Wave Operators for the Surface Maryland Model

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Abstract

We study scattering properties of the discrete Laplacian H on the half-space $\mathbf{Z}_+^{d+1} = \mathbf{Z}^d \times \mathbf{Z}_+$ with the boundary condition $\psi(n, -1) = \lambda \tan(\pi\alpha \cdot n + \theta)\psi(n, 0)$, where $\alpha \in [0, 1]^d$. We denote by H_0 the Dirichlet Laplacian on $\mathbf{Z}_+^{d+1}$. Khoruzenko and Pastur [KP] have shown that if α has typical Diophantine properties then the spectrum of H on $\mathbf{R} \setminus \sigma(H_0)$ is pure point and that corresponding eigenfunctions decay exponentially. In [JM1] it was shown that for every α independent over rationals the spectrum of H on $\sigma(H_0)$ is purely absolutely continuous. In this paper, we continue the analysis of H on $\sigma(H_0)$ and prove that whenever α is independent over rationals, the wave operators $\Omega^{\pm}(H, H_0)$ exist and are complete on $\sigma(H_0)$. Moreover, we show that under the same conditions H has no surface states on $\sigma(H_0)$.

1 Introduction

This work is a continuation of our series of papers [JM1, JM2, JM3] which deals with spectral and scattering theory of the discrete Laplacian on the half-space with a quasi-periodic or random boundary condition. This program was initiated in [JMP], and its principal goal is to understand the formation and the propagation properties of surface states in regions with corrugated boundaries. The history of this problem and its physical aspects are discussed in [JMP, KP]. For some recent rigorous work on the subject we refer the reader to [AM, BS, G, JM1, JM2, JM3, JMP, JL1, JL2, KP, M1, P].

Let us recall the model. Let $d \geq 1$ be given and let $\mathbf{Z}_+^{d+1} := \mathbf{Z}^d \times \mathbf{Z}_+$, where $\mathbf{Z}_+ = \{0, 1, \ldots\}$. We denote the points in $\mathbf{Z}_+^{d+1}$ by $\mathbf{n} = (n, x)$, $n \in \mathbf{Z}^d$, $x \in \mathbf{Z}_+$. Let H be the discrete Laplacian on $\mathcal{H} := l^2(\mathbf{Z}_+^{d+1})$ with the boundary condition $\psi(n, -1) = V(n)\psi(n, 0)$. When $V = 0$ the operator H reduces to the Dirichlet Laplacian which we denote by H_0. The operator H acts as

$$(H\psi)(n,x) = \begin{cases} \sum_{|n-n'|_+ + |x-x'|=1} \psi(n', x') & \text{if } x > 0, \\ \psi(n, 1) + \sum_{|n-n'|_+=1} \psi(n', 0) + V(n)\psi(n, 0) & \text{if } x = 0, \end{cases}$$

where $|n|_+ = \sum_{j=1}^d |n_j|$. Note that operator H can be viewed as the Schrödinger operator

$$H = H_0 + V, \tag{1.1}$$

where the potential V acts only along the boundary $\partial\mathbf{Z}_+^{d+1} = \mathbf{Z}^d$, that is, $(V\psi)(n, x) = 0$ if $x > 0$ and $(V\psi)(n, 0) = V(n)\psi(n, 0)$. For many purposes it is convenient to adopt this point of view and we will do so in the sequel. We recall that the spectrum of H_0 is purely absolutely continuous and that

$$\sigma(H_0) = [-2(d + 1), 2(d + 1)].$$

The starting point of this paper is the following result proven in [JL1]: For any boundary potential V the wave operators

$$\Omega^{\pm} := \mathrm{s} - \lim_{t \to \mp\infty} \mathrm{e}^{\mathrm{i}tH} \mathrm{e}^{-\mathrm{i}tH_0} \tag{1.2}$$

exist. An obvious question is: under what conditions on V are the wave operators $\Omega^{\pm}$ complete on $\sigma(H_0)$? In this paper we answer this question if V is the Maryland potential. Some physical implications of the completeness of the wave operators are discussed below.

Before we introduce the surface Maryland model let us briefly recall the usual Maryland model. Let $\alpha = (\alpha_1, \ldots, \alpha_d) \in [0, 1]^d$ and $\theta \in [0, \pi]$ be given. *The Maryland potential* on $\mathbf{Z}^d$ is the function

$$V_{\alpha,\theta}(n) = \tan(\pi\alpha \cdot n + \theta), \qquad n \in \mathbf{Z}^d. \tag{1.3}$$

To avoid singular cases, we will always assume that for a given α, θ is chosen so that $\forall n$,

$$\pi\alpha \cdot n + \theta \not\equiv 0 \mod \pi/2. \tag{1.4}$$

We remark that θ is an auxiliary parameter which will play little role in what follows. The results described and proven in this paper hold for all θ which satisfy (1.4).

