The spectral gap for the Kawasaki dynamics at low temperature

Revised version

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Abstract. In this paper we analyze the convergence to equilibrium of Kawasaki dynamics for the Ising model in the phase coexistence region. First we show, in strict analogy with the non–conservative case that in any lattice dimension, for any boundary condition, any positive temperature and particle density, the spectral gap in a box of side L does not shrink faster than a negative exponential of the surface L^{d-1}. Then we prove that, in two dimensions and free boundary condition, the spectral gap in a box of side L is smaller than a negative exponential of L provided that the temperature is below the critical one and the particle density ρ satisfies $\rho \in (\rho^*_-, \rho^*_+)$, where $\rho^*_\pm$ are the particle density of the plus and minus phase respectively.

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1. Introduction

In this paper we analyze in some detail the dependence on the volume of the spectral gap of the generator of the usual Kawasaki dynamics (*i.e.* a nearest neighbors spin–exchange Markov process on the spin configuration space, reversible w.r.t. to the canonical Gibbs measure) for the standard nearest neighbor ferromagnetic Ising model in the phase coexistence region.

As it is well known, the conservation of the particle number (in the lattice gas picture, or of the magnetization in the usual ± 1 spin variables) makes the analysis of the relaxational properties of conservative dynamics much more difficult then in the non–conservative case of Glauber dynamics, even for very high temperature. For Glauber dynamics the general picture is relatively clear for a wide class of models both in the one phase and in the phase coexistence region with the notable exception of the critical point (see e.g. [M1] and references therein). In particular, for the two dimensional Ising case with zero external field, the spectral gap of a Glauber dynamics does not go to zero in the thermodynamic limit for any temperature above the critical one, while below the critical temperature the spectral gap in a box of side L and free boundary conditions becomes exponentially small in L with a precise rate related to the surface tension.

In the conservative case, instead, the basic results [LY], [Y] on the spectral gap and logarithmic Sobolev inequality of Kawasaki dynamics are restricted to the one phase region and state that, under a suitable mixing condition on the grandcanonical Gibbs measure which for the two dimensional Ising model holds for any temperature above the critical one, the spectral gap in a box of side L shrinks like L^{-2}. The proof of such interesting and far–reaching result required the development of a rather sophisticated and intricate technology and posed new, non trivial problems on the theory of canonical Gibbs measure and their equivalence to grandcanonical ones.

It seems therefore natural to ask whether, at least for a simple case like the two dimensional Ising model, one could obtain upper and lower bounds on the spectral gap of Kawasaki dynamics also in the phase coexistence region. Here we provide a first partial answer to the above question by proving the following two results.

First we show, in strict analogy with the non–conservative case (see lemma in [CMM]), that in any dimension, for any boundary condition, any positive temperature and particle density, the spectral gap in a box of side L does not shrinks faster than $\exp(-\alpha_1 L^{d-1})$. Then we prove that, in two dimensions and free b.c., the spectral gap in a box of side L is smaller than $\exp(-\alpha_2 L)$ provided that the temperature is below the critical one and the particle density ρ is satisfies $\rho \in (\rho^*_-, \rho^*_+)$, where $\rho^*_\pm = \frac{(1 \pm m^*)}{2}$ and m^* is the usual spontaneous magnetization at inverse temperature β. It is important to point out that in this case, contrary to the non conservative case, by no means our proof provides a good bound on the rate α_2 appearing in the exponential. Actually we do not even have a good guess for such a quantity and we think it would be quite nice to be able to compute it. Let us now explain in simple terms the strategy behind the proof of these results.

In order to prove the first lower bound, namely $\mathrm{gap}(Q_L) \geq C e^{-kL^{d-1}}$, we proceed recursively and prove that

$$\mathrm{gap}(Q_{2L}) \geq e^{-kL^{d-1}}\,\mathrm{gap}(Q_L) \tag{1.1}$$

Such an inequality is rather straightforward in the Glauber case. In this case, in fact, the grand-canonical measure in Q_{2L} has a density w.r.t. the product of two grandcanonical measures in each half of Q_{2L} which is bounded from above and from below by an exponential of the interaction through the interface separating the two halves. In other words, by paying a price not larger than $e^{kL^{d-1}}$, one can compare the the spectral gap of the Glauber dynamics in Q_{2L} with that of a product Glauber dynamics in each half of Q_{2L} and easily get (1.1). For a conservative dynamics like Kawasaki dynamics such a reasoning does not apply because the conservation of the number of particles introduces a global constraint in the system and even at infinite temperature the dynamics does not factorize into independent components. If however we fix in addition the number of particles in each halves of Q_{2L}, *i.e.* we consider a multicanonical Gibbs measure, then we can proceed exactly as in the non conservative case and succesfully compare the dynamics in Q_{2L} with a product dynamics in each of the two halves. Technically this is done via the formula of the conditional variance

$$\mathrm{Var}(f) = E\big(\mathrm{Var}(f|n)\big) + \mathrm{Var}\big(E(f|n)\big) \tag{1.2}$$

where $\mathrm{Var}(\cdot)$ and $E(\cdot)$ denote respectively the variance and expectation w.r.t. the canonical Gibbs measure on Q_{2L}, and n denotes the number of particles in e.g. the upper half of Q_{2L}. In order to control the effects produced by the above extra conservation law, *i.e.* to be able to bound the last term in the r.h.s. of (1.2), one is led naturally to study the distribution of the number of particles in half cube under the canonical Gibbs distribution in Q_{2L} and in particular to prove a Poincaré inequality for it. This turned out to be an interesting problem which is fully discussed in Section 4.

It is worthwhile to mention that the above simple strategy to prove a lower bound on the spectral gap gives the correct L^{-2} scaling at infinite temperature (simple exclusion model) even at the level of the logarithmic Sobolev constant and it looks quite promising also to treat the high temperature regime [M2].

Let us now turn to our second result. Here the goal is to prove an upper bound on the spectral gap which is exponentially small in L and thus it is natural follow the "look for a bottleneck" approach *i.e.* to look for a suitable trial function with a very small Dirichlet form and a comparatively large variance to plug into the variational characterization of the gap (2.14). Let us denote by N the total number of particles in a square Q_L of side L and let us assume that $\rho \in (\rho^*_-, \rho^*_+)$ where $\rho = \frac{N}{L^2}$ and $\rho^*_\pm$ have been defined above. Then our choice is to take as test function the characteristic function of the event that the number of particles in the set U described in Fig. 1 (Section 5) is less than $\frac{N}{2}$. In order to explain such apparently weird choice, it is useful first to recall the shape of the typical configurations of the canonical Ising Gibbs measure in a square Q_L with N particles and free b.c. when the temperature is below the critical value.

Let $m_\rho = 2\rho - 1$ be the usual magnetization associated to the given particle density. Then, as discussed in [Sh] (see also [CGMS] and Section 5 below), there exists $0 < m_1 < m^*$ such that

(i) if $m_\rho \in (-m_1, m_1)$ then the typical configurations show phase segregation between a high/low density ($\approx \rho^*_\pm$) regions that are roughly two horizontal (vertical) rectangles of appropriate area separated by an horizontal (vertical) interface of length L.

(ii) if $m_\rho \in (-m^*, m^*)\backslash(-m_1, m_1)$ then the typical configurations show phase segregation between a high/low density ($\approx \rho^*_\pm$), regions one of which is a quarter of a Wulff shape (see [DKS]) of appropriate area and centered in one of the four vertices of Q_L.

What is important for us is that in both cases the typical configurations of the canonical measure show a discrete symmetry described by rotations of $k\frac{\pi}{2}$, $k = 0, 1\dots$ around the center of Q_L and that the critical value m_1 is such that for each typical configuration the particle density in the set U, ρ_U, is either below or above ρ. In particular, if the dynamics starts from one typical configuration, for which e.g. $\rho_U < \rho$, then, in order to relax to equilibrium, it must necessarily cross the unlikely region in the configuration space in which $\rho_U = \rho$. Thus in order to conclude the argument it is sufficient to show that the canonical probability of seeing $\rho_U = \rho$ is exponentially small in L. Such bound is proved in Sections 5, 6 and 7 for any temperature below the critical one and any $\rho \in (\rho^*_-, \rho^*_+)$. Its proof, which is unfortunately rather technical, requires in particular a rather sharp lower bound on the Ising grandcanonical probability of having exactly N particles in Q_L (see Section 7). We have been able to obtain such a bound by adapting and partially extending to free boundary condition the recent techniques introduced in [IS] where plus b.c. are treated.

We conclude by observing that the above discrete symmetry of the typical configurations is peculiar of free b.c. If instead one works with e.g plus b.c., then the typical configurations for the canonical measure when m_ρ is slightly below m^* consist of a low density Wulff bubble centered somewhere in the bulk of Q_L, immersed in a sea of high density. In this case we suspect that the continuous degeneracy of the typical configurations caused by the arbitrariness of the location of the center of the Wulff bubble prevents the spectral gap to be exponentially small in L. In particular one may argue that the slowest mode of the system is due to the random walk motion of the center of gravity of the unique Wulff bubble. If that is true then simple heuristics shows that the spectral gap should shrink as L^{-3}. Although at the moment we cannot prove a lower bound of this type, it is not too difficult to show that gap $\leq$ const L^{-3}. This inequality is obtained by plugging into the variational characterization of the spectral gap (2.14) a slowly (on scale L) varying function of the center of gravity of the Wulff bubble. Such a choice of the test function is dictated by the above heuristics.

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2. Notation and results

The lattice. We consider the d dimensional lattice $\mathbb{Z}^d$ with *sites* $x = (x_1, \dots, x_d)$ and norms

$$|x|_p = (\sum_{i=1}^{d} |x_i|^p)^{1/p} \quad p \geq 1 \quad \text{and} \qquad |x| = |x|_\infty = \max_{i \in \{1,\dots,d\}} |x_i| \,.$$

The associated distance functions are denoted by $d_p(\cdot,\cdot)$ and $d(\cdot,\cdot)$. By Q_L we denote the cube of all $x = (x_1, \dots, x_d) \in \mathbb{Z}^d$ such that $x_i \in \{0, \dots, L-1\}$. If $x \in \mathbb{Z}^d$, $Q_L(x)$ stands for $Q_L + x$. We also let B_L be the ball (w.r.t $d(\cdot,\cdot)$) of radius L centered at the origin, *i.e.* $B_L = Q_{2L+1}((-L, \dots, -L))$. If Λ is a finite subset of $\mathbb{Z}^d$ we write $\Lambda \subset\subset \mathbb{Z}^d$. The cardinality of Λ is denoted by $|\Lambda|$. $\mathbb{F}$ is the set of all nonempty finite subsets of $\mathbb{Z}^d$.

$[x, y]$ is the *closed segment* with endpoints x and y The *edges* of $\mathbb{Z}^d$ are those $e = [x, y]$ with x, y nearest neighbors in $\mathbb{Z}^d$. The *boundary of an edge* $e = [x, y]$ is $\delta e = \{x, y\}$ The *boundary of a set of edges* α is the set $\delta\alpha$ of all sites that belong to an odd number of edges of α. A set of edges is called *closed* if its boundary is empty. We denote by $\mathcal{E}_\Lambda$ the set of all edges such that both endpoints are in Λ and by $\bar{\mathcal{E}}_\Lambda$ the set of all edges with at least one endpoint in Λ. Viceversa, for a set of edges X, $\mathcal{V}(X)$ stands for the set of all sites which are endpoints of at least one edge in X. When $d = 2$ we consider also the dual lattice $\mathbb{Z}^2_* = \mathbb{Z}^2 + (1/2, 1/2)$. Given an edge e of $\mathbb{Z}^2$ ($\mathbb{Z}^2_*$), e^* is the unique edge in $\mathbb{Z}^2_*$ ($\mathbb{Z}^2$) that intersects e. Given $\Lambda \subset \mathbb{Z}^2$ we let Λ^* as the set of all $x \in \mathbb{Z}^2_*$ such that $d_2(x, \Lambda) = 1/\sqrt{2}$. The set of the dual edges is defined as

$$\widetilde{\mathcal{E}}_\Lambda = \{e^* : e \in \bar{\mathcal{E}}_\Lambda\} \,.$$

Notice that, in general, $\widetilde{\mathcal{E}}_\Lambda \subset \mathcal{E}_{\Lambda^*}$ (the equality holds, for instance, in the case of rectangles).

Given $\Lambda \subset \mathbb{Z}^d$ we define its interior and exterior boundaries as respectively, $\partial\Lambda = \{x \in \Lambda : d(x, \Lambda^c) \leq 1\}$ and $\partial^+\Lambda = \{x \in \Lambda^c : d(x, \Lambda) \leq 1\}$, and more generally we define the boundaries of width n as $\partial_n\Lambda = \{x \in \Lambda : d(x, \Lambda^c) \leq n\}$, $\partial^+_n\Lambda = \{x \in \Lambda^c : d(x, \Lambda) \leq n\}$. For $\Lambda \subset \mathbb{Z}^d$ we also let

$$\delta\Lambda = \{\, e^* : e = [x, y],\ x \in \Lambda,\ y \in \Lambda^c,\ |x - y|_1 = 1 \,\} \qquad (2.1)\,.$$

The configuration space. Our *configuration space* is $\Omega = S^{\mathbb{Z}^d}$, where $S = \{-1, 1\}$, or $\Omega_V = S^V$ for some $V \subset \mathbb{Z}^d$. Sometimes the *lattice gas* point of view will be more convenient, so we also consider the space $\Omega' = \{0, 1\}^{\mathbb{Z}^d}$ and its natural one–to–one correspondence with Ω. The single spin space S is endowed with the discrete topology and Ω with the corresponding product topology. Given $\sigma \in \Omega$ and $\Lambda \subset \mathbb{Z}^d$ we denote by σ_Λ the natural projection over Ω_Λ. If U, V are disjoint, $\sigma_U\tau_V$ is the configuration on $U \cup V$ which is equal to σ on U and τ on V. Given $V \in \mathbb{F}$ we define the *unnormalized magnetization* $M_V : \Omega \mapsto \mathbb{Z}$ and the *number of particles* $N_V : \Omega' \mapsto \mathbb{N}$ as

$$M_V(\sigma) = \sum_{x \in V} \sigma(x)\,, \qquad N_V(\eta) = \sum_{x \in V} \eta(x) \qquad (2.2)$$

while the *normalized magnetization* is given by $m_V = M_V/|V|$.

If f is a function on Ω, Λ_f denotes the smallest subset of $\mathbb{Z}^d$ such that $f(\sigma)$ depends only on σ_{Λ_f}. f is called *local* if Λ_f is finite. $\mathcal{F}_\Lambda$ stands for the $\sigma-$algebra generated by the set of projections $\{\pi_x\}$, $x \in \Lambda$, from Ω to $\{-1,1\}$, where $\pi_x : \sigma \mapsto \sigma(x)$. When $\Lambda = \mathbb{Z}^d$ we set $\mathcal{F} = \mathcal{F}_{\mathbb{Z}^d}$ and $\mathcal{F}$ coincides with the Borel $\sigma-$algebra on Ω with respect to the topology introduced above. By $\|f\|_\infty$ we mean the supremum norm of f. The *gradient* of a function f is defined as

$$(\nabla_x f)(\sigma) = f(\sigma^x) - f(\sigma)$$

where $\sigma^x \in \Omega$ is the configuration obtained from σ, by flipping the spin at the site x.

The interaction and the Gibbs measures. Given $V \in \mathbb{F}$ we define the Hamiltonian $H_V : \Omega \mapsto \mathbb{R}$ by

$$-H_V(\sigma) = \sum_{[x,y]\in\bar{\mathcal{E}}_V} \sigma(x)\sigma(y)\,. \tag{2.3}$$

For $\sigma, \tau \in \Omega$ we also let $H_V^\tau(\sigma) = H_V(\sigma_V\tau_{V^c})$ and τ is called the *boundary condition.* Sometimes we take $\tau \in \bar{\Omega} = \{0,-1,+1\}^{\mathbb{Z}^d}$ and we call it a *generalized boundary condition.* As a particular case we have the *free boundary conditions*, given by $\tau = 0$. The corresponding Hamiltonian is denoted by

$$-H_V^\emptyset(\sigma) = \sum_{[x,y]\in\mathcal{E}_V} \sigma(x)\sigma(y)\,. \tag{2.4}$$

For each $V \in \mathbb{F}$, $\tau \in \Omega$ the (finite volume) Gibbs measure on $(\Omega, \mathcal{F})$, is given by

$$\mu_V^{\beta,\tau}(\sigma) = \begin{cases} \left(Z_V^{\beta,\tau}\right)^{-1} \exp[-\beta H_V^\tau(\sigma)\,] & \text{if } \sigma(x) = \tau(x) \text{ for all } x \in V^c \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

where $Z_V^{\beta,\tau}$ is the proper normalization factor called partition function. We will often omit the superscript β in all quantities, since in all our results the value of β can be considered a fixed parameter. Given a measurable bounded function f on Ω, $\mu_V f$ denotes the function $\sigma \mapsto \mu_V^\sigma(f)$. Analogously, if $X \in \mathcal{F}$, $\mu_V(X) = \mu_V \mathbb{I}_X$, where $\mathbb{I}_X$ is the characteristic function on X. $\mu(f,g)$ stands for the covariance (with respect to μ) of f and g. The set of measures (2.5) satisfies the DLR compatibility conditions

$$\mu_\Lambda(\mu_V(X)) = \mu_\Lambda(X) \qquad \forall\, X \in \mathcal{F} \qquad \forall\, V \subset \Lambda \subset\subset \mathbb{Z}^d\,. \tag{2.6}$$

A probability measure μ on $(\Omega, \mathcal{F})$ is called a *Gibbs measure* if

$$\mu(\mu_V(X)) = \mu(X) \qquad \forall\, X \in \mathcal{F} \qquad \forall\, V \in \mathbb{F}\,. \tag{2.7}$$

It is well known that, for the interaction (2.3), the sequences $\mu_{Q_L}^{\beta,+}$ and $\mu_{Q_L}^{\beta,-}$ converge weakly, as $L \to \infty$, to the Gibbs measures $\mu^{\beta,+}$ and $\mu^{\beta,-}$ respectively. We call *spontaneous magnetization* the function $m^* : \mathbb{R}_+ \mapsto [0,1]$, given by

$$m^*(\beta) = \mu^{\beta,+}(\sigma(0)) = \lim_{L\to\infty} \mu_{Q_L}^{\beta,+}(\sigma(0))\,. \tag{2.8}$$

The critical inverse temperature β_c is defined as the supremum of all β's such that $m^*(\beta) = 0$. When $d = 2$ it is well known that $\beta_c = (1/2)\log(1+\sqrt{2})$. We introduce the *canonical Gibbs measures* on $(\Omega, \mathcal{F})$ defined as

$$\nu^\tau_{\Lambda,N} = \mu^\tau_\Lambda(\cdot \mid N_\Lambda = N) \qquad N \in \{0, 1, \ldots, |\Lambda|\} \tag{2.9}$$

where N_Λ is the number of particles (*i.e.* spins equal to $+1$) in Λ.

The dynamics. We consider the so–called Kawasaki dynamics in which particles (spins with $\sigma(x) = +1$) can jump to nearest neighbor empty ($\sigma(x) = -1$) locations, keeping the total number of particles constant. For $\sigma \in \Omega$, let σ^{xy} be the configuration obtained from σ by exchanging the spins $\sigma(x)$ and $\sigma(y)$. Let $t_{xy}\sigma = \sigma^{xy}$ and define $(T_{xy}f)(\sigma) = f(t_{xy}\sigma)$ The stochastic dynamics we want to study is determined by the Markov generators L_V, $V \subset\subset \mathbb{Z}^d$, defined by

$$(L_V f)(\sigma) = \sum_{[x,y]\in\mathcal{E}_V} c_{xy}(\sigma)\,(\nabla_{xy}f)(\sigma) \qquad \sigma\in\Omega\,, \quad f:\Omega\mapsto\mathbb{R} \tag{2.10}$$

where $\nabla_{xy} = T_{xy} - \mathbb{1}$. The nonnegative real quantities $c_{xy}(\sigma)$ are the *transition rates* for the process.

The general assumptions on the transition rates are

(1) *Nearest neighbor interactions.* $c_{xy}(\sigma)$ depends only on the spins that are nearest neighbors of either x or y

(2) *Detailed balance.* For all $\sigma \in \Omega$ and $[x, y] \in \mathcal{E}_{\mathbb{Z}^d}$

$$\exp\big[-\beta H_{\{x,y\}}(\sigma)\big]\,c_{xy}(\sigma) = \exp\big[-\beta H_{\{x,y\}}(\sigma^{xy})\big]\,c_{xy}(\sigma^{xy}) \tag{2.11}$$

(3) *Positivity and boundedness.* For all $\beta > 0$ there exist positive real numbers $c_m(\beta)$ $c_M(\beta)$ such that

$$c_m \le c_{xy}(\sigma) \le c_M \qquad \forall x, y \in \mathbb{Z}^d\,, \sigma \in \Omega\,. \tag{2.12}$$

We denote by $L^\tau_{V,N}$ the operator L_V acting on $L^2(\Omega, \nu^\tau_{V,N})$ (this amounts to choosing τ as the boundary condition and N as the number of particles). Assumptions (1), (2) and (3) guarantee that there exists a unique Markov process whose generator is $L^\tau_{V,N}$, and whose semigroup we denote by $(T^{V,N,\tau}_t)_{t\ge0}$. $L^\tau_{V,N}$ is a bounded operator on $L^2(\Omega,\nu^\tau_{V,N})$ and $\nu^\tau_{V,N}$ is its unique invariant measure. Moreover $\nu^\tau_{V,N}$ is *reversible* with respect to the process, *i.e.* $L^\tau_{V,N}$ is self–adjoint on $L^2(\Omega, \nu^\tau_{V,N})$.

