A new variational principle for a nonlinear Dirac equation on the Schwarzschild metric

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Abstract. In this paper, we prove the existence of infinitely many solutions of a stationary nonlinear Dirac equation on the Schwarzschild metric, outside a massive ball. These solutions are the critical points of a strongly indefinite functional. Thanks to a concavity property, we are able to construct a reduced functional, which is no longer strongly indefinite. We find critical points of this new functional using the Symmetric Mountain Pass Lemma. Note that, as A. Bachelot-Motet conjectured, these solutions vanish as the radius of the massive ball tends to the horizon radius of the metric.

1 Introduction

The goal of this paper is to give a partial answer to a conjecture of A. Bachelot-Motet [1] concerning nonlinear Dirac fields on the Schwarzschild metric. Many results about the nonlinear Dirac equation with mass $m > 0$

$$i\gamma^\mu \nabla_\mu \Psi - m\Psi = \lambda(\bar{\Psi}\Psi)\Psi \tag{1.1}$$

have been obtained in the Minkowski space-time. This equation is related to several models in particle physics (see [9]). In [1], A. Bachelot-Motet is looking towards a special pattern of solutions, related to the extended fermion model proposed by Soler [10] and Wakano [12], that we call *Soler-type* solutions:

$$\Psi(t,x) = e^{i\omega t} \begin{pmatrix} v(r) \\ 0 \\ iu(r)\cos\theta \\ iu(r)\sin\theta e^{i\varphi} \end{pmatrix}, \, \omega > 0\,. \tag{1.2}$$

In the Minkowski space-time, the existence of infinitely many Soler-type solutions (1.2) was established for a wide class of nonlinearities, first by using

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shooting methods ([2], [3], [4], [8]), then by a variational method ([6]). The existence of such solutions in the Schwarzschild metric

$$g_{\mu\nu}dx^\mu dx^\nu = \left(1-\frac{2M}{r}\right)dt^2 - \left(1-\frac{2M}{r}\right)^{-1}dr^2 - r^2(d\theta^2+\sin^2\theta d\varphi^2) \tag{1.3}$$

was then investigated by A. Bachelot-Motet. She proved ([1]) that there is no Soler-type solution with finite energy on the Schwarzschild manifold $\mathbb{R}_t \times (2M,\infty)_r \times S^2$ and carried out numerical experiments to put in evidence such solutions outside a massive sphere of radius $r_0 > 1$, with MIT-bag boundary conditions. In this case, (1.1) becomes a nonautonomous planar differential system in (v,u),

$$fu' + \frac{u}{r}\{f + f^{\frac{1}{2}}\} = v\left[\lambda(v^2-u^2) - (f^{\frac{1}{2}}m - \omega)\right] \tag{1.4}$$

$$fv' + \frac{v}{r}\{f - f^{\frac{1}{2}}\} = u\left[\lambda(v^2-u^2) - (f^{\frac{1}{2}}m + \omega)\right] \tag{1.5}$$

where $f(r) = 1-\frac{1}{r}$, taking the normalization $2M = 1$. The MIT-bag boundary condition becomes the following initial data condition

$$u(r_0) = -v(r_0)\,. \tag{1.6}$$

These numerical experiments led A. Bachelot-Motet to the following conjecture:

Conjecture 1.1 ([1]). *For every $r_0 > 1$ there is an integer $N(r_0)$ and an increasing sequence $(x_n)_{n\geq N(r_0)}$ of strictly positive numbers such that if we denote by (v_n,u_n) the solution of (1.4-1.5-1.6) with $v(r_0) = x_n$, we have for every $n \geq N(r_0)$:*

1. *the solution (v_n,u_n) is defined on $[r_0,\infty)$;*
2. *the function v_n (respectively u_n) has exactly n (respectively $n+1$) zeros in $[r_0,\infty)$;*
3. *(v_n,u_n) converges to $(0,0)$ when r goes to infinity.*

Moreover $N(r_0) = 0$ for r_0 large enough, and $N(r_0)$ tends to infinity when r_0 goes to 1.

In this paper, we take advantage of the variational structure of the system (1.4-1.5). In [6], Esteban and Séré found infinitely many solutions of a nonlinear Dirac equation by an *infinite dimensional* linking. Our method is different and simpler. Thanks to a concavity argument, we may replace the initial functional F_{r_0} with a reduced one, called G_{r_0}, which is even and has 0 as a trivial critical point. We can use a Mountain-Pass-type method based on the Krasnoselski *genus* (see [11] for a complete reference) to find out infinitely many critical points. The property used here, called in [11] the Symmetric Mountain Pass

Lemma, is equivalent to a *finite dimensional* linking. As r_0 goes to 1, the lowest min-max values converge to 0 by pair, creating then a negative direction for the Hessian of the functional in 0.

More precisely, $s_n := \frac{1}{\sqrt{2}}(u_n+v_n)$ is found as a critical point of the functional G_{r_0}, whose precise definition will be given in (2.7). We obtain the following result:

Theorem 1.2. *For every $r_0 > 1$, given $\omega \in (\frac{f(r_0)^{\frac{1}{2}}}{2r_0^2}, m)$, there exists an integer $N(r_0)$ and a sequence of non-trivial solutions of (1.4-1.5-1.6), denoted by $(v_n, u_n)_{n \geq N(r_0)}$ such that, for every $n \geq N(r_0)$:*

(i) (v_n, u_n) is defined on $[r_0, \infty)$ and converges exponentially to $(0,0)$ when r goes to infinity;

(ii) $s_n = \frac{1}{\sqrt{2}}(u_n+v_n)$ has Morse index at most $n+1$ relatively to the functional G_{r_0} described in (2.7).

Moreover $N(r_0) = 0$ for r_0 large enough, and $N(r_0)$ tends to infinity when r_0 goes to 1.

Remark 1.3. *There should be a relationship between the Morse index of s_n and the number of zeros of u_n and v_n: this is an open problem.*

To our knowledge, this is the first Mountain-Pass-type characterization of critical points for non-linear Dirac functionals. The limitation $\omega > \frac{f(r_0)^{\frac{1}{2}}}{2r_0^2}$ is technical, it is useful in an a priori estimate (see Lemma 4.1). Note that in her computations, Bachelot-Motet always considered the case $\omega > \frac{1}{2}$ covered by our assumption.

Section 2 is devoted to the introduction of the variational framework, in particular the new functional G_{r_0}. In Section 3, we solve suitable approximate problems, by the means of the Symmetric Mountain Pass Lemma. In Section 4, the convergence of the solutions found in Section 3 to solutions of the exact problem is proved, using a Pohozaev-type estimate. The proof of some important lemmas is postponed in Section 5.

2 A new variational principle for the nonlinear Dirac equation

Define now the variational framework. In order to insert the initial data condition (1.6) in the functional scheme, we change variables (v, u) symplectically into (s,d) with

$$\begin{aligned} s &= \frac{1}{\sqrt{2}}(u+v)\,, \\ d &= \frac{1}{\sqrt{2}}(u-v)\,. \end{aligned}$$

System (1.4-1.5) is now

$$fd' + \left(\frac{f}{r} - mf^{\frac{1}{2}}\right) d + \left(\frac{f^{\frac{1}{2}}}{r} - \omega\right) s = 2\lambda s d^2\,, \tag{2.1}$$

$$fs' + \left(\frac{f}{r} + mf^{\frac{1}{2}}\right) s + \left(\frac{f^{\frac{1}{2}}}{r} + \omega\right) d = -2\lambda s^2 d\,, \tag{2.2}$$

and condition (1.6) then becomes

$$s(r_0) = 0\,. \tag{2.3}$$

We introduce the following functional

$$F_{r_0}(s,d) = A_{r_0}(s,d) + B_{r_0}(s,d) + C_{r_0}(s,d)\,,$$

with

$$\begin{aligned} A_{r_0}(s,d) &= \int_{r_0}^{\infty} \left[-s'd - \frac{sd}{r} + \frac{f^{-\frac{1}{2}}}{2r}(s^2 - d^2)\right] r^2 dr\,, \\ B_{r_0}(s,d) &= \int_{r_0}^{\infty} \left[-mf^{-\frac{1}{2}} sd - \frac{1}{2}\omega f^{-1}(s^2 + d^2)\right] r^2 dr\,, \\ C_{r_0}(s,d) &= \int_{r_0}^{\infty} -\lambda f^{-1} s^2 d^2 r^2 dr\,. \end{aligned}$$

Functional F_{r_0} is well defined and continuously differentiable in the Hilbert space

$$E = H \times L = H_0^1((r_0,+\infty), r^2dr) \times L^2((r_0,+\infty), r^2dr)\,.$$