The usual Maryland model is a family of operators on $l^2(\mathbf{Z}^d)$ of the form $h_{\lambda,\alpha,\theta} = h_0 + \lambda V_{\alpha,\theta}$, where λ is a real parameter and h_0 the discrete Laplacian on $l^2(\mathbf{Z}^d)$. This model has been extensively studied in [FP, FGP, FGP1, PRF, S1, S2]. We say that $\alpha = (\alpha_1, \dots, \alpha_d)$ is independent over rationals if for any choice of rational numbers $r_1, \dots, r_d \in \mathbf{Q}$,

$$\sum r_k \alpha_k \notin \mathbf{Q}.$$

We say that α has typical Diophantine properties if there exist constants $C, k > 0$ such that

$$|n \cdot \alpha - m| > C|n|^{-k}, \tag{1.5}$$

for all $n \in \mathbf{Z}^d$ and $m \in \mathbf{Z}$. The set of α's in $[0,1]^d$ for which (1.5) holds has Lebesgue measure 1. If α has typical Diophantine properties then for all $\lambda \neq 0$, $\sigma(h_{\lambda,\alpha,\theta}) = \mathbf{R}$, the spectrum is pure point, the eigenvalues of $h_{\lambda,\alpha,\theta}$ are simple and the corresponding eigenfunctions decay exponentially. (See [CFKS, FP].) Thus, in any dimension and for typical α, the potential (1.3) is strongly localizing.

The surface Maryland model is the family of operators on $l^2(\mathbf{Z}_+^{d+1})$ defined by

$$H_{\lambda,\alpha,\theta} := H_0 + \lambda V_{\alpha,\theta}, \tag{1.6}$$

where $V_{\alpha,\theta}$ acts only along the boundary $\partial \mathbf{Z}_+^{d+1} = \mathbf{Z}^d$. It follows from the existence of wave operators (1.2) that for any λ and α, $\sigma(H_0) \subset \sigma_{\rm ac}(H_{\lambda,\alpha,\theta})$.

Notation. In the sequel, we use the shorthand $c_d = 2(d+1)$, so $\sigma(H_0) = [-c_d, c_d]$.

To the best of our knowledge, the model (1.6) was first studied in [KP], where the following result was proven.

Theorem 1.1 *Assume that α has typical Diophantine properties. Then, for all $\lambda \neq 0$, $\sigma(H_{\lambda,\alpha,\theta}) = \mathbf{R}$ and the spectrum of H on the set $\mathbf{R} \setminus (-c_d, c_d)$ is pure point. On this set, the eigenvalues are simple and the corresponding eigenfunctions decay exponentially.*

In [JM1] we have proven the following result

Theorem 1.2 *Assume that $\alpha \in [0,1]^d$ is independent over rationals. Then, for all λ, the spectrum of $H_{\lambda,\alpha,\theta}$ on $(-c_d, c_d)$ is purely absolutely continuous.*

We now turn to the subject of this paper, namely the scattering theory for $H_{\lambda,\alpha,\theta}$ on $\sigma(H_0)$. We first recall some basic facts. Let A and B be self-adjoint operators on a Hilbert space $\mathfrak{H}$. We denote by $\mathbf{1}_\Theta(A)$ the spectral projection of A onto the Borel set Θ. Assume that for a given Borel set Θ the wave operators

$$W^\pm := \mathrm{s} - \lim_{t\to\mp\infty} \mathrm{e}^{\mathrm{i}tB}\mathrm{e}^{-\mathrm{i}tA}\mathbf{1}_\Theta(A) \tag{1.7}$$

exist. Note that for any real s, $\mathrm{e}^{\mathrm{i}sB}W^\pm = W^\pm \mathrm{e}^{\mathrm{i}sA}$, which yields that for any bounded Borel function f, $f(B)W^\pm = W^\pm f(A)$. In particular, $\mathrm{Ran}W^\pm \subset \mathrm{Ran}\mathbf{1}_\Theta(B)$. The wave operators $W^\pm$ are *complete* on Θ if $\mathrm{Ran}W^\pm = \mathrm{Ran}\mathbf{1}_\Theta(B)$. One can easily show that the wave operators $W^\pm$ are complete on Θ iff the wave operators

$$U^\pm := \mathrm{s} - \lim_{t\to\mp\infty} \mathrm{e}^{\mathrm{i}tA}\mathrm{e}^{-\mathrm{i}tB}\mathbf{1}_\Theta(B) \tag{1.8}$$

exist.