A fundamental quantity associated with the dynamics of a reversible system is the gap of the generator, *i.e.*

$$\mathrm{gap}(L^\tau_{V,N}) = \inf\mathrm{spec}\,(-L^\tau_{V,N} \restriction \mathbb{1}^\perp)$$

where $\mathbb{1}^\perp$ is the subspace of $L^2(\Omega, \nu^\tau_{V,N})$ orthogonal to the constant functions. We let $\mathcal{E}$ be the Dirichlet form associated with the generator $L^\tau_{V,N}$,

$$\mathcal{E}^\tau_{V,N}(f,f) = \langle f, -L^\tau_{V,N}f\rangle_{L^2(\Omega,\nu^\tau_{V,N})} = \frac{1}{2}\sum_{[x,y]\in\mathcal{E}_V} \nu^\tau_{V,N}\left[\,c_{xy}\,(\nabla_{xy}f)^2\,\right] \tag{2.13}$$

and $\mathrm{Var}^\tau_{V,N}$ is the variance relative to the probability measure $\nu^\tau_{V,N}$. The gap can also be characterized as

$$\mathrm{gap}(L^\tau_{V,N}) = \inf_{f\in L^2(\Omega,\nu^\tau_{V,N}),\, \mathrm{Var}^\tau_{V,N}(f)\neq 0} \frac{\mathcal{E}^\tau_{V,N}(f,f)}{\mathrm{Var}^\tau_{V,N}(f)}\,. \tag{2.14}$$

where

Our main results are the following upper and lower bounds on the spectral gap of the generator in a finite volume.

Theorem 2.1.

(a) *For all integers $d \geq 2$ there exists $\alpha_1(\beta) > 0$ such that, for all $\beta > 0$, for all $\tau \in \Omega$, for all large enough L and for all $N \in \{0, \ldots, |Q_L|\}$, we have*

$$\mathrm{gap}(L^\tau_{Q_L,N}) \geq e^{-\alpha_1 L^{d-1}} \tag{2.15}$$

(b) *Let $d = 2$, $\beta > \beta_c$, and $m \in (-m^*(\beta), m^*(\beta))$. Then there exists $\alpha_2(\beta, m) > 0$ such that, if $\mathcal{N}^m_L = \lfloor (1+m)|Q_L|/2 \rfloor$ (i.e. $\mathcal{N}^m_L$ is the number of particles which gives a magnetization approximately equal to m), then, for large enough L*

$$\mathrm{gap}(L^\emptyset_{Q_L,\mathcal{N}^m_L}) \leq e^{-\alpha_2 L} \tag{2.16}$$

Remarks. While the lower bound is a quite general statement, the upper bound (2.16) is special to free boundary conditions, $\beta > \beta_c$ and $|m| < m^*$. In [LY] it has been shown that if one has a "complete analyticity" type of condition at a certain inverse temperature β and *for all magnetic fields h*, then $\mathrm{gap}(L^{\beta,\emptyset}_{Q_L,\mathcal{N}^m_L}) \sim L^{-2}$ follows. In the case of the 2-dimensional Ising model, this condition is known to be true when $\beta < \beta_c$. One might expect a polynomial law if $\beta > \beta_c$ and $|m| > m^*$ (this is certainly true if e.g there is only one particle !).

The Glauber dynamics. Even though our results deal with the gap associated to the Kawasaki dynamics, we sometimes need to use results for the Glauber dynamics, defined in terms of transition rates $c_x(\sigma)$ which is the rate of the transition $\sigma \to \sigma^x$, where σ^x is the configuration obtained from σ by flipping the spin at x. The generators of these processes are denoted by $\mathcal{L}_V$, defined as

$$(\mathcal{L}_V f)(\sigma) = \sum_{x\in V} c_x(\sigma)(\nabla_x f)(\sigma) \qquad \sigma \in \Omega \tag{2.17}$$

3. Preliminary results

We collect in this section some preliminary results that will be used in the future. The reader can skip them during a first reading and come back to this section when these results are needed.

Proposition 3.1. *Let $\Lambda \subset\subset \mathbb{Z}^d$. Then, for all $\beta > 0$, $\tau \in \Omega$ and for all $k \in \{0, \dots, |\Lambda| - 1\}$ we have*

$$\frac{|\Lambda| - k}{k+1} e^{-4d\beta} \le \frac{\mu_\Lambda^\tau\{N_\Lambda = k+1\}}{\mu_\Lambda^\tau\{N_\Lambda = k\}} \le \frac{|\Lambda| - k}{k+1} e^{+4d\beta}$$

Proof. Define $\eta(x) = (1 + \sigma(x))/2$, $N_\Lambda(\sigma) = \sum_{x\in\Lambda}(1 + \sigma(x))/2$, then

$$\mu_\Lambda^\tau\{N_\Lambda = k+1\} = \frac{\sum_{\sigma: N_\Lambda(\sigma)=k+1} e^{-\beta H_\Lambda^\tau(\sigma)} \sum_{x\in\Lambda}\eta(x)}{(k+1)Z_\Lambda^\tau} =$$

$$= \frac{1}{k+1}\sum_{x\in\Lambda} \mu_\Lambda^\tau\left(e^{-\beta(\nabla_x H_\Lambda^\tau)(\sigma)}(1-\eta(x))\mathbb{1}_{\{N_\Lambda = k\}}\right) \le \frac{|\Lambda| - k}{k+1} e^{4d\beta} \mu_\Lambda^\tau\{N_\Lambda = k\}$$

where we used the change of variable $\sigma \mapsto \sigma^x$ to obtain the second equality and $\|e^{-\beta\nabla_x H_\Lambda^\tau}\|_\infty \le e^{4d\beta}$ for the last inequality. The lower bound is analogous. $\square$

Proposition 3.2. *Let ρ be a probability measure on $\Omega = \{0, 1, \dots, N\}$, and assume that*

$$\rho(i) = \rho(N - i)\,. \tag{3.1}$$

Then, for all functions f on Ω we have

$$\mathrm{Var}(f) \le C_\rho \sum_{i=1}^N (\rho(i) \wedge \rho(i-1))\left[f(i) - f(i-1)\right]^2$$

where

$$C_\rho = 4\,(N+1)^2 \left[\sup_{i \le N/2,\, j\le i} \frac{\rho(j)}{\rho(i)}\right]^2$$

Proof. We consider a continuous time Markov chain with transition rates

$$c(i,j) = \begin{cases} \frac{\rho(j)}{\rho(i)} \wedge 1 & \text{if } j = i \pm 1 \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

Since the rates satisfy the detailed balance condition

$$\rho(i)\, c(i,j) = \rho(j)\, c(j,i) \tag{3.3}$$

the probability measure ρ is reversible with respect to the chain. The associated Dirichlet form is given by

$$\mathcal{E}_\rho(f,f) = \sum_{i=1}^N \rho(i) c(i, i-1)\,[f(i-1) - f(i)]^2 =$$

$$= \sum_{i=1}^N (\rho(i) \wedge \rho(i-1))\left[f(i) - f(i-1)\right]^2 .$$

If we denote by λ the spectral gap of the generator of the chain, we have

$$\mathrm{Var}_\rho(f) \leq \frac{1}{\lambda}\,\mathcal{E}_\rho(f,f) = \frac{1}{\lambda}\sum_{i=1}^{N}(\rho(i)\wedge\rho(i-1))\,\big[f(i)-f(i-1)\big]^2 .$$

To conclude the proof we need a lower bound for λ. Cheeger's inequality (see Theorem 2.1 in [LS]) states that

$$\lambda \geq \frac{I^2}{8M}$$

where $M = \sup_i\ (c(i,i+1)+c(i,i-1))$ and

$$I = \min_{A\subset\Omega}\frac{\sum_{(j,k)\in A\times A^c} c(j,k)\rho(j)}{\rho(A)\,(1-\rho(A))}$$

With the choice (3.2) $M \leq 2$. As the state space Ω is countable and connected, by Corollary 4.4 in [LS], the minimum can be taken over all subsets $A\subset\Omega$ such that A and A^c are connected. Using the symmetry between A and A^c we can also impose $\rho(A)\leq 1/2$. We can thus write, using (3.1), (3.3) and (3.2),

$$\frac{1}{I} \leq \sup_{i\leq[\frac{N-1}{2}]}\frac{\sum_{j\leq i}\rho(j)}{\rho(i)\wedge\rho(i+1)} \leq \left[\frac{N+1}{2}\right]\sup_{i\leq[\frac{N+1}{2}],\ j\leq i}\frac{\rho(j)}{\rho(i)} . \tag{3.4}$$

As, by equation (3.1), $\rho(\frac{N+1}{2}) = \rho(\frac{N-1}{2})$, in (3.4) we can take $i\leq N/2$. We thus obtain

$$\frac{1}{\lambda} \leq C_\rho = 4\ (N+1)^2\,\Big[\sup_{i\leq\frac{N}{2},\ j\leq i}\frac{\rho(j)}{\rho(i)}\Big]^2 \quad \square$$

In the following proposition we need to introduce a *magnetic field*, so we also define

$$H_V^h(\sigma) = H_V(\sigma) - h\sum_{x\in V}\sigma(x) . \tag{3.5}$$

The corresponding Gibbs measure is denoted by $\mu_\Lambda^{h,\tau}$.

Proposition 3.3. *Let $\Lambda\subset\subset\mathbb{Z}^d$, $n\in\{1,\dots,|\Lambda|-1\}$, and let β,h be such that $\mu_\Lambda^{h,\tau}(N_\Lambda) = n$. Consider a Glauber–type dynamics with transition rates $c_x(\sigma)$, generator $\mathcal{L}_\Lambda^{h,\tau}$, and let $c_M^g = \sup_{x,\sigma} c_x(\sigma)$. Then*

$$\mu_\Lambda^{h,\tau}\{N_\Lambda = n\} \geq \left(\frac{\mathrm{gap}(\mathcal{L}_\Lambda^{h,\tau})}{|\Lambda|\,c_M^g}\wedge 1\right)\frac{1}{(n+2)\,(|\Lambda|-n+1)}$$

Proof. Consider trial function

$$f_{\Lambda,n}(\sigma) = \mathbb{1}_{\{N_\Lambda(\sigma)>n\}}$$

where, as usual, N_Λ is the number of particles in Λ. The Dirichlet form associated to the Glauber dynamics is bounded by

$$\mathcal{E}_{\Lambda,h}(f,f) \le |\Lambda|\, c_M^g\, \mu_\Lambda^{h,\tau}\{N_\Lambda = n\}$$

while the variance is given by

$$\mathrm{Var}_{\Lambda,h}(f) = \mu_\Lambda^{h,\tau}\{N_\Lambda \le n\}\, \mu_\Lambda^{h,\tau}\{N_\Lambda > n\}$$

so that, by the analogous of (2.14) for the Glauber dynamics we have

$$\mu_\Lambda^{h,\tau}\{N_\Lambda = n\} \ge \left(\frac{\mathrm{gap}(\mathcal{L}_\Lambda^{h,\tau})}{|\Lambda|\, c_M^g}\right) \left[\, \mu_\Lambda^{h,\tau}\{N_\Lambda \le n\}\, \mu_\Lambda^{h,\tau}\{N_\Lambda > n\}\,\right]. \tag{3.6}$$

We now bound the two probabilities in the RHS of (3.6). For the first one we have

$$n = \mu_\Lambda^{h,\tau}(N_\Lambda) \ge (n+1)\ \mu_\Lambda^{h,\tau}\{N_\Lambda > n\} = (n+1)\left(1 - \mu_\Lambda^{h,\tau}\{N_\Lambda \le n\}\right)$$

thus

$$\mu_\Lambda^{h,\tau}\{N_\Lambda \le n\} \ge \frac{1}{n+1}$$

Analogously we obtain

$$\mu_\Lambda^{h,\tau}\{N_\Lambda < n\} \le \frac{|\Lambda| - n}{|\Lambda| - (n-1)}$$

so that for the second one

$$\mu_\Lambda^{h,\tau}\{N_\Lambda > n\} \ge \frac{1}{|\Lambda| - (n-1)} - \mu_\Lambda^{h,\tau}\{N_\Lambda = n\}$$

and the result follows. $\square$

For $x, z \in \mathbb{Z}^d$, we define the events

$$E_{xz} = \{\sigma \in \Omega : \sigma(x) = 1,\ \sigma(z) = 0\}. \tag{3.7}$$

Proposition 3.4. *Let $\Lambda \subset\subset \mathbb{Z}^d$, $\Lambda = V \cup W$ with $V \cap W = \emptyset$. Let $\nu = \nu_{\Lambda,N}^{\beta,\tau}$ and $\rho_i = \nu\{N_V = i\}$. Let also $c_i = i(|W| - N + i)$ and $c_i' = i(|V| - N + i)$. Then, for all functions f on Ω we have*

$$\begin{aligned}
&\nu(f \,|\, N_V = i) - \nu(f \,|\, N_V = i-1) = \\
&\quad -\frac{1}{c_i} \sum_{\substack{x\in V\\ z\in W}} \nu\left[(\nabla_{xz} f)\mathbb{1}_{E_{xz}} \,|\, N_V = i\right] + \frac{1}{c_i}\frac{\rho_{i-1}}{\rho_i} \sum_{\substack{x\in V\\ z\in W}} \nu\left[e^{-\beta\nabla_{xz}H_\Lambda}\mathbb{1}_{E_{zx}}, f \,|\, N_V = i-1\right] = \\
&\quad = \frac{1}{c'_{N-i+1}} \sum_{\substack{x\in V\\ z\in W}} \nu\left[(\nabla_{xz} f)\mathbb{1}_{E_{zx}} \,|\, N_V = i-1\right] - \frac{1}{c'_{N-i+1}}\frac{\rho_i}{\rho_{i-1}} \sum_{\substack{x\in V\\ z\in W}} \nu\left[e^{-\beta\nabla_{xz}H_\Lambda}\mathbb{1}_{E_{xz}}, f \,|\, N_V = i\right]
\end{aligned}$$

Remark. A similar statement is contained in Lemma 3.1 in [LY].

Proof. For $\sigma \in \Omega$, let $\pi'_x(\sigma) = (\sigma(x)+1)/2$. Adding and subtracting $T_{xz} f$ we can write

$$\nu(f \,|\, N_V = i) =$$
$$\frac{1}{c_i} \sum_{\substack{x\in V\\ z\in W}} \nu\left[(f - T_{xz}f)\pi'_x(1-\pi'_z) \,|\, N_V = i\right] \; + \frac{1}{c_i} \sum_{\substack{x\in V\\ z\in W}} \nu\left[(T_{xz}f)\pi'_x(1-\pi'_z) \,|\, N_V = i\right] \tag{3.8}$$

After the change of variable $\eta \mapsto \eta^{xz}$, using the equality $\nu(fg) = \nu(f,g) + \nu(f)\nu(g)$, we can write the second term in (3.8) as

$$\begin{aligned} &\frac{1}{c_i} \frac{\nu\{N_V = i-1\}}{\nu\{N_V = i\}} \sum_{\substack{x\in V\\ z\in W}} \nu\left[e^{-\beta\nabla_{xz}H_\Lambda}(1-\pi'_x)\pi'_z, f \,|\, N_V = i-1\right] \; + \\ &\frac{1}{c_i} \frac{\nu\{N_V = i-1\}}{\nu\{N_V = i\}} \sum_{\substack{x\in V\\ z\in W}} \nu\left[e^{-\beta\nabla_{xz}H_\Lambda}(1-\pi'_x)\pi'_z \,|\, N_V = i-1\right] \; \nu\left(f \,|\, N_V = i-1\right) \end{aligned} \tag{3.9}$$

Taking $f = 1$ in equation (3.8) we obtain that the term multiplying $\nu\left(f \,|\, N_V = i-1\right)$ in (3.9) is equal to one and the result is obtained.

The second equality follows from the first by interchanging V and W □

4. Proof of the lower bound

Here we prove the lower bound (2.15) of the main Theorem 2.1. In the first subsection we show how, at least when the boundary condition if *free* and the side L of the cube Q_L is a power of two, the lower bound is a consequence of a key inequality stated in Lemma 4.1. The second subsection is where our hands get dirty and we prove of Lemma 4.1. Next we show how to extend the results to an arbitrary boundary condition and all values of L (this is very simple). Finally we show how, with our approach, we recover the well known result for $\beta = 0$.

4.1 Reduction to the key inequality (4.5)

The core of the proof is to show that $\mathrm{gap}(2\Lambda) \geq \mathrm{gap}(\Lambda)\exp(-kL^{d-1})$. To be more precise, for even L, consider the parallelepipeds Q_L^j for $j = 0, 1, \ldots, d$, defined as the set of all $x \in Q_L$ such that $0 \leq x_i \leq L/2 - 1$ for $i = j+1, \ldots, d$. Thus Q_L^d coincides with Q_L, while $Q_L^0 = Q_{L/2}$. Moreover Q_L^{j+1} is the disjoint union of Q_L^j and a translate of Q_L^j. Let now

$$a(\Lambda) = \sup_{N \in \{0,\ldots,|\Lambda|\}} \mathrm{gap}(L^{\emptyset}_{\Lambda,N})^{-1}\,.$$

What we want to show is that for some k which depends only on β we have

$$a(Q_L^{j+1}) \leq a(Q_L^j)\, e^{kL^{d-1}} \qquad \forall j = 0, \ldots, d-1\,. \tag{4.1}$$

One can then iterate this inequality and obtain the result of Theorem 2.1 for $L = 2^n$, and free boundary condition. In subsection 3 we remove these limitations.

In the following we will often consider a volume Λ which is a disjoint union of two subsets

$$\Lambda = V \cup W \qquad V \cap W = \emptyset \tag{4.2}$$

and we consider a a modified Ising model where all interactions between V and W have been turned off, *i.e.* we define an interaction

$$J_{xy} = \begin{cases} 0 & \text{if } x \in V \text{ and } y \in W \text{ (or viceversa)} \\ 1 & \text{otherwise.} \end{cases} \tag{4.3}$$

Then we define the following "dotted" quantities for the "decoupled" system

$$\dot\nu = \nu^{J,\emptyset}_{\Lambda,N}\,, \quad \dot L = L^{J,\emptyset}_{\Lambda,N}\,, \quad \dot{\mathcal{E}}(f,f) = \dot\nu(-f\,\dot L f)\,. \tag{4.4}$$

The main ingredient for proving (4.1) is the following

Lemma 4.1. *With reference to the notation introduced in (4.2), (4.3), (4.4), if W is a translate of V and $d_1(V,W) = 1$ (there is at least one edge connecting the two sets), then there exists $k(\beta) > 0$ such that*

$$\mathrm{Var}_{\dot\nu}\left(\dot\nu(f \mid N_V)\right) \leq \exp(k|\Lambda|^{\frac{d-1}{d}}) \left[\dot{\mathcal{E}}(f,f) + \dot\nu\left(\mathrm{Var}_{\dot\nu}(f \mid N_V)\right)\right]\,. \tag{4.5}$$

We prove this result in the next subsection, while below we show that Lemma 4.1 implies (4.1). Let $V = Q_L^j$, $\Lambda = Q_L^{j+1}$ and le W be the unique translate of V such that $V \cap W = \emptyset$ and $V \cup W = \Lambda$.

Thus Λ, V and W satisfy the hypotheses of Lemma 4.1. Let $\dot\nu = \nu^{J,\emptyset}_{\Lambda,N}$. Then for all functions $f : \Omega_\Lambda \mapsto \mathbb{R}$ we have

$$\mathrm{Var}_{\dot\nu}(f) = \dot\nu(\mathrm{Var}_{\dot\nu}(f \mid N_V)) + \mathrm{Var}_{\dot\nu}(\dot\nu(f \mid N_V)) \tag{4.6}$$

The key fact now is that, since there is no interaction between V and W, the conditional measure $\dot\nu(\cdot \mid N_V = k)$ is a *product measure*, *i.e.*

$$\dot\nu(\cdot \mid N_V = k) = \nu^{\emptyset}_{V,k} \times \nu^{\emptyset}_{W,N-k} \,.$$

Moreover $\dot\nu(\cdot \mid N_V = k)$ is the reversible measure of a Markov process whose generator is the sum of the two commuting generators $L^{\emptyset}_{V,k} + L^{\emptyset}_{W,N-k}$. By consequence we have that

$$\begin{aligned}
&\mathrm{Var}_{\dot\nu}(f \mid N_V = k) \le \\
&\le \max\{\, \mathrm{gap}(L^{\emptyset}_{V,k})^{-1} , \mathrm{gap}(L^{\emptyset}_{W,N-k})^{-1} \,\} \left[\dot\nu(-f L^{\emptyset}_{V,k} f \mid N_V = k) + \dot\nu(-f L^{\emptyset}_{W,N-k} f \mid N_V = k) \right] \le \\
&\le a(Q^j_L) \left[\dot\nu(-f L^{\emptyset}_{V,k} f \mid N_V = k) + \dot\nu(-f L^{\emptyset}_{W,N-k} f \mid N_V = k) \right]
\end{aligned}$$

Therefore we get

$$\dot\nu(\mathrm{Var}_{\dot\nu}(f \mid N_V)) \le a(Q^j_L)\, \dot\nu(-f\, \dot L f) = a(Q^j_L)\, \dot{\mathcal{E}}(f,f)\,. \tag{4.7}$$

From (4.6), (4.5) and (4.7) we obtain

$$\mathrm{Var}_{\dot\nu}(f) \le e^{kL^{d-1}} [\, 1 + 2a(Q^j_L)\,]\, \dot{\mathcal{E}}(f,f) \le e^{k_1 L^{d-1}} a(Q^j_L)\, \dot{\mathcal{E}}(f,f)\,. \tag{4.8}$$

All we need to do at this point is to "remove the dot", *i.e.* restore the original interaction where $J_{xy} = 1$ for all x and y. This is very simple since, by a straightforward computations, we find that there is a constant k_2 such that

$$e^{-k_2 \beta d L^{d-1}} \le \frac{\dot\nu(\sigma)}{\nu^{\emptyset}_{\Lambda,N}(\sigma)} \le e^{k_2 \beta d L^{d-1}} \qquad \forall \sigma \in \Omega$$

which, combined with (4.8), yields

$$\mathrm{Var}^{\emptyset}_{\Lambda,N}(f) \le e^{(k_1 + 3k_2) L^{d-1}} a(Q^j_L)\, \mathcal{E}^{\emptyset}_{\Lambda,N}(f,f)\,. \tag{4.9}$$

Since f is arbitrary (4.1) follows from (2.14). In this way we have proven the lower bound (2.15) with free boundary condition and $L = 2^n$ for some integer n. $\square$

4.2 Proof of Lemma 4.1

Throughtout this subsection we use the notation (4.2), (4.3), (4.4). Thanks to the spin–flip symmetry we can assume that the number of particles is $N \le |\Lambda|/2 = |V|$. Choose $f : \Omega \mapsto \mathbb{R}$ and let

$$g(n) = \dot\nu(f \mid N_V = n) \qquad \rho_n = \dot\nu\{N_V = n\} \qquad \forall\, n = 0, \dots, N\,. \tag{4.10}$$

By Proposition 3.2 we get

$$\mathrm{Var}_\rho(g) \le 4\,(N+1)^2 \Big(\sup_{i \le N/2,\, j \le i} \frac{\rho_j}{\rho_i} \Big)^2 \sum_{i=1}^{N} (\rho_i \wedge \rho_{i-1}) \left[g(i) - g(i-1) \right]^2$$

The result of Lemma 4.1 then follows if we can prove the following two Propositions:

Proposition 4.2. *With respect to the notation introduced above, if $|\Lambda|$ is large enough,*

$$\Big(\sup_{i\le N/2,\, j\le i}\frac{\rho_j}{\rho_i}\Big)^2 \le \exp\Big[\,k|\Lambda|^{\frac{d-1}{d}}\,\Big]$$

for some k which depends on β.