We will use the following notations for Sobolev spaces of functions defined on $(r_0,+\infty)$ corresponding to radial functions on $\mathbb{R}^3 \setminus B_{r_0}$: given an integer $s \geq 1$ and $1 \leq p \leq \infty$, we set

$$W^{s,p}((r_0,+\infty), r^2dr) \equiv W^{s,p}_{rad}\,,$$

$$W^{s,p}_0((r_0,+\infty), r^2dr) \equiv W^{s,p}_{0,rad}\,,$$

$$L^p((r_0,+\infty), r^2dr) \equiv L^p_{rad}\,.$$

By classical arguments, we have the following compact Sobolev embeddings:

$$\forall p < q,\ W^{1,p}_{rad} \hookrightarrow L^q_{rad}\,.$$

Moreover, by a staightforward computation, we get the following property:

$$\forall s \in W^{1,p}_{rad},\, \exists K \in \mathbb{R}, /\, \forall r \in (r_0, \infty),\, s(r) \leq \frac{K}{r^{\frac{2}{p}}}\,. \tag{2.4}$$

A simple computation shows that critical points of F give rise to the solutions of (2.1-2.2), satisfying boundary conditions (2.3) and

$$\lim_{r \to +\infty} s(r) = \lim_{r \to +\infty} d(r) = 0\,. \tag{2.5}$$

In order to find such critical points, we point out that functional F_{r_0} has a concavity property in the d direction. Indeed, we get

$$\begin{aligned} F_{r_0}(s,d) &= \int_{r_0}^{\infty} \left(-\left[\frac{f^{-\frac{1}{2}}}{2r} + \frac{\omega f^{-1}}{2} + \lambda f^{-1} s^2 \right] d^2 \right. \\ &\quad \left. - \left[s' + \left(\frac{1}{r} + m f^{-\frac{1}{2}} \right) s \right] d + \left(\frac{f^{-\frac{1}{2}}}{2r} - \frac{\omega f^{-1}}{2} \right) s^2 \right) r^2 dr\,. \end{aligned} \tag{2.6}$$

Thus, given $s \in H$, there exists a unique $d^*(s) \in L$ such that

$$\frac{\partial F_{r_0}}{\partial d}(s, d^*(s)) = 0\,.$$

This property allows us to introduce the new functional G_{r_0},

$$\begin{aligned} G_{r_0}(s) &= 2F_{r_0}(s, d^*(s)) \\ &= \int_{r_0}^{\infty} \left[\frac{\left[s' + s\left(\frac{1}{r} + m f^{-\frac{1}{2}}\right)\right]^2}{\left(\frac{f^{-\frac{1}{2}}}{r} + \omega f^{-1}\right) + 2\lambda f^{-1} s^2} + \left(\frac{f^{-\frac{1}{2}}}{r} - \omega f^{-1} \right) s^2 \right] r^2 dr\,, \end{aligned} \tag{2.7}$$

where G_{r_0} is a C^2 and *even* functional. The main interest of G_{r_0} is that this functional is *no longer strongly indefinite.* Now we use a procedure based on the Symmetric Mountain Pass Lemma, which is a corollary of the well-known Mountain Pass Theorem:

Proposition 2.1. *Let E be an infinite dimensional Banach space and let $I \in C^1(E, \mathbb{R})$ be even, satisfy the Cerami condition, and $I(0) = 0$. If $E = V \oplus X$, where V is finite dimensional, and I satisfies*

- *(i) there are constants α, $\rho > 0$ such that $I|_{\partial B_\rho \cap X} \geq \alpha$, and*
- *(ii) there exists a sequence of subspaces $(X_k)_{k \in \mathbb{N}} \subset X$ with finite dimension k, such that $X_k \subset X_{k+1}$, and $R_k \geq 0$ such that, if we denote by E_k the finite dimensional subspace $X_k \oplus V$, we have $I \leq 0$ on $E_k \setminus B_{R_k}$,*

then I possesses an unbounded sequence of critical values, defined by:

$$c_k = \inf_{h \in \Gamma_k} \max_{u \in E_k} I(h(u)),$$

with the invariant classes Γ_k defined by

$$\Gamma_k = \{h \in C^0(E, E),\, h \text{ odd},\, \forall j \leq k,\, u \in E_j, ||u|| \geq R_j \Rightarrow h(u) = u\}\,.$$

Unfortunately, we do not know if G_{r_0} satisfies the Cerami condition. So we will study approximate functionals, as in [6].

3 Solution of approximate problems

Given $p > 4$ and $\varepsilon > 0$, we define now the following functionals:

$$\begin{aligned} F_{r_0,\varepsilon}(s, d) &= F_{r_0}(s, d) - \varepsilon \int_{r_0}^{\infty} s^p r^2 dr, \\ G_{r_0,\varepsilon}(s) &= 2F_{r_0,\varepsilon}(s, d^*(s)) = G_{r_0}(s) - 2\varepsilon \int_{r_0}^{\infty} s^p r^2 dr\,. \end{aligned}$$

These functionals are C^2 respectively in $H \times L$ and H. We have to prove a compactness result (i.e. convergence, up to a subsequence, of Cerami sequences) for these approximate functionals; then, we need to verify that functional $G_{r_0,\varepsilon}$ satisfies the assumptions of Proposition 2.1.

Lemma 3.1. *Given $r_0 > 1$ and $\varepsilon > 0$, the functional $G_{r_0,\varepsilon}$ satisfies the Cerami condition, i.e. any sequence (s_n) satisfying*

$$\begin{aligned} \lim_{n \to +\infty} G_{r_0,\varepsilon}(s_n) &= l \in \mathbb{R} \\ \lim_{n \to +\infty} (1 + ||s_n||_H)\, ||G'_{r_0,\varepsilon}(s_n)||_{H'} &= 0 \end{aligned}$$

has a convergent subsequence. Moreover, the limit, denoted by s, satisfies an exponential decay estimate: given any $q \geq 2$, we have

$$||s||_{W^{1,q}((R,\infty), r^2 dr)} \leq K e^{-\sigma_q R}\,. \tag{3.1}$$

Proof. This proof is postponed in Section 5.⋄

To check that the assumptions of Proposition 2.1 are satisfied, we need the following lemma:

Lemma 3.2. *Given $0 < \omega < m$, there exists a non-increasing integral function $N : (1, +\infty) \to \mathbb{N}$ which satisfies the following conditions:*

- $\lim_{r_0 \to 1} N(r_0) = +\infty$ *and there exists $\bar{r}_0 \in (1, +\infty)$ such that $N(\bar{r}_0) = 0$;*

- *there exists a finite dimensional subspace $V(r_0) \subset H$, with dimension $N(r_0)$, and $P(r_0)$ an orthogonal supplement of $V(r_0)$ in H such that for every $\varepsilon > 0$, $G''_{r_0,\varepsilon}(0)$ is a positive definite quadratic form on $P(r_0)$, and there exists $\alpha(r_0) > 0$ such that*

$$G''_{r_0,\varepsilon}(0).X.X \geq \alpha(r_0)||X||^2,\ \forall X \in P(r_0)\,.$$

Proof. This result is based on the computation of the second order derivative of G_{r_0} and a classical rescaling argument. For details, see Section 5. $\diamond$

Thanks to this result, we see that condition (i) is fulfilled if we put $V = V(r_0)$, $X = P(r_0)$. Condition (ii) is a consequence of the following lemma:

Lemma 3.3. *There exists sequences $(X_k)_{k\in\mathbb{N}} \subset X$ with $\dim X_k = k$, such that $X_k \subset X_{k+1}$, and $(R_k)_{k\in\mathbb{N}} \geq 0$ such that, if we denote by E_k the finite dimensional subspace $X_k \oplus V$, we have $G_{r_0} \leq 0$ on $E_k \setminus B(0, R_k)$.*

Proof. As $V = V(r_0)$ is finite dimensional, the unit ball in V is compact for the strong topology induced from H. This leads to the following uniform estimate: given $\eta > 0$ there exists $R > 0$ such that

$$\forall v \in V,\ \int_R^\infty (v^2 + v'^2) r^2 dr \leq \eta ||v||_H^2\,.$$

Let $\phi_1, ...\phi_k, ...$ be a basis of $W_0^{1,2}((R,\infty), r^2dr)$. We may suppose that every ϕ_k has the C^2 regularity. Prolonging by zero on the interval (r_0, R), we consider these functions as elements of H. Let $\tilde{X}_k = Span(\phi_1, ..., \phi_k)$. We denote by P_X (resp. P_V) the projection on X (resp. V) with respect to V (resp. X) and we get, for $f \in \tilde{X}_k$

$$\begin{aligned} ||f - P_X f||_H^2 = (f, P_V f)_H &= \int_R^\infty (f(P_V f) + f'(P_V f)')\, r^2 dr \\ &\leq ||f||_H \left(\int_R^\infty \left[(P_V f)^2 + (P_V f)'^2 \right] r^2 dr \right)^{\frac{1}{2}} \\ &\leq \sqrt{\eta} ||f||_H^2 \end{aligned}$$

