On the eigenvalues of operators with gaps.

Application to Dirac operators.

Jean Dolbeault, Maria J. Esteban and Eric Séré

CEREMADE (UMR C.N.R.S. 7534)
Université Paris IX-Dauphine
Place du Maréchal de Lattre de Tassigny
75775 Paris Cedex 16 - France
E-mail: dolbeaul, esteban or sere@ceremade.dauphine.fr

This paper is devoted to a general min-max characterization of the eigenvalues in a gap of the essential spectrum of a self-adjoint unbounded operator. We prove an abstract theorem, then we apply it to the case of Dirac operators with a Coulomb-like potential. The result is optimal for the Coulomb potential.

AMS Subject Classification: 81Q10; 49R05, 47A75, 35Q40, 35Q75

Keywords: Variational methods, self-adjoint operators, quadratic forms, spectral gaps, eigenvalues, min-max, Rayleigh-Ritz quotients, Dirac operators, Hardy's inequality.

1. INTRODUCTION

The aim of this paper is to prove a very general result on the variational characterization of the eigenvalues of operators with gaps in the essential spectrum. More precisely, let $\mathcal{H}$ be a Hilbert space and $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ a self-adjoint operator. We denote by $\mathcal{F}(A)$ the form-domain of A. Let $\mathcal{H}_+$, $\mathcal{H}_-$ be two orthogonal Hilbert subspaces of $\mathcal{H}$ such that $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. We denote Λ_+, Λ_- the projectors on $\mathcal{H}_+$, $\mathcal{H}_-$. We assume the existence of a core F (i.e. a subspace of $D(A)$ which is dense for the norm $\|.\|_{D(A)}$), such that :

(i) $F_+ = \Lambda_+ F$ and $F_- = \Lambda_- F$ are two subspaces of $\mathcal{F}(A)$.
(ii) $a = \sup_{x_- \in F_- \setminus \{0\}} \frac{(x_-, Ax_-)}{\|x_-\|^2_{\mathcal{H}}} < +\infty$.

We consider the sequence of min-max levels

$$\lambda_k = \inf_{\substack{V \text{ subspace of } F_+ \\ \dim V = k}} \sup_{x \in (V \oplus F_-) \setminus \{0\}} \frac{(x, Ax)}{||x||^2_{\mathcal{H}}} , \qquad k \geq 1. \tag{1}$$

Our last assumption is

(iii) $\qquad \lambda_1 > a$.

Now, let $b = \inf\ (\sigma_{\text{ess}}(A) \cap (a, +\infty)) \in [a, +\infty]$. For $k \geq 1$, we denote by μ_k the k^{th} eigenvalue of A in the interval (a, b), counted with multiplicity, *if this eigenvalue exists.* If there is no k^{th} eigenvalue, we take $\mu_k = b$. The main result of this note is

THEOREM 1.1. *With the above notations, and under assumptions* $(i) - (ii) - (iii)$,

$$\lambda_k \ = \ \mu_k \ , \quad \forall k \geq 1 \ .$$

As a consequence, $b = \lim_{k \to \infty} \lambda_k \ = \sup_k \lambda_k \ > \ a$.

Such a min-max approach was first proposed by Talman [15] and Datta-Deviah [2] in the particular case of Dirac operators with a potential, to compute numerically their first positive eigenvalue. In that case, the decomposition of $\mathcal{H}$ was very convenient for practical purposes: each 4-spinor was decomposed in its upper and lower parts. Note that in the Physics litterature, other min-max approaches were proposed, for the study of the eigenvalues of Dirac operators with a potential (see for instance [4], [10]).

A rigorous min-max procedure was then considered by Esteban and Séré in [6] for Dirac operators $H_0 + V$, V being a Coulomb-like potential. This time, $\mathcal{H}_+$ and $\mathcal{H}_-$ were the positive and negative spectral spaces of the free Dirac operator H_0.

To our knowledge, the first abstract theorem on the variational principle (1) is due to Griesemer and Siedentop [8]. These authors proved an analogue of Theorem 1.1, under conditions (i), (ii), and two additional hypotheses instead of (iii): they assumed that $(Ax, x) > a\|x\|^2$ for all $x \in F_+ \setminus \{0\}$, and they required the operator $(|A|+1)^{1/2}P_-\Lambda_+$ to be bounded. Here, Λ_+ is the orthogonal projection of $\mathcal{H}$ on $\mathcal{H}_+$ and P_- is the spectral projection of A for the interval $(-\infty, a]$, *i.e.* $P_- = \chi_{(-\infty,a]}(A)$.

Then, Griesemer and Siedentop applied their abstract result to the Dirac operator with potential. They proved that the min-max procedure proposed by Talman and Datta-Deviah was mathematically correct for a particular class of bounded potentials. In this case, the restrictions on the potentials were necessary in order to fulfill the requirement $(Ax, x) > a\|x\|^2$, $\forall x \in F_+ \setminus \{0\}$. Such a hypothesis excludes the Coulomb potentials which appear in atomic models. Griesemer and Siedentop also applied their theorem to the min-max of [6], but the boundedness of $(|A|+1)^{1/2}P_-\Lambda_+$ seems difficult to check in the case of Coulomb potentials. See the recent work [7], where this problem is partially solved.

In [3], we extended the result of [6] to a larger class of Coulomb-like potentials and introduced a minimization approach to define the first positive eigenvalue of $H_0 + V$.

The present work is motivated by the abstract result of Griesemer and Siedentop [8]. Our Theorem 1.1 contains, as particular cases, the results on the min-max principle for the Dirac operator of [6], [8], [3], [7]. It also applies to the Talman and Datta-Deviah procedure for atomic Coulomb potentials, under optimal conditions. However, Griesemer-Siedentop's abstract result is not a consequence of Theorem 1.1. Indeed, their hypothesis $(Ax_+, x_+) > a\|x_+\|^2 \quad (\forall x_+ \in F_+ \setminus \{0\})$ does not imply (iii).

In Section 2 of this paper we prove Theorem 1.1. The arguments are based on an abstract version of those in [3] (§4: the minimization procedure).

When appling Theorem 1.1 in practical situations, the main difficulty is to check assumption (iii). For that purpose, an abstract continuation principle (Theorem 3.1) will be given in Section 3.

In Section 4 we use Theorems 1.1 and 3.1 to justify two variational procedures for the eigenvalues of Dirac operators $H_0 + V$: first, Talman's and Datta-Deviah's procedure; then, the min-max principle of [3]. In both cases we cover a large class of potentials V including Coulomb potentials $-Z\alpha/|x|$, as long as $Z\alpha < 1$. This condition is optimal since it is well-known that when $Z\alpha \to 1^-$, the first eigenfunction "disappears". For each min-max, we obtain new Hardy-type inhomogeneous inequalities as by-products of the proof.

2. PROOF OF THEOREM 1.1.

The inequality $\lambda_k \leq \mu_k$ is an easy consequence of conditions (i) and (ii) (see [8] for the proof in a similar situation). It remains to prove that $\lambda_k \geq \mu_k$ for all k. The additional assumption (iii) will be needed, but for the moment, we only assume (i) and (ii).