Proposition 4.3. *Let $\Lambda \subset\subset \mathbb{Z}^d$, $V\subset\Lambda$ and $W=\Lambda\backslash V$ (W is not necessarily a translate of V). Then, with respect to the notation introduced in (4.10), there exists some C which depends on β such that*

(1)

$$\sum_{i=1}^{N}(\rho_i\wedge\rho_{i-1})\,\big[g(i)-g(i-1)\big]^2 \le C\,\Big[\,|V|\,|W|\,|\Lambda|\,\dot{\mathcal{E}}(f,f)+|\Lambda|^8\,\dot\nu\,(\mathrm{Var}_{\dot\nu}(f\,|\,N_V))\,\Big]$$

(2) If $\Lambda=Q_L^{j+1}$, $V=Q_L^j$ (so that $W=\Lambda\backslash V$ is a translate of V) we have

$$\sum_{i=1}^{N}(\rho_i\wedge\rho_{i-1})\,\big[g(i)-g(i-1)\big]^2 \le C\,\Big[\,\frac{L^{d+2}}{(N+1)\,(|\Lambda|-N+1)}\,\dot{\mathcal{E}}(f,f)+|\Lambda|^8\,\dot\nu\,(\mathrm{Var}_{\dot\nu}(f\,|\,N_V))\,\Big]$$

The rest of this subsection is then devoted to proving these two propositions.

Proof of Proposition 4.2. Let $q^h(i)=\mu_V^{h,\emptyset}\{N_V=i\}$ (remember (3.5)), and, for simplicity let $q(i)=q^0(i)$. Since V and W are decoupled, and since W is a translation of V, we have

$$\rho_i=\mu_\Lambda^{J,\emptyset}(N_V=i\,|\,N_\Lambda=N)=\frac{\mu_V^{\emptyset}\{N_V=i\}\,\mu_W^{\emptyset}\{N_W=N-i\}}{\mu_\Lambda^{J,\emptyset}\{N_\Lambda=N\}}=\frac{q(i)\,q(N-i)}{\mu_\Lambda^{J,\emptyset}\{N_\Lambda=N\}}$$

Hence

$$\frac{\rho_j}{\rho_i}=\frac{q(j)\,q(N-j)}{q(i)\,q(N-i)}\,. \tag{4.11}$$

Since

$$\mu_V^{h,\emptyset}(f)=\frac{\mu_V^{\emptyset}(e^{2hN_V}f)}{\mu_V^{\emptyset}(e^{2hN_V})}$$

for all functions f, we can write for $i=1,\dots,|V|$ and for all $h\in\mathbb{R}$,

$$q(i)=e^{-2hi}\,\mu_V^{\emptyset}(e^{2hN_V}\;\mathbb{1}\{N_V=i\})=e^{-2hi}\,\mu_V^{\emptyset}(e^{2hN_V})\,q^h(i)\,.$$

For all $a\in(0,|V|)$, define now $\tilde h(a)$ by

$$q^{\tilde h(a)}(N_V)=a \tag{4.12}$$

(this is always possible because $q^h(N_V)$ is an increasing continuous function of h with range $(0,|V|)$. Let also

$$I(a)=2\tilde h(a)a-\log\mu_V^{\emptyset}(e^{2\tilde h(a)N_V})$$

Thus we obtain

$$q(i) = e^{-I(i)}\, q^{\tilde h(i)}(i) \qquad \forall i = 1, \dots, |V| - 1\,.$$

A straightforward computation shows that $I'(a) = 2\tilde h(a)$, so that $I''(a) = 2\tilde h'(a) \geq 0$. From the convexity of I it is easy to show that

$$I(j) + I(N-j) - (I(i) + I(N-i)) \geq 0 \qquad \forall i, j \text{ such that } i \leq N/2,\ j \leq i\,.$$

Hence, if $i \leq N/2$ and $j \leq i$, we have

$$\frac{\rho_j}{\rho_i} \leq \frac{q^{\tilde h(j)}(j)\, q^{\tilde h(N-j)}(N-j)}{q^{\tilde h(i)}(i)\, q^{\tilde h(N-i)}(N-i)} \leq \frac{1}{q^{\tilde h(i)}(i)\, q^{\tilde h(N-i)}(N-i)}\,. \tag{4.13}$$

On the other side, thanks to Proposition 3.3 (recall (4.12)), we know that

$$q^{\tilde h(i)}(i) \geq \left(\frac{\mathrm{gap}(\mathcal{L}_V^{\tilde h(i), \emptyset})}{|V|\, c_M^g} \wedge 1 \right) \frac{1}{(i+1)\,(|V| - i + 1)}\,. \tag{4.14}$$

We need at this point a lower bound on the gap of the generator of a suitable Glauber dynamics. In Theorem 4.12 of [CMM] the following *universal lower bound* was proved:

Theorem [CMM]. *Consider an arbitrary finite range interaction $\mathcal{J} = \{\mathcal{J}_A\}_{A \in \mathbb{F}}$ with range equal to r. Let $\|\mathcal{J}\|_x = \sum_{A \ni x} |\mathcal{J}_A|$ and $\|\mathcal{J}\|_\Lambda = \sup_{x \in \Lambda} \|\mathcal{J}\|_x$ (the interaction is not necessarily translation invariant). Let $\mathcal{L}_\Lambda^\tau$ be the generator of a Glauber dynamics with finite range transition rates $c_x(\sigma)$ which satisfy the detailed balance principle w.r.t. the Gibbs measure $\mu_\Lambda^{\mathcal{J}}$. Assume also that $c_x(\sigma) \geq c_m^g e^{-\kappa_1 \|\mathcal{J}\|_x}$. Then there exist $k(d, r, \kappa_1)$, such that, for each $\Lambda \subset\subset \mathbb{Z}^d$ and for each $\tau \in \Omega$, we have*

$$\mathrm{gap}(\mathcal{L}_\Lambda^\tau) \geq c_m^g \exp\bigl[-k\, \beta \|\mathcal{J}\|_\Lambda\, |\Lambda|^{(d-1)/d} \bigr] \tag{4.15}$$

It is easy to verify that with the same proof one can obtain a slightly stronger result, where in the RHS of (4.15) the quantity $\|\mathcal{J}\|_\Lambda$ is replaced by $\|\mathcal{J}^{\geq 2}\|_\Lambda$, where $\mathcal{J}^{\geq 2}$ is the set of all interactions involving at least 2 spins, *i.e. excluding the magnetic field.* Choose then the generator $\mathcal{L}_V^{h,\emptyset}$ of the Glauber dynamics with *Heat–Bath* transition rates given by

$$c_x(\sigma) = \Bigl[1 + e^{(\nabla_x H_{\{x\}})(\sigma)} \Bigr]^{-1}\,.$$

In this case we can take $c_m^g = 1/2$, $\kappa_1 = 2$ and $c_M^g = 1$. Furthermore we also have for the nearest nieghbor Ising model $\|\mathcal{J}^{\geq 2}\|_x = 2d$, so we find

$$\mathrm{gap}(\mathcal{L}_V^{h,\tau}) \geq \frac{1}{2} \exp\bigl[-2d\beta k |V|^{(d-1)/d} \bigr] \qquad \text{for all } h \in \mathbb{R}\,. \tag{4.16}$$

Replacing into (4.14), we have, for large enough $|V|$

$$q^{\tilde h(i)}(i) \geq \exp\bigl[-\beta k'\, |V|^{(d-1)/d} \bigr]$$

which, together with (4.13) implies the Proposition $\square$

Remark. The reader may be puzzled by the fact that we have used a "dynamical" argument to prove Proposition 4.2 which is a statement about the equilibrium measure. A closer look at the proof of Theorem CMM reveals however that there is really little dynamics involved, since the spectral gap is just an analytical tool which allows us to write the Poincare inequality

$$\mathrm{Var}(f) \leq \mathrm{gap}(\mathcal{L})^{-1} \mathcal{E}(f,f) \,.$$

The dynamics enters in the arguments only through its reversible measure. Of course one may still wonder if there is an easier proof of the inequality

$$\mu_V^{\tilde{h}(i)}\{N_V = i\} \geq \exp\left[-\beta k' \, |V|^{(d-1)/d}\right] .$$

Such a proof (which we did not find) would eliminate the necessity of Proposition 3.3 and Theorem CMM.

Proof of Proposition 4.3.

Throughout this proof we will use the notation (4.10). The idea is first of all to use the identities 3.4. Unfortunately, in order to have sharp enough estimates, we need to use either the first or the second one, depending on the value of N_V. Let then

$$u = \frac{(N+1)\,(|V|+1)}{\Lambda + 2} \,.$$

By Proposition 3.4 and the Schwarz inequality, we get

$$\sum_{i=1}^{N} (\rho_i \wedge \rho_{i-1}) \left[g(i) - g(i-1)\right]^2 \leq 2(A+B) \tag{4.17}$$

where

$$\begin{aligned} A &= \sum_{i=1}^{\lfloor u \rfloor} (\rho_i \wedge \rho_{i-1})\, A_-^2(i) + \sum_{i=\lfloor u \rfloor + 1}^{N} (\rho_i \wedge \rho_{i-1})\, A_+^2(i) \\ B &= \sum_{i=1}^{\lfloor u \rfloor} (\rho_i \wedge \rho_{i-1})\, B_-^2(i) + \sum_{i=\lfloor u \rfloor + 1}^{N} (\rho_i \wedge \rho_{i-1})\, B_+^2(i) \end{aligned} \tag{4.18}$$

and, letting $c_i = i\,(|W| - N + i)$, $c'_i = i\,(|V| - N + i)$ and $g_{xz} = e^{-\beta \nabla_{xz} H_\Lambda} 1\!\!1_{E_{xz}}$,

$$\begin{aligned} A_+(i) &= \frac{1}{c_i} \sum_{(x,z)\in V\times W} \dot{\nu}\left[\,(\nabla_{xz} f) 1\!\!1_{E_{xz}} \,|\, N_V = i\,\right] \\ A_-(i) &= \frac{1}{c'_{N-i+1}} \sum_{(x,z)\in V\times W} \dot{\nu}\left[\,(\nabla_{xz} f) 1\!\!1_{E_{zx}} \,|\, N_V = i-1\,\right] \\ B_+(i) &= \frac{1}{c_i} \frac{\rho_{i-1}}{\rho_i} \sum_{(x,z)\in V\times W} \dot{\nu}\left[\,g_{zx}\,,\, f \,|\, N_V = i-1\,\right] \\ B_-(i) &= \frac{1}{c'_{N-i+1}} \frac{\rho_i}{\rho_{i-1}} \sum_{(x,z)\in V\times W} \dot{\nu}\left[\,g_{xz}\,,\, f \,|\, N_V = i\,\right] . \end{aligned} \tag{4.19}$$

Estimate of the term B in (4.17). In order to estimate $B_+(i)$ we write

$$\left(\dot\nu\,[\,g_{zx}\,,\,f\,|\,N_V=i-1\,]\right)^2 \le \mathrm{Var}_{\dot\nu}(f\,|\,N_V=i-1)\ \mathrm{Var}_{\dot\nu}(g_{zx}\,|\,N_V=i-1) \tag{4.20}$$

and we notice that

$$\mathrm{Var}_{\dot\nu}(g_{zx}\,|\,N_V=i-1) \le \|g_{zx}\|_\infty^2 \le \|e^{-2\beta\nabla_{xz}H_\Lambda}\|_\infty \le e^{2\beta\|\nabla_{xz}H_\Lambda\|_\infty} \le e^{8d\beta}\,. \tag{4.21}$$

Moreover, recalling (4.11) and using Proposition 3.1, we find

$$\frac{\rho_{i-1}}{\rho_i} \le \frac{i}{|V|-i+1}\,\frac{|W|-N+i}{N-i+1}e^{8d\beta} \le |V|\,|W|e^{8d\beta}\,.$$

A straightforward calculation gives

$$\begin{aligned} c_u &= \frac{(N+1)\,(|V|+1)\,\big[\,(|W|-N)\,(|V|+1+|W|+1)+(N+1)\,(|V|+1)\,\big]}{(|\Lambda|+2)^2} = \\ &= (N+1)\,(|V|+1)\,(|W|+1)\,(|\Lambda|-N+1)\,(|\Lambda|+2)^{-2}\,. \end{aligned} \tag{4.22}$$

Moreover, c_i is increasing in i. So, using the Schwarz inequality, we obtain, for $i \ge \lfloor u \rfloor + 1$,

$$\begin{aligned} B_+^2(i) &\le c_u^{-2}\left(|V|\,|W|\,e^{8d\beta}\right)^2(|V|\,|W|)\sum_{(x,z)\in V\times W}\left(\dot\nu\,[\,g_{zx}\,,\,f\,|\,N_V=i-1\,]\right)^2 \le \\ &\le c_u^{-2}\,(|V|\,|W|)^4\,e^{32d\beta}\ \mathrm{Var}_{\dot\nu}(f\,|\,N_V=i-1) \le \\ &\le \frac{(|V|\,|W|)^2\,(|\Lambda|+2)^4}{(N+1)^2\,(|\Lambda|-N+1)^2}\,e^{32d\beta}\ \mathrm{Var}_{\dot\nu}(f\,|\,N_V=i-1)\,. \end{aligned}$$

The term $B_-(i)$ can be estimated in the same way. So, we have

$$B \le C_1(\beta)\frac{|\Lambda|^8}{(N+1)^2\,(|\Lambda|-N+1)^2}\,\dot\nu\,(\,\mathrm{Var}_{\dot\nu}(f\,|\,N_V)\,)\ . \tag{4.23}$$

Estimate of the term A in (4.17). Before starting the estimate of the term A in (4.17), we need the following definition.

Definition 4.4. Given a finite connected subset Λ of $\mathbb{Z}^d$ a *path choice in* Λ is a collection $\lambda = \{\lambda_{xz} : (x,z) \in \Lambda\times\Lambda\}$ such that λ_{xz} is a self–avoiding path from x to z inside Λ.

Given a path choice λ in Λ, and $V \subset \Lambda$ we let $W = \Lambda\backslash V$ and

$$\begin{aligned} \mathcal{G}_V(\lambda) &= \max_{e\in\mathcal{E}_\Lambda}\#\{(x,z)\in V\times W : \lambda_{xz}\sqsupset e\} \\ \mathcal{D}_V(\lambda) &= \max_{(x,z)\in V\times W}|\lambda_{xz}| \end{aligned}$$

Now we claim that for some constant $C_2(\beta)$, the quantity A defined in (4.18) satisfies, for all path choices,

$$A \le C_2(\beta)\, \frac{|\Lambda|^2}{(N+1)\,(|V|+1)\,(|W|+1)\,(|\Lambda|-N+1)}\, \mathcal{D}_V(\lambda)\, \mathcal{G}_V(\lambda) \sum_{e \sqsubset \mathcal{E}_\Lambda} \dot\nu[(\nabla_e f)^2] \tag{4.24}$$

Once we prove inequality (4.24), the proof of Proposition 4.3 (and so of Lemma 4.1) easily follows: For statement (1) we just observe that the big fraction in (4.24) is less than 1, and that trivially $\mathcal{G}_V(\lambda) \le |V|\,|W|$, $\mathcal{D}_V(\lambda) \le |\Lambda|$, for any path choice. Statement (2) is a bit more complicated. If the geometry is like in statement (2) we then use the following path choice. Given $x \in V$ and $z \in W$ start increasing (or decreasing) the first coordinate of x until it is equal to the first coordinate of z. Then adjust the second coordinate and so on until you get to z. With this particular path choice it is easy to see that $\mathcal{G}_V(\lambda) \le L^{d+1}$. Assume in fact that the path λ_{xz} contains the edge $e=[u,v]$ where u and v differ in the j^{th} coordinate. This means that $x_i = u_i$ for all $i > j$ and $z_i = u_i$ for all $i < j$. So the number of possible pairs (x,z) is not greater than $L^j\, L^{d-j+1} = L^{d+1}$. Moreover with this geometry the big fraction is less than $4\,[(N+1)\,(|\Lambda|-N+1)]^{-1}$ and $\mathcal{D}_V(\lambda) \le dL$. So we are left with the

Proof of inequality (4.24). Since

$$\frac{1}{c_i} \sum_{(x,z)\in V\times W} \dot\nu\,[\,E_{xz}\,|\,N_V = i\,] = 1$$

we can use the Schwarz inequality and obtain

$$\begin{aligned} A_+(i)^2 &= \Big[\frac{1}{c_i} \sum_{(x,z)\in V\times W} \dot\nu\,[\,\nabla_{xz} f\,|\,N_V = i, E_{xz}\,]\,\dot\nu\,[\,E_{xz}\,|\,N_V = i\,]\Big]^2 \le \\ &\le \frac{1}{c_i} \sum_{(x,z)\in V\times W} \dot\nu\,[\,\nabla_{xz} f\,|\,N_V = i, E_{xz}\,]^2\,\dot\nu\,[\,E_{xz}\,|\,N_V = i\,] \end{aligned} \tag{4.25}$$

Moreover c_i and c_i' are increasing in i. Thus, by (4.25) and the corresponding inequality for $A_-(i)^2$, we obtain

$$A \le \frac{1}{c_u} \sum_{i=1}^{\lfloor u \rfloor} \rho_{i-1}\, A_-^2(i) + \frac{1}{c_u} \sum_{i=\lfloor u \rfloor+1}^{N} \rho_i\, A_+^2(i) \le \frac{1}{c_u} \sum_{(x,z)\in V\times W} Y_{xz} \tag{4.26}$$

where

$$Y_{xz} = \sum_{i=0}^{N} \dot\nu\,[\,\nabla_{xz} f\,|\,N_V = i, E_{xz}\,]^2\,\dot\nu\,[\,E_{xz}, N_V = i\,] + \sum_{i=0}^{N} \dot\nu\,[\,\nabla_{xz} f\,|\,N_V = i, E_{zx}\,]^2\,\dot\nu\,[\,E_{zx}, N_V = i\,]$$

If we denote by $\mathcal{B}_{xz}$ the sigma–algebra generated by π_x, π_z and N_V we can use the Schwarz inequality and we find

$$Y_{xz} = \dot\nu\,\big[\,\dot\nu(\nabla_{xz} f\,|\,\mathcal{B}_{xz})^2\,\big] \le \dot\nu\,\big[\,\dot\nu((\nabla_{xz} f)^2\,|\,\mathcal{B}_{xz})\,\big] = \dot\nu\,\big[\,(\nabla_{xz} f)^2\,\big] \tag{4.27}$$

Let $t_{x\to z}:\Omega\mapsto\Omega$ be the trasformation that moves a particle from x to z if this is possible, *i.e.*

$$t_{x\to z}\sigma=\begin{cases} t_{xy}\sigma & \text{if } \sigma\in E_{xz}\\ \sigma & \text{otherwise}\ ,\end{cases}$$

and let $T_{x\to z}f=f\circ t_{x\to z}$. Then we have $(\nabla_{xz}f)^2=(T_{x\to z}f-f)^2+(T_{z\to x}f-f)^2$, thus

$$Y_{xz}=\dot\nu\left[\,(T_{x\to z}f-f)^2\,\right]+\dot\nu\left[\,(T_{z\to x}f-f)^2\,\right]\ .$$

Let now λ be any path choice. Thanks to Lemma 4.3 in [Y] we get that there exists $C_3(\beta)$ such that

$$\dot\nu\left[\,(T_{x\to z}f-f)^2\,\right]\le C_3(\beta)\,|\lambda_{xz}|\sum_{e\sqsubset\lambda_{xz}}\dot\nu[(\nabla_e f)^2]$$

which, together with (4.26), (4.22) and the definition of $\mathcal{G}_V(\lambda)$ and $\mathcal{D}_V(\lambda)$, proves inequality (4.24) and by consequence Proposition 4.3 and Proposition 4.1. $\square$

4.3 Extension to all boundary conditions and all values of L

Arbitrary boundary condition. Let $\tau\in\Omega$ be an arbitrary boundary condition. Then, letting $h^\tau=\exp(H^\emptyset_\Lambda-H^\tau_\Lambda)$, we have for all $\sigma\in\Omega_\Lambda$ such that $N_\Lambda(\sigma)=N$,

$$\frac{\nu^\tau_{\Lambda,N}(\sigma)}{\nu^\emptyset_{\Lambda,N}(\sigma)}=\frac{\mu^\tau_\Lambda(\sigma)}{\mu^\emptyset_\Lambda(\sigma)}=\frac{h^\tau(\sigma)}{\mu^\emptyset_\Lambda(h^\tau)}\le e^{2\beta\,\|H^\tau_\Lambda-H^\emptyset_\Lambda\|_\infty}\le e^{4d\beta|\partial\Lambda|}\,.$$

Analogously we find

$$\frac{\nu^\tau_{\Lambda,N}(\sigma)}{\nu^\emptyset_{\Lambda,N}(\sigma)}\ge e^{-4d\beta|\partial\Lambda|}$$

therefore, from (2.14) we get

$$\mathrm{gap}(L^\tau_{\Lambda,N})\ge e^{-12d\beta|\partial\Lambda|}\,\mathrm{gap}(L^\emptyset_{\Lambda,N})\,.$$

Thus changing the boundary condition from free to τ amounts to changing the constant α_1 in the lower bound (2.15)

Arbitrary values of L. The recursive inequality (4.1) shows that for all integers L

$$a(Q_{2L})\le e^{kL^{d-1}}a(Q_L)\,.$$

By an oversimplified version of the arguments leading to (4.1), one easily proves that for any connected set Λ and any $x\notin\Lambda$ such that $\Lambda\cap\{x\}$ is connected

$$\sup_N\mathrm{gap}(L^\emptyset_{\Lambda\cap\{x\},N})^{-1}\le C\sup_N\mathrm{gap}(L^\emptyset_{\Lambda,N})^{-1}\tag{4.28}$$

for a suitable constant C. Thus, by simple iteration of (4.28), one gets immediately

$$a(Q_{2L+1})\le e^{k'L^{d-1}}a(Q_{2L})\tag{4.29}$$

for a new constant k_1 depending on C.
Define now $a_n \equiv \sup_{2^n \le L < 2^{n+1}} a(Q_L)$. Then, thanks to (4.1) and (4.29), we have

$$a_{n+1} \le e^{(k \wedge k_1) 2^{n(d-1)}} a_n$$

so that $a_n \le C' e^{k_2 2^{n(d-1)}}$ and part (b) of Theorem 2.1 follows.

