{t+x} \qquad \text{and} \qquad \omega \equiv t+x \tag{32}$$

and denoting with $'$ the derivation respect to k, one has

$$\begin{aligned}|b(\mathbf{x},t)| &= \frac{1}{(2\pi)^{\frac{3}{2}}}\left|\int_{\mathbb{R}^3} e^{i\omega\chi(k)}\widehat{\Psi}_{\rm out}(\mathbf{k})\frac{1}{4\pi\alpha+ik}\frac{1}{x}d^3k\right| \\ &= \frac{1}{(2\pi)^{\frac{3}{2}}}\frac{1}{x}\left|\int\frac{1}{\omega\chi'}\left[\frac{d}{dk}e^{-i\omega\chi}\right]\widehat{\Psi}_{\rm out}(\mathbf{k})\frac{1}{4\pi\alpha+ik}k^2\,dk\,d\Omega_{\mathbf{k}}\right|\end{aligned}$$

Integrating by parts and observing that the boundary term vanishes for $k = 0$ since $\chi'(k) \geq \min(1, 2k)$, we obtain

$$\begin{aligned}\sup_{\mathbf{x}\in S_R}|b(\mathbf{x},t)| &\leq \frac{1}{R(R+t)}\int\left|\frac{d}{dk}\left[\frac{1}{\chi'}\widehat{\Psi}_{\rm out}(\mathbf{k})\frac{1}{4\pi\alpha+ik}k^2\right]\right|\,dk\,d\Omega_{\mathbf{k}} \\ &\leq \frac{C}{R(R+t)} \qquad (33)\end{aligned}$$

The function $b(x,t)$ is continuosly differentiable in $\mathbb{R}^3\backslash\{0\}$ and for every $\mathbf{x} \neq 0$ one has

$$(\nabla b)(\mathbf{x},t) = -\frac{1}{(2\pi)^{\frac{3}{2}}}\frac{\omega_{\mathbf{x}}}{x}\int e^{-i(k^2t+kx)}\frac{\widehat{\Psi}_{\text{out}}(\mathbf{k})}{4\pi\alpha+ik}\left(\frac{1}{x}+ik\right)d^3k. \tag{34}$$

The first term is given by $-\frac{\omega_{\mathbf{x}}}{x}b(\mathbf{x},t)$ and the second term is similar to $b(\mathbf{x},t)$ when we replace $\widehat{\Psi}_{\text{out}}(\mathbf{k})$ by $k\widehat{\Psi}_{\text{out}}(\mathbf{k})$. We obtain

$$\sup_{\mathbf{x}\in S_R}|\nabla b(\mathbf{x},t)| \le \frac{c}{R(R+t)}\,. \tag{35}$$

Using (31), (33) and (35) it is now easy to verify (30) and then to prove the thesis.
Case II. In the case $\alpha = 0$ we proceed in a slightly different way. Denoting $\Psi(k) := \int_{S^2}\widehat{\Psi}_{\text{out}}(\mathbf{k})d\Omega_{\mathbf{k}}$ we write

$$b(x,t) = \frac{c}{x}\int_0^{+\infty} e^{-ik^2t-ixk}\;k\Psi(k)\;dk.$$

In order to isolate the contribution arising from the value of $\widehat{\Psi}_{\text{out}}$ in zero, we write $b(x,t) = b_1(x,t) + b_2(x,t)$ where

$$\begin{aligned} b_1(x,t) &:= \frac{c}{x}\int_0^{+\infty} e^{-ik^2t-ixk}\;k\left(\Psi(k)-\Psi(0)e^{-k^2}\right)\;dk \\ b_2(x,t) &:= \frac{c}{x}\Psi(0)\int_0^{+\infty} e^{-ik^2t-ixk}\;k\;e^{-k^2}\;dk \end{aligned}$$