We recall the notation $a = \sup_{x_- \in F_- \setminus \{0\}} \dfrac{(x_-, Ax_-)}{||x_-||^2_{\mathcal{H}}} < +\infty$. For $E > a$ and $x_+ \in F_+$, let us define

$$\begin{aligned} \varphi_{E,x_+} :\quad & F_- \to \mathbb{R} \\ & y_- \mapsto \varphi_{E,x_+}(y_-) = \Big((x_+ + y_-), A(x_+ + y_-)\Big) - E||x_+ + y_-||^2_{\mathcal{H}} . \end{aligned}$$

From assumption (ii), $N(y_-) = \sqrt{(a+1)||y_-||^2_{\mathcal{H}} - (y_-, Ay_-)}$ is a norm on F_-. Let $\overline{F}_-^N$ be the completion of F_- for this norm. Since $||.||_{\mathcal{H}} \leq N$ on F_-, we have $\overline{F}_-^N \subset \mathcal{H}_-$. For all $x_+ \in F_+$, there is an $x \in F$ such that $\Lambda_+ x = x_+$. If we consider the new variable $z_- = y_- - \Lambda_- x$, we can define

$$\psi_{E,x}(z_-) := \varphi_{E,\Lambda_+ x}(z_- + \Lambda_- x) = (A(x+z_-), x+z_-) - E(x+z_-, x+z_-) .$$

Since F is a subspace of $D(A)$, $\psi_{E,x}$ (hence φ_{E,x_+}) is well-defined and continuous for N, uniformly on bounded sets. So, φ_{E,x_+} has a unique continuous extension $\overline{\varphi}_{E,x_+}$ on $\overline{F}_-^N$, which is continuous for the extended norm $\overline{N}$. It is well-known (see e.g. [12]) that there is a unique self-adjoint operator $B : D(B) \subset \mathcal{H}_- \to \mathcal{H}_-$ such that $D(B)$ is a subspace of $\overline{F}_-^N$, and

$$\overline{N}(x_-)^2 \;=\; (a+1)||x_-||^2_{\mathcal{H}} + (x_-, Bx_-) \;, \quad \forall x_- \in D(B) \;. \tag{2}$$

Now, $\overline{\varphi}_{E,x_+}$ is of class C^2 on $\overline{F}_-^N$ and

$$\begin{aligned} D^2\overline{\varphi}_{E,x_+}(x_-)\cdot(y_-, y_-) \;&=\; -2(y_-, By_-) - 2E||y_-||^2_{\mathcal{H}} \\ &\leq\; -2\min\left(1, (E-a)\right) \overline{N}(y_-)^2 \;. \end{aligned} \tag{3}$$

So $\overline{\varphi}_{E,x_+}$ has a unique maximum, at the point $y_- = L_E(x_+)$. The Euler-Lagrange equations associated to this maximization problem are :

$$\Lambda_- A x_+ - (B+E) y_- = 0\,. \tag{4}$$

In the sequel of this note, we shall use the notation $\mathcal{X}'$ for the dual of a Hilbert space $\mathcal{X}$. Note that $(B+E)^{-1}$ is well-defined and bounded from $(\overline{F_-}^N)'$ to $\overline{F_-}^N$, since $E > a$ and $(y_-, (B+a)y_-) \geq 0,\ \forall y_- \in D(B)$. Moreover, $x_+ \in F_+ = \Lambda_+ F$ is of the form $x_+ = \Lambda_+ x = x - \Lambda_- x$ for some $x \in F \subset D(A)$. By assumption (i), $\Lambda_- x \in \mathcal{F}(A) \cap (\overline{F_-}^N)$, and finally $\Lambda_- A(x_+) = \Lambda_- A x - \Lambda_- A \Lambda_- x \in (\overline{F_-}^N)'$. So the expression $(B+E)^{-1}\Lambda_- x_+$ is meaningful, and we have

$$L_E = (B+E)^{-1}\ \Lambda_- A\ . \tag{5}$$

Remark 2.1. *The unique maximizer of* $\overline{\psi}_{E,x} := \overline{\varphi}_{E,\Lambda_+ x}(\cdot + \Lambda_- x)$ *is the vector* $z_- = M_E x := L_E \Lambda_+ x - \Lambda_- x$ *and one has the following equation for* $M_E x$*:*

$$M_E x = (B+E)^{-1}\Lambda_-(A-E)x\,. \tag{6}$$

This expression is well-defined, since $x \in D(A)$.

The above arguments allow us, for any $E > a$, to define a map

$$\begin{aligned} Q_E :\ & F_+ \to \mathbb{R} \\ & x_+ \mapsto Q_E(x_+) = \sup_{x_- \in F_-} \varphi_{E,x_+}(x_-) = \overline{\varphi}_{E,x_+}(L_E x_+) \\ & \qquad = (x_+, (A-E)x_+) + \big(\Lambda_- A x_+, (B+E)^{-1}\Lambda_- A x_+\big)\ . \end{aligned} \tag{7}$$

Note that for any $x \in F$,

$$\begin{aligned} Q_E(\Lambda_+ x) = (x, Ax)\ & + 2\,\mathrm{Re}\,(Ax, M_E x) \\ & - (M_E x, B M_E x) - E\|x + M_E x\|^2\,. \end{aligned} \tag{8}$$

It is easy to see that Q_E is a quadratic form with domain $F_+ \subset \mathcal{H}_+$.
We may also, for $E > a$ given, define the norm

$$n_E(x_+) = ||x_+ + L_E x_+||_{\mathcal{H}}\ . \tag{9}$$

The following lemma gives some useful inequalities involving n_E and Q_E, and a new formulation of (iii) :

LEMMA 2.1. *Assume that (i) and (ii) are satisfied. If $a < E < E'$, then*

$$\|\cdot\|_{\mathcal{H}} \leq n_{E'} \leq n_E \leq \frac{E'-a}{E-a} n_{E'} \ , \tag{10}$$

$$(E'-E)n_{E'}^2 \leq Q_E - Q_{E'} \leq (E'-E)n_E^2 \ . \tag{11}$$

Moreover, for any $E > a$:

$$\lambda_1 > E \quad \textit{if and only if} \quad Q_E(x_+) > 0\,, \quad \forall\, x_+ \in F_+ \ .$$

$$\lambda_1 \geq E \quad \textit{if and only if} \quad Q_E(x_+) \geq 0\,, \quad \forall\, x_+ \in F_+ \ .$$

As a consequence, (iii) is equivalent to

(iii') *For some $E > a$,* $Q_E(x_+) \geq 0\,, \quad \forall\, x_+ \in F_+$.

