Discrete Lagrangian reduction, discrete Euler–Poincaré equations, and semidirect products

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Abstract

A discrete version of Lagrangian reduction is developed in the context of discrete time Lagrangian systems on $G \times G$, where G is a Lie group. We consider the case when the Lagrange function is invariant with respect to the action of an isotropy subgroup of a fixed element in the representation space of G. In this context the reduction of the discrete Euler–Lagrange equations is shown to lead to the so called discrete Euler–Poincaré equations. A constrained variational principle is derived. The Legendre transformation of the discrete Euler–Poincaré equations leads to discrete Hamiltonian (Lie–Poisson) systems on a dual space to a semiproduct Lie algebra.

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1 Introduction

Dynamical systems with symmetry play an important role in mathematical modelling of a vast variety of physical and mechanical processes. A Hamiltonian approach to such systems is nowadays a well–established theory [A], [AM], [CB], [MR], [MRW], [RSTS]. More recently, also a variational (Lagrangian) description of systems with symmetries also attracted much attention [MS], [HMR], [CHMR]. In particular, in the last two papers the corresponding theory was developed for Lagrangian systems on Lie groups, i.e. for Lagrangians defined on tangent bundles TG of Lie groups. A symmetry of the Lagrangian with respect to a subgroup action leads to a reduced system on a semidirect product described by the so called Euler–Poincaré equation.

In the present paper we develop a discrete analog of this theory, i.e. for Lagrangians defined on $G \times G$. We introduce the corresponding reduced systems and derive the discrete Euler–Poincaré equations. We establish symplectic properties of the corresponding discrete dynamical systems. The continuous time theory may be considered as a limiting case of the discrete time one. The important particular case, when the representation of G participating in the general theory is chosen to be the adjoint representation, is developed in [BS].

2 Lagrangian mechanics on TG and on $G \times G$

Recall that a continuos time Lagrangian system is defined by a smooth function $\mathbf{L}(g, \dot{g}) : TG \mapsto \mathbb{R}$ on the tangent bundle of a smooth manifold G. The function $\mathbf{L}$ is called the *Lagrange function.* We will be dealing here only with the case when G carries an additional structure of a *Lie grioup.* For an arbitrary function $g(t) : [t_0, t_1] \mapsto G$ one can consider the *action functional*

$$\mathbf{S} = \int_{t_0}^{t_1} \mathbf{L}(g(t), \dot{g}(t))dt \ . \tag{2.1}$$

A standard argument shows that the functions $g(t)$ yielding extrema of this functional (in the class of variations preserving $g(t_0)$ and $g(t_1)$), satisfy with necessity the *Euler–Lagrange equations.* In local coordinates $\{g^i\}$ on G they read:

$$\frac{d}{dt}\left(\frac{\partial \mathbf{L}}{\partial \dot{g}^i}\right) = \frac{\partial \mathbf{L}}{\partial g^i} \ . \tag{2.2}$$

The action functional S is independent of the choice of local coordinates, and thus the Euler–Lagrange equations are actually coordinate independent as well. For a coordinate–free description in the language of differential geometry, see [A], [MR].

Introducing the quantities

$$\Pi = \nabla_{\dot{g}} \mathbf{L} \in T_g^* G \ , \tag{2.3}$$

one defines the *Legendre transformation*:

$$(g, \dot{g}) \in TG \mapsto (g, \Pi) \in T^*G \ . \tag{2.4}$$

If it is invertible, i.e. if $\dot g$ can be expressed through (g,Π), then the the Legendre transformation of the Euler–Lagrange equations (2.2) yield a *Hamiltonian system* on T^*G with respect to the standard symplectic structure on T^*G and with the Hamilton function

$$H(g,\Pi)=\langle \Pi,\dot g\rangle-\mathbf{L}(g,\dot g)\;, \tag{2.5}$$

(where, of course, $\dot g$ has to be expressed through (g,Π)).

We now turn to the discrete time analog of these constructions, introduced in [V], [MV]. Our presentation is an adaptation of the Moser–Veselov construction for the case when the basic manifold is a Lie group. Almost all constructions and results of the continuous time Lagrangian mechanics have their discrete time analogs. The only exception is the existence of the "energy" integral (2.5).

Let $\mathbb{L}(g,\hat g)\;:\;G\times G$ be a smooth function, called the (discrete time) *Lagrange function.* For an arbitrary sequence $\{g_k\in G,\;k=k_0,k_0+1,\dots,k_1\}$ one can consider the *action functional*

$$\mathbb{S}=\sum_{k=k_0}^{k_1-1}\mathbb{L}(g_k,g_{k+1})\;. \tag{2.6}$$

Obviously, the sequences $\{g_k\}$ delivering extrema of this functional (in the class of variations preserving g_{k_0} and g_{k_1}), satisfy with necessity the *discrete Euler–Lagrange equations*: [3]

$$\nabla_1\mathbb{L}(g_k,g_{k+1})+\nabla_2\mathbb{L}(g_{k-1},g_k)=0\;. \tag{2.7}$$

Here $\nabla_1\mathbb{L}(g,\hat g)$ ($\nabla_2\mathbb{L}(g,\hat g)$) denotes the gradient of $\mathbb{L}(g,\hat g)$ with respect to the first argument g (resp. the second argument $\hat g$). So, in our case, when G is a Lie group and not just a general smooth manifold, the equation (2.7) is written in a coordinate free form, using the intrinsic notions of the Lie theory. As pointed out above, an invariant formulation of the Euler–Lagrange equations in the continuous time case is more sophisticated. This seems to underline the fundamental character of discrete Euler–Lagrange equations.