Then, there exists $\eta_0 > 0$ such that, given any $\eta \leq \eta_0$, the projector P_X is close enough to the identity to be an isomorphism. We put then $X_k = P_X(\tilde{X}_k)$. Then $\dim X_k = k$, and, given any $g \in X_k$, with $f = P_X^{-1}(g)$,

$$\begin{aligned} \int_{r_0}^R (g^2 + (g')^2) r^2 dr &= \int_{r_0}^R (g(f - P_V f) + g'(f - P_V f)') r^2 dr \\ &= \int_R^\infty (g(P_V f) + g'(P_V f)') r^2 dr \\ &\leq 2\eta^{\frac{1}{2}} ||g||_H^2\,. \end{aligned} \tag{3.2}$$

Since the elements of V are eigenvectors for the operator G''_{r_0} from H to H, they satisfy a second order differential equation with smooth coefficients (see Section 5.2, the coefficient of the second order derivative is φ_2, which is positive), so they are twice continuously differentiable. Moreover, as we supposed each ϕ_i with the C^2 regularity, their projection on X is also C^2, and X_k is a finite dimensional subspace of C^2 functions.

We fix $k \in \mathbb{N}$. To prove that E_k satisfies the properties given in the lemma, we proceed by contradiction and suppose that there exists a sequence $(\mu_n)_{n\in\mathbb{N}}$ going to infinity and a sequence $(s_n)_{n\in\mathbb{N}}$ in the unit sphere of E_k, such that, for every $n \in \mathbb{N}$,

$$G_{r_0}(\mu_n s_n) \geq 0 \,.$$

According to (2.7), the function $\mu \mapsto \frac{1}{\mu^2} G_{r_0}(\mu s)$ is nonincreasing. So we infer

$$\forall \mu \leq \mu_n,\; G_{r_0}(\mu s_n) \geq 0 \,. \tag{3.3}$$

Moreover, as the unit sphere in E_k is finite dimensional, we may extract a converging subsequence to the sequence (s_n), whose limit is a C^2 function denoted by $\bar{s}$. Passing to the limit in (3.3), we get

$$\forall \mu \geq 1,\; \frac{1}{\mu^2} G_{r_0}(\mu \bar{s}) \geq 0 \,.$$

The equivalence of all norms in E_k implies the equivalence of the L^2_{rad}-norm and the H-norm. This and (3.2) imply that the new function

$$\begin{aligned}
H(\mu) &= \int_{r_0}^{\infty} \left[\frac{\left[\bar{s}' + \bar{s}\left(\frac{1}{r} + m f^{-\frac{1}{2}}\right)\right]^2}{\left(\frac{f^{-\frac{1}{2}}}{r} + \omega f^{-1}\right) + 2\lambda f^{-1}\mu^2 \bar{s}^2} \right] r^2 dr \\
&\geq \int_{r_0}^{\infty} \left(\omega f^{-1} - \frac{f^{-\frac{1}{2}}}{r}\right) \bar{s}^2 r^2 dr \\
&\geq \int_{r_0}^{R} \left(\omega f^{-1} - \frac{f^{-\frac{1}{2}}}{r}\right) \bar{s}^2 r^2 dr + \int_{R}^{\infty} \left(\omega f^{-1} - \frac{f^{-\frac{1}{2}}}{r}\right) \bar{s}^2 r^2 dr \\
&\geq (-K\eta^{\frac{1}{2}} + K')||\bar{s}||_H
\end{aligned}$$

is bounded from below with a positive constant (here K and K' are positive constants), provided η is small enough. We claim this is impossible because of Lebesgue's Dominated Convergence Theorem. Indeed, putting $\mu = n$ and denoting by $h_n(r)$ the integrand of the last functional, we prove that h_n converges strongly to 0 in L^1_{rad}. We get trivially, for every $r \in (r_0, \infty)$, $h_\mu(r) \leq h_1(r)$ and $h_1 \in L^1_{rad}$. Now h_μ converges almost everywhere to 0 as μ goes to infinity. Indeed, define

$$\mathcal{H} = \{r \in (r_0, \infty),\, / \bar{s}(r) \neq 0 \text{ or } \bar{s}(r) = \bar{s}'(r) = 0\} \,.$$

It is clear that, given any $r \in \mathcal{H}$, $h_n(r)$ goes to 0 as n goes to infinity. Moreover, the complement of $\mathcal{H}$ in (r_0, ∞) is the set

$$\{r \in (r_0, \infty),\ / \bar{s}(r) = 0 \text{ and } \bar{s}'(r) \neq 0\}$$

which is a reunion of isolated points, since $\bar{s}$ is a C^2 function, hence its Lebesgue measure is zero, and the contradiction is obtained.◇

Since $G_{r_0,\varepsilon}(s) \leq G_{r_0}(s)$ for every $\varepsilon \geq 0$ and $s \in H$, we obtain that $G_{r_0,\varepsilon}$ satisfies the property of Lemma 3.3 uniformly in ε. So we may use Proposition 2.1 with functionals $G_{r_0,\varepsilon}$: this gives rise to a sequence $(s_n^\varepsilon)_{n>N(r_0)}$ of critical points for $G_{r_0,\varepsilon}$, whose corresponding critical values are unbounded. Thanks to the concavity property, we can associate a $d_n^\varepsilon \in L$ to each s_n^ε, such that $(s_n^\varepsilon, d_n^\varepsilon)$ is a critical point for $F_{r_0,\varepsilon}$, for any $n > N(r_0)$. Moreover, since the constants α, ρ and R_k do not depend on ε, we get an uniform estimate for the critical levels of the perturbed functionals: there exists two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ going to $+\infty$ as n goes to infinity, such that, given any $\varepsilon > 0$, we get

$$a_n \leq F_{r_0,\varepsilon}(s_n^\varepsilon, d_n^\varepsilon) \leq b_n\,. \tag{3.4}$$

The next section is devoted to find uniform estimates on the corresponding critical points in order to pass to the limit as ε goes to 0, and to prove that this limiting procedure leads to an infinite sequence of solutions of (1.4-1.5).

4 Limiting procedure

A priori estimates for the sequence $(s_n^\varepsilon, d_n^\varepsilon)$ are necessary to pass to the limit as ε goes to 0. The following one is the consequence of a Pohozaev-type inequality:

Lemma 4.1. *Given $\omega \in (\frac{f(r_0)^{\frac{1}{2}}}{2r_0^2}, m)$, $\varepsilon \in (0, \bar{\varepsilon})$ and $n > N(r_0)$, we have*

$$B_{r_0}(s_n^\varepsilon, d_n^\varepsilon) \geq$$
$$\left(\frac{r_0}{r_0-1} - \frac{f^{-\frac{1}{2}}(r_0)}{2\omega r_0^2}\right)^{-1}\left[-Cb_n + \int_{r_0}^{\infty} m s_n^\varepsilon d_n^\varepsilon \left(\frac{f^{-\frac{1}{2}}(r_0) f^{-\frac{1}{2}} r^2}{2\omega r_0^2} - \frac{f^{-\frac{3}{2}} r}{2} + f^{-\frac{1}{2}} r^2\right) dr\right].$$

Proof. This inequality is obtained in Section 5.◇

Fix $n > N(r_0)$ and let $(\varepsilon_q)_{q\in\mathbb{N}}$ be a sequence of positive real numbers converging to 0. To show that $(s_n^{\varepsilon_q}, d_n^{\varepsilon_q})$ is bounded in E, we proceed by contradiction, and suppose that, up to a subsequence,

$$||(s_n^{\varepsilon_q}, d_n^{\varepsilon_q})||_E \to \infty\,.$$

In order to simplify the notations, we omit the dependence on n and we just write (s_q, d_q) for $(s_n^{\varepsilon_q}, d_n^{\varepsilon_q})$. We denote by $(\bar{s}_q, \bar{d}_q)$ the normalization of this sequence, i.e.:

$$\bar{s}_q(r) = \frac{s_q(r)}{||(s_q, d_q)||_E} \text{ and } \bar{d}_q(r) = \frac{d_q(r)}{||(s_q, d_q)||_E}\,.$$