As we have already remarked, it is known that the wave operators

$$\Omega^\pm := \mathrm{s} - \lim_{t\to\mp\infty} \mathrm{e}^{\mathrm{i}tH_{\lambda,\alpha,\theta}}\mathrm{e}^{-\mathrm{i}tH_0},$$

exist for all λ and α. If α is not independent over rationals, $V_{\alpha,\theta}$ is periodic and the wave operators $\Omega^\pm$ in general are not complete on $\sigma(H_0)$ – an additional scattering channel associated to the surface states may overlap $\sigma(H_0)$. The simplest case where this happens is $\alpha = 0$. Then, $V_{\alpha,\theta}(n) = \tan\theta$ is a constant boundary potential and the operator $H_{\lambda,0,\theta}$ is easily diagonalized by separation of variables. Set $a := \lambda\tan\theta$. If $|a| > 1$ then

$$\mathcal{H} = \mathcal{H}_{\mathrm{ac}}(H_{\lambda,0,\theta}) = \mathcal{H}_{\mathrm{ac}}^{(1)} \oplus \mathcal{H}_{\mathrm{ac}}^{(2)},$$

where both subspaces $\mathcal{H}_{\mathrm{ac}}^{(1)}$ and $\mathcal{H}_{\mathrm{ac}}^{(2)}$ are invariant under H and

$$\sigma(H|_{\mathcal{H}_{\mathrm{ac}}^{(1)}}) = \sigma(H_0), \tag{1.9}$$

$$\sigma(H|_{\mathcal{H}_{\mathrm{ac}}^{(2)}}) = [-2d, 2d] + a + a^{-1}. \tag{1.10}$$

The generalized eigenfunctions associated to the channel (1.9) do not decay in any direction (bulk waves) while the generalized eigenfunctions associated to the channel (1.10) decay exponentially in the x-direction (surface waves). Moreover,

$$\mathrm{Ran}\Omega^\pm = \mathcal{H}_{\mathrm{ac}}^{(1)}.$$

Thus if the channels (1.9) and (1.10) overlap then the wave operators $\Omega^\pm$ are not complete on $\sigma(H_0)$.

If α is independent over rationals, the natural question is whether there exists a non-trivial scattering channel on $\sigma(H_0)$. Our first result is

Theorem 1.3 *Assume that α is independent over rationals, Then, for all λ, the wave operators $\Omega^\pm$ are complete on $(-c_d, c_d)$.*

Theorem 1.3 implies that for the surface Maryland model the non-trivial scattering channel on $\sigma(H_0)$ may exist only in the periodic case. It also suggests that the surface states with energies in $\sigma(H_0)$ may exists only in the periodic case, and we turn to this question now.

Physically, the surface states are wave packets which are concentrated near the surface of the medium for all time. The bulk states are the wave packets which propagate away from the surface of the medium. There are obviously many different ways to make these heuristic notions mathematically precise (see e.g. [JMP, DS] for alternative definitions). We adopt the definition proposed in [JL2]. Let $R \geq 0$ be a positive integer and

$$\Gamma_R = \{(n, x) \in \mathbf{Z}_+^{d+1} : 0 \leq x \leq R\}.$$

We denote by $\mathbf{1}_R$ the characteristic function of the set Γ_R and we use the same symbol for the corresponding multiplication operator.

Let V be an arbitrary boundary potential and $H = H_0 + V$. For any $\psi \in \mathcal{H}$ we set

$$P(R, T, \psi) := \frac{1}{2T} \int_{-T}^{T} \left\| \mathbf{1}_R \mathrm{e}^{-\mathrm{i}tH} \psi \right\|^2 \mathrm{d}t.$$

The above heuristic description of the bulk and surface states can be quantified as follows: We say that the vector ψ is a bulk state if

$$\forall R, \qquad \lim_{T \to \infty} P(R, T, \psi) = 0. \tag{1.11}$$

and that it is a surface state if

$$\lim_{R \to \infty} \liminf_{T \to \infty} P(R, T, \psi) = \|\psi\|^2.$$

We denote by $\mathcal{H}_{\mathrm{b}}$ the set of all bulk states, and by $\mathcal{H}_{\mathrm{s}}$ the set of all surface states of the operator H. These sets have the following properties:

Proposition 1.4 *Let V be an arbitrary boundary potential and $H = H_0 + V$. Then,*
(i) *$\mathcal{H}_{\mathrm{b}}$ and $\mathcal{H}_{\mathrm{s}}$ are closed subspaces invariant under H.*
(ii) *$\mathcal{H}_{\mathrm{b}} \perp \mathcal{H}_{\mathrm{s}}$.*
(iii) *$\mathrm{Ran}\mathbf{1}_{\mathbf{R}\setminus\sigma(H_0)}(H) \subset \mathcal{H}_{\mathrm{s}}$, $\mathcal{H}_{\mathrm{b}} \subset \mathrm{Ran}\mathbf{1}_{\sigma(H_0)}(H)$.*

This proposition is proven in [JL2]. We remark that Proposition 1.4 will not be used in the sequel, except for the obvious fact that $\mathcal{H}_{\mathrm{b}}$ is a closed set.

With the above preliminaries, we can state our second result.