The equation (2.7) is an implicit equation for g_{k+1}. In general, it has more than one solution, and therefore defines a correspondence (multi–valued map) $(g_{k-1},g_k)\mapsto(g_k,g_{k+1})$. To discuss symplectic properties of this correspondence, one defines:

$$\Pi_k=\nabla_2\mathbb{L}(g_{k-1},g_k)\in T^*_{g_k}G\;. \tag{2.8}$$

[3]For the notations from the Lie groups theory used in this and subsequent sections see Appendix B. In particular, for an arbitrary smooth function $f:G\mapsto\mathbb{R}$ its right Lie derivative $d'f$ and left Lie derivative df are functions from G into $\mathfrak{g}^*$ defined via the formulas

$$\langle df(g),\eta\rangle=\frac{d}{d\epsilon}f(e^{\epsilon\eta}g)\Big|_{\epsilon=0}\;,\qquad \langle d'f(g),\eta\rangle=\frac{d}{d\epsilon}f(ge^{\epsilon\eta})\Big|_{\epsilon=0}\;,\qquad \forall\eta\in\mathfrak{g}\;,$$

and the gradient $\nabla f(g)\in T^*_gG$ is defined as

$$\nabla f(g)=R^*_{g^{-1}}\,df(g)=L^*_{g^{-1}}\,d'f(g).$$

Then (2.7) may be rewritten as the following system:

$$\left\{ \begin{array}{l} \Pi_k = -\nabla_1 \mathbb{L}(g_k\,, g_{k+1}) \\ \\ \Pi_{k+1} = \nabla_2 \mathbb{L}(g_k\,, g_{k+1}) \end{array} \right. \tag{2.9}$$

This system defines a (multivalued) map $(g_k, \Pi_k) \mapsto (g_{k+1}, \Pi_{k+1})$ of T^*G into itself. More precisely, the first equation in (2.9) is an implicit equation for g_{k+1}, while the second one allows for the explicit and unique calculation of Π_{k+1}, knowing g_k and g_{k+1}. As demonstrated in [V], [MV], this map $T^*G \mapsto T^*G$ is symplectic with respect to the standard symplectic structure on T^*G.

3 Left symmetry reduction

We want to consider the Lagrangian reduction procedure, in the case when the Lagrange function is symmetric with respect to the action of a certain subgroup of G (precise formulations will follow). It turns out to be convenient to perform this reduction in two steps. The first one of them is not related to any symmetry, and is quite general.

3.1 Left trivialization

The tangent bundle TG does not appear in the discrete time context at all. On the contrary, the cotangent bundle T^*G still plays an important role in the discrete time theory, as the phase space with the canonical invariant symplectic structure. When working with the cotangent bundle of the Lie group, it is convenient to trivialize it, translating all covectors to the group unit by left or right multiplication. This subsection is devoted to the constructions related to the left trivialization of the cotangent bundle T^*G:

$$(g_k, M_k) \in G \times \mathfrak{g}^* \ \mapsto \ (g_k, \Pi_k) \in T^*G\;, \tag{3.1}$$

where

$$\Pi_k = L^*_{g_k^{-1}} M_k \quad \Leftrightarrow \quad M_k = L^*_{g_k} \Pi_k\;. \tag{3.2}$$

Consider also the map

$$(g_k, W_k) \in G \times G \ \mapsto \ (g_k, g_{k+1}) \in G \times G\;, \tag{3.3}$$

where

$$g_{k+1} = g_k W_k \quad \Leftrightarrow \quad W_k = g_k^{-1} g_{k+1}\;. \tag{3.4}$$

In the continuous limit the elements W_k lie in a neighborhood of the group unit e.

Denote the pull–back of the Lagrange function under (3.3) through

$$\mathbb{L}^{(l)}(g_k, W_k) = \mathbb{L}(g_k\,, g_{k+1})\;. \tag{3.5}$$

We want to find difference equations satisfied by the sequences $\{(g_k, W_k), k = k_0, \dots, k_1 - 1\}$ delivering extrema of the action functional

$$\mathbb{S}^{(l)} = \sum_{k_0}^{k_1-1} \mathbb{L}^{(l)}(g_k, W_k) \;, \tag{3.6}$$

and satisfying $W_k = g_k^{-1} g_{k+1}$. Admissible variations of $\{(g_k, W_k)\}$ are those preserving the values of g_{k_0} and $g_{k_1} = g_{k_1-1} W_{k_1-1}$. A more explicit description of admissible variations is given by the following statement.

Lemma 3.1 *The original variational problem for the functional* $\mathbb{S}$ (2.6) *is equivalent to finding extremals of the functional* $\mathbb{S}^{(l)}$ (3.6) *with admissible variations* $\{(\tilde{g}_k, \tilde{W}_k)\}$ *of the sequence* $\{(g_k, W_k)\}$ *of the form*

$$\tilde{g}_k = g_k e^{\eta_k} \;, \qquad \tilde{W}_k = W_k e^{\eta_{k+1} - \mathrm{Ad}\, W_k^{-1} \cdot \eta_k} \;, \tag{3.7}$$

where $\{\eta_k\}_{k=k_0}^{k_1}$ *is an arbitrary sequence of elements of the Lie algebra* $\mathfrak{g}$ *with* $\eta_{k_0} = \eta_{k_1} = 0$.