Then $(\bar{s}_q, \bar{d}_q)$ is a bounded sequence in E, which moreover satisfies the exponential decay estimates (3.1), as (s_q, d_q) satisfies it. Hence this sequence is precompact in every L^q_{rad}, for $q \geq 2$. Now, as (s_q, d_q) is a critical point for F_{r_0,ε_q}, $C_{r_0}(s_q, d_q)$ is uniformly bounded, and

$$C_{r_0}(\bar{s}_q, \bar{d}_q) = \frac{C_{r_0}(s_q, d_q)}{||(s_q, d_q)||_E^4} \to 0 \text{ as } q \to +\infty\,.$$

Keeping the same notations for a converging subsequence of $(\bar{s}_q, \bar{d}_q)$, this implies that the function $\bar{s}_q\bar{d}_q$ converges to 0 in L^α_{rad}, given any $\alpha \geq 1$. Using Lemma 4.1, we get

$$\int_{r_0}^\infty (s_q^2 + d_q^2) r^2 dr \leq K b_n - K' \int_{r_0}^\infty \zeta(r) s_q d_q r^2 dr\,,$$

with K, K' positive constants and ζ a bounded function independent of q. The normalization implies

$$\int_{r_0}^\infty (\bar{s}_q^2 + \bar{d}_q^2) r^2 dr \leq \frac{K b_n}{||(s_q, d_q)||_E^2} - K' \int_{r_0}^\infty \zeta(r) \bar{s}_q \bar{d}_q r^2 dr\,.$$

Taking the limit for $q \to +\infty$, we find that $\bar{s}_q$ and $\bar{d}_q$ converge strongly to 0 in L^2_{rad}, hence the contradiction, because of the normalization. So every sequence $(s_n^{\varepsilon_q}, d_n^{\varepsilon_q})$ is bounded in E, and by an elementary bootstrap argument, we obtain that this sequence is precompact in E, and the limit, denoted by (s_n^*, d_n^*) is a critical point for F_{r_0} with critical value $\frac{1}{2}c_n^* = F_{r_0}(s_n^*, d_n^*) > 0$.

We have to prove that the limiting procedure given here leads to an infinite number of critical points for F_{r_0}. Of course, it is sufficient to prove that $(s_n^*)_{n > N(r_0)}$ is an infinite sequence of critical points for G_{r_0}, and we will study G_{r_0} instead of F_{r_0}. To this aim, we follow a method used in [6] for a similar problem. If the sequence c_n^* is strictly increasing, the proof is complete. Let us assume therefore that there exists $\bar{n}$ such that

$$c_{\bar{n}}^* = c_{\bar{n}+1}^* = c\,, \tag{4.1}$$

and let us denote by K_c the set of critical points of G_{r_0} at the level c. If K_c is not compact, its cardinal is infinite, and the proof is over.

To treat the remaining possibilities, we need the following lemma

Lemma 4.2. *Suppose that (4.1) holds and that K_c is compact. Then* *(i)* *Let $\mathcal{U}$ and $\mathcal{V}$ two bounded open sets in H such that $K_c \subset \mathcal{U} \subset \bar{\mathcal{U}} \subset \mathcal{V}$. Then there exist constants δ_1, α, $\varepsilon_1 > 0$ such that for all $\varepsilon \in (0, \varepsilon_1]$, for all $s \in (\mathcal{V} \setminus \mathcal{U}) \cap \{s / G_{r_0}(s) \in [c - \delta_1, c + \delta_1]\}$, we have*

$$||G'_{r_0}(s)||_{H'} \geq \alpha \,.$$

(ii) *For all $\varepsilon \in (0, \varepsilon_1]$, if $s \notin \mathcal{U}$ and if $G_{r_0}(s) \in [c - \delta_1, c + \delta_1]$, then*

$$||G'_{r_0}(s)||_{H'} \geq \beta(\varepsilon) > 0 \,.$$

This lemma has the following consequence (whose proof is standard and will be omitted):

Corollary 4.3. *Under the above assumptions, there exists $\delta_2 > 0$ such that for any $0 < \varepsilon \leq \varepsilon_1$ there is a time $T(\varepsilon) > 0$ with*

$$G_{r_0}\left(h_\varepsilon\left(T(\varepsilon), s\right)\right) \leq c - \delta_2 \,, \tag{4.2}$$

for all $s \notin \mathcal{V}$ and such that $G_{r_0}(s) \leq c + \delta_2$, where we denote by h_ε the flow induced by $-\nabla G_{r_0}$.

Proof of Lemma 4.2. As $\mathcal{V}$ is bounded in H, and H is compactly embedded in every L^q_{rad}, for $q > 2$, the restriction of G_{r_0} to the domain $\mathcal{V} \setminus \mathcal{U}$ satisfies the Cerami conditions, and (i) follows.

From the exponential decay estimate and the same compact embedding as above, for $\varepsilon > 0$ small enough, any critical point s of $G_{r_o,\varepsilon}$ such that $G_{r_0,\varepsilon}(s) \in [c - \delta_1, c + \delta_1]$ must be in $\mathcal{U}$. But we know that $G_{r_0,\varepsilon}$ satisfies the Cerami condition, hence (ii).$\diamond$

Making the assumptions of Lemma 4.2, we take two bounded open neighborhoods of $K_c, \mathcal{U}, \mathcal{V}$ such that, denoting by γ the Krasnoselski genus, $\gamma(K_c) = \gamma(\mathcal{V})$ and $K_c \subset \mathcal{U} \subset \bar{\mathcal{U}} \subset \mathcal{V}$. By Corollary 4.3,

$$\begin{aligned} d_\varepsilon &= \gamma\left(h_\varepsilon\left(T(\varepsilon), \{s \,/\, G_{r_0,\varepsilon}(s) \leq c + \delta_2\} \setminus \mathcal{V}\right)\right) \\ &\leq l_\varepsilon = \gamma\left(\{s \,/\, G_{r_0,\varepsilon}(s) \leq c - \delta_2\}\right) \leq \bar{n} - 2 \,. \end{aligned}$$

But we also have $d_\varepsilon + \gamma(\mathcal{V}) \geq \bar{n}$. Therefore, $\gamma(K_c) \geq 2$ and K_c is an infinite set.

There remains to prove the property related to the Morse index of the critical points, i.e., (ii) of Theorem 1.2.

Lemma 4.4. *The Morse index of s^*_n as critical point for the functional G_{r_0} is at most $n + 1$.*

Proof. This result is classical, since the critical points are obtained by the Symmetric Mountain Pass Lemma, and the approximation scheme does not perturb the Morse index (for references, see [7]).$\diamond$

5 Estimates and auxiliary results

5.1 Proof of Lemma 3.1

The proof of the Cerami property for $G_{r_0,\varepsilon}$ is a bit more complicated than in the Minkowski case, because of the apparition of r-dependant coefficients in the system (2.1-2.2), but the a priori estimates are similar. Let $(s_n)_{n\in\mathbb{N}}$ be a Cerami sequence for $G_{r_0,\varepsilon}$ at the level $l \in \mathbb{R}$. Then, with $d_n = d^*(s_n)$, we obtain a Cerami sequence (s_n, d_n) for $F_{r_0,\varepsilon}$ at the same level. First we find an estimate of the superquadratic terms:

$$F_{r_0,\varepsilon}(s_n,d_n)-\frac{1}{2}F'_{r_0,\varepsilon}(s_n,d_n).(s_n,d_n) = \lambda\int_{r_0}^{\infty} f^{-1}s_n^2d_n^2r^2+\varepsilon\left(\frac{p}{2}-1\right)\int_{r_0}^{\infty} s_n^pr^2 \geq 0\,, \tag{5.1}$$

hence we get

$$\int_{r_0}^{\infty} f^{-1}s_n^2d_n^2r^2dr \leq M\,,\text{and} \tag{5.2}$$

$$\int_{r_0}^{\infty} s_n^pr^2dr \leq M\,. \tag{5.3}$$

In order to find other a priori estimates, we put in evidence in the following lemma some properties of the operators involved in the system (2.1-2.2).