4.4 The case $\beta = 0$

Here we show how to recover the well known result (see [LY] for the same result under much more general hypotheses) that if $\beta = 0$ then

$$\operatorname{gap}(L^{\tau}_{Q_L,N}) \ge C\, L^{-2} . \tag{4.30}$$

The proof of (4.30) is almost identical to the proof of the lower bound (2.15). We just need to be a little extra careful in certain estimates. First of all we notice that, since $\beta = 0$ we have

$$g_{xz} = e^{-\beta \nabla_{xz} H_\Lambda} \mathbb{1}_{E_{xz}} = \mathbb{1}_{E_{xz}}$$

so

$$\sum_{(x,z) \in V \times W} \dot\nu\, [\, g_{xz}\, ,\, f \mid N_V = i\,] = \dot\nu\, [\, (i\,(N-i))\, ,\, f \mid N_V = i\,] = 0\, .$$

Therefore the B term in (4.18) is zero, and so statement (2) of Proposition 4.3 becomes

$$\sum_{i=1}^{N} (\rho_i \wedge \rho_{i-1}) \left[g(i) - g(i-1) \right]^2 \le C \frac{L^{d+2}}{(N+1)\,(|\Lambda| - N + 1)} \dot{\mathcal{E}}(f,f)\, . \tag{4.31}$$

Assume now that we can prove that, when ρ_n is defined by (4.10), we have for all functions g on $\{0, 1, \dots, N\}$

$$\operatorname{Var}(g) \le 16\pi \frac{N\,(|\Lambda| - N)}{|\Lambda|} \sum_{i=1}^{N} (\rho_i \wedge \rho_{i-1}) \left[g(i) - g(i-1) \right]^2 . \tag{4.32}$$

Then we would be pretty much done. In fact, from (4.31) and (4.32) we get the following improvement of Lemma 4.1

$$\operatorname{Var}_{\dot\nu}\, (\dot\nu(f \mid N_V)) \le C\, L^2\, \dot{\mathcal{E}}(f,f)\, . \tag{4.33}$$

By consequence (4.8) becomes

$$\operatorname{Var}_{\dot\nu}(f) \le \left[a(Q_L^j) + C\, L^2 \right] \dot{\mathcal{E}}(f,f)$$

which implies $a(Q_L^{j+1}) \le a(Q_L^j) + C\, L^2$, and so

$$a(Q_{2L}) \le a(Q_L) + d\, C\, L^2\, .$$

If we let $b_L = a(Q_L)/L^2$ we then get

$$b_{2L} \le \frac{b_L}{4} + C'$$

which, iterated, show that the sequence b_L is bounded, *i.e.* that the inverse of the spectral gap is bounded by $C''\,L^2$. Thus we are left with the proof of (4.32) which is more or less a straightforward computation. We have $\Lambda = V \cup W$ with W a translate of V, so $|\Lambda|$ is even and we set $|\Lambda| = 2R$. We can assume, by symmetry, that $N \le R$. We proceed as in the proof of Proposition 3.2, by introducing a continuos time Markov chain with rates given by (3.2). So, with respect to the notation of Proposition 3.2 we get

$$\lambda^{-1} \le 8MI^{-2} = 16I^{-2} \tag{4.34}$$

where

$$I^{-1} \le \sup_{n \le N/2} \sum_{i=0}^{n} \frac{\rho_i}{\rho_n}\,.$$

Since $\beta = 0$ we have

$$\rho_i = \dot{\nu}\{N_V = i\} = \frac{\binom{R}{i}\binom{R}{N-i}}{\binom{2R}{N}}$$

thus, if we let $J = N/2$, $i = J - x$, and $D = J(R-J)/R$, we get

$$\begin{aligned}\frac{\rho_{i-1}}{\rho_i} &= \frac{i\,(R-N+i)}{(R-i+1)\,(N-i+1)} = \\ &= \frac{(J-x)\,(R-J-x)}{(J+x+1)\,(R-J+x+1)} \le \left(1-\frac{x}{J}\right)\left(1-\frac{x}{R-J}\right) \le e^{-(J-i)/D}\,.\end{aligned}$$

Therefore, for all $j \le J$,

$$\frac{\rho_{i-k}}{\rho_i} \le \exp\Big[-\frac{1}{D}\sum_{j=0}^{k-1}(J-i+j)\Big] \le \exp\Big[-\frac{1}{D}\sum_{j=0}^{k-1} j\Big] = e^{-k(k-1)/(2D)}\,.$$

Finally we can estimate I^{-1} by

$$\begin{aligned}I^{-1} &\le \sum_{k=0}^{n}\frac{\rho_{n-k}}{\rho_n} \le 2 + \sum_{k=2}^{n} e^{-k(k-1)/(2D)} = 2 + \sum_{k=1}^{n} e^{-k(k+1)/(2D)} \le \\ &\le 2 + \sum_{k=1}^{n} e^{-k^2/(2D)} \le 2 + \int_0^\infty e^{-x^2/(2D)} \le \sqrt{2\pi D} = \sqrt{\pi\,\frac{N\,(|\Lambda|-N)}{|\Lambda|}}\end{aligned}$$

which, together with (4.34) implies (4.32).

5. Geometric results

In this section we derive some geometric results that will be used in the proof of the upper bound (2.16) which appears in part (b) of Theorem 2.1. The main goal is to prove Proposition 5.4. Most of the results contained in this and in the next sections have analogous versions in the paper [PV] which addresses in great generality the problem of the large deviations in the 2–dimensional Ising model. Unfortunately, in most cases, we cannot make direct references to results contained in this paper since our setting is different.

We denote by $\mathcal{D}$ the set of all rectifiable curves $\gamma \in \mathbb{R}^2$ such that γ is either a closed curve inside the unit open square $Q = \{(x, y) \in \mathbb{R}^2 : 0 < x < 1\,, 0 < y < 1\}$ or it is an open curve, which, with the exception of its endpoints, is entirely contained in Q. The collection of (finite or countable) families of curves in $\mathcal{D}$ will be denoted by $\mathcal{D}^*$. Given a family $\underline{\gamma} \in \mathcal{D}^*$ we fix an arbitrary point $x_0 \in Q$ in such a way that x_0 does not lie on any curve of $\underline{\gamma}$, and we define the set $A_{\underline{\gamma}}$ as the union of those points $x \in Q$ such that any path connecting x with x_0 has a odd number of intersections (counting multiplicities) with $\underline{\gamma}$, provided that this number is finite. We also let $B_{\underline{\gamma}} = Q \setminus A_{\underline{\gamma}}$ and we can always assume that x_0 has been chosen such that

$$|A_{\underline{\gamma}}| \le |B_{\underline{\gamma}}|$$

where $|X|$ denotes the (Lebesgue) volume of any measurable subset $X \subset \mathbb{R}^2$. We define the *phase volume* of the family $\underline{\gamma}$ as

$$\mathbb{V}(\underline{\gamma}) = |A_{\underline{\gamma}}|\,.$$

Given $\beta > 0$ we also define the *Wulff functional* $W : \mathcal{D} \mapsto [0, \infty]$ as (we omit the dependence of W on β)

$$W(\gamma) = \int_\gamma \tau_\beta(\vec{n}_s)\, ds$$

where s is the length parameter of the curve γ, $\vec{n}_s$ is the unit normal vector to the curve γ at the point s, and $\tau_\beta(\vec{n}_s)$ is the *surface tension* (see for instance [DKS]) at inverse temperature β. We extend τ_β to a function on $\mathbb{R}^2$ by setting

$$\tau_\beta(x) = |x|_2\, \tau_\beta\Big(\frac{x}{|x|_2}\Big) \qquad x \in \mathbb{R}^2\,.$$

The surface tension of the two dimensional Ising model can be computed exactly [A], [ABSZ]. What we will use are the following two properties (the first one is actually a general consequence of the second Griffiths' inequality. See (6.29) in [P])

(1) For all $x, y \in \mathbb{R}^2$ we have $\tau_\beta(x + y) \le \tau_\beta(x) + \tau_\beta(y)$
(2) $\tau_\beta((1, 0)) \le \tau_\beta(x)$ for all x such that $|x|_2 = 1$.

We extend the Wulff functional from $\mathcal{D}$ to $\mathcal{D}^*$, by letting

$$W(\underline{\gamma}) = \sum_{\gamma \in \underline{\gamma}} W(\gamma) \qquad \underline{\gamma} \in \mathcal{D}^*\,.$$

We finally define the function $\bar\varphi(v)$, $0 < v \le \infty$, as

$$\bar\varphi(v) = \begin{cases} \inf\{W(\underline{\gamma}) : \underline{\gamma} \in \mathcal{D}^*, \mathbb{V}(\underline{\gamma}) = v\} & \text{if} \quad 0 < v \le \frac{1}{2} \\ \bar\varphi(1/2) & \text{if} \quad \frac{1}{2} < v \end{cases} \tag{5.1}$$

The value $\bar\varphi(v)$ can be computed exactly (see [Sh] where however there is a mistake in the expression for $\bar\varphi(v)$ due to a misprint) and the result is

$$\bar\varphi(v) = \frac{1}{2} w(\sqrt{v} \wedge \sqrt{v_0}) \tag{5.2}$$

where the constant w is defined as $w = W(\gamma_w)$ and γ_w is the *Wulff curve* (see e.g. [Sh] or [DKS]) which depends on β and it is characterized by the fact that it is the unique solution to the following variational problem

$$w = W(\gamma_w) = \min\{\, W(\gamma) : \gamma \text{ is a closed curve enclosing a unit area} \,\}\,. \tag{5.3}$$

The singularity point v_0 satisfies the equation

$$\frac{1}{2} w\sqrt{v_0} = \tau_\beta((1,0))\,. \tag{5.4}$$

It is fairly easy to show that the infimum in (5.1) is attained when $\underline{\gamma}$ is a single curve which corresponds to

(1) a quarter of a Wulff shape centered at one of the four corners of Q if $v \le v_0$
(2) a unit (horizontal or vertical) segment if $v \ge v_0$.

We set then

$$\Gamma_w(v) = \{\gamma \in \mathcal{D} : \mathbb{V}(\underline{\gamma}) = v,\, W(\gamma) = \bar\varphi(v)\}\,. \tag{5.5}$$

In the following proposition we prove a stability property of the variational problem (5.1) which will be used later.

Proposition 5.1. *Let $v \in (0, 1/2]$. For all $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon, v) > 0$ such that if $\underline{\gamma} \in \mathcal{D}^*$ satisfies $\mathbb{V}(\underline{\gamma}) = v$ and $W(\underline{\gamma}) \le \bar\varphi(v) + \delta$, then there exists $v' \in (0, 1/2]$ with $|v - v'| \le \varepsilon$ and $\gamma_1 \in \underline{\gamma}$ such that*

$$\inf_{\gamma \in \Gamma_w(v')} d_H(\gamma_1, \gamma) \le \varepsilon \quad \text{and} \quad \sum_{\gamma \in \underline{\gamma} :\, \gamma \ne \gamma_1} \mathbb{V}(\gamma) \le \varepsilon\,.$$

Proof. The proof relies on the following preliminary results, Lemma 5.2 and Lemma 5.3

Lemma 5.2. *Let $v_i \in (0, 1/2]$ for $i = 1, \ldots, n$, and let $v = \sum_{i=1}^n v_i$. Then*

$$\sum_{i=1}^n \bar\varphi(v_i) \ge \bar\varphi(v) \sqrt{\frac{v}{\max_i v_i}}$$

Proof. We can assume $v_1 \ge v_2 \ge \ldots \ge v_n$.

Case 1: $v_1 \le v_0$. In this case we can write

$$\sum_{i=1}^{n} \bar\varphi(v_i) = \frac{1}{2} w \sum_{i=1}^{n} \frac{v_i}{\sqrt{v_i}} \ge \frac{1}{2} w \frac{v}{\sqrt{v_1}} \ge \bar\varphi(v) \sqrt{\frac{v}{v_1}} \,.$$

Case 2: $v_1 > v_0$. Let s be such that $v_1 \ge \ldots \ge v_s \ge v_0 > v_{s+1} \ge \ldots \ge v_n$. Then

$$\sum_{i=1}^{n} \bar\varphi(v_i) = \frac{1}{2} w \Big[\sum_{i=1}^{s} \sqrt{v_0} + \sum_{i=s+1}^{n} \sqrt{v_i} \Big] \ge \frac{1}{2} w \sqrt{v_0} \sum_{i=1}^{n} \sqrt{\frac{v_i}{v_1}} \ge \frac{1}{2} w \sqrt{\frac{v_0 v}{v_1}} = \bar\varphi(v) \sqrt{\frac{v}{v_1}} \quad \square$$

Lemma 5.3. *For all $v \in (0, 1/2]$, all $\varepsilon > 0$ there exists $\delta'(\varepsilon, v) > 0$ such that for all $\gamma \in \mathcal{D}$ with $\mathbb{V}(\gamma) = v$ and $W(\gamma) \le \bar\varphi(v) + \delta'$ we have*

$$\inf_{\gamma' \in \Gamma_w(v)} d_H(\gamma, \gamma') \le \varepsilon \,.$$

Proof. This statement follows froma simple compactness argument. For a proof see Lemma 12.4 in [PV].

We can finally prove 5.1. Let $\underline{\gamma} = \{\gamma_1, \gamma_2, \ldots\}$, let $v_i = \mathbb{V}(\gamma_i)$ and assume $v_1 \ge v_2 \ge \ldots$. Let also

$$v' = \sum_i v_i \quad \text{and} \quad a = \sum_{i \ge 2} v_i \,.$$

Clearly $v' \ge v$. Thus, by Lemma 5.2, we have

$$W(\underline{\gamma}) = \sum_i W(\gamma_i) \ge \sum_i \bar\varphi(v_i) \ge \bar\varphi(v') \sqrt{1 + \frac{a}{v_1}} \ge \bar\varphi(v) \sqrt{1 + \frac{a}{v_1}} \ge \bar\varphi(v) \Big(1 + \frac{1}{4} (a \wedge \sqrt{a}) \Big) \,.$$

Let a_δ be defined by $\bar\varphi(v)(a_\delta \wedge \sqrt{a_\delta})/4 = \delta$. The hypothesis $W(\underline{\gamma}) \le \bar\varphi(v) + \delta$ then implies $a \le a_\delta$. Choose then $\delta > 0$ such that

(1) $a_\delta < \varepsilon$
(2) $\bar\varphi(v_1 + a_\delta) + \delta < \bar\varphi(v_1) + \delta'$, where δ' is the quantity defined in Lemma 5.3.

Since $v \le v' = v_1 + a$, and $a \le a_\delta$, we have

$$W(\gamma_1) \le \bar\varphi(v) + \delta \le \bar\varphi(v_1 + a_\delta) + \delta < \bar\varphi(v_1) + \delta'$$

and so we can conclude by using Lemma 5.3 and the fact that $|v - v_1| \le a_\delta < \varepsilon$. $\square$

We define now a subset $U \subset Q$. Our choice is somehow arbitrary. We could have chosen any other set for which Proposition 5.4 holds. Divide Q into 16 equal squares of side $1/4$ an call these squares

$$A1, A2, \ldots, B1, B2, \ldots, D4$$

as in a chessboard. Then we define (see Fig. 1)

$$U = A1 \cup B1 \cup B2 \cup C1 \cup C2 \cup D1 \cup D2 \cup D3\,. \tag{5.6}$$

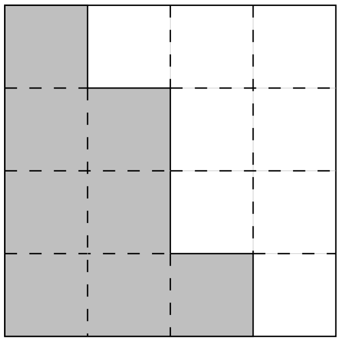

Fig. 1. The set U

We let $\mathbb{V}_U(\underline{\gamma}) = |A_{\underline{\gamma}} \cap U|$ and

$$V_U^*(v) = \{\mathbb{V}_U(\gamma) : \gamma \in \Gamma_w(v)\}$$

In other words $V_U^*(v)$ is the collections of all the possible values $|A_{\underline{\gamma}} \cap U|$ when $\underline{\gamma}$ is such that it minimizes the quantity $\bar{\varphi}(v)$. Let now

$$\bar{\psi}(v) = \begin{cases} \inf\{W(\underline{\gamma}) : \underline{\gamma} \in \mathcal{D}^*, \mathbb{V}(\underline{\gamma}) = v,\ \mathbb{V}_U(\underline{\gamma}) = v/2\} & \text{if} \quad 0 < v \le \frac{1}{2} \\ \bar{\psi}(1/2) & \text{if} \quad \frac{1}{2} < v \end{cases} \tag{5.7}$$

We now state the main result of this section.

Proposition 5.4. *For all $v \in (0, 1/2]$ we have $\bar{\psi}(v) > \bar{\varphi}(v)$.*

Proof. Proposition 5.4 follows from Proposition 5.5 given below plus the observation that for all $v \in (0, 1/2]$ there exists $\varepsilon(v)$ such that if $|v' - v| \le \varepsilon$ then $d(v/2, V_U^*(v')) > \varepsilon$.

Proposition 5.5. *Let $v \in (0, 1/2]$. For all $\varepsilon > 0$ there is a $\delta_1 = \delta_1(\varepsilon, v) > 0$ such that if $\underline{\gamma} \in \mathcal{D}^*$ satisfies $\mathbb{V}(\underline{\gamma}) = v$ and $W(\underline{\gamma}) \le \bar{\varphi}(v) + \delta_1$, then there exists $v' \in (0, 1/2]$ with $|v - v'| \le \varepsilon$ and $d(\mathbb{V}_U(\underline{\gamma}), V_U^*(v')) \le \varepsilon$.*

Proof. Is it easy to see that, for all γ_1, γ_2 in $\mathcal{D}$ we have

$$|\mathbb{V}_U(\gamma_1) - \mathbb{V}_U(\gamma_2)| \le k\, d_H(\gamma_1, \gamma_2)\, (|\gamma_1| \wedge |\gamma_2|)$$

for some universal constant k. So in particular, if $\gamma_2 \in \Gamma_w(v)$ then

$$|\mathbb{V}_U(\gamma_1) - \mathbb{V}_U(\gamma_2)| \le k_1\, d_H(\gamma_1, \gamma_2)$$

where k_1 equals k times the maximum length of a Wulff curve.

For $\varepsilon > 0$ let then $\delta_1(\varepsilon, v) = \delta(\varepsilon', v)$ where δ is defined in Proposition 5.1 and $\varepsilon' = \varepsilon/(1 + k_1)$. Assume that $\mathbb{V}(\underline{\gamma}) = v$ and $W(\underline{\gamma}) \le \bar{\varphi}(v) + \delta_1$, Thanks to Proposition 5.1 we know that there is

$\gamma_1 \in \underline{\gamma}$, $v' \in (v-\varepsilon', v+\varepsilon')$ and $\gamma_w \in \Gamma_w(v')$ such that $d_H(\gamma_1, \gamma_w) < \varepsilon'$ and $\sum_{\gamma \neq \gamma_1} \mathbb{V}(\gamma) \leq \varepsilon$. This implies

$$|\mathbb{V}_U(\gamma_1) - \mathbb{V}_U(\gamma_w)| \leq k_1 \varepsilon'$$

and

$$|\mathbb{V}_U(\underline{\gamma}) - \mathbb{V}_U(\gamma_1)| \leq \sum_{\gamma \in \underline{\gamma} : \gamma \neq \gamma_1} \mathbb{V}(\gamma) \leq \varepsilon \, .$$

Thus we get

$$|\mathbb{V}_U(\underline{\gamma}) - \mathbb{V}_U(\gamma_w)| \leq (1+k_1)\varepsilon' \leq \varepsilon \quad \square$$

We conclude this section with two results which will be used in the proof of Lemma 6.6.