Theorem 1.5 *Assume that α is independent over rationals. Then, for all λ, there exists a set $\mathcal{D}$, dense in $\mathrm{Ran}\mathbf{1}_{(-c_d,c_d)}(H_{\lambda,\alpha,\theta})$, such that for $\psi \in \mathcal{D}$,*

$$\int_{\mathbf{R}} \left\| \mathbf{1}_R \mathrm{e}^{-\mathrm{i}tH_{\lambda,\alpha,\theta}} \psi \right\|^2 \mathrm{d}t < \infty. \tag{1.12}$$

In particular, $\mathrm{Ran}\mathbf{1}_{(-c_d,c_d)}(H_{\lambda,\alpha,\theta}) \subset \mathcal{H}_{\mathrm{b}}$.

Remark 1. The estimate (1.12) is the main technical result of this paper. It immediately implies the absence of surface states with energies in $(-c_d, c_d)$. Also, we will prove Theorem 1.3 using this estimate and Kato's theory of smooth perturbations.

Remark 2. The estimate (1.12) is a stronger property then (1.11). For the various refinements of the notion of the bulk state we refer the reader to [JL2].

Theorems 1.1, 1.2, 1.3 and 1.5 complete the program of [JMP] for the surface Maryland model.

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2 The key estimate

Notation. In the sequel, whenever the meaning is clear within the context, we will drop the subscripts λ, α and θ. Thus, we write V for $V_{\alpha,\theta}$, H for $H_{\lambda,\alpha,\theta}$ etc. We will also use the shorthand $\mathcal{R}(z) := (H-z)^{-1}$.

The goal of this section is to prove

Theorem 2.1 *Assume that α is independent over rationals and that $[a,b] \subset (-c_d, c_d)$. Then, for all $m \in \mathbf{Z}^{\mathrm{d}}$, λ and $R \geq 0$,*

$$\sup_{\epsilon\neq 0, e\in[a,b]} \|\mathbf{1}_R \mathcal{R}(e+\mathrm{i}\epsilon)\delta_{(m,0)}\| < \infty.$$

For $\mathbf{n}, \mathbf{m} \in \mathbf{Z}_+^{d+1}$ we set

$$\mathcal{R}(\mathbf{m}, \mathbf{n}; z) := (\delta_{\mathbf{m}} | (H-z)^{-1}\delta_{\mathbf{n}}).$$

We first note that

$$\sup_{\epsilon\neq 0, e\in[a,b]} \|\mathbf{1}_R \mathcal{R}(e+\mathrm{i}\epsilon)\delta_{(m,0)}\| = \sup_{\epsilon\neq 0, e\in[a,b]} \sum_{\mathbf{n}\in\Gamma_R} |\mathcal{R}((m,0), \mathbf{n}; e+\mathrm{i}\epsilon)|^2 \tag{2.13}$$

Let $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$ be the unit circle and $\mathbf{T}^d$ the d-dimensional torus. We denote the points in $\mathbf{T}^d$ by $\phi = (\phi_1, \dots, \phi_d)$ and by $\mathrm{d}\phi$ the usual Lebesgue measure. Let $\mathbf{C}_\pm = \{z : \pm \mathrm{Im} z > 0\}$, $\Phi(\phi) = 2\sum_{k=1}^d \cos\phi_k$, and, for $z \in \mathbf{C}_\pm$, let $r(\phi, z)$ be the root of the quadratic equation

$$X + X^{-1} + \Phi(\phi) = z, \tag{2.14}$$

which satisfies $|r(\phi,z)|<1$. One easily verifies that $z\in\mathbf{C}_\pm \Rightarrow \pm\mathrm{Im}r(\phi,z)<0$. Explicitly, for $z\in\mathbf{C}_\pm$, $r(\phi,z)$ is given by

$$r(\phi,z)=\frac{1}{2}\left(\Phi(\phi)-z-\sqrt{(\Phi(\phi)-z)^2-4}_\pm\right) \tag{2.15}$$

where the branch of the square root is fixed by

$$\sqrt{w}_\pm=\sqrt{x+\mathrm{i}y}=\frac{\sqrt{2}}{2}\left(\sqrt{|w|+x}\pm \mathrm{i}\sqrt{|w|-x}\right), \qquad \pm\mathrm{Im}w>0.$$

Obviously, $r(\phi,z)$ has a well-defined continuous extension from $\mathbf{T}^d\times\mathbf{C}_\pm$ to $\mathbf{T}^d\times\overline{\mathbf{C}}_\pm$. The values of these extensions along the real axis we denote by $r(\phi,e\pm\mathrm{i}0)$.

We denote the other root of the equation (2.14) by $\tilde{r}(\phi,z)$. Clearly, $\tilde{r}(\phi,z)=1/r(\phi,z)$, $\pm\mathrm{Im}\tilde{r}(\phi,z)>0$ for $z\in\mathbf{C}_\pm$ etc.