Lemma 5.1. *With the notations*

$$\begin{aligned} L_1\,:\,d &\mapsto d'+\left(\frac{1}{r}-mf^{-\frac{1}{2}}(r)\right)d\,,\\ L_2\,:\,s &\mapsto s'+\left(\frac{1}{r}+mf^{-\frac{1}{2}}(r)\right)s\,,\end{aligned}$$

we have the following properties:

- *(i) L_1 is an invertible operator from L to H', where H' denotes the dual space of H, and its inverse is bounded;*
- *(ii) Given an integer $m \geq 1$ and $1 \leq q \leq \infty$, L_1 is an invertible bounded operator from $W^{m,q}_{rad}$ onto $W^{m-1,q}_{rad}$, and its inverse is bounded;*
- *(iii) Given an integer $m \geq 1$ and $1 \leq q \leq \infty$, L_2 is an invertible bounded operator from $W^{m,q}_{0,rad}$ onto $W^{m-1,q}_{rad}$, and its inverse is bounded.*

Proof. It is clear, by classical differential operator theory, that L_1 and L_2 are bounded operators with respect to the spaces defined in the lemma. To obtain invertibility, we make use of the following elementary proposition concerning asymptotic behaviour of primitive functions (see [5]):

Proposition 5.2. *Let f be a C^2 positive function which satisfies*

$$\lim_{x\to+\infty}\left(\frac{f'(x)}{f(x)}\right)=a\neq 0\,.$$

(*a*) *If $a>0$, then $\int_{r_0}^{\infty}f(x)dx$ is divergent and we get the following behaviour:*

$$\int_{r_0}^{r}f(x)dx\sim_{+\infty}\frac{f(r)}{a}\,.$$

(*b*) *If $a<0$, then $\int_{r_0}^{\infty}f(x)dx$ is convergent and we get the following behaviour:*

$$\int_{r}^{\infty}f(x)dx\sim_{+\infty}-\frac{f(r)}{a}\,.$$

To prove (i), suppose that

$$L_1(d)=d'+(\frac{1}{r}-mf^{-\frac{1}{2}})d=\mu\in H'\,.$$

Then, using the variation of the constant, we get

$$d(r)=-\int_{r}^{\infty}\mu(\tau)e^{G(\tau)-G(r)}d\tau\,.$$

First check that d belongs to L. Let $\varphi\in L$ and

$$\begin{aligned}(d,\varphi)_L &= -\int_{r_0}^{\infty}r^2\varphi(r)\int_{r}^{\infty}\mu(\tau)e^{G(\tau)-G(r)}d\tau dr\\ &= -\int_{r_0}^{\infty}\tau^2\mu(\tau)e^{G(\tau)}\frac{1}{\tau^2}\int_{r_0}^{\tau}r^2\varphi(r)e^{-G(r)}drd\tau\,,\end{aligned}$$

by the Fubini formula. Define

$$\Phi(\tau)=\frac{e^{G(\tau)}}{\tau^2}\int_{r_0}^{\tau}r^2\varphi(r)e^{-G(r)}dr\,.$$

We have to prove that $\Phi\in H$. To this end, we compute its L-norm:

$$\begin{aligned}||\Phi||_L^2 &= \int_{r_0}^{\infty}\tau^2\frac{e^{2G(\tau)}}{\tau^4}\left(\int_{r_0}^{\tau}r^2\varphi(r)e^{-G(r)}dr\right)^2d\tau\\ &\leq \int_{r_0}^{\infty}\tau^{-2}e^{2G(\tau)}\left(\int_{r_0}^{\tau}e^{-G(r)}r^2\varphi^2(r)dr\right)\left(\int_{r_0}^{\tau}e^{-G(r)}r^2dr\right)d\tau\,,\end{aligned}$$

by the Cauchy-Schwarz formula. Then, according to Proposition 5.2 (a), the last integral is divergent as τ goes to infinity and we obtain the following majoration:

$$\int_{r_0}^{\tau}e^{-G(r)}r^2dr\leq Ke^{-G(\tau)}\tau^2\,,$$

where K is a constant. Thus we get, using Fubini formula again:

$$||\Phi||_L^2 \le K \int_{r_0}^{\infty} e^{-G(r)} r^2 \varphi^2(r) \int_r^{\infty} e^{G((\tau)} d\tau dr \,.$$

The last integral is convergent and, using Proposition 5.2 (b), we obtain

$$||\Phi||_L^2 \le K' \int_{r_0}^{\infty} \varphi^2(r) dr = K' ||\varphi||_L^2 \,.$$

Moreover,

$$\Phi'(\tau) = G''(\tau)\Phi(\tau) - \frac{2}{\tau}\Phi(\tau) + \varphi(\tau)\,,$$

hence we proved that $||\Phi||_H \le K''||\varphi||_L$. This implies

$$(d,\varphi)_L = (\mu,\Phi)_{H'\times H} \le ||\mu||_{H'}||\Phi||_H \le K''||\mu||_{H'}||\phi||_L \,.$$

Then we have $||d||_L \le K''||\mu||_{H'}$ and the boundedness of L_1^{-1} is proved.

To prove (ii), we can limit the study to $m = 1$. As before, we write, using the variation of the constant

$$d(r) = -\int_r^{\infty} \mu(\tau) e^{G(\tau)-G(r)} d\tau \,,$$

with $\mu \in L^q((r_0,\infty), r^2 dr)$. To check that $d \in W^{1,q}_{rad}$, we compute its norm in this space:

$$\int_{r_0}^{\infty} r^2 \left(\int_r^{\infty} \mu(\tau) e^{G(\tau)-G(r)} d\tau\right)^q dr = \int_{r_0}^{\infty} r^2 e^{-qG(r)} \left(\int_r^{\infty} \mu(\tau) e^{G(\tau)} d\tau\right)^q dr \,.$$

By Hölder inequality, we get

$$\left(\int_r^{\infty} \mu(\tau) e^{G(\tau)} d\tau\right)^p \le \int_r^{\infty} \mu^p(\tau)\tau e^{G(\tau)} d\tau \left(\int_r^{\infty} \tau^{-\frac{q^*}{q}} e^{G(\tau)} d\tau\right)^{\frac{q}{q^*}} \,.$$

According to Proposition 5.2 (b), the last integral is convergent, and its upper bound leads to

$$\begin{aligned} \int_{r_0}^{\infty} r^2 \left(\int_r^{\infty} \mu(\tau) e^{G(\tau)-G(r)} d\tau\right)^q dr &\le K \int_{r_0}^{\infty} r e^{q(\frac{1}{q^*}-1)G(r)} \int_r^{\infty} \mu^q(\tau)\tau e^{G(\tau)} d\tau dr \\ &\le K \int_{r_0}^{\infty} \mu^q(\tau)\tau e^{G(\tau)} \int_{r_0}^{\tau} r e^{-G(r)} dr d\tau \,, \end{aligned}$$

by the Fubini formula. Proposition 5.2 (a) gives the estimate

$$||d||^q_{L^q_{rad}} \le K''||\mu||^q_{L^q_{rad}} \,.$$

Since $d' = -G'd + \mu$, we have proved that $||d||_{W^{1,p}_{rad}} \leq K||\mu||_{L^q_{rad}}$, and (ii) follows. The proof of (iii) is exactly the same as for (ii), exchanging in the proof the use of Proposition 5.2 (a) and (b). Lemma 5.1 is proved.$\diamond$

We are now ready to prove Lemma 3.1. Since $d_n = d^*(s_n)$, we have the exact equation:

$$s_n' + \left(\frac{1}{r} + mf^{-\frac{1}{2}}\right) s_n + \left(\frac{f^{-\frac{1}{2}}}{r} + \omega f^{-1}\right) d_n = -2\lambda f^{-1} s_n^2 d_n \,. \tag{5.4}$$

In order to prove precompactness for the Cerami sequence, we cut d_n into $\tilde{d}_n + \varepsilon_n$, where $\varepsilon_n = L_1^{-1}(\frac{\partial F_{r_0,\varepsilon}}{\partial s}(s_n, d_n))$. We obtain of course that $||\varepsilon_n||_L$ tends to 0 as n goes to infinity. Moreover, we can replace (2.1) with the following equation concerning $\tilde{d}_n$:

$$\begin{aligned} \tilde{d}_n' + \left(\frac{1}{r} - mf^{-\frac{1}{2}}\right) \tilde{d}_n - 2\lambda f^{-1} s_n d_n \tilde{d}_n &= \left(\omega f^{-1} - \frac{f^{-\frac{1}{2}}}{r}\right) s_n \\ &\quad + 2\lambda f^{-1} s_n d_n \varepsilon_n + \varepsilon p s_n^{p-1} \,. \end{aligned} \tag{5.5}$$

The left hand side of (5.5) is related to the derivative of $e^{G(r)+H_n(r)}\tilde{d}_n$ with

$$\begin{aligned} G'(r) &= \frac{1}{r} - mf^{-\frac{1}{2}} \,, \\ H_n'(r) &= -2\lambda f^{-1} s_n(r) d_n(r) \,. \end{aligned}$$