Proposition 5.6. *For all $v \in (0, 1/2]$, $\varepsilon > 0$ there exists $\delta > 0$ such that if $W(\underline{\gamma}) < \bar{\psi}(v) - \varepsilon$ then*

$$|\mathbb{V}(\underline{\gamma}) - v| + |\mathbb{V}_U(\underline{\gamma}) - v/2| > \delta$$

Sketch of the proof. Assume the statement to be false. Then there exists some $v \in (0, 1/2]$, $\varepsilon > 0$ such that for all $\delta > 0$ there is $\underline{\gamma} \in \mathcal{D}^*$ such that

$$|\mathbb{V}(\underline{\gamma}) - v| + |\mathbb{V}_U(\underline{\gamma}) - v/2| \leq \delta$$

But if this is true we can easily construct a new $\underline{\gamma}'$, by adding to $\underline{\gamma}$ two small closed curves of appropiate areas, such that

(1) $\mathbb{V}(\underline{\gamma}') = v$ and $\mathbb{V}_U(\underline{\gamma}') = v/2$
(2) $W(\underline{\gamma}') \leq W(\underline{\gamma}) + C\sqrt{\delta}$

So if δ is small enough we use the definition of $\bar{\psi}(v)$ and we get a contradiction. $\square$

For $v, w \in [0, 1/2]$ we define the quantities

$$\begin{aligned}
\hat{m}^+(v) &= m^*(1-2v) \\
\hat{m}^-(v) &= -m^*(1-2v) \\
\hat{m}_U^+(w) &= 2m^*(1/2-2w) \\
\hat{m}_U^-(w) &= -2m^*(1/2-2w)
\end{aligned}$$

To understand why we introduce these quantities, let $v = \mathbb{V}(\underline{\gamma})$ and let $w = \mathbb{V}_U(\underline{\gamma})$. If the set $A_{\underline{\gamma}}$ had a uniform "magnetization density" $-m^*$ and the set $B_{\underline{\gamma}}$ had a uniform magnetization density $+m^*$, then $\hat{m}^+(v)$ would be the magnetization in Q and $\hat{m}_U^+$ the magnetization in U. If viceversa we have $+m^*$ on $A_{\underline{\gamma}}$ and $-m^*$ on $B_{\underline{\gamma}}$ then we get $\hat{m}^-(v)$ and $\hat{m}_U^-(w)$ as the respective magnetizations on Q and U. For any $m \in [0, m^*]$ we let $\varphi(m) = \bar{\varphi}(v)$ and $\psi(m) = \bar{\psi}(v)$ where v is such that $m = \hat{m}^+(v)$.

Proposition 5.7. *Let $v \in (0, 1/2]$, $\delta > 0$, and let $m = \hat{m}^+(v)$. If either*

$$|m - \hat{m}^+(v')| + |m - \hat{m}_U^+(w')| \leq \delta$$

or

$$|m - \hat{m}^-(v')| + |m - \hat{m}_U^-(w')| \leq \delta$$

then we have

$$|v - v'| + |v/2 - w'| \leq \frac{2\delta}{m^*}$$

We omit the elementary proof.

6. Proof of the upper bound

In this section we prove the upper bound (2.16) which appears in part (b) of Theorem 2.1. The proof proceeds along the following steps: first we show that, roughly, (see (6.16) for a precise statement)

$$\mathrm{gap}(L^{\emptyset}_{Q_L,\mathcal{N}^m_L}) \leq C\,L\,\frac{\mu^{\beta,\emptyset}_{Q_L}\{\,m_{Q_L}(\sigma)\sim m,\; m_{U_L}(\sigma)\sim m\,\}}{\mu^{\beta,\emptyset}_{Q_L}\{\,m_{Q_L}(\sigma)\sim m\,\}}\,. \tag{6.1}$$

where U_L is the rescaled version of the set U defined in the previous section and, for a finite volume A, we let $m_A(\sigma) = |A|^{-1}\sum_{x\in A}\sigma(x)$. Inequality (6.1) comes from the variational characterization of the spectral gap given in (2.14) when one uses an appropriate "trial function". Next we show that

$$\mu^{\beta,\emptyset}_{Q_L}\{\,m_{Q_L}(\sigma)\sim m,\; m_{U_L}(\sigma)\sim m\,\} \leq e^{-\psi(m)\,L} \tag{6.2}$$

and that

$$\mu^{\beta,\emptyset}_{Q_L}\{\,m_{Q_L}(\sigma)\sim m\,\} \geq e^{-\varphi(m)\,L} \tag{6.3}$$

where φ and ψ were defined just before Proposition 5.7. Finally we use Proposition 5.4, and we are done. The upper bound (6.2) is proven along the lines of similar results in [I] and [CGMS]. The lower bound (6.3) is a local large deviation type of result and its proof requires a more recent technology developed in [IS].

6.1 Contours, Skeletons

Below we define *contours* and *skeletons* and summarize some of their properties. For a more detailed exposition see, for instance [CGMS].

Contours. We use the contour representation with the so called *splitting rules*. Given a configuration $\sigma\in\Omega$, a generalized boundary condition $\eta\in\bar\Omega$ (see Section 2), and $\Lambda\subset\subset\mathbb{Z}^d$ we define $\mathcal{B}^\eta_\Lambda(\sigma)$ as the set of all unsatisfied edges in $\widetilde{\mathcal{E}}_\Lambda$, *i.e.* , if $\rho=\sigma_\Lambda\eta_{\Lambda^c}$, then

$$\mathcal{B}^\eta_\Lambda(\sigma) = \{\,e^* = [x,y]^*\in\widetilde{\mathcal{E}}_\Lambda : \rho(x)\rho(y) = -1\,\}$$

If the b.c. η is $+$ (or $-$), then $\mathcal{B}^\eta_\Lambda(\sigma)$ is closed, while in general it has a nonempty boundary. It is useful to decompose $\mathcal{B}^\eta_\Lambda(\sigma)$ and, in general, any arbitrary set X of dual edges as a collection of *contours* γ_i

$$X = \gamma_1\cup\cdots\cup\gamma_n \tag{6.4}$$

which have the advantage that they can be associated with simple self–avoiding (open or closed) curves in $\mathbb{R}^2$. If $\mathcal{B}^\eta_\Lambda(\sigma)=\gamma_1\cup\cdots\cup\gamma_n$ is the decomposition in contours, we let

$$\mathcal{G}^\eta_\Lambda(\sigma) = \{\gamma_1,\ldots,\gamma_n\}\,. \tag{6.5}$$

Decomposition (6.4) is intuitively obtained by cutting all three and four edges meetings, along the south–west to north–east direction (see [CGMS] for details). Fig. 2 shows a set of edges on the left and the corresponding collection of contours on the right.

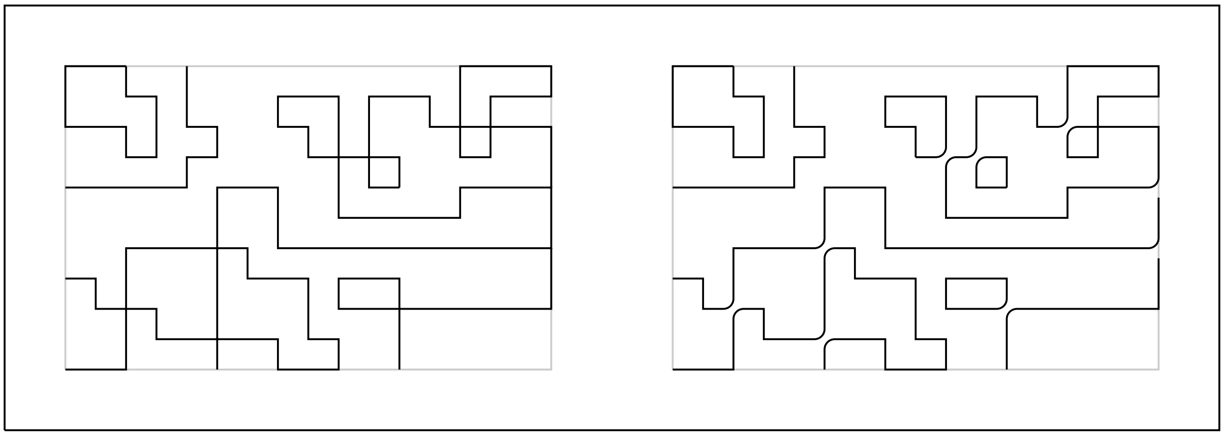

Fig. 2. A collection of edges (left) is split into a collection of contours (right)

Given a set E of edges (or dual edges), a *contour* γ in E is a finite sequence of sites $\gamma = (x_1, \dots, x_n)$ with $[x_i, x_{i+1}] \in E$ and subject to certain *selfavoiding* constraints [CGMS]. If $x_1 = x_n$ the contour is called *closed*, otherwise it is *open*. The *boundary* $\delta\gamma$ of a contour is given by the usual boundary of γ when γ is thought of as a set of edges. Thus $\delta\gamma$ can either be empty or consist of pair of sites. Given a collection of contours $\underline{\gamma} = \{\gamma_1, \dots, \gamma_n\}$, its *boundary* $\delta\underline{\gamma}$ is defined as

$$\delta\underline{\gamma} = \bigcup_{i=1}^{n} \delta\gamma_i \,.$$

A notion of *compatibility* between contours can be introduced in such a way that, for each simply connected finite volume Λ and for each $\tau \in \bar{\Omega}$, $\tau \neq 0$, the mapping $\mathcal{G}_\Lambda^\tau$ is one–to–one from Ω_Λ onto the set of all compatible collections of contours with a given boundary determined by τ. If, on the other side, one considers free boundary conditions then the mapping $\mathcal{G}_\Lambda^\emptyset$ is two–to–one from Ω_Λ onto the set of all compatible collections of contours. We let $C^*_{\Lambda,\tau}$ be the image of $\mathcal{G}_\Lambda^\tau$, *i.e.* the set of all compatible collections of contours, in the volume Λ with generalized boundary condition τ. In particular we have that $C^*_{\Lambda,\emptyset}$ consists of all compatible collections of contours $\underline{\gamma}$ such that (remember definition (2.1) and the definition of $\mathcal{V}(\cdot)$ given in Section 2)

(1) $\underline{\gamma} \subset \widetilde{\mathcal{E}}_\Lambda \backslash \delta\Lambda$
(2) For all $\gamma \in \underline{\gamma}$, we have that γ is either closed or open with $\delta\gamma \subset \mathcal{V}(\delta\Lambda)$

A compatible collection of contours $\underline{\gamma}$ splits Λ into the disjoint union $\Lambda = A_{\underline{\gamma}} \cup B_{\underline{\gamma}}$ such that if σ is the configuration corresponding to $\underline{\gamma}$ (*i.e.* if $\mathcal{G}_\Lambda^\eta(\sigma) = \underline{\gamma}$), then σ is constant on both $A_{\underline{\gamma}}$ and $B_{\underline{\gamma}}$. We assume that $|A_{\underline{\gamma}}| \leq |B_{\underline{\gamma}}|$. It can be shown that for each $\underline{\gamma} \in C^*_{\Lambda,\emptyset}$ there exists $\Delta^s(\underline{\gamma}) \subset \Lambda$ such that

$$\begin{aligned}\{\, \underline{\gamma} \subset \mathcal{G}_\Lambda^\emptyset \,\} = \{\, &\sigma(x) = +1 \ \forall x \in A_{\underline{\gamma}} \cap \Delta^s(\underline{\gamma}) \text{ and } \sigma(x) = -1 \ \forall x \in B_{\underline{\gamma}} \cap \Delta^s(\underline{\gamma}) \,\} \cup \\ \cup \{\, &\sigma(x) = -1 \ \forall x \in A_{\underline{\gamma}} \cap \Delta^s(\underline{\gamma}) \text{ and } \sigma(x) = +1 \ \forall x \in B_{\underline{\gamma}} \cap \Delta^s(\underline{\gamma}) \,\}\end{aligned} \tag{6.6}$$

The set $\Delta^s(\underline{\gamma})$ consists of all sites x such that $d_2(x, \underline{\gamma}) = 1/2$ plus some of the sites x such that $d_2(x, \underline{\gamma}) = 1/\sqrt{2}$ [CGMS]. We then define

$$A^\circ_{\underline{\gamma}} = A_{\underline{\gamma}} \backslash \Delta^s(\underline{\gamma}) \quad B^\circ_{\underline{\gamma}} = B_{\underline{\gamma}} \backslash \Delta^s(\underline{\gamma}) \,. \tag{6.7}$$

In other words, knowing that $\underline{\gamma}$ is a subset of the set of all contours is equivalent to knowing that $\sigma = +1$ on one "side" of $\underline{\gamma}$ and $\sigma = -1$ on the opposite side, or viceversa. A contour γ is said to be *s–large* if $\mathrm{diam}(\gamma) > s$. For any compatible collection of s–large contours $\underline{\gamma}$ in Λ, we let

$$E_{\underline{\gamma}} = \{ \sigma \in \Omega_\Lambda : \underline{\gamma} \text{ is the set of all } s\text{–large contours of } \sigma \} \tag{6.8}$$

We also define

$$\mu^\tau_{\Lambda,s} = \mu^\tau_\Lambda(\cdot \,|\, E_{\underline{\gamma}} = \emptyset) \tag{6.9}$$

i.e. the subscript s means that the measure is conditioned to have only s–short contours. A generalized boundary condition η is called *of type* $(s,+)$ if $\eta \geq 0$ and there is a sequence of consecutive $+1$ spins in $\partial^+\Lambda$ whose length is at least s. Given a generalized boundary condition η and a volume Λ we define another (generalized) b.c. $\eta^{+,\Lambda}$ (or $\eta^{-,\Lambda}$), by

$$\eta^{\pm,\Lambda}(x) = \begin{cases} \eta(x) & \text{if } x \in \Lambda^c \\ \pm & \text{if } x \in \Lambda \end{cases} \tag{6.10}$$

Skeletons. Given a sequence $(u_1, \dots, u_n)$ of dual sites, we denote by $P = [u_1, \dots, u_n]$ the polygon

$$[u_1, \dots, u_n] = \bigcup_{i=2}^n [u_{i-1}, u_i]$$

If $u_1 = u_n$ P is said to be closed ($\delta P = \emptyset$), otherwise it is open and its boundary is $\delta P = \{u_1, u_n\}$. P is called a *s–skeleton* if

$$\frac{s}{2} \leq |u_{i+1} - u_i|_\infty \leq 2s \qquad \forall\, i = 1, \dots, n-1$$

Given now a contour γ we say that γ is *consistent* with an s–skeleton $P = [u_1, \dots, u_n]$, and we write $\gamma \sim P$, if

- (i) All vertices of P lie on γ
- (ii) $\delta\gamma = \delta P$
- (iii) $d_H(\gamma, P) \leq s$

It is easy to check that for each contour γ with $\mathrm{diam}(\gamma) \geq s$, there always exists a s–skeleton S consistent with γ. In particular, it is always possible to associate a particular s–skeleton S to any s–large contour γ. We assume that a definite choice has been made once and for all, and $\hat{S}_s(\gamma)$ denotes *the* skeleton of γ. We extend the mapping $\hat{S}_s$ to collections of contours by

$$\hat{S}_s(\underline{\gamma}) = \{ \hat{S}_s(\gamma) : \gamma \text{ is } s\text{–large} \}\,. \tag{6.11}$$

Given $\Lambda \in \mathbb{F}$ and $\eta \in \bar{\Omega}$ we also (improperly) define, for $\sigma \in \Omega_\Lambda$, $\hat{S}_s(\sigma) = \hat{S}_s(\mathcal{G}^\eta_\Lambda(\sigma))$. Finally, given a set of skeletons $\mathcal{S} = \{S_1, \dots, S_r\}$ we let

$$E_s(\mathcal{S}) = \{\sigma \in \Omega_\Lambda : \hat{S}_s(\sigma) = \mathcal{S}\}$$

We also set

$$W(\mathcal{S}) = \sum_{i=1}^{r} W(S_i)$$

where, for any s–skeleton $S = [x_1, \ldots, x_k]$,

$$W(S) \equiv \sum_{i=2}^{k} \tau_\beta(x_i - x_{i-1})$$

In the following two propositions we collect some results which will be used later.

Proposition 6.1. *Let $\Lambda \subset\subset \mathbb{Z}^d$ and let $\underline{\gamma} \in C^*_{\Lambda,\emptyset}$ be a compatible collection of s–large contours. Then if $F_A \in \mathcal{F}_{A^\circ_{\underline{\gamma}}}$ and $F_B \in \mathcal{F}_{B^\circ_{\underline{\gamma}}}$ we have (remember (6.10))*

$$\mu^{\emptyset}_{\Lambda}(F_A \cap F_B \,|\, E_{\underline{\gamma}}) = \frac{1}{2}\Big[\, \mu^{\eta^{+,\Lambda}}_{A^\circ_{\underline{\gamma}},s}(F_A)\, \mu^{\eta^{-,\Lambda}}_{B^\circ_{\underline{\gamma}},s}(F_B) \; + \mu^{\eta^{-,\Lambda}}_{A^\circ_{\underline{\gamma}},s}(F_A)\, \mu^{\eta^{+,\Lambda}}_{B^\circ_{\underline{\gamma}},s}(F_B) \,\Big]$$

Proof. The result easlily follows from the identity (6.6) and from the DLR compatibility conditions.

Proposition 6.2. *Let $\Lambda \subset\subset \mathbb{Z}^d$, $V \subset \Lambda$, $V^\circ = V \setminus \partial_{2s} V$ (remember the definition preceding (2.1)) and $F \in \mathcal{F}_{V^\circ}$. Then*

$$\min_{\bullet\in\pm} \; \min_{W: V^\circ \subset W \subset V} \mu^{\bullet}_{W,s}(F) \le \mu^{\eta}_{\Lambda,s}(F) \le \max_{\bullet\in\pm} \; \max_{W: V^\circ \subset W \subset V} \mu^{\bullet}_{W,s}(F)\,.$$

If, moreover, η is of type $(s,+)$, then

$$\min_{W: V^\circ \subset W \subset V} \mu^{+}_{W,s}(F) \le \mu^{\eta}_{\Lambda,s}(F) \le \max_{W: V^\circ \subset W \subset V} \mu^{+}_{W,s}(F)\,.$$

Proof. Given a site $x \in \mathbb{Z}^d$ we let x^{nw}, x^{ne}, x^{sw}, x^{se} be the 4 next–to–nearest neighbors of x, where nw means north–west and so on. The set $(x_1, \cdots, x_n)$ is called a $\nearrow$–path if either x_{i+1} is a nearest neighbor of x_i, or $x_{i+1} = x_i^{sw}$ or $x_{i+1} = x_i^{ne}$. Let

$$\bar{\partial} V = \partial^+ V \cup \bigcup_{x \in \partial V} \{x^{sw}, x^{ne}\}$$

Now we claim that either (see Fig. 3)

(i) there is a $\nearrow$–path of $+1$ spins in $\partial_{2s} V \cup \bar{\partial} V$ encircling V°, or
(ii) there is a $\nearrow$–path of -1 spins in $\partial_{2s} V \cup \bar{\partial} V$ encircling V°

In fact, if neither (i) nor (ii) hold then there are both a $\nearrow$–path of $+1$ spins and a $\nearrow$–path of -1 spins connecting $\partial^+ V$ with V°. We then observe that the splitting rules assure that a $\nearrow$–path of $+1$ (or -1) spins never crosses a contour. This implies that there exists a contours which separates the two paths connecting $\partial^+ V$ with V°. Thus this contours has a diameter of at least $2s$ which is

$\mu^{\eta}_{\Lambda,s}$–almost surely forbidden. Therefore we define $\mathcal{C}(\sigma)$ as the most external $\nearrow$–path of $+1$ or -1 spins which surrounds V° and we can write

$$\inf_{\mathcal{C}} \mu^{\eta}_{\Lambda,s}(F \,|\, \mathcal{C}(\sigma) = \mathcal{C}) \leq \mu^{\eta}_{\Lambda,s}(F) \leq \sup_{\mathcal{C}} \mu^{\eta}_{\Lambda,s}(F \,|\, \mathcal{C}(\sigma) = \mathcal{C})\,.$$

The proof is then completed by a standard argument based on the DLR property of the Gibbs measures. In order to prove the last statement we only need to observe that if the boundary condition is of type $(s,+)$ then the alternative (ii) is not allowed, because otherwise there would be a long contour separating the $\nearrow$–path of -1 spins from a stretch of consecutive $+1$ spins in the boundary condition whose length is at least s. $\square$