Proof. Obviously, the first formula in (3.7) gives a generic variation of g_k. We have:

$$\tilde{W}_k = \tilde{g}_k^{-1} \tilde{g}_{k+1} = e^{-\eta_k} W_k e^{\eta_{k+1}} = W_k e^{-\mathrm{Ad}\, W_k^{-1} \cdot \eta_k} e^{\eta_{k+1}} \;.$$

Supposing now that all η_k are small (of order ϵ), we find, in the first order in ϵ, the second formula in (3.7). (Clearly, for variational purposes only the first order terms are essntial.) ■

Proposition 3.2 *The difference equations for extremals of the functional* $\mathbb{S}^{(l)}$ *read:*

$$\begin{cases} \mathrm{Ad}^* \, W_k^{-1} \cdot M_{k+1} = M_k + d'_g \mathbb{L}^{(l)}(g_k, W_k) \;, \\ \\ g_{k+1} = g_k W_k \;, \end{cases} \tag{3.8}$$

where

$$M_k = d'_W \mathbb{L}^{(l)}(g_{k-1}, W_{k-1}) \in \mathfrak{g}^* \;. \tag{3.9}$$

If the "Legendre transformation"

$$(g_{k-1}, W_{k-1}) \in G \times G \;\mapsto\; (g_k, M_k) \in G \times \mathfrak{g}^* \;, \tag{3.10}$$

where $g_k = g_{k-1} W_{k-1}$*, is invertible, then* (3.8) *defines a map* $(g_k, M_k) \mapsto (g_{k+1}, M_{k+1})$ *which is symplectic with respect to the following Poisson bracket on* $G \times \mathfrak{g}^*$*:*

$$\{f_1, f_2\} = -\langle d'_g f_1, \nabla_M f_2 \rangle + \langle d'_g f_2, \nabla_M f_1 \rangle + \langle M, [\nabla_M f_1, \nabla_M f_2] \rangle \;. \tag{3.11}$$

The first proof. The simplest way to derive (3.8) is to pull back the equations (2.7) under the map (3.3). To do this, first rewrite (2.7) as

$$d_1'\mathbb{L}(g_k, g_{k+1}) + d_2'\mathbb{L}(g_{k-1}, g_k) = 0 \ . \tag{3.12}$$

We have to express these Lie derivatives in terms of (g, W). The answer is this:

$$d_2'\mathbb{L}(g_{k-1}, g_k) = d_W'\mathbb{L}^{(l)}(g_{k-1}, W_{k-1}) \ , \tag{3.13}$$

$$d_1'\mathbb{L}(g_k, g_{k+1}) = d_g'\mathbb{L}^{(l)}(g_k, W_k) - d_W\mathbb{L}^{(l)}(g_k, W_k) \ . \tag{3.14}$$

Indeed, let us prove, for example, the (less obvious) (3.14). We have:

$$\begin{aligned}
\langle d_1'\mathbb{L}(g_k, g_{k+1}), \eta\rangle &= \frac{d}{d\epsilon}\mathbb{L}(g_k e^{\epsilon\eta}, g_{k+1})\bigg|_{\epsilon=0} = \frac{d}{d\epsilon}\mathbb{L}^{(l)}(g_k e^{\epsilon\eta}, e^{-\epsilon\eta}W_k)\bigg|_{\epsilon=0} \\
&= \langle d_g'\mathbb{L}^{(l)}(g_k, W_k), \eta\rangle - \langle d_W\mathbb{L}^{(l)}(g_k, W_k), \eta\rangle \ .
\end{aligned}$$

It remains to substitute (3.13), (3.14) into (3.12). Taking into account that

$$d_W\mathbb{L}^{(l)}(g_k, W_k) = \mathrm{Ad}^*\, W_k^{-1} \cdot d_W'\mathbb{L}^{(l)}(g_k, W_k) \ ,$$

we find (3.8). Finally, notice that the notation (3.9) is consistent with the definitions (2.8), (3.2). Indeed, from these definitions it follows: $M_k = d_2'\mathbb{L}(g_{k-1}, g_k)$, and the reference to (3.13) proves (3.9). The bracket (3.11) is the pull–back of the standard symplectic bracket on T^*G under the trivialization map (3.2). ■

The second proof. This proof will show us that (3.8) describe extremals (g_k, W_k) of a constrained variational principle, with the functional (3.6), with the admissible variations given in (3.7). Indeed, introducing the small parameter ϵ explicitly, i.e. replacing η_k by $\epsilon\eta_k$, and writing $(g_k(\epsilon), W_k(\epsilon))$ for $(\tilde{g}_k, \tilde{W}_k)$, we are looking for the extremum of the functional

$$\mathbb{S}^{(l)}(\epsilon) = \sum_{k_0}^{k_1-1} \mathbb{L}^{(l)}(g_k(\epsilon), W_k(\epsilon)) \ .$$

Considering the necessary condition $d\mathbb{S}^{(l)}(\epsilon)/d\epsilon|_{\epsilon=0} = 0$, we find:

$$\begin{aligned}
0 &= \sum_k \left\langle d_g'\mathbb{L}^{(l)}(g_k, W_k)\,, \eta_k \right\rangle + \sum_k \left\langle d_W'\mathbb{L}^{(l)}(g_k, W_k)\,, \eta_{k+1} - \mathrm{Ad}\, W_k^{-1} \cdot \eta_k \right\rangle \\
&= \sum_k \left\langle d_g'\mathbb{L}^{(l)}(g_k, W_k) + d_W'\mathbb{L}^{(l)}(g_{k-1}, W_{k-1}) - \mathrm{Ad}^*\, W_k^{-1} \cdot d_W'\mathbb{L}^{(l)}(g_k, W_k), \eta_k \right\rangle \ .
\end{aligned}$$

A reference to the arbitraryness of the sequence $\{\eta_k\}$ finishes the proof. ■

3.2 Reduction of left invariant Lagrangians

Let us describe the context leading to the (discrete) Euler–Poincaré equations.