By (5.2), H_n is a positive and uniformly bounded function. Thus we can write

$$\begin{aligned} \tilde{d}_n(r) = & -\int_r^\infty e^{G(\rho)-G(r)+H_n(\rho)-H_n(r)} \left[\left(\omega f^{-1}(\rho) - \frac{f^{-\frac{1}{2}}(\rho)}{\rho}\right) s_n \right. \\ & \left. + 2\lambda f^{-1}(\rho) s_n d_n \varepsilon_n + \varepsilon p s_n^{p-1}\right] d\rho \,, \end{aligned}$$

and we get

$$\begin{aligned} |\tilde{d}_n(r)| &\leq C \int_r^\infty e^{G(\rho)-G(r)} \left[\left|\omega f^{-1}(\rho) - \frac{f^{-\frac{1}{2}}(\rho)}{\rho}\right| |s_n| + \varepsilon p |s_n|^{p-1} \right. \\ &\quad \left. + 2\lambda f^{-1}(\rho)|s_n d_n \varepsilon_n|\right] d\rho \,. \\ &\leq C L_1^{-1} \left[\left|\omega f^{-1}(\rho) - \frac{f^{-\frac{1}{2}}(\rho)}{\rho}\right| |s_n| + \varepsilon p |s_n|^{p-1} \right. \\ &\quad \left. + 2\lambda f^{-1}(\rho)|s_n d_n \varepsilon_n|\right] d\rho \,. \end{aligned} \tag{5.6}$$

From (5.6) we infer that $\tilde{d}_n$ is bounded in L^q_{rad} for all $q > p$ and in L^∞. Applying L_1^{-1} to (5.5), we find that $\tilde{d}_n$ is the sum of the following functions:

- $L_1^{-1}\left(2\lambda f^{-1} s_n d_n \tilde{d}_n\right)$, which is bounded in $W^{1,2}_{rad}$ since $s_n d_n$ is bounded in L^2_{rad} and $\tilde{d}_n$ is bounded in L^∞;

- $L_1^{-1}\left(2\lambda f^{-1}s_n d_n \varepsilon_n\right)$, which is bounded in $W^{1,1}_{rad}$, since ε_n is bounded in L^2_{rad};
- $L_1^{-1}\left((\omega f^{-1} - \frac{f^{-\frac{1}{2}}}{r})s_n\right)$, which is bounded in $W^{1,p}_{rad}$;
- $L_1^{-1}\left(\varepsilon p s_n^{p-1}\right)$, which is bounded in $W^{1,\frac{p}{p-1}}_{rad}$.

Hence, by Sobolev embedding, $\tilde{d}_n$ is precompact in every L^q_{rad}, for $q > p$, and we have the following uniform estimate:

$$\exists K_1,\, \forall n \in \mathbb{N},\, \forall r \in (r_0, \infty),\, |\tilde{d}_n(r)| \leq \frac{K_1}{r^{\frac{2}{p}}}\,.$$

We denote by $\tilde{d}$ its limit, up to a subsequence. Applying L_2^{-1} to (5.4), we obtain a similar decomposition of s_n, which implies that s_n is also precompact in every L^q_{rad}, for $q > p$ and the same uniform estimate:

$$\exists K_2,\, \forall n \in \mathbb{N},\, \forall r \in (r_0, \infty),\, |s_n(r)| \leq \frac{K_2}{r^{\frac{2}{p}}}\,.$$

We denote by s its limit, up to a subsequence.

We now have to prove that convergence is obtained in the right spaces, that is, respectively H for s_n and L for d_n. To this aim, we prove an exponential decay estimate for the couple $(s_n, \tilde{d}_n)$. We denote by $\mathcal{L}$ the following differential operator:

$$\mathcal{L}\begin{pmatrix} s \\ d \end{pmatrix} = \begin{pmatrix} s' \\ d' \end{pmatrix} + A\begin{pmatrix} s \\ d \end{pmatrix},$$

with the matrix A given by

$$A = \begin{pmatrix} m & \omega \\ -\omega & -m \end{pmatrix}.$$

This matrix has eigenvalues $\pm\sqrt{m^2 - \omega^2}$, and if we denote by $\Lambda_\pm$ the projectors on the corresponding eigenspaces, $\mathcal{L}$ has a bounded inverse from $L^q((k,\infty), r^2 dr)$ on to $W^{1,q}_0((k,\infty), r^2 dr)$:

$$(\mathcal{L}^{-1}\Psi)(r) = \int_k^r e^{-A(r-\rho)}\Lambda_-\Psi(\rho)d\rho - \int_r^\infty e^{-A(r-\rho)}\Lambda_+\Psi(\rho)d\rho\,.$$

Of course, we cannot hope any exponential decay property for the Cerami sequence. Nevertheless, we prove that this sequence is the sum of two terms: the first one satisfies an exponential decay property while the second one vanishes. Using the preceding notations, we can rewrite system (5.4-5.5) as follows:

$$\mathcal{L}\begin{pmatrix} s_n \\ \tilde{d}_n \end{pmatrix} = B(r)\begin{pmatrix} s_n \\ \tilde{d}_n \end{pmatrix} + \begin{pmatrix} -2\lambda f^{-1}s_n^2\tilde{d}_n \\ 2\lambda f^{-1}s_n\tilde{d}_n^2 + \varepsilon p s_n^{p-1} \end{pmatrix} + \begin{pmatrix} -2\lambda f^{-1}s_n^2\varepsilon_n \\ 2\lambda f^{-1}s_n\varepsilon_n(2\tilde{d}_n + \varepsilon_n) \end{pmatrix},$$
(5.7)

where $B(r)$ is a matrix with all its coefficients going to 0 as r goes to infinity. Denoting the couple $(s_n, \tilde{d}_n)$ by φ_n, we put

$$\psi_n = \varphi_n - \mathcal{L}^{-1}\begin{pmatrix} -2\lambda f^{-1} s_n^2 \varepsilon_n \\ 2\lambda f^{-1} s_n \varepsilon_n (2\tilde{d}_n + \varepsilon_n) \end{pmatrix}. \tag{5.8}$$

To prove an exponential decay estimate on ψ_n, we will use a method inspired of [6]. Define now a smooth cut-off function θ_k^δ with $\delta > 0$, $k \in \mathbb{N}$, $k \geq 1$:

- $\theta \in C^\infty(\mathbb{R}, [0,1])$;
- $\theta(r) \equiv 0$ for $r \in (-\infty, 0)$;
- $\theta(r) \equiv 1$ for $r \in (1, \infty)$;
- $\theta'(r) \in (0,2)$ for $r \in (0,1)$;
- $\theta_k^\delta(r) = \theta\left(\frac{r-r_0}{\delta} - k\right)$.

We put $\theta_k^\delta \psi_n = \psi_{n,k}$. From (5.7) and the properties of $\mathcal{L}^{-1}$, we infer that, given any $q > p$,

$$||\psi_{n,k}||_{W^{1,q}_{rad}} \overset{k\to\infty}{\longrightarrow} 0\,.$$

Then, we write

$$\begin{aligned} \mathcal{L}\psi_{n,k} &= \theta_k^\delta \mathcal{L}\psi_n + (\theta_k^\delta)'\psi_n \\ &= \left(\theta_k^\delta B(r) + (\theta_k^\delta)'\right)\psi_n + \theta_k^\delta \begin{pmatrix} -2\lambda f^{-1} s_n^2 \tilde{d}_n \\ 2\lambda f^{-1} s_n \tilde{d}_n^2 + \varepsilon p s_n^{p-1} \end{pmatrix} \\ &\quad + \theta_k^\delta B(r) \mathcal{L}^{-1} \begin{pmatrix} -2\lambda f^{-1} s_n^2 \varepsilon_n \\ 2\lambda f^{-1} s_n \varepsilon_n (2\tilde{d}_n + \varepsilon_n) \end{pmatrix}. \end{aligned}$$

We have to give a L^q_{rad} estimation for the last three terms. For the first one, we get easily, with the properties satisfied by θ_k^δ:

$$||\left(\theta_k^\delta B(r) + (\theta_k^\delta)'\right)\psi_n||_{L^q_{rad}} \leq \left(\frac{C}{\delta^{\frac12} k^{\frac12}} + \frac{2}{\delta}\right)||\psi_{n,k}||_{L^q_{rad}}\,. \tag{5.9}$$

For the second one, we have

$$||\theta_k^\delta \begin{pmatrix} -2\lambda f^{-1} s_n^2 \tilde{d}_n \\ 2\lambda f^{-1} s_n \tilde{d}_n^2 + \varepsilon p s_n^{p-1} \end{pmatrix}||_{L^q_{rad}} \leq \frac{C}{\delta^{\frac12} k^{\frac12}} ||\theta_k^\delta \varphi_n||_{L^q_{rad}}\,. \tag{5.10}$$