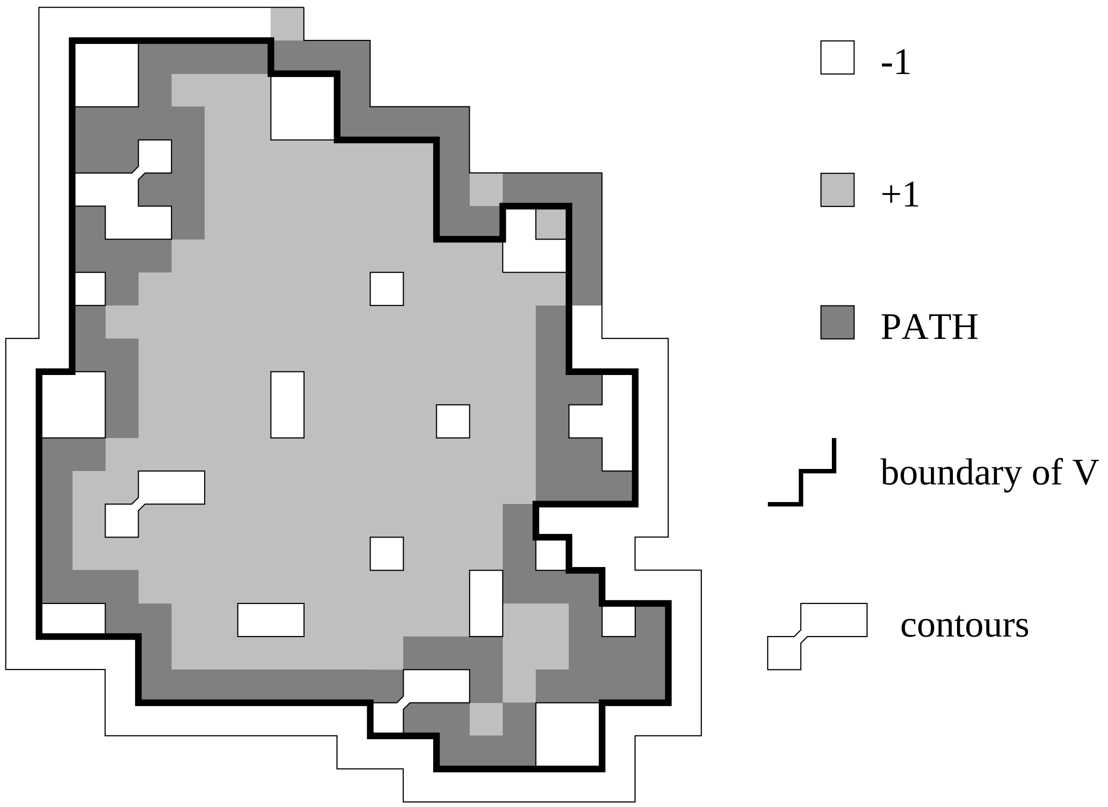

Fig. 3. An example of how to construct the $\nearrow$–path of Proposition 6.2

6.2 Proof of the upper bound

Throughout this subsection we assume to have chosen once and for all the skeleton scale as

$$s(L) = L^{\rho} \qquad \rho < \frac{1}{2}\,. \tag{6.12}$$

For simplicity we assume L to be a multiple of four. In this way there no ambiguity when we define the macroscopically rescaled version of the set U defined in (5.6)

$$U_L = \{(Lx, Ly) - (\frac{1}{2}, \frac{1}{2}) : \, (x,y) \in U\} \cap \mathbb{Z}^2$$

The volume of U_L is $|U_L| = L^2/2$. Consider then the trial function

$$f(\sigma) = 1\!\!\mathrm{I}\{N_{U_L}(\sigma) \le \lfloor \mathcal{N}_L^m/2 \rfloor\} \tag{6.13}$$

If we plug f_L in the variational characterization of the gap (2.14) we get that the Dirichlet form can be bounded from above by

$$\mathcal{E}^{\emptyset}_{Q_L,\mathcal{N}_L^m}(f,f) \le 4c_M\, L\, \nu^{\beta,\emptyset}_{Q_L,\mathcal{N}_L^m}\{\, N_{U_L}(\sigma) = \lfloor \mathcal{N}_L^m/2 \rfloor \,\} \tag{6.14}$$

while the variance $\mathrm{Var}^{\emptyset}_{Q_L,\mathcal{N}_L^m}(f_L)$ tends to $1/4$ as $L \to \infty$. Define now the event

$$\mathcal{M}_L^m = \{\, |m_{Q_L}(\sigma) - m| \le 10/L^2,\ |m_{U_L}(\sigma) - m| \le 10/L^2 \,\} \tag{6.15}$$

From equation (6.14) we get

$$\begin{aligned} \mathrm{gap}(L^{\emptyset}_{Q_L,\mathcal{N}_L^m}) &\le C\, L\, \frac{\mu^{\beta,\emptyset}_{Q_L}\{\, N_{Q_L}(\sigma) = \mathcal{N}_L^m,\ N_{U_L}(\sigma) = \lfloor \mathcal{N}_L^m/2 \rfloor \,\}}{\mu^{\beta,\emptyset}_{Q_L}\{\, N_{Q_L}(\sigma) = \mathcal{N}_L^m \,\}} \\ &\le C\, L\, \frac{\mu^{\beta,\emptyset}_{Q_L}(\mathcal{M}_L^m)}{\mu^{\beta,\emptyset}_{Q_L}\{\, N_{Q_L}(\sigma) = \mathcal{N}_L^m \,\}} \end{aligned} \tag{6.16}$$

for a suitable positive constant C.

Remark. In [PV] the authors consider the conditional probability measures $\mu^{\beta,\tau}_{Q_L}(\,\cdot \mid A(m,c))$ where

$$A(m,c) = \big\{ |\, N_{Q_L}(\sigma) - \mathcal{N}_L^m \,| \le |Q_L|\, L^{-c} \big\}$$

$c \in (0, 1/4)$ and τ is a fairly arbitrary bonundary condition. Their results are however insufficent for our purposes, since we need to condition to the event of having a precise number of particles.

The upper bound of Theorem 2.1 then follows form Proposition 5.4 and from Theorems 6.3, Proposition 6.4 given below.

Theorem 6.3. *Let $\beta > \beta_c$, $m \in (-m^*(\beta), m^*(\beta))$, and let $\mathcal{N}_L^m = \lfloor (1+m)|Q_L|/2 \rfloor$ Then*

$$\lim_{L\to\infty} -\frac{1}{L} \log\left[\mu^{\beta,\emptyset}_{Q_L}\{\, N_{Q_L} = \mathcal{N}_L^m \}\right] = \varphi(m)$$

The proof is given in the next section.

Proposition 6.4. *Let $\beta > \beta_c$ and $m \in (-m^*(\beta), m^*(\beta))$. Then*

$$\liminf_{L\to\infty} -\frac{1}{L} \log \mu^{\beta,\emptyset}_{Q_L}(\mathcal{M}_L^m) \ge \psi(m)$$

Proof. Given now $0 < \delta < \psi(m)$, we define the event

$$K_\delta = \{\, \sigma : W(\hat{S}_s(\sigma)) \ge (\psi(m) - \delta) L \,\}$$

and write

$$\mu^{\beta,\emptyset}_{Q_L}(\mathcal{M}^m_L) \le \mu^{\beta,\emptyset}_{Q_L}(\mathcal{M}^m_L \,|\, K^c_\delta\,) + \mu^{\beta,\emptyset}_{Q_L}(K_\delta) \tag{6.17}$$

In order to estimate the second term we can use Lemma 5.2 in [CGMS] which says that

$$\liminf_{L\to\infty} -\frac{1}{L}\log \mu^{\beta,\emptyset}_{Q_L}(K_\delta) \ge \psi(m) - \delta$$

Next we want to prove that

$$\liminf_{L\to\infty} -\frac{1}{L}\log \mu^{\beta,\emptyset}_{Q_L}(\mathcal{M}^m_L \,|\, K^c_\delta\,) = +\infty \tag{6.18}$$

We denote by $R_{\delta,L}$ the set of all compatible collections of (free boundary conditions) s–large contours $\underline{\gamma}$ in Q_L, such that $W(\hat{S}_s(\underline{\gamma})) < (\psi(m)-\delta)L$. We also let

$$E_{\underline{\gamma}} = \{\ \sigma\in\Omega_{Q_L} : \underline{\gamma} \text{ is the set of all } s\text{–large contours for } \sigma\ \}$$

We can then write

$$\mu^{\beta,\emptyset}_{Q_L}(\mathcal{M}^m_L \,|\, K^c_\delta) \le \sup_{\underline{\gamma}\in R_{\delta,L}} \mu^{\beta,\emptyset}_{Q_L}(\mathcal{M}^m_L \,|\, E_{\underline{\gamma}}) \tag{6.19}$$

Bound on (6.19) when $\underline{\gamma}\neq\emptyset$.

Assume $\underline{\gamma}\neq\emptyset$.

Definition 6.5. Given a positive integer L and $\delta>0$ and a collection of large contours $\underline{\gamma}$, we say that $\sigma\in E_{\underline{\gamma}}$ produces a *δ–natural magnetization pattern* (in Q_L) if either (remember (2.2))

(1) $\big|\, M_{A^\circ_{\underline{\gamma}}} + m^*|A^\circ_{\underline{\gamma}}|\,\big| + \big|\, M_{A^\circ_{\underline{\gamma}}\cap U_L} + m^*|A^\circ_{\underline{\gamma}}\cap U_L|\,\big| + \big|\, M_{B^\circ_{\underline{\gamma}}} - m^*|B^\circ_{\underline{\gamma}}|\,\big| + \big|\, M_{B^\circ_{\underline{\gamma}}\cap U_L} - m^*|B^\circ_{\underline{\gamma}}\cap U_L|\,\big| \le \delta L^2$, *or*

(2) $\big|\, M_{A^\circ_{\underline{\gamma}}} - m^*|A^\circ_{\underline{\gamma}}|\,\big| + \big|\, M_{A^\circ_{\underline{\gamma}}\cap U_L} - m^*|A^\circ_{\underline{\gamma}}\cap U_L|\,\big| + \big|\, M_{B^\circ_{\underline{\gamma}}} + m^*|B^\circ_{\underline{\gamma}}|\,\big| + \big|\, M_{B^\circ_{\underline{\gamma}}\cap U_L} + m^*|B^\circ_{\underline{\gamma}}\cap U_L|\,\big| \le \delta L^2$

We let $E^{\delta-nat}_{\underline{\gamma}}$ the subset of all the configurations in $E_{\underline{\gamma}}$ which correspond to a δ–natural magnetization pattern.

The following two results, Lemma 6.6 and Lemma 6.7 are now sufficient for finding an appropriate upper bound on the RHS of (6.19).

Lemma 6.6. *For all $\delta>0$, $m\in[0,m^*(\beta))$ there exist $\delta'=\delta'(\beta,m,\delta)$ and $L_0=L_0(\beta,m,\delta)$ such that for all $L\ge L_0$ and for all $\underline{\gamma}\in R_{\delta,L}$, we have*

$$\mathcal{M}^m_L\cap E_{\underline{\gamma}} \subset (E^{\delta'-nat}_{\underline{\gamma}})^c$$

Lemma 6.7. *For all $\delta>0$, $m\in[0,m^*(\beta))$ there exist $c=c(\beta,m,\delta)>0$ and $L_0=L_0(\beta,m,\delta)$ such that for all $L\ge L_0$ and for all $\underline{\gamma}\in R_{\delta,L}$*

$$\mu^{\beta,\emptyset}_{Q_L}((E^{\delta-nat}_{\underline{\gamma}})^c \,|\, E_{\underline{\gamma}}) \le e^{-cL^2/s(L)^2}\,. \tag{6.20}$$

Proof of Lemma 6.6. Assume $\underline{\gamma} \in R_{\delta,L}$, $\underline{\gamma} \neq \emptyset$ and let $\mathcal{S} = \hat{S}_s(\underline{\gamma})$. Since $W(\mathcal{S}) \leq (\psi(m) - d)L$ we have that the length of $\mathcal{S}$ cannot exceed $C_1 L$ (C_1 depends on β and m). But since $d_H(\mathcal{S}, \underline{\gamma}) \leq s(L)$, we get that the length of $\underline{\gamma}$ cannot exceed $C_2 s(L) L$, so

$$|\Delta^s(\underline{\gamma})| \leq C_3 s(L) L \tag{6.21}$$

Assume now to have a δ'–natural magnetization pattern for some $\delta' > 0$. For simplicity we assume that statement (1) of Definition 6.5 holds. The case when (2) holds is analogous. Let then

$$v = \frac{1}{2}\left(1 - \frac{m}{m^*}\right) \quad \text{so that} \quad m = \hat{m}^+(v) \tag{6.22}$$

and

$$v_1 = |A^\circ_{\underline{\gamma}}|/L^2, \quad w_1 = |A^\circ_{\underline{\gamma}} \cap U_L|/L^2, \qquad v_2 = |A_{\mathcal{S}}|/L^2, \quad w_2 = |A_{\mathcal{S}} \cap U_L|/L^2 \tag{6.23}$$

Then we have

$$\begin{aligned} M_{Q_L} &= M_{A^\circ_{\underline{\gamma}}} + M_{B^\circ_{\underline{\gamma}}} + M_{\Delta^s(\underline{\gamma})} = M_{A_{\mathcal{S}}} + M_{B_{\mathcal{S}}} \\ M_{U_L} &= M_{A^\circ_{\underline{\gamma}} \cap U_L} + M_{B^\circ_{\underline{\gamma}} \cap U_L} + M_{\Delta^s(\underline{\gamma}) \cap U_L} = M_{A_{\mathcal{S}} \cap U_L} + M_{B_{\mathcal{S}} \cap U_L} \end{aligned} \tag{6.24}$$

From (6.23), (6.24), and from the assumption of having a δ'–natural magnetization pattern, we obtain

$$|m_{Q_L} - \hat{m}^+(v_1)| + |m_{U_L} - \hat{m}^+_U(w_1)| \leq \delta' + C_3 s(L)/L$$

If $|m_{Q_L} - m| \leq 10/L^2$ and $|m_{U_L} - m| \leq 10/L^2$ and L is large enough, the previous inequality becomes

$$|m - \hat{m}^+(v_1)| + |m - \hat{m}^+_U(w_1)| \leq 2\delta' \tag{6.25}$$

From Proposition 5.7 we find now

$$|v - v_1| + |v/2 - w_1| \leq \frac{4\delta'}{m^*} \tag{6.26}$$

On the other side, since $W(\mathcal{S}) < \bar{\psi}(m) - \delta$, by rescaling and by Proposition 5.6 we get that there exists $\delta'' = \delta''(m, \delta)$ such that

$$|v - v_2| + |v/2 - w_2| > \delta'' \tag{6.27}$$

Finally from the definition of skeletons it follows (see for instance Lemma 5.13 of [DKS]) that

$$|v_1 - v_2| \leq C_4 \frac{s(L)}{L} \quad \text{and} \quad |w_1 - w_2| \leq C_4 \frac{s(L)}{L} \tag{6.28}$$

for some universal constant C_4. Inequalities (6.26), (6.27) and (6.28) clearly generate a contradiction if, say, $4\delta'/m^* \leq \delta''/10$ and $C_4 s(L)/L \leq \delta''/10$. $\square$

Proof of Lemma 6.7. We can write

$$E^{\delta,nat}_{\underline{\gamma}} = Y_1 \cup Y_2$$

where Y_i is the event that statement (i) of the Definition 6.5 holds. Also, we have

$$(Y_i)^c \subset \cup_{j=1}^4 G_j^i$$

where $G_1^1 = \{\, |M_{A^\circ_{\underline{\gamma}}} + m^*|A^\circ_{\underline{\gamma}}|\,| \geq \delta L^2/4\,\}$ and so on. Thus

$$(E_{\underline{\gamma}}^{\delta,nat})^c = (Y_1)^c \cap (Y_2)^c = \cup_{i=1}^4 \cup_{j=1}^4 G_i^1 \cap G_j^2$$

So, in order to estimate the LHS of (6.20) it is enough to find an appropriate upper bound for each term

$$\mu_{Q_L}^{\beta,\emptyset}(G_j^1 \cap G_j^2 \,|\, E_{\underline{\gamma}})$$

We show now how to deal with one of them, say the $G_1^1 \cap G_4^2$ term, since the others can be treated analogoulsy. Using Proposition 6.1 and the spin flip symmetry, we get

$$\begin{aligned}
&\mu_{Q_L}^{\beta,\emptyset}(G_1^1 \cap G_4^2 \,|\, E_{\underline{\gamma}}) = \\
&= \mu_{Q_L}^{\beta,\emptyset}\Big(\{\, |M_{A^\circ_{\underline{\gamma}}} + m^*|A^\circ_{\underline{\gamma}}|\,| \geq \delta L^2/4\,\} \cap \{\, |M_{B^\circ_{\underline{\gamma}}\cap U_L} + m^*|B^\circ_{\underline{\gamma}} \cap U_L|\,|\,\} \geq \delta L^2/4 \,\Big|\, E_{\underline{\gamma}} \Big) = \\
&= \frac{1}{2}\,\mu_{A^\circ_{\underline{\gamma}},s}^{\beta,\emptyset^{+,Q_L}}\Big(|M_{A^\circ_{\underline{\gamma}}} + m^*|A^\circ_{\underline{\gamma}}|\,| \geq \delta L^2/4\Big)\,\mu_{B^\circ_{\underline{\gamma}},s}^{\beta,\emptyset^{-,Q_L}}\Big(|M_{B^\circ_{\underline{\gamma}}\cap U_L} + m^*|B^\circ_{\underline{\gamma}} \cap U_L|\,| \geq \delta L^2/4\Big) + \\
&+ \frac{1}{2}\,\mu_{A^\circ_{\underline{\gamma}},s}^{\beta,\emptyset^{-,Q_L}}\Big(|M_{A^\circ_{\underline{\gamma}}} + m^*|A^\circ_{\underline{\gamma}}|\,| \geq \delta L^2/4\Big)\,\mu_{B^\circ_{\underline{\gamma}},s}^{\beta,\emptyset^{+,Q_L}}\Big(|M_{B^\circ_{\underline{\gamma}}\cap U_L} + m^*|B^\circ_{\underline{\gamma}} \cap U_L|\,| \geq \delta L^2/4\Big) \leq \\
&\leq \frac{1}{2}\,\mu_{B^\circ_{\underline{\gamma}},s}^{\beta,\emptyset^{-,Q_L}}\Big(|M_{B^\circ_{\underline{\gamma}}\cap U_L} + m^*|B^\circ_{\underline{\gamma}} \cap U_L|\,| \geq \delta L^2/4\Big) + \frac{1}{2}\,\mu_{A^\circ_{\underline{\gamma}},s}^{\beta,\emptyset^{-,Q_L}}\Big(|M_{A^\circ_{\underline{\gamma}}} + m^*|A^\circ_{\underline{\gamma}}|\,| \geq \delta L^2/4\Big) = \\
&= \frac{1}{2}\,\mu_{B^\circ_{\underline{\gamma}},s}^{\beta,\emptyset^{+,Q_L}}\Big(|M_{B^\circ_{\underline{\gamma}}\cap U_L} - m^*|B^\circ_{\underline{\gamma}} \cap U_L|\,| \geq \delta L^2/4\Big) + \frac{1}{2}\,\mu_{A^\circ_{\underline{\gamma}},s}^{\beta,\emptyset^{+,Q_L}}\Big(|M_{A^\circ_{\underline{\gamma}}} - m^*|A^\circ_{\underline{\gamma}}|\,| \geq \delta L^2/4\Big).
\end{aligned}$$

From this inequality and similar ones for the other terms we get

$$\begin{aligned}
\mu_{Q_L}^{\beta,\emptyset}((E_{\underline{\gamma}}^{\delta-nat})^c \,|\, E_{\underline{\gamma}}) \leq 8 \sum_{D=A^\circ_{\underline{\gamma}},B^\circ_{\underline{\gamma}}} \Big[&\mu_{D,s}^{\beta,\emptyset^{+,Q_L}}\{\, \big|M_D - m^*|D|\big| \geq (\delta/4)L^2\,\} + \\
&+\mu_{D,s}^{\beta,\emptyset^{+,Q_L}}\{\, \big|M_{D\cap U_L} - m^*|D \cap U_L|\big| \geq (\delta/4)L^2\,\} \Big]
\end{aligned} \tag{6.29}$$

Notice then that since $\underline{\gamma}$ is a collection of s–large contours, the boundary condition $\emptyset^{+,Q_L}$ which appears in the Gibbs measures on the right hand side of (6.29) is of type $(s,+)$. Also we can assume that each of the four sets $A^\circ_{\underline{\gamma}}$, $B^\circ_{\underline{\gamma}}$, $A^\circ_{\underline{\gamma}} \cap U_L$ and $B^\circ_{\underline{\gamma}} \cap U_L$ has a volume at least $(\delta/8)L^2$, otherwise the corresponding term in (6.29) would be zero. Finally, since $\underline{\gamma} \in R_{\delta,L}$ the length of $\underline{\gamma}$ cannot exceed $C_1 L s(L)$ and, by consequence, the length of the boundary of each of the four sets under consideration is bounded by $C_2 L s(L)$. So we can apply Proposition 6.8 given below and we obtain

$$\mu_{Q_L}^{\beta,\emptyset}((E_{\underline{\gamma}}^{\delta-nat})^c \,|\, E_{\underline{\gamma}}) \leq 32\, e^{-cL^2/s(L)^2} \quad \square$$

Proposition 6.8. *Let $\alpha, \varepsilon > 0$ and let L be a positive integer. Consider a finite volume Λ with a boundary condition η type $(s(L),+)$. Let also $V \subset \Lambda$ be such that* *(i)* $|V| \geq \alpha L^2$,

(ii) $|\partial V| \leq c_1 L^\gamma / s(L)$ *where* $\gamma < 2$ *and* c_1 *is a positive constant. Then there exists* $c = c(\alpha, \varepsilon, \gamma, \beta) > 0$ *such that if* L *is big enough*

$$\mu_{\Lambda,s}^{\beta,\eta}\{|m_V(\sigma) - m^*| \geq \varepsilon\} \leq e^{-cL^2/s(L)^2} \tag{6.30}$$