Proof. Inequality (10) is easily proved using the spectral decomposition of B, the formula

$$n_E(x_+)^2 = ||x_+||_{\mathcal{H}}^2 + ||(B+E)^{-1}\Lambda_- A x_+||_{\mathcal{H}}^2$$

and the standard inequality

$$1 \leq \frac{t+u}{t+v} \leq \frac{u}{v} \ , \quad \forall t \geq 0\,,\ u \geq v > 0 \ .$$

On the other hand, (11) is a consequence of

$$Q_{E'}(x_+) \geq \overline{\varphi}_{E',x_+}(L_E(x_+))\,, \text{ for all } E, E' > a \ .$$

Finally, the definition of λ_1 implies that $Q_E(x_+) > 0$ for all $x_+ \in F_+ \setminus \{0\}$ and $a < E < \lambda_1$. But (10) and (11) imply that

$$Q_{\lambda_1}(x_+) \geq Q_E(x_+) + (E-\lambda_1)\frac{(\lambda_1-a)^2}{(E-a)^2}\, n_{\lambda_1}^2(x_+) \ .$$

Passing to the limit $E \to \lambda_1$, we obtain $Q_{\lambda_1}(x_+) \geq 0$.
In the case $E > \lambda_1$, it follows from the definition of λ_1 that for some $x_+ \in F_+ \setminus \{0\}$ and some $\varepsilon > 0$,

$$(x_+ + x_-, A(x_+ + x_-)) \leq (E-\varepsilon)||x_+ + x_-||^2 \ , \quad \forall\, x_- \in F_- \ .$$

Hence

$$\varphi_{E,x_+}(x_-) \leq -\varepsilon ||x_+ + x_-||^2 \ , \quad \forall\, x_- \in F_-$$

and $Q_E(x_+) \leq -\varepsilon ||x_+||^2 < 0$. This ends the proof of Lemma 2.1.

□

We are now going to give a new definition of the numbers λ_k, equivalent to formula (1). First of all, let us recall the standard definitions and results on Rayleigh-Ritz quotients (see e.g. [13]).
Let T be a self-adjoint operator on a Hilbert space X, with domain $D(T)$ and form-domain $\mathcal{F}(T)$. If T is bounded from below, we may define a sequence of min-max levels,

$$\ell_k(T) \ = \inf_{\substack{Y \text{ subspace of } \mathcal{F}(T) \\ \dim\ Y=k}} \ \sup_{x\in Y\setminus\{0\}} \frac{(x,Tx)}{||x||_X^2}.$$

To each k we also associate the (possibly infinite) multiplicity number

$$m_k(T) \ = \ \mathrm{card}\Big\{k' \geq 1 \ , \ \ell_{k'}(T) = \ell_k(T)\Big\} \ \geq \ 1 \ .$$

Then $\ell_k(T) \leq \inf \sigma_{\rm ess}(T)$. In the case $\ell_k(T) < \inf \sigma_{\rm ess}(T)$, ℓ_k is an eigenvalue of T with multiplicity $m_k(T)$.

As a consequence, if $\mathcal{C} \subset \mathcal{F}(T)$ is a form-core for T (i.e. a dense subspace of $\mathcal{F}(T)$ for $||.||_{\mathcal{F}(T)}$), then there is a sequence (Z_n) of subspaces of $\mathcal{C}$, with dim $(Z_n) = m_k(T)$ and

$$\sup_{\substack{z\in Z_n \\ ||z||_X=1}} \quad ||Tz - \ell_k(T)z||_{(\mathcal{F}(T))'} \quad \underset{n\to\infty}{\longrightarrow} \quad 0 \ .$$

Coming back to our situation, we consider the completion X of F_+ for the norm n_E. By (10), X does not depend on $E > 0$. We denote by $\overline{n}_E$ the extended norm, and by $< \cdot, \cdot >_E$ its polar form:

$$< x_+, x_+ >_E = (\overline{n}_E(x_+))^2 \ , \ \forall x_+ \in X \ .$$

Since $n_E(x_+) \geq ||x_+||_{\mathcal{H}}$, X is a subspace of $\mathcal{H}_+$.

We now assume that (iii) *is satisfied, i.e.* $\lambda_1 > a$. We may define another norm on F_+ by

$$\mathcal{N}_E(x_+) = \sqrt{Q_E(x_+) + (K_E+1)(n_E(x_+))^2}$$

with $K_E = \max\left(0, \frac{(E-a)^2(E-\lambda_1)}{(\lambda_1-a)^2}\right)$.

From (10) and (11), $\mathcal{N}_E$ is well-defined and satisfies $\mathcal{N}_E \geq n_E$. Indeed, in the case $a < E \leq \lambda_1$, Lemma 2.1 implies $Q_E(x_+) \geq 0$ for all $x_+ \in F_+$. When $E \geq \lambda_1$, again from Lemma 2.1, we have

$$Q_E \geq Q_{\lambda_1} + (\lambda_1 - E)\, n_{\lambda_1}^2 \geq -K_E\, n_E^2 \,. \tag{12}$$

Note that for any $a < E < E'$, Lemma 2.1 implies the existence of two positive constants, $0 < c(E,E') < 1 < C(E,E')$, such that

$$c(E,E')\, \mathcal{N}_{E'} \leq \mathcal{N}_E \leq C(E,E')\, \mathcal{N}_{E'} \,. \tag{13}$$

Let us consider the completion G of F_+ for the norm $\mathcal{N}_E$. Since $\mathcal{N}_E \geq n_E$, G is a subspace of X, dense for the extended norm $\bar{n}_E$. From (13), G does not depend on E. The extension $\bar{Q}_E$ of Q_E to G is a closed quadratic form with form-domain G. So (see e.g. [12]) there is a unique self-adjoint operator $T_E \;:\; D(T_E) \subset X \to X$ with form-domain $\mathcal{F}(T_E) = G$, such that $\bar{Q}_E(x_+) =< x_+, T_E x_+ >_E$, for any $x_+ \in D(T_E)$. Then F_+ is a form-core of T_E. The min-max levels $\ell_k(T_E)$ are given by

$$\ell_k(T_E) \;=\; \inf_{\substack{V \text{ subspace of } G \\ \dim\, Y = k}} \;\sup_{x_+ \in V \setminus \{0\}} \frac{\bar{Q}_E(x_+)}{(\bar{n}_E(x_+))^2} \,. \tag{14}$$

The next lemma explains the relashionship between $\ell_k(T_E)$ and the min-max principle (1) for A.

Lemma 2.2. *Under assumptions (i), (ii), (iii) :*
(a) *for any $x_+ \in F_+ \setminus \{0\}$, the real number*

$$\lambda(x_+) := \sup_{x \in (Span(x_+) \oplus F_-) \setminus \{0\}} \frac{(x, Ax)}{||x||_{\mathcal{H}}^2}$$

is the unique solution in $(a, +\infty)$ of the nonlinear equation

$$Q_\lambda(x_+) = 0 \,. \tag{15}$$

This equation may be written

$$\lambda \|x_+\|_{\mathcal{H}}^2 = (x_+, Ax_+) + (\Lambda_- A x_+, (B + \lambda)^{-1} \Lambda_- A x_+) \,. \tag{16}$$

(b) *The min-max principle (1) is equivalent to*

$$\lambda_k = \inf_{\substack{V \text{ subspace of } F_+ \\ \dim\, V = k}} \;\sup_{x_+ \in V \setminus \{0\}} \lambda(x_+) \,, \qquad k \geq 1. \tag{17}$$

(c) *For any $k \geq 1$, the level λ_k defined by (1) is the unique solution in $(a, +\infty)$ of the nonlinear equation*

$$\ell_k(T_\lambda) \ = \ 0 \ . \tag{18}$$

In other words, 0 is the k^{th} min-max level for the Rayleigh-Ritz quotients of T_{λ_k}, and this determines λ_k in a unique way. Moreover, for $a < \lambda \neq \lambda_k$, the signs of $\lambda_k - \lambda$ and $\ell_k(T_\lambda)$ are the same.