The term $b_1(x,t)$ can be dealt with by the same technique used in the case $\alpha \neq 0$, so we analyse the term $b_2(x,t)$. Using the formula (valid for $\text{Re}\xi > 0, \eta \in C$)

$$\int_0^{+\infty} w\exp\left(-\xi w^2-\eta w\right)dw = -\frac{1}{2}\sqrt{\frac{\pi}{\xi}}\frac{\partial}{\partial\eta}\left[\exp\left(\frac{\eta^2}{4\xi}\right)\text{erfc}\left(\frac{\eta}{2\sqrt{\xi}}\right)\right]$$

one obtains that (posing $\xi := 1-it$ and $R = |x|$)

$$b_2(x,t) = -\frac{1}{4}\frac{c}{R\,\xi^{3/2}}\left(iR\sqrt{\pi}e^{-\frac{R^2}{4\xi}} - 2\sqrt{\xi} - iR\sqrt{\pi}e^{-\frac{R^2}{4\xi}}\text{erf}\left(\frac{iR}{2\sqrt{\xi}}\right)\right).$$

Then we conclude that

$$|b(x,t)| \le \frac{C_1}{R(R+t)} + \frac{C_2}{R\,t}. \tag{36}$$

Concerning the gradient we have

$$\begin{aligned} \nabla b_2(x,t) &= -\frac{c}{x^2}\,\omega_x\int_0^{+\infty} e^{-ik^2t-ixk}\;ke^{-k^2}\;dk + \\ &- i\frac{c}{x}\,\omega_x\int_0^{+\infty} e^{-ik^2t-ixk}\;k^2e^{-k^2}\;dk \end{aligned} \tag{37}$$

From (37) one easily deduces the estimate

$$|\nabla b(x,t)| \le \frac{C_1}{R(R+t)} + \frac{C_2}{R^2t}. \tag{38}$$

Estimates (36) and (38) allow us to apply the dominated convergence theorem, and then to prove (30).

Remark 2. Using (4), (3) and (2) one obtains that

$$\begin{aligned}\widehat{\Psi}_{\mathrm{out}}(\mathbf{k}) &= (\mathcal{F}_{+}\Psi_0)(\mathbf{k}) = \int_{\mathbb{R}^3} \Phi_{+}(\mathbf{x},\mathbf{k})^{*}\,\Psi_0(\mathbf{x})\; d^3x \\ &= \int_{\mathbb{R}^3} \left(e^{-i\mathbf{k}\cdot\mathbf{x}} + \frac{1}{(4\pi\alpha - ik)} \frac{e^{ikx}}{x} \right) \Psi_0(\mathbf{x})\; d^3x \\ &= \widehat{\Psi}_0(\mathbf{k}) + \frac{1}{4\pi\alpha - ik} \int_{\mathbb{R}^3} \frac{e^{ikx}}{x} \Psi_0(\mathbf{x})\; d^3x.\end{aligned}$$

The above expression shows that, in presence of zero-energy resonances (i.e. for $\alpha = 0$), and if $\widehat{\Psi}_0$ is regular, the asymptotic outgoing state has a singularity in the origin of the momentum space, unless the initial state Ψ_0 belongs to the linear subspace

$$\mathcal{W} = \left\{ \Psi \in L^2(\mathbb{R}^3) : \int_{\mathbb{R}^3} \frac{1}{x} \Psi(\mathbf{x})\; d^3x = 0 \right\}$$

We underline that the set $\mathcal{W}$ is not closed in $L^2(\mathbb{R}^3)$. Note that, if $\Psi_0 \in \mathcal{W}$, then the problematic term $\mathbf{A}_0$ can be estimated taking the absolute value. The condition $\Psi_0 \in \mathcal{W}$ can be read as a condition of pseudo-orthogonality between Ψ_0 and the resonance function $\Psi_{\mathrm{res}}(\mathbf{x}) = \frac{1}{|\mathbf{x}|} \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$. It is then clear that an assumption on the smoothness of $\widehat{\Psi}_{\mathrm{out}}$ is related to this condition of pseudo-orthogonality on the initial state. Since in Theorem 1 we don't need such restrictive assumption on Ψ_0 we conclude that the method of the propagator allows a better result. Nevertheless the method cannot be extended to more general situations while, on the other hand, this seems to be the case for the method of generalized eigenfunctions.

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