Proof. The hypothesis (i) and (ii) on V guarantee that, for L large enough,

$$|V^\circ| = |V \setminus \partial_s V| \geq \frac{3\alpha}{4} L^2 \tag{6.31}$$

and

$$|m_V(\sigma) - m_{V^\circ}(\sigma)| \leq \frac{\varepsilon}{2} \tag{6.32}$$

we thus have that the LHS of (6.30) can be bounded from above by

$$\mu_{\Lambda,s}^{\beta,\eta}\{|m_{V^\circ}(\sigma) - m^*| \geq \frac{\varepsilon}{2}\} \leq \max_{W : V^\circ \subset W \subset V} \mu_{W,s}^{\beta,+}\{|m_{V^\circ}(\sigma) - m^*| \geq \frac{\varepsilon}{2}\} \tag{6.33}$$

where in the second inequality we used Proposition 6.2. By construction the set W where the maximum is attained satisfies $|W| \geq \frac{3\alpha}{4} L^2$ and $|W \setminus V^\circ| \leq |\partial_s V| \leq c_1 L^\gamma$ so that, for L large enough,

$$|m_W(\sigma) - m_{V^\circ}(\sigma)| \leq \frac{\varepsilon}{4}$$

and the RHS of (6.33) can be bounded from above by

$$\begin{aligned} &\leq \mu_{W,s}^{\beta,+}\{|m_W(\sigma) - m^*| \geq \frac{\varepsilon}{4}\} \\ &\leq \mu_{W,s}^{\beta,+}\{m_W(\sigma) \leq m^* - \frac{\varepsilon}{4}\} + \frac{\mu_W^{\beta,+}\{m_W(\sigma) \geq m^* + \frac{\varepsilon}{4}\}}{\mu_W^{\beta,+}\{\text{all contours are small}\}}\,. \end{aligned} \tag{6.34}$$

Using Lemma 3.1 of [I], we have that there exists a constant $c' = c'(\alpha, \varepsilon) > 0$ such that

$$\mu_{W,s}^{\beta,+}\{m_W(\sigma) \leq m^* - \frac{\varepsilon}{4}\} \leq \exp\{-c' \frac{L^2}{s(L)^2}\}\,. \tag{6.35}$$

We now bound the other term. By construction $|\partial W| \leq |\partial_s V| + |\partial V| \leq c_2 L^\gamma$ where c_2 is a suitable positive constant, for any L large enough, we can use inequality (4.8) in [I] and get that there exists a constant $c'' = c''(\alpha, \varepsilon, \gamma, \beta) > 0$ such that

$$\mu_W^{\beta,+}\{m_W(\sigma) \geq m^* + \frac{\varepsilon}{4}\} \leq \exp\{-c'' L^2\}\,. \tag{6.36}$$

We finally estimate the denominator in (6.34). We have

$$\begin{aligned} &\mu_W^{\beta,+}\{\text{all contours are small}\} \\ &\geq 1 - \mu_W^{\beta,+}\{\text{there exists a } * \text{-path of } - \text{ spins with diameter } \geq s(L)\} \\ &\geq 1 - \mu^{\beta,+}\{\text{there exists a } * \text{-path of } - \text{ spins with diameter } \geq s(L)\} \end{aligned} \tag{6.37}$$

where we have used the FKG property [FKG] in the second inequality and $\mu^{\beta,+}$ denotes the infinite–volume plus phase. Finally, using the result of [CCS], we have that the right hand side of (6.37) tends to one as $L \to \infty$. If we now combine (6.35)-(6.37) we obtain (6.30). $\square$

Bound on (6.19) when $\underline{\gamma} = \emptyset$. In this case we write

$$\mu^{\beta,\emptyset}_{Q_L}(\mathcal{M}^m_L \mid E_\emptyset) = \mu^{\beta,\emptyset}_{Q_L,s}(\mathcal{M}^m_L) \le \mu^{\beta,\emptyset}_{Q_L,s}\{\, |m_{Q_L} - m| \le 10/L^2 \,\}\,. \tag{6.38}$$

Let $Q^\circ_L = Q_L \backslash \partial_{2s} Q_L$ Then, is L is large enough, we have

$$\mu^{\beta,\emptyset}_{Q_L,s}\{\, |m_{Q_L} - m| \le 10/L^2 \,\} \le \mu^{\beta,\emptyset}_{Q_L,s}\{\, |m_{Q^\circ_L} - m| \le 9s(L)/L \,\}\,. \tag{6.39}$$

Thanks to Proposition 6.2 we obtain

$$\begin{aligned} \text{RHS of (6.39)} \le& \max_{W: Q^\circ_L \subset W \subset Q_L} \mu^+_{W,s}\{\, |m_{Q^\circ_L} - m| \le 9s(L)/L \,\} + \\ &+ \max_{W: Q^\circ_L \subset W \subset Q_L} \mu^-_{W,s}\{\, |m_{Q^\circ_L} - m| \le 9s(L)/L \,\}\,. \end{aligned}$$

We can now use Proposition 6.8 plus spin–flip symmetry and the fact that $m \in [0, m^*)$, and we get

$$\mu^{\beta,\emptyset}_{Q_L,s}\{\, |m_{Q_L} - m| \le 10/L^2 \,\} \le e^{-cL^2/s(L)^2} \quad \square$$

7. A local large deviation result

Here we prove Theorem 6.3 following [IS]. We start with a result on the decay of the connectivity function for the $d = 2$, $q = 2$ FK random cluster model, which improves the results in [CCS].

7.1 Decay of connectivity for the FK measures with wired boundary conditions

We consider here the FK "random cluster" model. We refer the reader to [FK] (see also [Pi] where some useful properties are discussed).

Given $\mathcal{G} \subset \mathcal{E}_{\mathbb{Z}^2}$ and an element ω of the FK–configuration space $\Omega_{\mathcal{G}}^{FK} = \{0,1\}^{\mathcal{G}}$, we say that an edge $e \in \mathcal{G}$ is occupied (empty) for ω, or simply FK occupied (FK empty), if $\omega(e) = 1$ ($\omega(e) = 0$). We then denote by $\mu_{\mathcal{G}}^{FK,\beta,\emptyset}$ $\left(\mu_{\mathcal{G}}^{FK,\beta,w}\right)$ the standard FK–measure on $\Omega_{\mathcal{G}}^{FK}$ with parameters $p = 1-\exp(-2\beta)$, $q = 2$ and with free (wired) boundary conditions (see e.g. [Pi] for more details). To simplify the notation, for any $\Lambda \subset\subset \mathbb{Z}^2$, we will sometimes write $\mu_{\Lambda}^{FK,\beta,\emptyset}$ instead of $\mu_{\mathcal{E}_\Lambda}^{FK,\beta,\emptyset}$ and $\mu_{\Lambda}^{FK,\beta,w}$ instead of $\mu_{\bar{\mathcal{E}}_\Lambda}^{FK,\beta,w}$. The following duality property of the FK model plays an important role in the arguments of [IS]. Given $\omega \in \Omega_{\mathcal{G}}^{FK}$, let $\omega^* \in \Omega_{\mathcal{G}^*}^{FK}$ be such that

$$\omega^*(e^*) = 1 - \omega(e) \qquad \forall\, e \in \mathcal{G}\,.$$

Using the above mapping, any FK-measure on $\Omega_{\mathcal{G}}^{FK}$ can be identified with a FK-measure on $\Omega_{\mathcal{G}^*}^{FK}$. In particular we have

$$\mu_{\mathcal{G}}^{FK,\beta,\emptyset}(\omega) = \mu_{\mathcal{G}^*}^{FK,\beta^*,w}(\omega^*) \quad \text{and} \quad \mu_{\mathcal{G}}^{FK,\beta,w}(\omega) = \mu_{\mathcal{G}^*}^{FK,\beta^*,\emptyset}(\omega^*) \tag{7.1}$$

where the dual inverse temperature β^* is given by the Krammer–Wannier equation

$$e^{2\beta} = \tanh \beta^*\,.$$

Notice that $\beta^* < \beta_c$ as long as $\beta > \beta_c$. We will also use the following monotonicity property of the FK measures: let $U \subset V \subset\subset \mathbb{Z}^d$, and let f be a nondecreasing function on $\Omega_{\mathcal{E}_U}^{FK}$. Then (see, for instance [Pi] for a proof)

$$\mu_U^{FK,\emptyset}(f) \le \mu_V^{FK,\emptyset}(f) \le \mu_V^{FK,w}(f) \le \mu_U^{FK,w}(f)\,. \tag{7.2}$$

For $\Lambda \subset\subset \mathbb{Z}^d$ and $x, y \in \Lambda$ we define $\{x \overset{\Lambda}{\longleftrightarrow} y\}$ as the event that x is connected to y by a path of bonds in $\mathcal{E}_\Lambda$ which are FK–occupied. What we want to prove is the exponential decay of the connectivity function

$$\tau_\Lambda^{\beta,w}(x,y) = \mu_\Lambda^{FK,\beta,w}\{x \overset{\Lambda}{\longleftrightarrow} y\}\,.$$

Notice that $\{x \overset{\Lambda}{\longleftrightarrow} y\}$ means that there is a connection which makes no use of the "boundary cluster", otherwise the result would clearly be false. In [CCS] it has been proven that the *free b.c.* connectivity function, being equal to the two–point correlation function of the Ising model, has exponential decay for all $\beta < \beta_c$. The wired case needs however a different treatment (the FKG inequality says that $\tau^{\beta,w} \ge \tau^{\beta,\emptyset}$). We also point out that the following result has no "obvious" extension to the case $d \ge 3$.

Theorem 7.1. *For all $\beta < \beta_c$ there exist $C(\beta)$, $m(\beta) > 0$ such that for all L*

$$\tau_{Q_L}^{\beta,w}(x,y) \le C\, e^{-m\,|x-y|} \qquad \forall x,y \in Q_L\,.$$

Proof. We can assume that L is greater than some $L_0(\beta)$. For $l < L$, denote by $A_l(x)$ the event that the top side of the square $Q_l(x)$ is FK–connected to the lower half of Q_l, *i.e.* to the set $\{y \in Q_l(x) : y_2 \le x_2 + \lfloor \frac{1}{2} l \rfloor\}$, by a path of FK–occupied bonds entirely contained in $\mathcal{E}_{Q_l(x)}$. Let also $\hat{A}_l(x)$ be the event that the $A_l(x)$ occurs modulo a rotation by $k\frac{\pi}{2}$, $k = 0,1,2,3$, *i.e.* that any side of $Q_l(x)$ is connected to the "furthest" half of the square. Let $n = \lfloor |x-y|/(2l) \rfloor$. It is then easy to see that the event $\{x \overset{Q_L}{\longleftrightarrow} y\}$ is contained in the event $\mathcal{A}_n(l)$ defined as follows: there exists $j \ge n$ and a sequence of disjoint squares $\{Q_l(x^i)\}_{i=1}^{j}$ inside Q_L such that

(a) $x \in Q_l(x^1)$ and $y \in Q_l(x^j)$
(b) $3 < d\big(Q_l(x^i), Q_l(x^{i+1})\big) \le l \quad \forall i = 1 \dots j-1$
(c) $\hat{A}_l(x^i)$ occurs for each $i = 1 \dots j$.

Thus by an elementary Peierl's argument we get

$$\tau_{Q_L}^{\beta,w}(x,y) \le \sum_{j=n}^{\infty} (3l)^{2j} \mu_{Q_L}^{FK,\beta,w}(\,\cap_{i=1}^{j} \hat{A}_l(x^i)\,) \tag{7.3}$$

and, using (7.2)

$$\mu_{Q_L}^{FK,\beta,w}(\,\cap_{i=1}^{j} \hat{A}_l(x^i)\,) \le \mu_{\cup_i Q_l(x^i)}^{FK,\beta,w}(\,\cap_{i=1}^{j} \hat{A}_l(x^i)\,) = \prod_{i=1}^{j} \mu_{Q_l(x^i)}^{FK,\beta,w}(\,\hat{A}_l(x^i)\,)\,. \tag{7.4}$$

Thanks to Lemma 7.2 given below we can now choose $l = l_0(\beta)$ such that

$$(3l)^2 \mu_{Q_l(x^i)}^{FK,\beta,w}(\,\hat{A}_l(x^i)\,) \le e^{-1}\,. \tag{7.5}$$

In this way we obtain

$$\tau_{Q_L}^{\beta,w}(x,y) \le \sum_{j=n}^{\infty} e^{-j} \le 2e^{-n} = 2e^{-\lfloor |x-y|/(2l_0) \rfloor}\,. \quad \square$$

Lemma 7.2. *Let $\beta < \beta_c$. Then, for any positive n*

$$\lim_{l\to\infty} l^n \mu_{Q_l}^{FK,\beta,w}(\hat{A}_l) = 0\,.$$

Proof. The proof is, in turn, based on the following lemma quite similar to an analogous result proved in [MOS] for general finite range two dimensional lattice spin systems.

Lemma 7.3. *Let*

$$R_l = \{x = (x_1, x_2) \in Q_l \,:\, x_2 \le \sqrt{l}\}\,.$$

Consider the event B_l that there is a connection inside R_l between the top side of R_l and the bottom half of R_l, defined as

$$\{x = (x_1, x_2) \in Q_l \,:\, x_2 \le \lfloor \sqrt{l}/2 \rfloor\}\,.$$

If $\beta < \beta_c$ there exist positive constants ε and l_0 such that

$$\mu_{R_l}^{FK,\beta,w}(B_l) \le 1 - \varepsilon \qquad \forall\, l \ge l_0\,.$$

We postpone the proof and complete instead the proof of Lemma 7.2. Assume $l \ge 9$ and denote by R_l^i the vertical translate of R_l by the amount $\lfloor 2i\sqrt{l} \rfloor$, $i = 0, 1 \ldots$. Notice that the rectangles R_l^{i-1} and R_l^i are separated by a strip of width larger than 2. Then clearly the occurrence of the event A_l implies the occurence of the event B_l for each rectangle R_l^i with $i = 0, 1 \ldots \lfloor \frac{1}{3}\sqrt{l} \rfloor$. Thus, if we use Lemma 7.3, we get that

$$\mu_{Q_l}^{FK,\beta,w}(A_l) \le \mu_{R_l}^{FK,\beta,w}(B_l)^{\frac{1}{3}\sqrt{l}-1} \le (1-\varepsilon)^{\frac{1}{3}\sqrt{l}-1}$$

for any $l \ge l_0 \wedge 9$, where l_0 and ε are the constants given in Lemma 7.3. Thus $\mu_{Q_l}^{FK,\beta,w}(A_l)$ decays at least as a negative exponential of the square root of the side l and the proposition follows at once from the trivial inequality $\mu_{Q_l}^{FK,\beta,w}(\hat{A}_l) \le 4\mu_{Q_l}^{FK,\beta,w}(A_l)$. $\square$

Proof of Lemma 7.3. Let $M = \{x \in R_l \,:\; x_2 = \lfloor \frac{1}{2}\sqrt{l} \rfloor\}$ be the horizontal segment which splits R_l into roughly two equal part and, for any given positive integer $\delta < l/2$, let us divide M into a middle part M_{middle} and a lateral part M_{lat} as follows

$$\begin{aligned} M_{\text{lat}} &= M \setminus M_{\text{middle}} \\ M_{\text{middle}} &= \{x \in M \,:\; \delta \le x_1 \le l - \delta\}\,. \end{aligned}$$

Accordingly we can write the event B as $B = B_{\text{lat}} \cup B_{\text{middle}}$ where B_{lat} is the event that the lateral part of M, M_{lat}, is FK–connected to the top side of R_l and similarily for the event B_{middle}. Then, thanks to the FKG inequality, we get

$$\mu_{R_l}^{FK,\beta,w}(B) = 1 - \mu_{R_l}^{FK,\beta,w}\big(B_{\text{lat}}^c \cap B_{\text{middle}}^c\big) \le 1 - \mu_{R_l}^{FK,\beta,w}(B_{\text{lat}}^c)\mu_{R_l}^{FK,\beta,w}(B_{\text{middle}}^c)\,. \tag{7.6}$$

We now find lower bounds for both factors in the RHS of (7.6). The term $\mu_{R_l}^{FK,\beta,w}(B_{\text{lat}}^c)$ is clearly bounded from below by the probability that all vertical bonds with one vertex on M_{lat} are not FK–occupied, that is

$$\mu_{R_l}^{FK,\beta,w}(B_{\text{lat}}^c) \ge e^{-C(\beta)\delta}$$

for a suitable constant $C(\beta)$. Using the exponential decay of connectivity for any $\beta < \beta_c$ proved in [CCS], the term $\mu_{R_l}^{FK,\beta,w}(B_{\text{middle}}^c)$, is bounded from below by

$$\begin{aligned} \mu_{R_l}^{FK,\beta,w}(B_{\text{middle}}^c) &\ge 1 - \sum_{x \in M_{\text{middle}}} \mu_{R_l}^{FK,\beta,w}\big(x \text{ is FK–connected to the boundary of } R_l\big) \\ &\ge 1 - C_1(\beta) \sum_{x \in M_{\text{middle}}} e^{-m(\beta) d(x, R_l^c)} \ge \frac{1}{2} \end{aligned} \tag{7.7}$$

provided that both δ and l are taken large enough depending on β. In conclusion the RHS of (7.6) is larger than $1-\frac{1}{2}e^{-C(\beta)\delta}$ for any l and δ large enough. The proof of the lemma, and thus of Theorem 7.1, is complete. $\square$

7.2 Proof of Theorem 6.3

For $\beta>\beta_c$, let $m_0(\beta)=m^*(\beta)(1-2v_0)$, where v_0 is defined in (5.4). In Proposition 2.3 of [CGMS] (and in the following remark) it was already proved that, under the same assumptions of Theorem 6.3 we have (remember that $\mathcal{N}_L^m=\lfloor(1+m)|Q_L|/2\rfloor$),

$$\lim_{L\to\infty}-\frac{1}{L}\log\mu_{Q_L}^{\beta,\emptyset}\{\,N_{Q_L}=\mathcal{N}_L^m\}\geq\varphi(m)$$

with equality if $m\leq m_0$. Thus we only have to consider the case $m\in(m_0,m^*)$ and we must prove a lower bound of the form

$$\mu_{Q_L}^{\beta,\emptyset}\{\,N_{Q_L}=\mathcal{N}_L^m\}\geq e^{-L[\varphi(m)+\delta(L)]}\tag{7.8}$$

with $\lim_{L\to\infty}\delta(L)=0$. For this purpose we choose some $m\in(m_0,m^*)$ once and for all, and we establish some useful notation. We set $\bar{Q}_L=\{x\in\mathbb{R}^2:d_\infty(x,Q_L)\leq1/2\}=[0,L]^2-(1/2,1/2)$ and we define $\mathcal{D}_L^{lb}$ as the set of all rectifiable curves $\gamma\in\bar{Q}_L$ such that γ is open with one endpoint on the left side of $\bar{Q}_L$ and the other one on the bottom side of $\bar{Q}_L$. Let $\mathcal{B}_L^{lb}$ be the set of all subsets λ of $\mathcal{E}_{Q_L}$ such that λ connects the left side of Q_L with its bottom side. Similarly, let $\mathcal{B}_L^{*,lb}$ be the set of all subsets λ of $\widetilde{\mathcal{E}}_{Q_L}$ such that λ connects the left side of $\bar{Q}_L$ with its bottom side. Since $m\in(m_0,m^*)$, the set Γ_w defined in (5.5) consists of 4 curves and each of them is a quarter of Wulff shape centered at one of the four corners of Q. One of these curves suitably rescaled belongs to $\mathcal{D}_L^{lb}$ and we denote it by γ_L^w. We then let $w_L=L^{1/4}\sqrt{\log L}$ and define the sets

$$\begin{aligned}\widetilde{T}_L&=\{x\in\bar{Q}_L:d(x,\gamma_L^w)\leq w_L/2\}\\T_L&=\{x\in\bar{Q}_L:d(x,\gamma_L^w)\leq w_L\}\,.\end{aligned}\tag{7.9}$$

The difference $T_L\backslash\widetilde{T}_L$ consists of two disjoint "corridors", which we denote by T_L^i (the "internal" one, *i.e.* closer to the origin) and T_L^e (the "external" one). With this notation we then define the key event E_L^k as the set of all $\sigma\in\Omega_{Q_L}$ such that

(E_1) $\mathcal{G}_{Q_L}^\emptyset(\sigma)$ (remember (6.5)) contains exactly one open contour γ such that $\gamma\in\mathcal{B}_L^{*,lb}$, $\gamma\subset T_L$ and $|\gamma|\leq kL\log L$

(E_2) and all the other contours have length smaller than $k\log L$.

Then, following [IS], we write

$$\mu_{Q_L}^{\beta,\emptyset}\{\,N_{Q_L}=\mathcal{N}_L^m\}\geq\mu_{Q_L}^{\beta,\emptyset}(E_L^k)\,\mu_{Q_L}^{\beta,\emptyset}(\,N_{Q_L}=\mathcal{N}_L^m\,|\,E_L^k)\tag{7.10}$$

and we estimate separately the two factors in the RHS of (7.10).

Proposition 7.4. *Let $\beta > \beta_c$. Then we can choose the constant k in such a way that there exists a positive constant $c(\beta)$ such that, for any L large enough*

$$\mu_{Q_L}^{\beta,\emptyset}(E_L^k) \ge e^{-L\varphi(m) - c(\beta)\sqrt{L}\log L} .$$

Proof. The proof follows word by word the proof of the analogous statement discussed in Section 3.4 of [IS], with some modifications due to the fact that our geometry and our boundary conditions are different from theirs. However, in our context, only two key points of the proof of [IS] have to be reproved, namely Lemma 7.5 and Lemma 7.6 which appear below. Once these results are available, the rest of the proof is an almost literally transposition to our case of their arguments and, in order to avoid useless repetitions, it will be skipped.

The strategy envisaged in [IS] to prove Proposition 7.4 can be explained in simple terms:

(1) One first shows (see lemma 7.5 below) that with probability larger than $e^{-L\varphi(m)-c(\beta)\sqrt{L}\log L}$ there exists $\lambda \in \mathcal{B}_L^{*,lb}$ such that $\lambda \subset \widetilde{T}_L$ and λ^* is FK–empty.