Proof.
(a) From Lemma 2.1, $Q_\lambda(x_+)$ is a decreasing continuous function of λ, such that $Q_{\lambda_1}(x_+) \geq 0$ and $\lim\limits_{\lambda \to +\infty} Q_\lambda(x_+) = -\infty$. So the equation $Q_\lambda(x_+) = 0$ has one and only one solution $\tilde{\lambda}(x_+)$, which lies in the interval $[\lambda_1, +\infty)$. Equation (16) is equivalent to (15) by easy calculations. Now, if $\lambda < \tilde{\lambda}(x_+)$, then $Q_\lambda(x_+) > 0$, hence $\lambda(x_+) := \sup\limits_{x \in (Span(x_+) \oplus F_-) \setminus \{0\}} \dfrac{(x, Ax)}{||x||^2_{\mathcal{H}}} > \lambda$. Similarly, $\lambda > \tilde{\lambda}(x_+)$ implies $\lambda(x_+) < \lambda$. So we get

$$\tilde{\lambda}(x_+) = \lambda(x_+) \ .$$

(b) Since $\lambda(x_+) = \sup\limits_{\substack{x \in Span(x_+) \oplus F_- \\ x \neq 0}} \dfrac{(x, Ax)}{||x||^2_{\mathcal{H}}}$, (1) is obviously equivalent to (17).

(c) We follow the same arguments as in the proof of (a). From Lemma 2.1, the map $\lambda \to \ell_k(T_\lambda)$ is continuous, and $\ell_k(T_{\lambda_1}) \geq 0$, $\lim\limits_{\lambda \to +\infty} \ell_k(T_\lambda) = -\infty$. As a consequence, the equation $\ell_k(T_\lambda) = 0$ has at least one solution $\tilde{\lambda}_k$ which lies in the interval $[\lambda_1, +\infty)$. Now, if $\lambda < \tilde{\lambda}_k$ then from Lemma 2.1, $\ell_k(T_\lambda) > 0$. Hence $\sup\limits_{x \in (V \oplus F_-) \setminus \{0\}} \dfrac{(x, Ax)}{||x||^2_{\mathcal{H}}} > \lambda$ for any k-dimensional subspace V of F_+. Similarly, $\lambda > \tilde{\lambda}_k$ implies $\sup\limits_{x \in (V \oplus F_-) \setminus \{0\}} \dfrac{(x, Ax)}{||x||^2_{\mathcal{H}}} < \lambda$ for some k-dimensional subspace V of F_+. So, we get $\tilde{\lambda}_k = \lambda_k$.

□

As already mentioned, F_+ is a form-core of T_E and G is its form-domain. From Lemma 2.2 (c), $\lambda_k = \lambda_{k'}$ if and only if $l_{k'}(T_{\lambda_k}) = 0$. So, denoting $m_k := \text{card } \{k' \geq 1 \ ; \ \lambda_k = \lambda_{k'}\}$, there is a sequence (Z_n) of subspaces of F_+, of dimension m_k, such that

$$\sup_{\substack{x_+ \in Z_n \\ ||x_+||^2_{\mathcal{H}} + ||L_{\lambda_k} x_+||^2_{\mathcal{H}} = 1}} \left\| T_{\lambda_k} x_+ \right\|_{G'} \quad \underset{n \to \infty}{\longrightarrow} \quad 0 \ .$$

Using the explicit expressions of Q_E and L_E on F_+ (see (5),(7)), we obtain

$$\sup_{\substack{\tilde{x}\in(\mathbf{I}+\mathrm{L}_{\lambda_k})(\mathrm{Z_n})\\ ||\tilde{x}||_{\mathcal{H}}=1}} \quad \sup_{\substack{\tilde{y}\in(\mathbf{I}+\mathrm{L}_{\lambda_k})(\mathrm{F}_+)\\ \tilde{y}\neq 0}} \frac{|\mathcal{A}(\tilde{x},\tilde{y})-\lambda_k(\tilde{x},\tilde{y})_{\mathcal{H}}|}{\left((K_{\lambda_k}+1)\,||\tilde{y}||^2_{\mathcal{H}}+Q_{\lambda_k}(\Lambda_+\tilde{y})\right)^{1/2}} \underset{n\to\infty}{\longrightarrow} 0\ , \tag{19}$$

where $\mathcal{A}(\tilde{x},\tilde{y}) := (x, Ay) + (Ax, M_{\lambda_k}y) + (M_{\lambda_k}x, Ay) - (BM_{\lambda_k}x, M_{\lambda_k}y)$, with $x, y \in F \subset D(A)$ such that $\Lambda_+x = \Lambda_+\tilde{x}$, $\Lambda_+y = \Lambda_+\tilde{y}$ and $M_{\lambda_k}x = L_{\lambda_k}\Lambda_+x - \Lambda_-x$. Note that the value of $\mathcal{A}(\tilde{x},\tilde{y})$ does not depend on the choice of x and y. Indeed, $\mathcal{A}$ is the polar form of the quadratic form $\tilde{y} \mapsto \overline{Q}_{\lambda_k}(\Lambda_+\tilde{y}) + \lambda_k||\tilde{y}||^2_{\mathcal{H}}$.

Denote $\tilde{Z}_n = (\mathbf{I}+\mathrm{L}_{\lambda_k})(\mathrm{Z_n})$. Take $y \in F$, and let $\tilde{y} = (\mathbf{I}+\mathrm{L}_{\lambda_k})(\Lambda_+\mathrm{y})$. There is a constant $C(\lambda_k)$ such that

$$(K_{\lambda_k}+1)\,||\tilde{y}||^2_{\mathcal{H}} + Q_{\lambda_k}(\Lambda_+y) \le C(\lambda_k)\,||y||^2_{D(A)}\ . \tag{20}$$

Indeed, by Remark 2.1,

$$\begin{aligned} &Q_{\lambda_k}(\Lambda_+y) = ((A-\lambda_k)y, y + M_{\lambda_k}y)\\ &\quad\le (1+|\lambda_k|)||y||_{D(A)}\,(||y||_{\mathcal{H}}+||M_{\lambda_k}y||_{\mathcal{H}}) \le (1+|\lambda_k|)\left(1+\tfrac{1+|\lambda_k|}{\lambda_k-a}\right)||y||^2_{D(A)}. \end{aligned}$$

Moreover, for any $x\in F_+$, and any $z_-\in\mathcal{F}(B)$, by (6) we have : $((Ax - BM_{\lambda_k}x) - \lambda_k(x + M_{\lambda_k}x), z_-) = 0$. As a consequence, (19) is equivalent to

$$\sup_{\substack{\tilde{x}\in\tilde{Z}_n\\ ||\tilde{x}||_{\mathcal{H}}=1}} \quad \sup_{y\in F\setminus\{0\}} \frac{|(\tilde{x}, Ay-\lambda_k y)|}{||y||_{D(A)}} \quad \underset{n\to\infty}{\longrightarrow} \quad 0\ .$$

So, by the standard spectral theory of self-adjoint operators, we obtain an alternative: either $\lambda_k \in \sigma_{\mathrm{ess}}(A)\cap(a,+\infty)$, or λ_k is an eigenvalue of A in the interval $(a,+\infty)$, with multiplicity greater than or equal to m_k.