Let $\Phi : G \times V \mapsto V$ be a representation of a Lie group G in a linear space V; we denote it by

$$\Phi(g) \cdot v \quad \text{for} \quad g \in G\ , \ \ v \in V\ .$$

We denote also by ϕ the corresponding representation of the Lie algebra $\mathfrak{g}$ in V:

$$\phi(\xi) \cdot v = \frac{d}{d\epsilon}\Big(\Phi(e^{\epsilon\xi}) \cdot v\Big)\Big|_{\epsilon=0} \qquad \text{for} \quad \xi \in \mathfrak{g}\ , \ \ v \in V\ . \tag{3.15}$$

The map $\phi^* : \mathfrak{g} \times V^* \mapsto V^*$ defined by

$$\langle \phi^*(\xi) \cdot y, v \rangle = \langle y, \phi(\xi) \cdot v \rangle \qquad \forall v \in V\ ,\ y \in V^*\ ,\ \xi \in \mathfrak{g}\ , \tag{3.16}$$

is an anti–representation of the Lie algebra $\mathfrak{g}$ in V^*. We shall use also the bilinear operation $\diamond : V^* \times V \mapsto \mathfrak{g}^*$ introduced in [HMR, CHMR] and defined as follows: let $v \in V$, $y \in V^*$, then

$$\langle y \diamond v, \xi \rangle = -\langle y, \phi(\xi) \cdot v \rangle \qquad \forall \xi \in \mathfrak{g}\ . \tag{3.17}$$

(Notice that the pairings on the left–hand side and on the right–hand side of the latter equation are defined on different spaces).

Fix an element $a \in V$, and consider the isotropy subgroup $G^{[a]}$ of a, i.e.

$$G^{[a]} = \{h\ :\ \Phi(h) \cdot a = a\} \subset G\ . \tag{3.18}$$

Suppose that the Lagrange function $\mathbb{L}(g, \hat{g})$ is invariant under the action of $G^{[a]}$ on $G \times G$ induced by *left* translations on G:

$$\mathbb{L}(hg, h\hat{g}) = \mathbb{L}(g, \hat{g})\ , \quad h \in G^{[a]}\ . \tag{3.19}$$

The corresponding invariance property of $\mathbb{L}^{(l)}(g, W)$ is expressed as:

$$\mathbb{L}^{(l)}(hg, W) = \mathbb{L}^{(l)}(g, W)\ , \quad h \in G^{[a]}\ . \tag{3.20}$$

We want to reduce the Euler–Lagrange equations with respect to this left action. As a section $(G \times G)/G^{[a]}$ we choose the set $G \times O_a$, where O_a is the orbit of a under the action Φ:

$$O_a = \{\Phi(g) \cdot a\ ,\ g \in G\} \subset V\ . \tag{3.21}$$

The reduction map is

$$(g, W) \in G \times G\ \mapsto\ (W, P) \in G \times O_a\ , \qquad \text{where} \qquad P = \Phi(g^{-1}) \cdot a\ , \tag{3.22}$$

so that the reduced Lagrange function $\Lambda^{(l)}\ :\ G \times O_a \mapsto \mathbb{R}$ is defined as

$$\Lambda^{(l)}(W, P) = \mathbb{L}^{(l)}(g, W)\ , \quad \text{where} \quad P = \Phi(g^{-1}) \cdot a\ . \tag{3.23}$$

The reduced Lagrangian $\Lambda^{(l)}(W,P)$ is well defined, because from

$$P=\Phi(g_1^{-1})\cdot a=\Phi(g_2^{-1})\cdot a$$

there follows $\Phi(g_2g_1^{-1})\cdot a=a$, so that $g_2g_1^{-1}\in G^{[a]}$, and $\mathbb{L}^{(l)}(g_1,W)=\mathbb{L}^{(l)}(g_2,W)$.

Theorem 3.3 a) *Consider the reduction $(g,W)\mapsto(W,P)$. The reduced Euler–Lagrange equations* (3.8) *become the following* **discrete Euler–Poincaré equations***:*

$$\begin{cases} \mathrm{Ad}^*\,W_k^{-1}\cdot M_{k+1}=M_k+\nabla_P\Lambda^{(l)}(W_k,P_k)\diamond P_k\ , \\ \\ P_{k+1}=\Phi(W_k^{-1})\cdot P_k\ , \end{cases} \tag{3.24}$$

where

$$M_k=d'_W\Lambda^{(l)}(W_{k-1},P_{k-1})\in\mathfrak{g}^*\ . \tag{3.25}$$

They describe extremals of the constrained variational principle, with the functional

$$S^{(l)}=\sum_{k_0}^{k_1-1}\Lambda^{(l)}(W_k,P_k)\ , \tag{3.26}$$

and the admissible variations $\{(\widetilde{W}_k,\widetilde{P}_k)\}$ of $\{(W_k,P_k)\}$ of the form

$$\widetilde{W}_k=W_ke^{\eta_{k+1}-\mathrm{Ad}\,W_k^{-1}\cdot\eta_k}\ ,\qquad \widetilde{P}_k=P_k-\phi(\eta_k)\cdot P_k\ , \tag{3.27}$$

where $\{\eta_k\}_{k=k_0}^{k_1}$ is an arbitrary sequence of elements of the Lie algebra $\mathfrak{g}$ with $\eta_{k_0}=\eta_{k_1}=0$.

b) *If the "Legendre transformation"*

$$(W_{k-1},P_{k-1})\in G\times O_a\mapsto(M_k,P_k)\in\mathfrak{g}^*\times O_a\ , \tag{3.28}$$

where $P_k=\Phi(W_{k-1}^{-1})\cdot P_{k-1}$, is invertible, then (3.24) *define a map $(M_k,P_k)\mapsto(M_{k+1},P_{k+1})$ of $\mathfrak{g}^*\times O_a$ which is Poisson with respect to the Poisson bracket*

$$\{F_1,F_2\}=\langle M,[\nabla_MF_1,\nabla_MF_2]\,\rangle+\langle\nabla_PF_1,\phi(\nabla_MF_2)\cdot P\,\rangle-\langle\nabla_PF_2,\phi(\nabla_MF_1)\cdot P\,\rangle \tag{3.29}$$

for two arbitrary functions $F_{1,2}(M,P):\mathfrak{g}^*\times O_a\mapsto\mathbb{R}$.