The third term may be estimated by the use of Sobolev embeddings:

$$\begin{aligned} &||\theta_k^\delta B(r)\mathcal{L}^{-1}\begin{pmatrix} -2\lambda f^{-1} s_n^2 \varepsilon_n \\ 2\lambda f^{-1} s_n \varepsilon_n (2\tilde{d}_n + \varepsilon_n) \end{pmatrix}||_{L^q_{rad}} \leq \\ &\leq \frac{C_0}{\delta^{\frac12} k^{\frac12}}\left[||\mathcal{L}^{-1}\begin{pmatrix} 0 \\ 2\lambda f^{-1} s_n \varepsilon_n^2 \end{pmatrix}||_{W^{1,1}_{rad}} + ||\mathcal{L}^{-1}\begin{pmatrix} -2\lambda f^{-1} s_n^2 \varepsilon_n \\ 4\lambda f^{-1} s_n \tilde{d}_n \varepsilon_n \end{pmatrix}||_{W^{1,2}_{rad}}\right] \\ &\leq \frac{1}{\delta^{\frac12} k^{\frac12}}\left[C_1 ||2\lambda f^{-1} s_n \varepsilon_n^2||_{L^1_{rad}} + C_2 ||2\lambda f^{-1} s_n^2 \varepsilon_n||_{L^2_{rad}} + C_2||4\lambda f^{-1} s_n \tilde{d}_n \varepsilon_n||_{L^2_{rad}}\right] \\ &\leq \frac{C_3}{\delta^{\frac12} k^{\frac12}} ||\varepsilon_n||^2_{L^2_{rad}} ||\varphi_n||_{L^\infty}\,. \end{aligned} \tag{5.11}$$

With the definition of ψ_n, we then obtain

$$||\varphi_n|| \leq ||\psi_n|| + \frac{C_3}{\delta^{\frac{1}{2}} k^{\frac{1}{2}}} ||\varepsilon_n||^2_{L^2_{rad}} ||\varphi_n||_{L^\infty} ,$$

in the L^∞- or the L^q_{rad}-norm. This implies that, for n great enough, we get

$$||\varphi_n||_{L^\infty} \leq 2||\psi_n||_{L^\infty} \leq 2C_4||\psi_n||_{W^{1,q}_{rad}} \tag{5.12}$$

and

$$||\varphi_n||_{L^q_{rad}} \leq \left(1 + \frac{C_3}{\delta^{\frac{1}{2}} k^{\frac{1}{2}}} ||\varepsilon_n||^2_{L^2_{rad}}\right) ||\psi_n||_{W^{1,q}_{rad}} \leq 2||\psi_n||_{W^{1,q}_{rad}} . \tag{5.13}$$

Then, (5.10) and (5.13) lead to an estimate of the second term:

$$||\theta^\delta_k \begin{pmatrix} -2\lambda f^{-1} s_n^2 \tilde{d}_n \\ 2\lambda f^{-1} s_n \tilde{d}_n^2 + \varepsilon p s_n^{p-1} \end{pmatrix} ||_{L^q_{rad}} \leq \frac{C_5}{\delta^{\frac{1}{2}} k^{\frac{1}{2}}} ||\psi_{n,k}||_{W^{1,q}_{rad}} , \tag{5.14}$$

and (5.11) with (5.12) give the following estimate for the third term:

$$||\theta^\delta_k B(r) \mathcal{L}^{-1} \begin{pmatrix} -2\lambda f^{-1} s_n^2 \varepsilon_n \\ 2\lambda f^{-1} s_n \varepsilon_n (2\tilde{d}_n + \varepsilon_n) \end{pmatrix} ||_{L^q_{rad}} \leq \frac{C_6 ||\varepsilon_n||_{L^2_{rad}}}{\delta^{\frac{1}{2}} k^{\frac{1}{2}}} ||\psi_{n,k}||_{W^{1,q}_{rad}} . \tag{5.15}$$

Finally, joining (5.9) with (5.14) and (5.15), we obtain, given n, k and δ great enough,

$$||\psi_n||_{W^{1,q}((r_0+(k+1)\delta,\infty),r^2dr)} \leq ||\psi_n, k||_{W^{1,q}_{rad}} \leq \frac{1}{2} ||\psi_n||_{W^{1,q}((r_0+k\delta,\infty),r^2dr)} ,$$

which implies an exponential decay estimate for $||\psi_n||_{W^{1,q}((r_0+k\delta,\infty),r^2dr)}$:

$$\forall q > p,\ \exists \sigma_q > 0 \,/\, ||\psi_n||_{W^{1,q}((r_0+k\delta,\infty),r^2dr)} \leq e^{-\sigma_q k} . \tag{5.16}$$

Using Hölder's inequality, this estimate may be checked for $q > 1$. Now, we can pass to the limit in (5.8), in L^q_{rad}: ψ_n is a bounded sequence in $W^{1,q}_{rad}$, for every $q \geq 2$ Its limit, denoted by $\begin{pmatrix} s \\ \tilde{d} \end{pmatrix}$, satisfies the estimate (5.16). We deduce that ψ_n has a convergent subsequence in $W^{1,q}_{rad}$, for every $q \geq 2$. This implies in particular that $\begin{pmatrix} s \\ \tilde{d} \end{pmatrix} \in W^{1,q}_{rad}$ for every $q \geq 2$, and moreover these functions satisfy the same exponential decay estimate, which ends the proof. $\diamond$

5.2 Proof of Lemma 3.2

Since the functional $G_{r_0,\varepsilon}$ is smooth, we compute its second order derivative at 0. Remark that adding superquadratic terms in the functional does not change this quadratic form, denoted by $Q_{r_0,\varepsilon}$. After tedious but straightforward computations, we find, for every $h \in H$,

$$Q_{r_0,\varepsilon}(h) = \int_{r_0}^{\infty} \left[\varphi_2(r)h'^2 + \varphi_1(r)h^2\right] dr \,, \tag{5.17}$$

with

$$\varphi_2(r) = \frac{2r^2 f}{\frac{f^{\frac{1}{2}}}{r} + \omega} \,, \tag{5.18}$$

$$\begin{aligned}\varphi_1(r) = {} & \frac{d}{dr}\left(\frac{2rf^{\frac{1}{2}} + mr^2}{\frac{1}{r} + \omega f^{-\frac{1}{2}}}\right) + \frac{2r^2 f^{\frac{1}{2}}}{\frac{1}{r} + \omega f^{-\frac{1}{2}}}\left(\frac{1}{r} + mf^{-\frac{1}{2}}\right)^2 \\ & -2r^2 f^{-\frac{1}{2}}\left(-\frac{1}{r} + \omega f^{-\frac{1}{2}}\right).\end{aligned} \tag{5.19}$$

Fix $r_0 > 1$. As φ_1 and φ_2 are positive everywhere, excepted in a compact set in (r_0, ∞), $Q_{r_0,\varepsilon}$ is a compact perturbation of a positive symmetric operator, so we claim that $Q_{r_0,\varepsilon}$ has a finite number of nonpositive directions. Moreover, this number is increasing as r_0 goes to 1 because of the following monotonicity: if h_{r_0} satisfies $Q_{r_0,\varepsilon}(h_{r_0}) \le 0$ then, for all $r' \in (1, r_0)$, h_{r_0} extends to a unique $\tilde{h}_{r'}$ and

$$Q_{r',\varepsilon}(\tilde{h}_{r'}) = Q_{r_0,\varepsilon}(h_{r_0}) \le 0\,.$$

If we denote by $N(r_0)$ the number of nonpositive directions of $Q_{r_0,\varepsilon}$, this proves that N is nondecreasing as r_0 goes to 1 and $N(r_0) = 0$ for large r_0.