Let

$$\hat{\mathcal{R}}((m,0),(\phi,x);z):=(2\pi)^{-\frac{d}{2}}\sum_{n\in\mathbf{Z}^d}\mathcal{R}((m,0),(n,x);z)\mathrm{e}^{\mathrm{i}n\phi}.$$

Lemma 2.2 *Assume that $z\in\mathbf{C}_\pm$. Then for any $x\geq 0$,*

$$\hat{\mathcal{R}}((m,0),(\phi,x);z)=\hat{\mathcal{R}}((m,0),(\phi,0);z)r(\phi,z)^x. \tag{2.16}$$

In particular, Theorem 2.1 holds if and only if for all $m\in\mathbf{Z}^d$ and λ,

$$\sup_{\epsilon\neq 0, e\in[a,b]}\int_{\mathbf{T}^d}|\hat{\mathcal{R}}((m,0),(\phi,0);e+\mathrm{i}\epsilon)|^2\mathrm{d}\phi<\infty.$$

Proof. The proof of Relation (2.16) is elementary, see e.g. [JM2] or [JL1]. Clearly,

$$\sum_{\mathbf{n}\in\Gamma_R}|\mathcal{R}((m,0),\mathbf{n};z)|^2\geq(2\pi)^{-d}\int_{\mathbf{T}^d}|\hat{\mathcal{R}}((m,0),(\phi,0);z)|^2\mathrm{d}\phi,$$

and since $|r(\phi,z)|<1$, the relation (2.16) yields

$$\sum_{\mathbf{n}\in\Gamma_R}|\mathcal{R}((m,0),\mathbf{n};z)|^2\leq\frac{R+1}{(2\pi)^d}\int_{\mathbf{T}^d}|\hat{\mathcal{R}}((m,0),(\phi,0);z)|^2\mathrm{d}\phi.$$

These relations and (2.13) yield the second part of the lemma. □

In the sequel we adopt the shorthand

$$\hat{\mathcal{R}}_m(\phi;z):=\hat{\mathcal{R}}((m,0),(\phi,0);z).$$

The following result also follows from a simple computation. For the proof we refer the reader to [JM2] or [JL1].

Proposition 2.3 *Assume that $\lambda = 0$. Then, for all $m \in \mathbf{Z}^d$,*

$$\hat{\mathcal{R}}_m(\phi, z) = -(2\pi)^{-\frac{d}{2}} \mathrm{e}^{\mathrm{i}m\phi} r(\phi, z).$$

In particular, for $\lambda = 0$ Theorem 2.1 holds.

In the sequel we will assume that $\lambda \neq 0$.

We remark that all the results described so far are valid for an arbitrary boundary potential V. To proceed, we have to use the particular structure of the Maryland potential. Let

$$h_m(\phi) := (2\pi)^{-\frac{d}{2}} \mathrm{e}^{\mathrm{i}m\phi}(1 + \mathrm{e}^{-\mathrm{i}(2\theta + 2\pi\alpha)}).$$

The following lemma was proven in [JM1].

Lemma 2.4 *Assume that $z \in \mathbf{C}_\pm$. Then, for $\phi \in \mathbf{T}^d$,*

$$\mathrm{e}^{-2\mathrm{i}\theta} \hat{\mathcal{R}}_m(\phi - 2\pi\alpha; z) \left(\lambda\mathrm{i} - \tilde{r}(\phi - 2\pi\alpha, z)\right) - \hat{\mathcal{R}}_m(\phi; z)(\lambda\mathrm{i} + \tilde{r}(\phi, z)) = h_m(\phi). \tag{2.17}$$

In what follows we distinguish two cases, depending whether λ and $\mathrm{Im} z$ have the same sign or not:
Case 1. $\pm\lambda > 0$, $z \in \mathbf{C}_\pm$.
Case 2. $\pm\lambda > 0$, $z \in \mathbf{C}_\mp$.
We set

$$\tilde{\mathcal{R}}_m(\phi; z) := \hat{\mathcal{R}}_m(\phi; z) \left(\lambda\mathrm{i} \pm \tilde{r}(\phi, z)\right),$$

and

$$\gamma(\phi, z) := \frac{\lambda\mathrm{i} \mp \tilde{r}(\phi, z)}{\lambda\mathrm{i} \pm \tilde{r}(\phi, z)},$$

where we take $\pm = +$ in the Case 1 and $\pm = -$ in the Case 2. The signs are chosen so that for $\lambda \neq 0$ and $z \in \mathbf{C}_\pm$,

$$|\gamma(\phi, z)| < 1.$$

It follows from Lemma 2.4 that in the Case 1,

$$\mathrm{e}^{-2\mathrm{i}\theta} \tilde{\mathcal{R}}_m(\phi - 2\pi\alpha; z)\gamma(\phi - 2\pi\alpha, z) - \tilde{\mathcal{R}}_m(\phi; z) = h_m(\phi), \tag{2.18}$$

and that in the Case 2,

$$\mathrm{e}^{-2\mathrm{i}\theta} \tilde{\mathcal{R}}_m(\phi - 2\pi\alpha; z) - \tilde{\mathcal{R}}_m(\phi; z)\gamma(\phi, z) = h_m(\phi). \tag{2.19}$$

These two equations will play a key role in what follows.