Proof is a consequence of the following formula: if $f:G\mapsto\mathbb{R}$ is a pull–back of the function $F:O_a\mapsto\mathbb{R}$, i.e.

$$f(g)=F(P)=F(\Phi(g^{-1})\cdot a)\ ,$$

then

$$d'f(g)=\nabla_PF(P)\diamond P\ . \tag{3.30}$$

(Indeed,

$$\langle d'f(g),\xi\rangle = \frac{d}{d\epsilon}f(ge^{\epsilon\xi})\bigg|_{\epsilon=0} = \frac{d}{d\epsilon}F(\Phi(e^{-\epsilon\xi})\cdot P)\bigg|_{\epsilon=0} = -\langle\nabla_P F(P),\phi(\xi)\cdot P\rangle = \langle\nabla_P F(P)\diamond P,\xi\rangle\ ;$$

the last equality is the definition (3.17)). In particular, plugging

$$d'_g\mathbb{L}^{(l)} = \nabla_P\Lambda^{(l)}\diamond P\ , \qquad d'_W\mathbb{L}^{(l)} = d'_W\Lambda^{(l)}$$

into the first equation in (3.8), we come to the first equation in (3.24). Similarly, the Poisson bracket (3.11) turns into (3.29) by use of (3.30). ■

Remark 1. The formula (3.29) defines a Poisson bracket not only on $\mathfrak{g}^*\times O_a$, but on all of $\mathfrak{g}^*\times V$. Rewriting this formula as

$$\{F_1,F_2\} = \langle M,[\nabla_M F_1,\nabla_M F_2]\,\rangle + \langle P,\phi^*(\nabla_M F_2)\cdot\nabla_P F_1 - \phi^*(\nabla_M F_1)\cdot\nabla_P F_2\,\rangle \tag{3.31}$$

one immediately identifies this bracket with the Lie–Poisson bracket of the semiproduct Lie algebra $\mathfrak{g}\ltimes V^*$ corresponding to the representation $-\phi^*$ of $\mathfrak{g}$ in V^*.

Remark 2. In an important particular case of constructions of this section, the vector space is chosen as the Lie algebra of our basic Lie group: $V=\mathfrak{g}$, the group representation is the adjoint one: $\Phi(g)\cdot v = \mathrm{Ad}\,\mathrm{g}\cdot v$, so that $\phi(\xi)\cdot v = \mathrm{ad}\,\xi\cdot v = [\xi,v]$, and the bilinear operation $\diamond$ is nothing but the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$: $y\diamond v = \mathrm{ad}^*\,v\cdot y$. This is the framework, e.g., for the heavy top mechanics. The model studied in [BS] belongs to this class.

4 Right symmetry reduction

All constructions here are parallel to those of the previous section, so we restrict ourselves to formulations of the basic results only.

4.1 Right trivialization

Consider the right trivialization of the cotangent bundle T^*G:

$$(g_k,m_k)\in G\times\mathfrak{g}^*\ \mapsto\ (g_k,\Pi_k)\in T^*G\ , \tag{4.1}$$

where

$$\Pi_k = R^*_{g_k^{-1}}m_k \quad\Leftrightarrow\quad m_k = R^*_{g_k}\Pi_k\ . \tag{4.2}$$

Consider also the map

$$(g_k,w_k)\in G\times G\ \mapsto\ (g_k,g_{k+1})\in G\times G\ , \tag{4.3}$$

where

$$g_{k+1} = w_k g_k \quad \Leftrightarrow \quad w_k = g_{k+1} g_k^{-1} \ . \tag{4.4}$$

Denote the pull–back of the Lagrange function under (4.3) through

$$\mathbb{L}^{(r)}(g_k, w_k) = \mathbb{L}(g_k\,, g_{k+1}) \ . \tag{4.5}$$

Proposition 4.1 *Consider the functional*

$$\mathbb{S}^{(r)} = \sum_{k_0}^{k_1-1} \mathbb{L}^{(r)}(g_k, w_k) \ ,$$

and its extremals with respect to variations $\{(\tilde{g}_k, \tilde{w}_k)\}$ of $\{(g_k, w_k)\}$ of the form

$$\tilde{g} {=} e^{\eta_k} g_k \ , \qquad \tilde{w}_k = e^{\eta_{k+1} - \mathrm{Ad}\, w_k \cdot \eta_k} w_k \ ,$$

where $\{\eta_k\}_{k=k_0}^{k_1}$ is an arbitrary sequence of elements of $\mathfrak{g}$ with $\eta_{k_0} = \eta_{k_1} = 0$. The difference equations for extremals of this constrained variational problem read:

$$\left\{ \begin{array}{l} \mathrm{Ad}^* \, w_k \cdot m_{k+1} = m_k + d_g \mathbb{L}^{(r)}(g_k, w_k) \ , \\ \\ g_{k+1} = w_k g_k \ , \end{array} \right. \tag{4.6}$$

where

$$m_k = d_w \mathbb{L}^{(r)}(g_{k-1}, w_{k-1}) \in \mathfrak{g}^* \ . \tag{4.7}$$