We have thus proved the inequality $\lambda_k \ge \mu_k,\ \forall k\ge 1$. This ends the proof of Theorem 1.1. □

3. AN ABSTRACT CONTINUATION PRINCIPLE.

This section is devoted to a general method for checking condition (iii) of Theorem 1.1. It applies to 1-parameter families of self-adjoint operators of the form $A_\nu = A_0 + \mathcal{V}_\nu$, with $\mathcal{V}_\nu$ bounded. The idea is to prove (iii) for all A_ν knowing that one of them satisfies it, and having spectral information on every A_ν.

More precisely, we start with a self-adjoint operator $A_0 : D(A_0) \subset \mathcal{H} \to \mathcal{H}$. We denote by $\mathcal{F}(A_0)$ the form-domain of A_0.

For I an interval containing 0, let $\nu \mapsto \mathcal{V}_\nu$ a map whose values are bounded self-adjoint operators and which is continuous for the usual norm of bounded operators

$$|||\mathcal{V}||| = \sup_{x \in \mathcal{H} \setminus \{0\}} \frac{||\mathcal{V}x||_{\mathcal{H}}}{||x||_{\mathcal{H}}} .$$

In order to have consistent notations, we also assume that $\mathcal{V}_0 = 0$.

Since A_0 is self-adjoint and $\mathcal{V}_\nu$ symmetric and bounded, the operator A_ν is self-adjoint with $\mathcal{D}(A_\nu) = \mathcal{D}(A_0)$, $\mathcal{F}(A_\nu) = \mathcal{F}(A_0)$. Let $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ be an orthogonal splitting of $\mathcal{H}$, and Λ_+ , Λ_- the associated projectors, as in Section 1. We assume the existence of a core F (i.e. a subspace of $D(A_0)$ which is dense for the norm $\|\cdot\|_{D(A_0)}$), such that :

(j) $F_+ = \Lambda_+ F$ and $F_- = \Lambda_- F$ are two subspaces of $\mathcal{F}(A_0)$.

(jj) There is $a_- \in \mathbb{R}$ such that for all $\nu \in I$,

$$a_\nu := \sup_{x_- \in F_- \setminus \{0\}} \frac{(x_-, A_\nu x_-)}{||x_-||^2_{\mathcal{H}}} \le a_- .$$

For $\nu \in I$, let $b_\nu := \inf(\sigma_{\rm ess}(A_\nu) \cap (a_\nu, +\infty))$, and for $k \ge 1$, let $\mu_{k,\nu}$ be the k-th eigenvalue of A_ν in the interval (a_ν, b_ν), counted with multiplicity, if it exists. If it does not exist, take $\mu_{k,\nu} := b_\nu$. Our next assumption is

(jjj) There is $a_+ > a_-$ such that for all $\nu \in I$, $\mu_{1,\nu} \ge a_+$.

Finally, we define the levels

$$\lambda_{k,\nu} := \inf_{\substack{V \text{ subspace of } F_+ \\ \dim V = k}} \ \sup_{x \in (V \oplus F_-) \setminus \{0\}} \frac{(x, A_\nu x)}{||x||^2_{\mathcal{H}}} , \qquad k \ge 1 , \tag{21}$$

and our last assumption is

(jv) $\quad \lambda_{1,0} > a_-$.

The main result of this section is

Theorem 3.1. *Under conditions* (j) *to* (jv), A_ν *satisfies the assumptions* (i) *to* (iii) *of Theorem 1.1 for all* $\nu \in I$, *and* $\lambda_{k,\nu} = \mu_{k,\nu} \ge a_+$, *for all* $k \ge 1$.

Note that the boundedness assumption on $\mathcal{V}_\nu$ is rather restrictive. However, as it will be seen in Section 4, unbounded perturbations can also be dealt with, thanks to a regularization argument.

Proof of Theorem 3.1. Assumptions (i), (ii) of Theorem 1.1 are of course satisfied for all $\nu \in I$: see (j), (jj). From formula (21), it is clear that for all $\nu, \nu' \in I$,

$$|\lambda_{1,\nu} - \lambda_{1,\nu'}| \leq |||\mathcal{V}_\nu - \mathcal{V}_{\nu'}||| \,.$$

So the map $\nu \in I \to \lambda_{1,\nu}$ is continuous. The set

$$P := \{\nu \in I : \lambda_{1,\nu} \geq a_+\}$$

is thus closed in I, and the set

$$P' := \{\nu \in I : \lambda_{1,\nu} > a_-\}$$

is open. Obviously, $P \subset P'$. But if $\nu \in P'$ then A_ν satisfies (iii), so it follows from Theorem 1.1 that

$$\lambda_{k,\nu} = \mu_{k,\nu} \geq a_+ \,, \quad \text{for all } k \geq 1 \,,$$

hence $\nu \in P$. As a consequence, $P = P'$, and P is open and closed in I. But P is nonempty : it contains 0. So, P coincides with I. □

4. APPLICATIONS AND REMARKS : DIRAC OPERATORS.

With the notations of the preceding sections, let us define $\mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C}^4)$, Let $F = C_0^\infty(\mathbb{R}^3, \mathbb{C}^4)$ be the space of smooth, compactly supported functions from $\mathbb{R}^3$ to $\mathbb{C}^4$.
The free Dirac operator is $H_0 = -i\alpha \cdot \nabla + \beta$, with

$$\alpha_1, \alpha_2, \alpha_3, \beta \in \mathcal{M}_{4\times 4}(\mathbb{C}), \ \beta = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{pmatrix} \,, \ \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \,,$$

σ_i being the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Let V be a scalar potential satisfying

$$V(x) \underset{|x| \to +\infty}{\longrightarrow} 0 \,, \tag{22}$$

$$-\frac{\nu}{|x|} - c_1 \leq V \leq c_2 = \sup(V) \,, \tag{23}$$

with $\nu \in (0,1)$, $c_1, c_2 \in \mathbb{R}$.