Lemma 2.5 *Assume that for all $m \in \mathbf{Z}^d$ and $\lambda \neq 0$*

$$\sup_{\epsilon \neq 0, e \in [a,b]} \int_{\mathbf{T}^d} |\tilde{\mathcal{R}}_m(\phi; e + \mathrm{i}\epsilon)|^2 \mathrm{d}\phi < \infty.$$

Then Theorem 2.1 holds.

Proof. It follows from the definition of $\tilde{\mathcal{R}}_m$ that

$$|\tilde{\mathcal{R}}_m(\phi, z)| \leq |\lambda|^{-1} |\hat{\mathcal{R}}_m(\phi, z)|.$$

This observation and Lemma 2.5 yield the statement. $\square$

For $e \in \mathbf{R}$ and $\delta > 0$ we set

$$D(e, \delta) := \{z : \mathrm{Re} z \in (e - \delta, e + \delta), 0 < |\mathrm{Im} z| \leq 1\}.$$

We will need

Lemma 2.6 *Let $e_0 \in (-c_d, c_d)$ be given. Then there exist $\delta > 0$ and an open set $\mathcal{O} \subset \mathbf{T}^d$, such that*

$$\sup |\gamma(\phi, z)| < 1, \tag{2.20}$$

where supremum is taken over $\phi \in \mathcal{O}$ and $z \in D(e_0, \delta)$.

Proof. It follows from the definition of $\gamma(\phi, z)$ that it suffices to show that

$$\inf |\mathrm{Im} \tilde{r}(\phi, z)| > 0, \tag{2.21}$$

where the infimum is taken as in (2.20). Note that it follows from (2.15) that for any $e_0 \in (-c_d, c_d)$ there exists $\phi_0 \in \mathbf{T}^d$ such that,

$$|\mathrm{Im} \tilde{r}(\phi_0, e_0 \pm \mathrm{i}0)| > 0. \tag{2.22}$$

Since the function $\tilde{r}$ is continuous on the sets $\mathbf{T}^d \times \overline{\mathbf{C}}_{\pm}$, the estimate (2.21) follows from (2.22). $\square$

In the sequel we fix $m \in \mathbf{Z}^d$ and $\lambda \neq 0$. Our next result is an improvement of the key estimate in [JM1].

Proposition 2.7 *Let $e_0 \in (-c_d, c_d)$ be given and assume that α is independent over rationals. Then there exist $\delta > 0$ such that*

$$\sup \int_{\mathbf{T}^d} |\tilde{\mathcal{R}}_m(\phi; z)| \mathrm{d}\phi < \infty,$$

where the supremum is taken over $z \in D(e_0, \delta)$.

Remark. This proposition (see [JM1]), implies that the spectrum of H on $(-c_d, c_d)$ is purely absolutely continuous.

Proof. We will consider the Case 1. One argues similarly in the Case 2. It follows from the equation (2.18) that for all $z \in \mathbf{C}_{\pm}$,

$$|\tilde{\mathcal{R}}_m(\phi; z)| \leq 2(2\pi)^{-\frac{d}{2}} + |\gamma(\phi - 2\pi\alpha, z)||\tilde{\mathcal{R}}_m(\phi - 2\pi\alpha; z)|.$$

Integrating over $\mathbf{T}^d$ we derive

$$\int_{\mathbf{T}^d} |\tilde{\mathcal{R}}_m(\phi; z)| \mathrm{d}\phi \leq C_0 + \int_{\mathbf{T}^d} |\gamma(\phi, z)||\tilde{\mathcal{R}}_m(\phi; z)| \mathrm{d}\phi, \tag{2.23}$$

where $C_0 = 2(2\pi)^{\frac{d}{2}}$. Now let δ and $\mathcal{O}$ be as in Lemma 2.6. Splitting the integrals in (2.23) over $\mathcal{O}$ and $\mathbf{T}^d \setminus \mathcal{O}$ we derive

$$\begin{aligned} \textstyle\int_{\mathcal{O}}(1 - |\gamma(\phi; z))|\tilde{\mathcal{R}}_m(\phi; z)| \mathrm{d}\phi &\leq C_0 + \textstyle\int_{\mathcal{O}}(|\gamma(\phi; z)| - 1)|\tilde{\mathcal{R}}_m(\phi; z)| \mathrm{d}\phi \\ &\leq C_0, \end{aligned}$$

where we used that $|\gamma(\phi, z)| < 1$. It now follows from Lemma 2.6 that there exists a constant C such that

$$\sup \int_{\mathcal{O}} |\tilde{\mathcal{R}}_m(\phi; z)| \mathrm{d}\phi < C, \tag{2.24}$$

where the supremum is taken over $z \in D(e_0, \delta)$.