If the "Legendre transformation"

$$(g_{k-1}, w_{k-1}) \in G \times G \mapsto (g_k, m_k) \in G \times \mathfrak{g}^* \ , \tag{4.8}$$

where $g_k = w_{k-1} g_{k-1}$, is invertible, then (4.6) *define a map $(g_k, m_k) \mapsto (g_{k+1}, m_{k+1})$ which is symplectic with respect to the following Poisson bracket on $G \times \mathfrak{g}^*$:*

$$\{f_1, f_2\} = -\langle d_g f_1, \nabla_m f_2 \rangle + \langle d_g f_2, \nabla_m f_1 \rangle - \langle m, [\nabla_m f_1, \nabla_m f_2] \rangle \ . \tag{4.9}$$

Proof. This time the discrete Euler–Lagrange equations (2.7) are rewritten as

$$d_1 \mathbb{L}(g_k\,, g_{k+1}) + d_2 \mathbb{L}(g_{k-1}\,, g_k) = 0 \ , \tag{4.10}$$

and the expressions for these Lie derivatives in terms of (g, w) read:

$$d_2 \mathbb{L}(g_{k-1}\,, g_k) = d_w \mathbb{L}^{(r)}(g_{k-1}, w_{k-1}) \ , \tag{4.11}$$

$$d_1 \mathbb{L}(g_k\,, g_{k+1}) = d_g \mathbb{L}^{(r)}(g_k, w_k) - d_w' \mathbb{L}^{(r)}(g_k, w_k) = d_g \mathbb{L}^{(r)}(g_k, w_k) - \mathrm{Ad}^* \, w_k \cdot d_w \mathbb{L}^{(r)}(g_k, w_k) \ . \tag{4.12}$$

Finally, the expression (4.7) is consistent with the definitions (2.8), (4.2), which imply that $m_k = d_2 \mathbb{L}(g_{k-1}\,, g_k)$, and a reference to (4.11) finishes the proof. ■

4.2 Reduction of right invariant Lagrangians

Assume that the function $\mathbb{L}^{(r)}$ is invariant under the action of $G^{[a]}$ on $G \times G$ induced by *right* translations on G:

$$\mathbb{L}^{(r)}(gh, w) = \mathbb{L}^{(r)}(g, w) \;, \quad h \in G^{[a]} \;. \tag{4.13}$$

Define the reduced Lagrange function $\Lambda^{(r)} : G \times O_a \mapsto \mathbb{R}$ as

$$\Lambda^{(r)}(w, p) = \mathbb{L}^{(r)}(g, w) \;, \quad \text{where} \quad p = \Phi(g) \cdot a \;. \tag{4.14}$$

Theorem 4.2 a) *Consider the reduction $(g, w) \mapsto (w, p)$. The reduced Euler–Lagrange equations* (4.6) *become the following* **discrete Euler–Poincaré equations***:*

$$\left\{ \begin{array}{l} \mathrm{Ad}^* \, w_k \cdot m_{k+1} = m_k - \nabla_p \Lambda^{(r)}(w_k, p_k) \diamond p_k \;, \\[2ex] p_{k+1} = \Phi(w_k) \cdot p_k \;, \end{array} \right. \tag{4.15}$$

where

$$m_k = d_w \Lambda^{(r)}(w_{k-1}, p_{k-1}) \in \mathfrak{g}^* \;. \tag{4.16}$$

They describe extremals of the constrained variational principle, with the functional

$$S^{(r)} = \sum_{k_0}^{k_1 - 1} \Lambda^{(r)}(w_k, p_k) \;, \tag{4.17}$$

and the admissible variations $\{(\widetilde{w}_k, \widetilde{p}_k)\}$ of $\{(w_k, p_k)\}$ of the form

$$\widetilde{w}_k = e^{\eta_{k+1} - \mathrm{Ad}\, w_k \cdot \eta_k} w_k \;, \qquad \widetilde{p}_k = p_k + \phi(\eta_k) \cdot p_k \;, \tag{4.18}$$

where $\{\eta_k\}_{k=k_0}^{k_1}$ is an arbitrary sequence of elements of the Lie algebra $\mathfrak{g}$ with $\eta_{k_0} = \eta_{k_1} = 0$.

b) *If the "Legendre transformation"*

$$(w_{k-1}, p_{k-1}) \in G \times O_a \mapsto (m_k, p_k) \in \mathfrak{g}^* \times O_a \;, \tag{4.19}$$

where $p_k = \Phi(w_{k-1}) \cdot p_{k-1}$, is invertible, then (4.15) *define a map $(m_k, p_k) \mapsto (m_{k+1}, p_{k+1})$ of $\mathfrak{g}^* \times O_a$ which is Poisson with respect to the bracket*

$$\{F_1, F_2\} = -\langle m, [\nabla_m F_1, \nabla_m F_2] \rangle - \langle \nabla_p F_1, \phi(\nabla_m F_2) \cdot p \rangle + \langle \nabla_p F_2, \phi(\nabla_m F_1) \cdot p \rangle \tag{4.20}$$

for two arbitrary functions $F_{1,2}(m, p) : \mathfrak{g}^* \times O_a \mapsto \mathbb{R}$. This formula indeed defines a Poisson bracket on all of $\mathfrak{g}^* \times V$, the Lie–Poisson bracket of the semiproduct Lie algebra $\mathfrak{g} \ltimes V^*$ corresponding to the representation $-\phi^*$ of $\mathfrak{g}$ in V^*.

Proof is based on the following simple result: if $f(g) = F(\Phi(g) \cdot a)$, then

$$df(g) = -\nabla_p F(p) \diamond p \ .$$

Notice that the brackets (3.29) and (4.20) essentially coincide (differ only by a sign). ■

A table summarizing the unreduced and reduced Lagrangian equations of motion, both in the continuous and discrete time formulations, is put in Appendix A. The continuous time results were obtained in [HMR],[CHMR]; they also may be derived by taking a continuous limit of our present results.