(2) Then one proves that with large probability there exist two FK-clusters $\lambda_i, \lambda_e \in B_L^{lb}$, such that $\lambda_i \in T_L^i$ and $\lambda_e \in T_L^e$.

When both events in (1) and (2) happen then λ_i and λ_e belong to different FK–clusters and therefore the associated Ising spins will have opposite sign with probability 1/2. Thus, conditioned to the existence of a set λ as in (1), with large probability there will be a Peierls contour $\gamma \in B_L^{*,lb}$ with $\gamma \subset T_L$. We are now ready to state our first result. Let $\mathcal{C}_L$ be the event

$$\mathcal{C}_L = \{\ \exists \lambda \in \mathcal{B}_L^{*,lb} \text{ such that } \lambda \subset \widetilde{T}_L \text{ and } \lambda^* \text{ is FK–empty }\}. \tag{7.11}$$

Then we have (see Lemma 3.3.1 in [IS] for an analogous statement in the case of plus boundary conditions)

Lemma 7.5. *Let $\beta > \beta_c$. Then there exists a positive constant $c(\beta)$ such that, for any L large enough*

$$\mu_{Q_L}^{FK,\beta,\emptyset}(\mathcal{C}_L) \ge e^{-L\varphi(m)-c(\beta)\sqrt{L}\log L} .$$

Proof. On the lattice $\mathbb{Z}^2$ define the three mappings ϑ_{vert}, ϑ_{hor} and ϑ_{diag} to be the reflections around the lines in $\mathbb{R}^2$ given by $x = -\frac{1}{2}$, $y = -\frac{1}{2}$ and $y = -x-1$ respectively. For simplicity we denote with the same symbol the induced mappings on the configuration space $\Omega^{FK} = \{0,1\}^{\mathcal{E}_{\mathbb{Z}^2}}$ given by $\vartheta_{\text{vert}}(\omega)([x,y]) = \omega([\vartheta_{\text{vert}}(x), \vartheta_{\text{vert}}(y)])$ and similarly for the others.

Let now $Q_L^{\#} \equiv \vartheta_{\#}(Q_L)$ where $\# = \text{vert}, \text{hor}, \text{diag}$, let $\Lambda = Q_L \cup Q_L^{\text{vert}} \cup Q_L^{\text{hor}} \cup Q_L^{\text{diag}}$ and let $\mathcal{G} = \mathcal{E}_{Q_L} \cup \mathcal{E}_{Q_L^{\text{vert}}} \cup \mathcal{E}_{Q_L^{\text{hor}}} \cup \mathcal{E}_{Q_L^{\text{diag}}}$. Notice that $\mathcal{G}$ contains all bonds in $\mathcal{E}_\Lambda$ minus those connecting different $Q_L^{\#}$'s. So, if A is a negative event in $\{0,1\}^{\mathcal{E}_{Q_L}}$, thanks to the monotonicity property (7.2) and the symmetry of the FK–measures w.r.t. rotations of multiples of $\pi/2$, we have

$$\begin{aligned} \mu_{Q_L}^{FK,\beta,\emptyset}(A) &= \mu_{\mathcal{G}}^{FK,\beta,\emptyset}\big(A \cap \vartheta_{\text{vert}}(A) \cap \vartheta_{\text{hor}}(A) \cap \vartheta_{\text{diag}}(A)\big)^{1/4} \\ &\ge \mu_{\Lambda}^{FK,\beta,\emptyset}\big(A \cap \vartheta_{\text{vert}}(A) \cap \vartheta_{\text{hor}}(A) \cap \vartheta_{\text{diag}}(A)\big)^{1/4} \\ &\ge \mu_{\Lambda}^{FK,\beta,w}\big(A \cap \vartheta_{\text{vert}}(A) \cap \vartheta_{\text{hor}}(A) \cap \vartheta_{\text{diag}}(A)\big)^{1/4} \end{aligned} \tag{7.12}$$

We now apply the above result in order to get a lower bound on $\mu_{Q_L}^{FK,\beta,\emptyset}(\mathcal{C}_L)$. Remember that the unique Wulff curve $\gamma_L^w \in \mathcal{D}_L^{lb}$ is a quarter of the Wulff curve $\hat{\gamma}_L^w$ in Λ, centered at the point $(-\frac{1}{2},-\frac{1}{2})$ and enclosing an area equal to $2L^2(m^*-m)/m^*$. Therefore, letting

$$\widetilde{T}_\Lambda = \{x \in \Lambda : \, d(x,\hat{\gamma}_L^w) \le w_L/2\}$$

we have that the event

$$\mathcal{C}_\Lambda = \{\, \exists \text{ a loop } \lambda \text{ of dual bonds, inside } \widetilde{T}_\Lambda \text{ around the origin, such that } \lambda^* \text{ is FK-empty} \,\} \tag{7.13}$$

is clearly contained in the event $\mathcal{C}_L \cap \vartheta_{\text{vert}}(\mathcal{C}_L) \cap \vartheta_{\text{hor}}(\mathcal{C}_L) \cap \vartheta_{\text{diag}}(\mathcal{C}_L)$. Thus, thanks to (7.12) and using Lemma 3.3.1 of [IS], we get

$$\mu_{Q_L}^{FK,\beta,\emptyset}(\mathcal{C}_L) \ge \mu_\Lambda^{FK,\beta,w}(\mathcal{C}_\Lambda)^{1/4} \ge e^{-L\varphi(m)-c(\beta)\sqrt{L}\log L} \tag{7.14}$$

for a suitable constant $c(\beta)$ and any L large enough. $\square$

The second estimate that we need is a bound on the dual FK connectivity function and it is a key ingredient to carry out the second step of the argument of [IS] (see the estimate above formula 3.4.5 there). Consider the events

$$F_L^\bullet = \{\ \exists \lambda \in \mathcal{B}_L^{lb} \text{ such that } \lambda \subset T_L^\bullet \text{ and } \lambda \text{ is FK–occupied}\ \} \qquad \bullet \in \{i,e\}\,.$$

Then we have the following result:

Lemma 7.6. *Let $\beta > \beta_c$. Then*

$$\lim_{L\to\infty} \mu_{Q_L}^{FK,\beta,\emptyset}(F_L^i) = 1 \quad \textit{and} \quad \lim_{L\to\infty} \mu_{Q_L}^{FK,\beta,\emptyset}(F_L^e) = 1\,.$$

Proof. We consider the event T_L^i, since the proof for T_L^e is identical. Let $Q_{L-1}^* = Q_{L-1} + (1/2,1/2)$ in such a way that $(\mathcal{E}_{Q_L})^* = \mathcal{E}_{Q_{L-1}^*}$. Thanks to the duality relationships (7.1), it is enough to prove that

$$\lim_{L\to\infty} \mu_{Q_{L-1}^*}^{FK,\beta^*,w}(((F_L^i)^c)^*) = 0$$

where the event $((F_L^i)^c)^*$ is the dual of the complement of F_L^i, and it obviously implies the occurrence of the event D_L given by

$$D_L = \left\{ \begin{array}{l} \text{there exists } x,y \in Q_{L-1}^* \text{ such that } d(x,y) \ge L^{1/4} \\ \text{and there is an FK–connection between } x \text{ and } y \end{array} \right\}.$$

Thanks to Theorem 7.1 we get

$$\mu_{Q_{L-1}^*}^{FK,\beta^*,w}(D_L) \le L^2 C e^{-mL^{1/4}}$$

which goes to zero as $L \to \infty$. $\square$

We are left with the problem of proving a lower bound on the last factor in the RHS of (7.10), $\mu_{Q_L}^{\beta,\emptyset}\{\, N_{Q_L} = \mathcal{N}_L^m \,|\, E_L^k \}$, which does not spoil the good lower bound we already have on $\mu_{Q_L}^{\beta,\emptyset}(E_L^k)$. As in [IS] we have

Proposition 7.7. *Let $\beta > \beta_c$ and k be fixed. Then there exists a constant $c(\beta,k)$ such that*

$$\mu_{Q_L}^{\beta,\emptyset}(\, N_{Q_L} = \mathcal{N}_L^m \mid E_L^k \,) \geq e^{-c(\beta,k)\sqrt{L}\,(\log L)^2}$$

for any L large enough.

Proof. Define a new scale for large contours as $s(L) = k \log L$. For any contour $\gamma \in B_L^{*,lb}$ such that $\gamma \subset T_L$, we set $\Lambda = A_\gamma \cup B_\gamma$ as explained in Section 6.1 and define the sets A_γ°, B_γ° as in (6.7). We can write the event E_γ (remember (6.8)) as the disjoint union $E_\gamma = E_\gamma^+ \cup E_\gamma^-$, where the superscript $+$ means that $\sigma(x) = +1$ for all $x \in \Delta_\gamma^s \cap B_\gamma$ and $\sigma(x) = -1$ for all $x \in \Delta_\gamma^s \cap A_\gamma$, and viceversa for E_γ^-. Thus we have

$$\mu_{Q_L}^{\beta,\emptyset}(\, N_{Q_L} = \mathcal{N}_L^m \mid E_L^k \,) \geq \inf_{\substack{\gamma \in B_L^{*,lb} \\ \gamma \subset T_L}} \mu_{Q_L}^{\beta,\emptyset}(\, N_{Q_L} = \mathcal{N}_L^m \mid E_\gamma \,) = \frac{1}{2} \inf_{\substack{\gamma \in B_L^{*,lb} \\ \gamma \subset T_L}} \mu_{Q_L}^{\beta,\emptyset}(\, N_{Q_L} = \mathcal{N}_L^m \mid E_\gamma^+ \,)\,.$$

By property (E_1) in the definition of E_L^k we know that $|\gamma| \leq kL \log L$. Therefore

$$|\Delta_\gamma^s| \leq 3kL \log L\,. \tag{7.15}$$

Furthermore, since $\gamma \subset T_L$ and since $A_{\gamma_L^w}^\circ = L^2(1 - m/m^*)/2$ and $B_{\gamma_L^w}^\circ = L^2(1+m/m^*)/2$, we have that, for constant C_1,

$$\begin{aligned} \big|\, |A_\gamma^\circ| - L^2(1 - m/m^*)/2 \,\big| &\leq C_1 L w_L \\ \big|\, |B_\gamma^\circ| - L^2(1 + m/m^*)/2 \,\big| &\leq C_1 L w_L\,. \end{aligned} \tag{7.16}$$

Let, for any finite volume V, $\mathcal{N}_V^m = \lfloor (1+m)|V|/2 \rfloor$, so that $\mathcal{N}_L^m = \mathcal{N}_{Q_L}^m$. By (7.16) and (7.15) one easily sees that, if we define $\vartheta(\gamma)$ by

$$\mathcal{N}_L^m = \mathcal{N}_{A_\gamma^\circ}^{-m^*} + \mathcal{N}_{B_\gamma^\circ}^{+m^*} + \vartheta_\gamma^a$$

then we have $|\vartheta_\gamma^a| \leq 3C_1 L w_L$. Moreover, we can write

$$N_{Q_L} = N_{A_\gamma^\circ} + N_{B_\gamma^\circ} + \vartheta_\gamma^b$$

where $\vartheta_\gamma^b = N_{\Delta_\gamma^s}$ is a number which does not depend on the specific configuration $\sigma \in E_\gamma^+$ and satisfies $\vartheta_\gamma^b \leq |\Delta_\gamma^s| \leq C_1 L w_L$. Therefore, by an obvious variation on Proposition 6.1, we get

$$\begin{aligned} \mu_{Q_L}^{\beta,\emptyset}(\, N_{Q_L} = \mathcal{N}_L^m \mid E_\gamma^+ \,) &= \mu_{Q_L}^{\beta,\emptyset}(\, N_{A_\gamma^\circ} + N_{B_\gamma^\circ} = \mathcal{N}_{A_\gamma^\circ}^{-m^*} + \mathcal{N}_{B_\gamma^\circ}^{+m^*} + \vartheta_\gamma^a - \vartheta_\gamma^b \mid E_\gamma^+ \,) \geq \\ &\geq \mu_{A_\gamma^\circ,s}^{\beta,\emptyset^{-,Q_L}}(\, N_{A_\gamma^\circ} = \mathcal{N}_{A_\gamma^\circ}^{-m^*} + \vartheta_\gamma^a \,)\, \mu_{B_\gamma^\circ,s}^{\beta,\emptyset^{+,Q_L}}(\, N_{B_\gamma^\circ} = \mathcal{N}_{B_\gamma^\circ}^{m^*} - \vartheta_\gamma^b \,)\,. \end{aligned} \tag{7.17}$$

We now have to find appropriate lower bounds on both factors of the RHS of (7.17). Since the treatment is identical (modulo spin–flip), we consider only the second factor. Let $B_\gamma^{\circ\circ} = B_\gamma^\circ \backslash \partial_{2s} B_\gamma^\circ$. The b.c. $\emptyset^{+,Q_L}$ is of type $(s,+)$, so we proceed as in the proof of Lemma 6.2 and show that to (almost) every σ we can associate a $\nearrow$–path $\mathcal{C}(\sigma)$ which is the most external $\nearrow$–path of $+1$ spins surrounding $B_\gamma^{\circ\circ}$. It also clear that this path, being the "most external", can be written as a disjoint union

$$\mathcal{C}(\sigma) = \mathcal{C}_1(\sigma) \cup \mathcal{C}_2(\sigma) \tag{7.18}$$

where $C_1(\sigma) \subset \Delta_\gamma^s$ and $\mathcal{C}_2(\sigma) \subset \partial_{2s} Q_L$. Therefore

$$|\mathcal{C}(\sigma)| = |\mathcal{C}_1(\sigma)| + |\mathcal{C}_2(\sigma)| \le |\Delta_g^s| + |\partial_{2s} Q_L| \le 3kL\log L + 8kL\log L = 11kL\log L\,. \tag{7.19}$$

We can also define the sets $X^i(\sigma)$ which is the set of all sites x surrounded by $\mathcal{C}(\sigma)$ and $X^e(\sigma) = B_\gamma^\circ \backslash X^i(\sigma)$. Then, again we have $X^e(\sigma) \subset \Delta_\gamma^s \cup \partial_{2s} Q_L$, so

$$|X^e(\sigma)| \le 11kL\log L\,, \qquad |\partial X^e(\sigma)| \le 11kL\log L\,, \qquad |\partial X^i(\sigma)| \le 11kL\log L\,.$$

The idea is then to condition on what happens "outside" $\mathcal{C}(\sigma)$, and to use the DLR property. In this way we get

$$\begin{aligned}
&\mu_{B_\gamma^\circ,s}^{\beta,\emptyset^+,Q_L}\left(N_{B_\gamma^\circ} = \mathcal{N}_{B_\gamma^\circ}^{m^*} - \vartheta_\gamma^b \right) \ge \inf_{X^e} \inf_{\sigma_{X^e}} \mu_{B_\gamma^\circ,s}^{\beta,\emptyset^+,Q_L}\left(N_{B_\gamma^\circ} = \mathcal{N}_{B_\gamma^\circ}^{m^*} - \vartheta_\gamma^b \,|\, X^e(\sigma) = X^e, \sigma_{X^e} \right) = \\
&= \inf_{X^e} \inf_{\sigma_{X^e}} \mu_{B_\gamma^\circ,s}^{\beta,\emptyset^+,Q_L} (N_{X^i} = \mathcal{N}_{B_\gamma^\circ}^{m^*} - \vartheta_\gamma^b - N_{X^e} \,|\, X^e(\sigma) = X^e, \sigma_{X^e}) \ge \\
&\ge \inf_{X^e} \inf_{\sigma_{X^e}} \mu_{X^i,s}^{\beta,+}(N_{X^i} = \mathcal{N}_{B_\gamma^\circ}^{m^*} - \vartheta_\gamma^b - N_{X^e}(\sigma)) \ge \\
&\ge \inf_{\substack{X\subset Q_L : X \supset B_\gamma^{\circ\circ} \\ |\partial X| \le 11kL\log L}} \inf_{|n| \le 2C_1 L w_l} \mu_{X,s}^{\beta,+}(N_X = \mathcal{N}_{B_\gamma^\circ}^{m^*} + n) \ge \\
&\ge \inf_{\substack{X\subset Q_L : |X| \ge C_2 L^2 \\ |\partial X| \le 11kL\log L}} \inf_{|n| \le 2C_1 L^{5/4}\log L} \mu_{X,s}^{\beta,+}(N_X = \mathcal{N}_{B_\gamma^\circ}^{m^*} + n)\,.
\end{aligned}$$

At this point we can use inequalities (1.1.1) and Lemma 2.3.3 in [IS], and the proposition is proven. □

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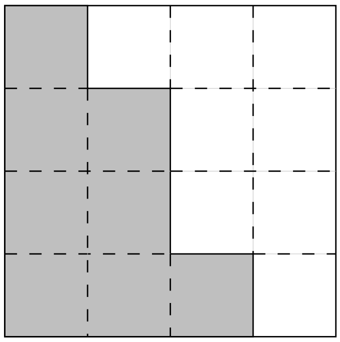

Fig. 1. The set U

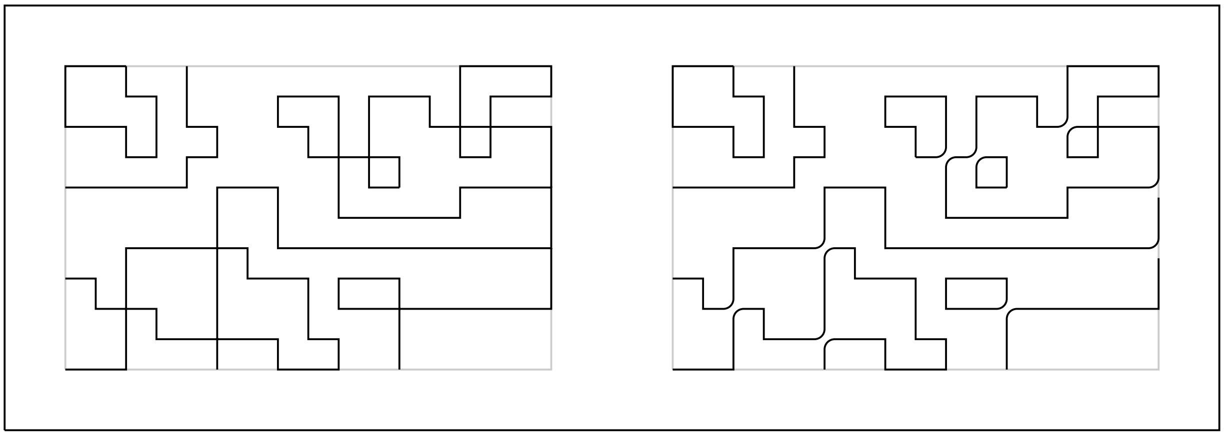

Fig. 2. A collection of edges (left) is split into a collection of contours (right)

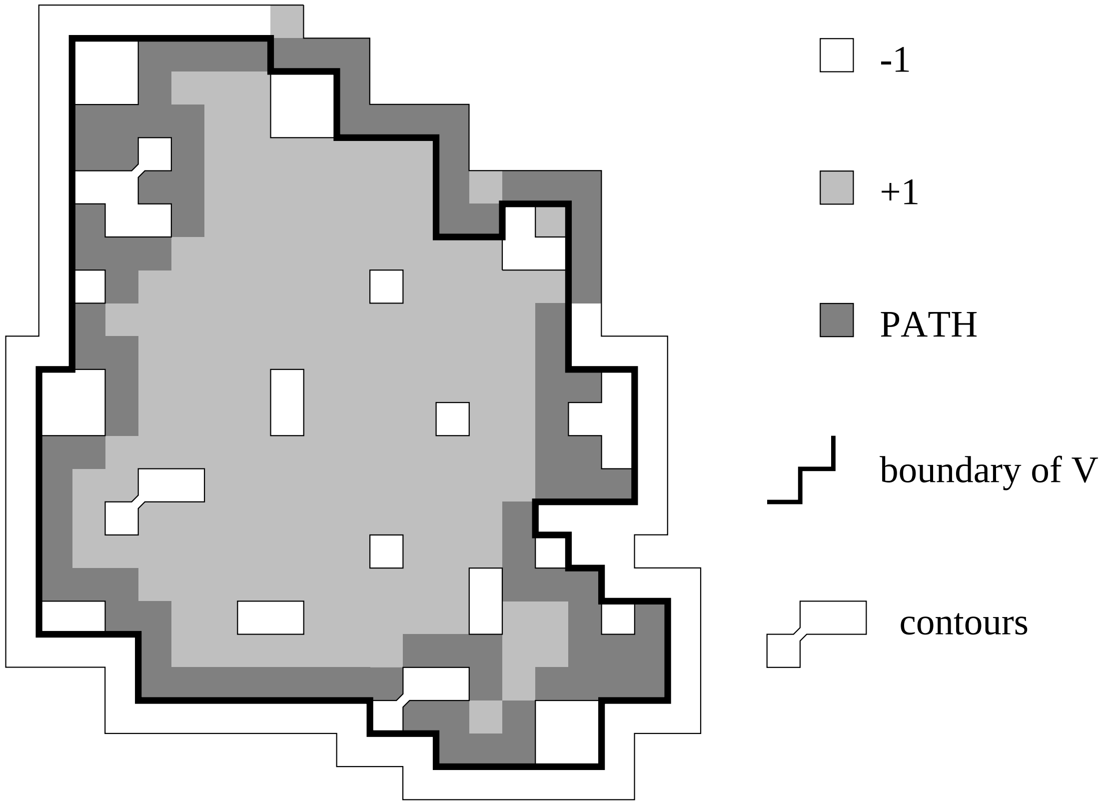

Fig. 3. An example of how to construct the $\nearrow$–path of Proposition 6.2

Figure captions

Fig. 1 The set U

Fig. 2 A collection of edges (left) is split into a collection of contours (right)

Fig. 3 An example of how to construct the $\nearrow$–path of Proposition 6.2