Let $T_\alpha : \mathbf{T}^d \mapsto \mathbf{T}^d$ be the translation map $T_\alpha(\phi) = \phi + 2\pi\alpha$. We set $\mathcal{O}_k = T_\alpha^k(\mathcal{O})$. It follows from the equation (2.18) and the estimate (2.24) that

$$\sup \int_{\mathcal{O}_1} |\tilde{\mathcal{R}}_m(\phi; z)| \mathrm{d}\phi < C_0 + C, \tag{2.25}$$

and inductively that for any k,

$$\sup \int_{\mathcal{O}_k} |\tilde{\mathcal{R}}_m(\phi; z)| \mathrm{d}\phi < kC_0 + C, \tag{2.26}$$

where the supremums are taken over $z \in D(e_0, \delta)$. Since α is independent over rationals, the translation T_α is an ergodic map, and the open sets $\mathcal{O}_k$ cover $\mathbf{T}^d$. Picking a finite subcover, we obtain the statement. □

We are now able to prove

Proposition 2.8 *Let $e_0 \in (-c_d, c_d)$ be given. Then there exists $\delta > 0$ such that*

$$\sup \int_{\mathbf{T}^d} |\tilde{\mathcal{R}}_m(\phi; z)|^2 \mathrm{d}\phi < \infty, \tag{2.27}$$

where supremum is taken over $z \in D(e_0, \delta)$.

Proof. We again consider the Case 1. It follows from the equation (2.18) that

$$\int_{\mathbf{T}^d} |\tilde{\mathcal{R}}_m(\phi;z)|^2 \mathrm{d}\phi \le 4 + 4(2\pi)^{-\frac{d}{2}} \int_{\mathbf{T}^d} |\gamma(\phi,z)||\tilde{\mathcal{R}}_m(\phi;z)|\mathrm{d}\phi + \int_{\mathbf{T}^d} |\gamma(\phi,z)|^2 |\tilde{\mathcal{R}}_m(\phi;z)|^2 \mathrm{d}\phi.$$

It follows from Proposition 2.7 that there exist $\delta > 0$ and a constant C, independent of z, such that for $z \in D(e_0, \delta)$,

$$\int_{\mathbf{T}^d} |\tilde{\mathcal{R}}_m(\phi;z)|^2 \mathrm{d}\phi \le C + \int_{\mathbf{T}^d} |\gamma(\phi,z)|^2 |\tilde{\mathcal{R}}_m(\phi;z)|^2 \mathrm{d}\phi.$$

From this point the proof follows line by line the proof of Proposition 2.7. □

We are now ready to finish the
Proof of Theorem 2.1. It follows from Proposition 2.8 that for any $e \in [a,b]$ we can find $\delta > 0$ so that the estimate (2.27) holds. Clearly, the open sets $(e-\delta, e+\delta)$ cover $[a,b]$. Picking a finite subcover, we derive the statement from Proposition 2.5. □

3 Dynamics

In this section we establish some dynamical consequences of Theorem 2.1 and prove Theorem 1.5. In the sequel we assume that the conditions of Theorem 2.1 are satisfied and we fix $R \ge 0$, $m \in \mathbf{Z}^d$, λ and α.

Proposition 3.1 *Let $\chi \in C_0^\infty(\mathbf{R})$ be such that* $\mathrm{supp}\chi \subset (-c_d, c_d)$*. Then*

$$\sup_{\epsilon \ne 0} \int_{\mathbf{R}} \|\mathbf{1}_R \mathcal{R}(e + \mathrm{i}\epsilon)\chi(H)\delta_{(m,0)}\|^2 \mathrm{d}e < \infty.$$

Proof. Let $[a,b]$ be an interval such that $\mathrm{supp}\chi \subset [a,b] \subset (-c_d, c_d)$, where the first inclusion is proper. Then there is a constant C such that $\forall e \in \mathbf{R} \setminus [a,b]$,

$$\sup_{\epsilon \ne 0} \|\mathbf{1}_R \mathcal{R}(e + \mathrm{i}\epsilon)\chi(H)\delta_{(m,0)}\| \le C/\mathrm{dist}(e, \mathrm{supp}\chi).$$

Thus, it suffices to show that

$$\sup_{\epsilon \ne 0, e \in [a,b]} \|\mathbf{1}_R \mathcal{R}(e + \mathrm{i}\epsilon)\chi(H)\delta_{(m,0)}\| < \infty.$$

Let $\tilde{\chi}$ be an almost analytic extension of χ. By the Helffer-Sjöstrand formula,

$$\chi(H) = \frac{1}{\pi} \int_{\mathbf{C}} \frac{\partial \tilde{\chi}(z)}{\partial \overline{z}} \mathcal{R}(z) \mathrm{d}x \mathrm{d}y.$$

For the basic facts about almost analytic extensions and the Helffer-Sjöstrand formula we refer the reader to [Da]. It follows that for any $w \in \mathbf{C}_{\pm}$,

$$\mathbf{1}_R \mathcal{R}(w)\chi(H)\delta_{(m,0)} = \frac{1}{\pi}\int_{\mathbf{C}} A(w,z)\mathrm{d}x\mathrm{d}y, \tag{3.28}$$

where

$$A(w,z) := \frac{\partial\tilde{\chi}(z)}{\partial\bar{z}}\frac{1}{z-w}\mathbf{1}_R(\mathcal{R}(w)-\mathcal{R}(z)). \tag{3.29}$$