Under the above assumptions, $H_0 + V$ has a distinguished self-adjoint extension A with domain $\mathcal{D}(A)$ such that

$$H^1(\mathbb{R}^3, \mathbb{C}^4) \subset \mathcal{D}(A) \subset H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)\,,$$

$$\sigma_{\rm ess}(A) \;=\; (-\infty, -1] \cup [1, +\infty)\;,$$

and F is a core for A (see [16], [14],[11], [9]). In the sequel, we shall denote this extension indifferently by A or H_0+V. We shall also denote $\mu_k(V)$ the k-th eigenvalue of H_0+V in the interval $(c_2-1,1)$, with the understanding that $\mu_k(V)=1$ whenever H_0+V has less than k eigenvalues in $(c_2-1,1)$.

In this section, we shall prove the validity of two different variational characterizations of the eigenvalues $\mu_k(V)$ corresponding to two different choices of the splitting $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, under conditions which are optimal for the Coulomb potential. In both cases, this will be done using Theorem 1.1. The main difficulty is to check assumption (iii) of this theorem. It will be sufficient to do it for the Coulomb potential $V_\nu := -\nu/|x|$. Then, by a simple comparison argument, all potentials satisfying (22), (23) with the additional condition

$$c_1, c_2 \geq 0,\;\; c_1 + c_2 - 1 < \sqrt{1-\nu^2} \tag{24}$$

will be covered by our results. The constant $\sqrt{1-\nu^2}$ is the smallest eigenvalue of $H_0 - \frac{\nu}{|x|}$ in the interval $(-1,1)$.

The Coulomb potential is not bounded. In order to apply Theorem 3.1, we shall use a regularization argument. The method will be the following: first replace $V_\nu = -\frac{\nu}{|x|}$ by $V_{\nu,\varepsilon} := -\frac{\nu}{|x|+\varepsilon}$, $\varepsilon > 0$. Then apply Theorem 3.1 to $A_{\nu,\varepsilon} := H_0 + V_{\nu,\varepsilon}$, for $\varepsilon > 0$ fixed and ν varying in $I = [0,1)$, and $a_+ = 0$, $a_- = -1$. Combined with Lemma 2.1, this theorem gives

$$Q_{0,\nu,\varepsilon}(x_+) \geq 0\,, \quad \forall\, x_+ \in F_+$$

where, following (6),

$$\begin{aligned} Q_{E,\nu,\varepsilon}(x_+) &:= \sup_{x_- \in F_-} \Big((x_+ + y_-), A_{\nu,\varepsilon}(x_+ + y_-)\Big) - E\|x_+ + y_-\|^2_{\mathcal{H}} \\ &= (x_+, (A_{\nu,\varepsilon} - E)x_+) + \big(\Lambda_- A_{\nu,\varepsilon} x_+, (B_{\nu,\varepsilon} + E)^{-1} \Lambda_- A_{\nu,\varepsilon} x_+\big)\;, \end{aligned}$$

and $B_{\nu,\varepsilon} : D(B_{\nu,\varepsilon}) \subset \mathcal{H}_- \to \mathcal{H}_-$ is a self-adjoint operator such that $(x_-, A_{\nu,\varepsilon} x_-) = -(x_-, B_{\nu,\varepsilon} x_-)$ for all $x_- \in F_-$: see §2, formula (2).

Passing to the limit $\varepsilon \to 0$ in the above inequality, we get

$$Q_{0,\nu,0}(x_+) \geq 0\,, \quad \forall x_+ \in F_+,$$

and by Lemma 2.1, this is equivalent to assumption (iii) of Theorem 1.1 for the operator $H_0 - \frac{\nu}{|x|}$.

4.1. The min-max of Talman and Datta-Deviah.

In this subsection, we choose the following splitting of $\mathcal{H}$:

$$\mathcal{H}_+^T = L^2(\mathbb{R}^3, \mathbb{C}^2) \otimes \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}, \ \mathcal{H}_-^T = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \otimes L^2(\mathbb{R}^3, \mathbb{C}^2) ,$$

so that, for any $\psi = \binom{\varphi}{\chi} \in L^2(\mathbb{R}^3, \mathbb{C}^4)$,

$$\Lambda_+^T \psi = \begin{pmatrix} \varphi \\ 0 \end{pmatrix}, \ \Lambda_-^T \psi = \begin{pmatrix} 0 \\ \chi \end{pmatrix} .$$

With this choice, let $\lambda_k^T(V)$ be the k-th min-max associated to $A = H_0 + V$ by formula (1). In the case $k = 1$, we have

$$\lambda_1^T(V) = \inf_{\varphi \neq 0} \sup_{\chi} \frac{(\psi, (H_0 + V)\psi)}{(\psi, \psi)} . \tag{25}$$

This is exactly the min-max principle of Talman ([15]) and Datta-Deviah ([2]). It is clear that under conditions (22)- (23), assumptions (i) and (ii) of Theorem 1.1 are satisfied, with

$$a = \sup_{x_- \in F_- \setminus \{0\}} \frac{(x_-, Ax_-)}{\|x_-\|_{\mathcal{H}}^2} = c_2 - 1 .$$

The main result of this subsection is

Theorem 4.1. *Let V a scalar potential satisfying (22)-(23)-(24). Then, for all $k \geq 1$,*

$$\lambda_k^T(V) = \mu_k(V) . \tag{26}$$

Moreover, $\lambda_k^T(V) = \mu_k(V)$ is given by

$$\lambda_k^T(V) = \inf_{\substack{Y \text{ subspace of } C_o^\infty(\mathbb{R}^3, \mathbb{C}^2) \\ \dim Y = k}} \ \sup_{\varphi \in Y \setminus \{0\}} \lambda^T(V, \varphi) , \tag{27}$$

where

$$\lambda^T(V, \varphi) := \sup_{\substack{\psi = \binom{\varphi}{\chi} \\ \chi \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^2)}} \frac{((H_0 + V)\psi, \psi)}{(\psi, \psi)} \tag{28}$$

is the unique number in $(c_2-1,+\infty)$ *such that*

$$\lambda^T(V,\varphi)\int_{\mathbb{R}^3}|\varphi|^2dx=\int_{\mathbb{R}^3}\Big(\frac{|(\sigma\cdot\nabla)\varphi|^2}{1-V+\lambda^T(V,\varphi)}+(1+V)|\varphi|^2\Big)dx \quad (29)$$

The maximizer of (28) in $\mathcal{H}_-^T$ *is*

$$\chi(V,\varphi):=\frac{-i(\sigma\cdot\nabla)\varphi}{1-V+\lambda^T(V,\varphi)}\,. \quad (30)$$

Remark 4.1. *In the case* $k=1$, *the min-max (27) reduces to*

$$\lambda_1^T(V)=\inf_{\varphi\in C_\circ^\infty(\mathbb{R}^3,\mathbb{C}^2)\setminus\{0\}}\lambda^T(V,\varphi)\,,$$

where $\lambda^T(V,\varphi)$ *is given by equation (29). This formulation is equivalent to the minimization principle of [3], §4, formula (4.16).*

Proof of Theorem 4.1.
Formulas (27), (29), (30) are simply those of Lemma 2.2 (a)-(b), rewritten in the context of the present subsection. So the only thing to prove is (26). For that purpose, we just have to check that condition (iii) of Theorem 1.1 is fulfilled by H_0+V. In view of Remark 4.1, this was already done in [3]. But the arguments can be made simpler and clearer, thanks to the formalism of Sections 2 and 3.