5 Conclusion

We consider the discrete time Lagrangian mechanics on Lie groups as an important source of symplectic and, more general, Poisson maps. Moreover, from some points of view the variational (Lagrangian) structure is even more fundamental and important than the Poisson (Hamiltonian) one (cf. [HMR], [MPS], where a similar viewpoint is represented). In particular, discrete Lagrangians on $G \times G$ may serve as models for the rigid body motion (cf. [WM]). The integrable cases of Euler (a free rotation of a rigid body fixed at the center of mass) and of Lagrange (symmetric spinning top) are discretized in this framework preserving the integrability property in [V],[MV] and in [BS], respectively. It would be interesting and important to apply the above theory to the infinite dimensional case, e.g. to discretization of ideal compressible fluids motion (see [HMR]).

A Euler–Lagrange and Euler–Poincaré equations

CONTINUOUS TIME	DISCRETE TIME
General Lagrangian systems	
$\mathbf{L}(g, \dot{g})$ $\begin{cases} \Pi = \nabla_{\dot{g}} \mathbf{L} \\ \dot{\Pi} = \nabla_g \mathbf{L} \end{cases}$	$\mathbb{L}(g_k, g_{k+1})$ $\begin{cases} \Pi_k = -\nabla_1 \mathbb{L}(g_k, g_{k+1}) \\ \Pi_{k+1} = \nabla_2 \mathbb{L}(g_k, g_{k+1}) \end{cases}$

Left trivialization, left symmetry reduction: $M = L_g^*\Pi\,,\ \ P = \Phi(g^{-1})\cdot a$	
$\mathbf{L}(g,\dot g) = \mathcal{L}^{(l)}(\Omega,P)$ $\Omega = L_{g^{-1}*}\dot g\,,\ \ P = \Phi(g^{-1})\cdot a$ $M = L_g^*\Pi = \nabla_\Omega \mathcal{L}^{(l)}$ $\begin{cases} \dot M = \mathrm{ad}^*\,\Omega\cdot M + \nabla_P\mathcal{L}^{(l)}\diamond P \\ \dot P = -\phi(\Omega)\cdot P \end{cases}$	$\mathbb{L}(g_k, g_{k+1}) = \Lambda^{(l)}(W_k,P_k)$ $W_k = g_k^{-1}g_{k+1}\,,\ \ P_k = \Phi(g_k^{-1})\cdot a$ $M_k = L_{g_k}^*\Pi_k = d'_W\Lambda^{(l)}(W_{k-1},P_{k-1})$ $\begin{cases} \mathrm{Ad}^*\,W_k^{-1}\cdot M_{k+1} = M_k + \nabla_P\Lambda^{(l)}(W_k,P_k)\diamond P_k \\ P_{k+1} = \Phi(W_k^{-1})\cdot P_k \end{cases}$
Right trivialization, right symmetry reduction: $m = R_g^*\Pi\,,\ \ p = \Phi(g)\cdot a$	
$\mathbf{L}(g,\dot g) = \mathcal{L}^{(r)}(\omega,p)$ $\omega = R_{g^{-1}*}\dot g\,,\ \ p = \Phi(g)\cdot a$ $m = R_g^*\Pi = \nabla_\omega \mathcal{L}^{(r)}$ $\begin{cases} \dot m = -\mathrm{ad}^*\,\omega\cdot m - \nabla_p\mathcal{L}^{(r)}\diamond p \\ \dot p = \phi(\omega)\cdot p \end{cases}$	$\mathbb{L}(g_k, g_{k+1}) = \Lambda^{(r)}(w_k,p_k)$ $w_k = g_{k+1}g_k^{-1}\,,\ \ p_k = \Phi(g_k)\cdot a$ $m_k = R_{g_k}^*\Pi_k = d_w\Lambda^{(r)}(w_{k-1},p_{k-1})$ $\begin{cases} \mathrm{Ad}^*\,w_k\cdot m_{k+1} = m_k - \nabla_p\Lambda^{(r)}(w_k,p_k)\diamond p_k \\ p_{k+1} = \Phi(w_k)\cdot p_k \end{cases}$

The relation between the continuous time and the discrete time equations is established, if we set

$$g_k = g\ , \qquad g_{k+1} = g + \varepsilon\dot g + O(\varepsilon^2)\ , \qquad \mathbb{L}(g_k, g_{k+1}) = \varepsilon\mathbf{L}(g,\dot g) + O(\varepsilon^2)\ ;$$
$$P_k = P\ , \qquad W_k = \mathbf{1} + \varepsilon\Omega + O(\varepsilon^2)\ , \qquad \Lambda^{(l)}(W_k,P_k) = \varepsilon\mathcal{L}^{(l)}(\Omega,P) + O(\varepsilon^2)\ ;$$
$$p_k = p\ , \qquad w_k = \mathbf{1} + \varepsilon\omega + O(\varepsilon^2)\ , \qquad \Lambda^{(r)}(w_k,p_k) = \varepsilon\mathcal{L}^{(r)}(\omega,p) + O(\varepsilon^2)\ .$$

B Notations

We fix here some notations and definitions used throughout the paper.