We use now a rescaling argument to show that N is not a constant, i.e. there exists negative directions, for $r_0 > 1$ small enough, and N tends to infinity as r_0 goes to 1. Let $\theta \in C^\infty(\mathbb{R}, [0,1])$ be a smooth function, whose compact support lays in $[0,1]$. Given $r_0 > 1$, we put

$$h_{r_0}(r) = \theta\left(\frac{r - r_0}{r_0 - 1}\right) \,.$$

Now, some properties of φ_1 and φ_2 are sufficient to show that, for r_0 small enough, $Q_{r_0,\varepsilon}(h_{r_0}) < 0$, and this leads to the results claimed. By computation, there exist k_1, $k_2 > 0$ such that

$$\varphi_2(r) \le k_2(r-1)\,, \tag{5.20}$$

$$\varphi_1(r) \le -k_1(r-1)^{-\frac{3}{2}}\,, \tag{5.21}$$

for $r > 1$ small enough. Then, with the same condition,

$$\begin{aligned}Q_{r_0,\varepsilon}(h_r) &= \int_{r_0}^{2r_0-1} \varphi_2 h'^2 + \varphi_1 h^2 \\ &\le k_2(2r_0-2)\int_{r_0}^{2r_0-1} h'^2 - k_1(2r_0-2)^{-\frac{3}{2}}\int_{r_0}^{2r_0-1} h^2 \\ &\le 2k_2\int_0^1 \theta'^2 - 2^{-\frac{3}{2}}k_1(r_0-1)^{-\frac{1}{2}}\int_0^1 \theta^2\,,\end{aligned}$$

so there exists $\bar{r}_0$ such that $Q_{r_0,\varepsilon}(h_{\bar{r}_0}) < 0$. Since this construction is possible from every $\theta \in C^\infty(\mathbb{R},[0,1])$, the proof is over.$\diamond$

5.3 Proof of Lemma 4.1

Since $(s_n^\varepsilon, d_n^\varepsilon)$ is a critical point for $F_{r_0,\varepsilon}$, it satisfies the system

$$
\begin{aligned}
(5.22) f(d_n^\varepsilon)' + (\frac{f}{r} - mf^{\frac{1}{2}})d_n^\varepsilon + (\frac{f^{\frac{1}{2}}}{r} - \omega)s_n^\varepsilon &= 2\lambda s_n^\varepsilon (d_n^\varepsilon)^2 + \varepsilon p (s_n^\varepsilon)^{p-1}\,,\\
(5.23) f(s_n^\varepsilon)' + (\frac{f}{r} + mf^{\frac{1}{2}})s_n^\varepsilon + (\frac{f^{\frac{1}{2}}}{r} + \omega)d_n^\varepsilon &= -2\lambda (s_n^\varepsilon)^2 d_n^\varepsilon\,.
\end{aligned}
$$

Following a classical bootstrap idea, we use Lemma 5.1 (i) and (ii) to find more regularity for the critical points, and we obtain, jointly with the exponential decay estimate

$$(r(s_n^\varepsilon)', r(d_n^\varepsilon)') \in H \times L\,.$$

Hence we can follow a Pohozaev-type idea and compute the following quantity, which is the equivalent in our case of the derivative of the functional relatively to the dilations, taken at the critical point $(s_n^\varepsilon, d_n^\varepsilon)$:

$$
\begin{aligned}
&\frac{\partial F_{r_0,\varepsilon}}{\partial d}(s_n^\varepsilon, d_n^\varepsilon).(r(d_n^\varepsilon)') + \frac{\partial F_{r_0,\varepsilon}}{\partial s}(s_n^\varepsilon, d_n^\varepsilon).(r(s_n^\varepsilon)') = 0\\
&= \int_{r_0}^\infty \Big[(s_n^\varepsilon (d_n^\varepsilon)' - d_n^\varepsilon (s_n^\varepsilon)') + f^{-\frac{1}{2}}(d_n^\varepsilon (d_n^\varepsilon)' - s_n^\varepsilon (s_n^\varepsilon)') + 2\lambda f^{-1} r((s_n^\varepsilon)^2 d_n^\varepsilon (d_n^\varepsilon)'\\
&\quad + s_n^\varepsilon (s_n^\varepsilon)'(d_n^\varepsilon)^2) + mf^{-\frac{1}{2}} r(s_n^\varepsilon (d_n^\varepsilon)' + d_n^\varepsilon (s_n^\varepsilon)') + \omega f^{-1} r(d_n^\varepsilon (d_n^\varepsilon)' + s_n^\varepsilon (s_n^\varepsilon)')\\
&\quad + \varepsilon p r(s_n^\varepsilon)^{p-1}(s_n^\varepsilon)'\Big]\, r^2 dr
\end{aligned}
$$

Using integration by parts, we may recognize in this integral some terms appearing in the functional $F_{r_0,\varepsilon}$. Indeed, we obtain

$$
\begin{aligned}
&2A_{r_0}(s_n^\varepsilon, d_n^\varepsilon) + 3B_{r_0}(s_n^\varepsilon, d_n^\varepsilon) + 3C_{r_0}(s_n^\varepsilon, d_n^\varepsilon) - 3\varepsilon \int_{r_0}^\infty (s_n^\varepsilon)^p r^2 d_n^\varepsilon r =\\
&= \int_{r_0}^\infty \Big[\frac{1}{4}((s_n^\varepsilon)^2 - (d_n^\varepsilon)^2) f^{-\frac{3}{2}} - m\frac{f^{-\frac{3}{2}}}{2} s_n^\varepsilon d_n^\varepsilon r - \frac{\omega f^{-2}}{2}((s_n^\varepsilon)^2 + (d_n^\varepsilon)^2) r\\
&\quad - \lambda f^{-2}(s_n^\varepsilon)^2 (d_n^\varepsilon)^2 r\Big]\, dr + \frac{1}{2} d_n^\varepsilon(r_0)^2 r_0^2 \left(f^{-\frac{1}{2}}(r_0) + \omega f^{-1}(r_0) r_0\right)\,.
\end{aligned}
$$

Then the superquadraticity inequality (5.3) leads to the following lower bound for B_{r_0}:

$$
\begin{aligned}
B_{r_0}(s_n^\varepsilon, d_n^\varepsilon) \quad \geq \quad & -b_n + \varepsilon \frac{p-4}{2}\int_{r_0}^\infty (s_n^\varepsilon)^p r^2 dr + \int_{r_0}^\infty \Bigg[\frac{f^{-\frac{3}{2}}}{4}\left((s_n^\varepsilon)^2 - (d_n^\varepsilon)^2\right)\\
& -\omega \frac{f^{-2} r}{2}\left((s_n^\varepsilon)^2 + (d_n^\varepsilon)^2\right)\Bigg]\, dr
\end{aligned}
$$

$$-\int_{r_0}^{\infty}\left(\frac{mf^{-\frac{3}{2}}r}{2}s_n^{\varepsilon}d_n^{\varepsilon}+\lambda f^{-2}r(s_n^{\varepsilon})^2(d_v^{\varepsilon})^2\right)dr \tag{5.24}$$

Now the uniform bound on the critical values $F_{r_0,\varepsilon}(s_n^{\varepsilon},d_n^{\varepsilon})$ gives an upper bound for $\varepsilon||s_n^{\varepsilon}||_{L^p_{rad}}^p$ and $\lambda||s_n^{\varepsilon}d_n^{\varepsilon}||_{L^2_{rad}}^2$. The use of the definition of B_{r_0} allows to get rid of the terms containing $(d_n^{\varepsilon})^2$, since we get

$$\begin{aligned}&\int_{r_0}^{\infty}-\frac{\omega f^{-2}r}{2}\left((s_n^{\varepsilon})^2+(d_n^{\varepsilon})^2\right)dr\geq\\ &\quad\frac{1}{r_0-1}\left(B_{r_0}(s_n^{\varepsilon},d_n^{\varepsilon})+\int_{r_0}^{\infty}mf^{-\frac{1}{2}}s_n^{\varepsilon}d_n^{\varepsilon}r^2dr\right),\end{aligned} \tag{5.25}$$

and

$$\begin{aligned}&\int_{r_0}^{\infty}-\frac{f^{-\frac{3}{2}}}{4}(d_n^{\varepsilon})^2dr\geq\\ &\quad\frac{f^{-\frac{1}{2}}(r_0)}{2\omega r_0^2}\left(B_{r_0}(s_n^{\varepsilon},d_n^{\varepsilon})+\int_{r_0}^{\infty}mf^{-\frac{1}{2}}s_n^{\varepsilon}d_n^{\varepsilon}r^2dr\right).\end{aligned} \tag{5.26}$$

With (5.25) and (5.26), we obtain from (5.24) a lower bound for B_{r_0}, which only depends on n and the function $s_n^{\varepsilon}d_n^{\varepsilon}$,

$$\begin{aligned}&\left(\frac{r_0}{r_0-1}-\frac{f^{-\frac{1}{2}}(r_0)}{2\omega r_0^2}\right)B_{r_0}(s_n^{\varepsilon},d_n^{\varepsilon})\geq\\ &\quad -b_n+\int_{r_0}^{\infty}ms_n^{\varepsilon}d_n^{\varepsilon}\left(\frac{f^{-\frac{1}{2}}(r_0)f^{-\frac{1}{2}}r^2}{2\omega r_0^2}-\frac{f^{-\frac{3}{2}}r}{2}+f^{-\frac{1}{2}}r^2\right),\end{aligned} \tag{5.27}$$

and the proof is over.◇

Acknowledgement. The author would like to thank Eric Séré for his patience, great competence and kindness. He also thanks the members of CEREMADE for stimulating discussions and their kind hospitality.

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