First of all, since λ_1 is monotonic in V, it is sufficient to check (iii) when $V_\nu=-\frac{\nu}{|x|}$, for all $\nu\in[0,1)$.
The key inequality that we use below is the following :

$$\mu_1(V)\geq 0 \quad \text{as soon as} \quad -\frac{\nu}{|x|}\leq V\leq 0\,,\ 0\leq\nu<1\,. \quad (31)$$

This inequality can be found in [18]. In the particular case of Coulomb potentials, it is well-known that

$$\mu_1(-\frac{\nu}{|x|})=\sqrt{1-\nu^2} \quad \text{for} \quad 0\leq\nu<1\,. \quad (32)$$

We proceed in two steps.

First step : for $\nu\in I:=[0,1)$ and $\varepsilon\geq 0$, let $V_{\nu,\varepsilon}:=-\frac{\nu}{|x|+\varepsilon}$. We now fix $\varepsilon>0$. The one-parameter family $\nu\in I\to A_{\nu,\varepsilon}:=H_0+V_{\nu,\varepsilon}$ and the projectors $\Lambda_\pm^T$ satisfy all the assumptions of Theorem 3.1, with $a_- = -1$ and $a_+ = 0$. In particular, (jjj) follows from (31). So we obtain

$$\lambda_1^T(V_{\nu,\varepsilon}) = \mu_1(V_{\nu,\varepsilon}) \geq 0\,,$$

for all $\nu \in [0,1)$. From Lemma 2.1, this can be written as

$$Q^T_{0,\nu,\varepsilon}(\varphi) \geq 0\,, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^2)\,, \tag{33}$$

with

$$Q^T_{E,\nu,\varepsilon}(\varphi) \;=\; \int_{\mathbb{R}^3} \Big(\frac{|(\sigma\cdot\nabla)\varphi|^2}{1+E-V_{\nu,\varepsilon}} + (1-E+V_{\nu,\varepsilon})|\varphi|^2 \Big)\, dx\,. \tag{34}$$

Second step : For $\nu \in [0,1)$ and $\varphi \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^2)$ fixed, we pass to the limit $\varepsilon \to 0$ in (33). We get :

$$Q^T_{0,\nu,0}(\varphi) \geq 0\,, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^2)\,. \tag{35}$$

So $A_{\nu,0} = H_0 + V_\nu$ satisfies criterion (iii') of Lemma 2.1, which is equivalent to (iii). By Theorem 1.1, we thus have

$$\lambda_1^T(V_\nu) = \mu_1(V_\nu) = \sqrt{1-\nu^2}\,,$$

for all $\nu \in (0,1)$. This ends the proof. □

Note that a by-product of Theorem 4.1 is that for all $\varphi \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^2)$, and all $\nu \in [0,1]$, the following Hardy-type inhomogeneous inequality holds

$$\nu \int_{\mathbb{R}^3} \frac{|\varphi|^2}{|x|} + \sqrt{1-\nu^2} \int_{\mathbb{R}^3} |\varphi|^2 \;\leq\; \int_{\mathbb{R}^3} \frac{|(\sigma\cdot\nabla)\varphi|^2}{\frac{\nu}{|x|} + 1 + \sqrt{1-\nu^2}} + \int_{\mathbb{R}^3} |\varphi|^2\,.$$

This is just the inequality $Q^T_{\sqrt{1-\nu^2},\nu,0}(\varphi) \geq 0$ in the case $0 \leq \nu < 1$, and the case $\nu = 1$ is obtained by passing to the limit.

Moreover, taking $\nu = 1$ and functions φ which concentrate near the origin, the above inequality yields, in the limit, the following homogeneous one :

$$\int_{\mathbb{R}^3} \frac{|\varphi|^2}{|x|}\, dx \leq \int_{\mathbb{R}^3} |x| |(\sigma\cdot\nabla)\varphi|^2\, dx \quad \text{for all} \quad \varphi \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^2)\,.$$

Actually, taking $\phi = \frac{\varphi}{|x|^{1/2}}$, this inequality is a direct consequence of the standard Hardy inequality

$$\int_{\mathbb{R}^3} \frac{|\phi|^2}{|x|^2} \;\leq\; 4 \int_{\mathbb{R}^3} |\nabla\phi|^2 \;=\; 4 \int_{\mathbb{R}^3} |(\sigma\cdot\nabla)\phi|^2\,.$$

4.2. The min-max associated with the free-energy projectors. Here we define the splitting of $\mathcal{H}$ as follows: $\mathcal{H} = \mathcal{H}_+^f \oplus \mathcal{H}_-^f$, with $\mathcal{H}_\pm^f = \Lambda_\pm^f \mathcal{H}$, where

$$\Lambda_+^f = \chi_{(0,+\infty)}(H_0) = \frac{1}{2}\left(\mathbf{I} + \frac{\mathrm{H}_0}{\sqrt{1-\Delta}}\right),$$

$$\Lambda_-^f = \chi_{(-\infty,0)}(H_0) = \frac{1}{2}\left(\mathbf{I} - \frac{\mathrm{H}_0}{\sqrt{1-\Delta}}\right).$$

As in Subsection 4.1, assumptions (i) and (ii) of Theorem 1.1 are satisfied, with the same choice $a = c_2 - 1$.

With the new splitting $\mathcal{H}_\pm^f$, and the operator $A = H_0 + V$, the min-max values given by formula (1) will be denoted by $\lambda_k^f(V)$. This min-max principle based on free-energy projectors was first introduced in [6]. Using some inequality proved in [1] and [17], we proved in [3] that $\lambda_k^f(V)$ is indeed equal to the eigenvalue $\mu_k(V)$ for all potentials V satisfying $-\frac{\nu}{|x|} \le V \le 0$, and all $0 \le \nu < 2\left(\frac{\pi}{2} + \frac{2}{\pi}\right)^{-1} \sim 0,9$. Here, we extend this result to cover all $0 \le \nu < 1$, and we obtain new inequalities as a by-product.

The main result of this subsection is the following

THEOREM 4.2. *Let V a scalar potential satisfying (22)-(23)-(24). Then, for all $k \ge 1$,*

$$\lambda_k^f(V) = \mu_k(V). \tag{36}$$

Proof : As in Subsection 4.1, we just have to consider the Coulomb potential V_ν, for $\nu \in [0,1)$.