Let G be a Lie group with the Lie algebra $\mathfrak{g}$, and let $\mathfrak{g}^*$ be a dual vector space to $\mathfrak{g}$. We identify $\mathfrak{g}$ and $\mathfrak{g}^*$ with the tangent space and the cotangent space to G in the group unit, respectively:

$$\mathfrak{g} = T_eG\ , \qquad \mathfrak{g}^* = T_e^*G\ .$$

The pairing between the cotangent and the tangent spaces T_g^*G and T_gG in an arbitrary point $g\in G$ is denoted by $\langle\cdot,\cdot\rangle$. The left and right translations in the group are the maps $L_g\,, R_g\ : G\mapsto G$ defined by

$$L_g\,h = gh\ , \qquad R_g\,h = hg \qquad \forall h\in G\ ,$$

and L_{g*}, R_{g*} stand for the differentials of these maps:

$$L_{g*}\ :\ T_hG \mapsto T_{gh}G\ , \qquad R_{g*}\ :\ T_hG \mapsto T_{hg}G\ .$$

We denote by

$$\mathrm{Ad}\, g = L_{g*}R_{g^{-1}*}\ :\mathfrak{g} \mapsto \mathfrak{g}$$

the adjoint action of the Lie group G on its Lie algebra $\mathfrak{g} = T_eG$. The linear operators

$$L_g^*\ :\ T_{gh}^*G \mapsto T_h^*G\ , \qquad R_g^*\ :\ T_{hg}^*G \mapsto T_h^*G$$

are conjugated to L_{g*}, R_{g*}, respectively, via the pairing $\langle\cdot,\cdot\rangle$:

$$\langle L_g^*\xi,\eta\rangle = \langle\xi, L_{g*}\eta\rangle \quad \text{for} \quad \xi\in T_{gh}^*G\ ,\ \ \eta\in T_hG\ ,$$

$$\langle R_g^*\xi,\eta\rangle = \langle\xi, R_{g*}\eta\rangle \quad \text{for} \quad \xi\in T_{hg}^*G\ ,\ \ \eta\in T_hG\ .$$

The coadjoint action of the group

$$\mathrm{Ad}^*\, g = L_g^*R_{g^{-1}}^*\ :\mathfrak{g}^* \mapsto \mathfrak{g}^*$$

is conjugated to $\mathrm{Ad}\, g$ via the pairing $\langle\cdot,\cdot\rangle$:

$$\langle \mathrm{Ad}^*\, g\cdot\xi,\eta\rangle = \langle\xi, \mathrm{Ad}\, g\cdot\eta\rangle \quad \text{for} \quad \xi\in\mathfrak{g}^*\ ,\ \ \eta\in\mathfrak{g}\ .$$

The differentials of $\mathrm{Ad}\, g$ and of $\mathrm{Ad}^*\, g$ with respect to g in the group unity e are the operators

$$\mathrm{ad}\,\eta\ :\mathfrak{g}\mapsto\mathfrak{g} \qquad \text{and} \qquad \mathrm{ad}^*\,\eta\ :\mathfrak{g}^*\mapsto\mathfrak{g}^*\ ,$$

respectively, also conjugated via the pairing $\langle\cdot,\cdot\rangle$:

$$\langle \mathrm{ad}^*\,\eta\cdot\xi,\zeta\rangle = \langle\xi, \mathrm{ad}\,\eta\cdot\zeta\rangle \qquad \forall\xi\in\mathfrak{g}^*\ ,\ \ \zeta\in\mathfrak{g}\ .$$

The action of ad is given by applying the Lie bracket in $\mathfrak{g}$:

$$\mathrm{ad}\,\eta\cdot\zeta = [\eta,\zeta]\ , \quad \forall\zeta\in\mathfrak{g}\ .$$

Finally, we need the notion of gradients of functions on vector spaces and on manifolds. If $\mathcal{X}$ is a vector space, and $f:\mathcal{X}\mapsto\mathbb{R}$ is a smooth function, then the gradient $\nabla f:\mathcal{X}\mapsto\mathcal{X}^*$ is defined via the formula

$$\langle\nabla f(x), y\rangle = \frac{d}{d\epsilon}\, f(x+\epsilon y)\bigg|_{\epsilon=0}\ , \qquad \forall y\in\mathcal{X}\ .$$

Similarly, for a function $f: G\mapsto\mathbb{R}$ on a smooth manifold G its gradient $\nabla f\ :\ G\mapsto T^*G$ is defined in the following way: for an arbitrary $\dot g\in T_gG$ let $g(\epsilon)$ be a curve in G through $g(0)=g$ with the tangent vector $\dot g(0)=\dot g$. Then

$$\langle\nabla f(g),\dot g\rangle = \frac{d}{d\epsilon}\, f(g(\epsilon))\bigg|_{\epsilon=0}\ .$$

If G is a Lie group, then two convenient ways to define a curve in G through g with the tangent vector $\dot{g}$ are the following:

$$g(\epsilon) = e^{\epsilon\eta} g \ , \quad \eta = R_{g^{-1}*}\, \dot{g} \ ,$$

and

$$g(\epsilon) = g e^{\epsilon\eta} \ , \quad \eta = L_{g^{-1}*}\, \dot{g} \ ,$$

which allows to establish the connection of the gradient ∇f with the (somewhat more convenient) notions of the left and the right Lie derivatives of a function $f : G \mapsto \mathbb{R}$:

$$\nabla f(g) = R_{g^{-1}}^{*}\, df(g) = L_{g^{-1}}^{*}\, d'f(g) \ .$$

Here $df : G \mapsto \mathfrak{g}^*$ and $d'f : G \mapsto \mathfrak{g}^*$ are defined via the formulas

$$\langle df(g), \eta \rangle = \frac{d}{d\epsilon} f(e^{\epsilon\eta} g) \Big|_{\epsilon=0} \ , \qquad \forall \eta \in \mathfrak{g} \ ,$$

$$\langle d'f(g), \eta \rangle = \frac{d}{d\epsilon} f(g e^{\epsilon\eta}) \Big|_{\epsilon=0} \ , \qquad \forall \eta \in \mathfrak{g} \ .$$

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