First Step : Let $\varepsilon > 0$ fixed and $V_{\nu,\varepsilon}$ as before. Thanks to (31), Theorem 3.1 applies to the one-parameter family $\nu \in [0,1) \to A_{\nu,\varepsilon} := H_0 + V_{\nu,\varepsilon}$ with the projectors $\Lambda_\pm^f$, and $a_- = -1$, $a_+ = 0$. So we get

$$\lambda_1^f(V_{\nu,\varepsilon}) = \mu_1(V_{\nu,\varepsilon}) \ge 0,$$

for all $\nu \in [0,1)$. By Lemma 2.1, this may be written

$$Q_{0,\nu,\varepsilon}^f(\psi_+) \ge 0, \quad \text{for all} \quad \psi_+ \in F_+^f := \Lambda_+^f\left(C_0^\infty(\mathbb{R}^3, \mathbb{C}^4)\right),$$

with

$$Q^f_{E,\nu,\varepsilon}(\psi_+) = ||\psi_+||^2_{H^{1/2}} - (\psi_+, (E - V_{\nu,\varepsilon})\psi_+) \\ + \left(\Lambda^f_-|V_{\nu,\varepsilon}|\psi_+, \left(\Lambda^f_-(\sqrt{1-\Delta} + E + |V_{\nu,\varepsilon}|)\Lambda^f_-\right)^{-1}\Lambda^f_-|V_{\nu,\varepsilon}|\psi_+\right). \tag{37}$$

Second step : Passing to the limit $\varepsilon \to 0$ in (37) with ψ_+ and ν fixed, we get

$$Q^f_{0,\nu,0}(\psi_+) \geq 0\ , \qquad \psi_+ \in F^f_+ \tag{38}$$

for all $\nu \in [0, 1)$. Then, applying Theorem 1.1 to $H_0 + V_\nu$, we obtain (36), and the theorem is proved. □

Finally, note that some inequalities can be derived from the free-energy min-max principle, as in the Talman case: for all $\nu \in [0, 1]$ and all functions $\psi_+ \in \Lambda^f_+\left(C^\infty_0(\mathbb{R}^3, \mathbb{C}^4)\right)$, we have

$$\nu\int_{\mathbb{R}^3}\frac{|\psi_+|^2}{|x|}\,dx + \sqrt{1-\nu^2}\int_{\mathbb{R}^3}|\psi_+|^2\,dx$$

$$\leq \int_{\mathbb{R}^3}(\psi_+, \sqrt{1-\Delta}\,\psi_+)\,dx$$

$$+\nu^2\int_{\mathbb{R}^3}\left(\Lambda^f_-\left(\frac{\psi_+}{|x|}\right), \left(\Lambda^f_-\left(\sqrt{1-\Delta} + \frac{\nu}{|x|} + \sqrt{1-\nu^2}\right)\Lambda^f_-\right)^{-1}\Lambda^f_-\left(\frac{\psi_+}{|x|}\right)\right)dx.$$

Moreover, taking functions with support near the origin, we find, after rescaling and passing to the limit, a new homogeneous Hardy-type inequality. This inequality involves the projectors associated with the zero-mass free Dirac operator:

$$\Lambda^{f,0}_\pm := \frac{1}{2}\left(\mathbf{1} \pm \frac{\alpha\cdot\hat{\mathrm{p}}}{|\hat{\mathrm{p}}|}\right)\ ,\ \hat{p} := -i\nabla\,.$$

It may be written as follows :

$$\int_{\mathbb{R}^3}\frac{|\psi_+|^2}{|x|}\,dx \leq \int_{\mathbb{R}^3}(\psi_+, |\hat{p}|\psi_+)\,dx \\ + \int_{\mathbb{R}^3}\left(\Lambda^{f,0}_-\left(\frac{\psi_+}{|x|}\right), \left(\Lambda^{f,0}_-\left(|\hat{p}| + \frac{1}{|x|}\right)\Lambda^{f,0}_-\right)^{-1}\Lambda^{f,0}_-\left(\frac{\psi_+}{|x|}\right)\right)dx\,,$$

for all $\psi_+ \in \Lambda^{f,0}_+\left(C^\infty_0(\mathbb{R}^3, \mathbb{C}^4)\right)$.

These two inequalities look like the ones obtained by Evans-Perry-Siedentop [5], Tix [17] and Burenkov-Evans [1], but they are not the same. We do not know whether they can be obtained by direct computations, as was the case in those works.

REFERENCES

1. V.I. Burenkov, W.D. Evans. On the evaluation of the norm of an integral operator associated with the stability of one-electron atoms. Proc. Roy. Soc. Edinburgh A, **128(5)** (1998). p. 993-1005.
2. S.N. Datta and G. Deviah. The minimax technique in relativistic Hartree-Fock calculations. Pramana, **30(5)** (1988), p. 387-405.
3. J. Dolbeault, M.J. Esteban and E. Séré. Variational characterization for eigenvalues of Dirac operators. Preprint mp-arc 98-177, to appear in Calc. Var. and P.D.E.
4. G.W.F. Drake and S.P. Goldman. Relativistic Sturmian and finite basis set methods in atomic physics. Adv. Atomic Molecular Phys. **25** (1988), p. 393-416.
5. W.D. Evans, P. Perry, H. Siedentop. The spectrum of relativistic one-electron atoms according to Bethe and Salpeter. Comm. Math. Phys. **178** (1996), p. 733-746.
6. M.J. Esteban, E. Séré. Existence and multiplicity of solutions for linear and nonlinear Dirac problems. *Partial Differential Equations and Their Applications.* CRM Proceedings and Lecture Notes, volume 12. Eds. P.C. Greiner, V. Ivrii, L.A. Seco and C. Sulem. AMS, 1997.
7. M. Griesemer, R.T. Lewis, H. Siedentop. A minimax principle in spectral gaps: Dirac operators with Coulomb potentials. Preprint 1999.
8. M. Griesemer, H. Siedentop. A minimax principle for the eigenvalues in spectral gaps. Preprint mp-arc 97-492, to appear in J. London Math. Soc.
9. M. Klaus and R. Wüst. Characterization and uniqueness of distinguished self-adjoint extensions of Dirac operators. Comm. Math. Phys. **64(2)** (1978-79), p. 171-176.
10. W. Kutzelnigg. Relativistic one-electron Hamiltonians "for electrons only" and the variational treatment of the Dirac equation. Chemical Physics **225** (1997), p. 203-222.
11. G. Nenciu. Self-adjointness and invariance of the essential spectrum for Dirac operators defined as quadratic forms. Comm. Math. Phys. **48** (1976), p. 235-247.
12. M. Reed and B. Simon. *Methods of modern mathematical physics, Vol. 1.* Academic Press, New York.
13. M. Reed and B. Simon. *Methods of modern mathematical physics, Vol. 4.* Academic Press, New York.
14. U.W. Schmincke. Distinguished self-adjoint extensions of Dirac operators. Math. Z. **129** (1972), p. 335-349.
15. J.D. Talman. Minimax principle for the Dirac equation. Phys. Rev. Lett. **57(9)** (1986), p. 1091-1094.
16. B. Thaller. *The Dirac Equation.* Texts and Monographs in Physics. Springer-Verlag, Berlin, 1st edition, 1992.
17. C. Tix. Strict Positivity of a relativistic Hamiltonian due to Brown and Ravenhall. Bull. London Math. Soc. **30(3)** (1998), p. 283-290.
18. R. Wüst. Dirac operators with strongly singular potentials. Math. Z. **152** (1977), p. 259-271.