In deriving (3.28) we used the resolvent identity and that

$$\int_{\mathbf{C}} \|A(w,z)\delta_{(m,0)}\|\mathrm{d}x\mathrm{d}y < \infty.$$

We recall that by the construction of the almost analytic extensions, $\mathrm{supp}\tilde{\chi}$ is a compact set and $\tilde{\chi}(z) = 0$ for $\mathrm{Re} z \notin \mathrm{supp}\chi$. We denote by $B(z_0, r)$ the ball of center z_0 and radius r. If $e \in [a,b]$ and $\epsilon \neq 0$, we derive from (3.28), (3.29) and Theorem 2.1 that there exist constants C and r, independent of e and ϵ, so that

$$\|\mathbf{1}_R\mathcal{R}(e+\mathrm{i}\epsilon)\chi(H)\delta_{(m,0)}\| \leq C\int_{B(0,r)}\frac{1}{|z-e-\mathrm{i}\epsilon|}\mathrm{d}x\mathrm{d}y.$$

This inequality yields

$$\sup_{\epsilon\neq 0, e\in[a,b]}\|\mathbf{1}_R\mathcal{R}(e+\mathrm{i}\epsilon)\chi(H)\delta_{(m,0)}\| \leq \sup_{z_0}\int_{B(z_0,r)}\frac{1}{|z|}\mathrm{d}x\mathrm{d}y < \infty.$$

□

Proposition 3.2 *Let $\chi \in C_0^\infty(\mathbf{R})$ be such that* $\mathrm{supp}\chi \subset (-c_d, c_d)$. *Then,*

$$\int_{\mathbf{R}}\left\|\mathbf{1}_R \mathrm{e}^{-\mathrm{i}tH}\chi(H)\delta_{(m,0)}\right\|^2 \mathrm{d}t < \infty.$$

Proof. Let $\epsilon > 0$ be given. By the well-known identity (see e.g. [RS], Section XIII.7),

$$2\pi\int_{\mathbf{R}} \mathrm{e}^{-2\epsilon|t|}\|\mathbf{1}_R\mathrm{e}^{-\mathrm{i}tH}\chi(H)\delta_{(m,0)}\|^2\mathrm{d}t = \sum_{\pm}\int_{\mathbf{R}}\|\mathbf{1}_R\mathcal{R}(e\pm\mathrm{i}\epsilon)\chi(H)\delta_{(m,0)}\|^2\mathrm{d}e. \tag{3.30}$$

The result follows from this identity, Proposition 3.1 and the Monotone convergence theorem.

Proof of Theorem 1.5. It is shown in [JL1] that the set $\{\delta_{(m,0)} : m \in \mathbf{Z}^d\}$ is cyclic for H. Let $\mathcal{D}$ be the linear span of the set

$$\{\chi(H)\delta_{(m,0)} : m \in \mathbf{Z}^d, \chi \in C_0^\infty(\mathbf{R}), \mathrm{supp}\chi \subset (-c_d, c_d)\}.$$

The set $\mathcal{D}$ is dense in $\mathrm{Ran}\mathbf{1}_{(-c_d,c_d)}(H)$, and for $\psi \in \mathcal{D}$ the relation

$$\int_{\mathbf{R}}\left\|\mathbf{1}_R\mathrm{e}^{-\mathrm{i}tH}\psi\right\|^2\mathrm{d}t < \infty,$$

holds by Proposition 3.2. □

4 Wave operators

In this section we prove Theorem 1.3. In the sequel we assume that the conditions of this theorem are satisfied and we fix λ and α.

The proof of Theorem 1.3 is based on Kato's theory of smooth perturbations. We refer the reader to [RS], Section XIII.7, for basic notions and results concerning this theory.

Lemma 4.1 *If the wave operators*

$$\tilde{\Omega}^{\pm} := \mathrm{s} - \lim_{t\to\mp\infty} \mathrm{e}^{\mathrm{i}tH_0}\mathrm{e}^{-\mathrm{i}tH}\mathbf{1}_{(-c_d,c_d)}(H) \tag{4.31}$$

exist then the wave operators $\Omega^{\pm}$ *are complete on* $(-c_d, c_d)$.

The proof of this lemma is elementary.

Lemma 4.2 *For any* $\psi \in \mathrm{Ran}\mathbf{1}_{(-c_d,c_d)}(H)$,

$$\lim_{|t|\to\infty} \mathbf{1}_0\mathrm{e}^{-\mathrm{i}tH}\psi = 0. \tag{4.32}$$

Proof. Let $\mathcal{D}$ be as in Theorem 1.5 and $\psi \in \mathcal{D}$. Let

$$w(t) := \mathrm{e}^{\mathrm{i}tH}\mathbf{1}_0\mathrm{e}^{-\mathrm{i}tH}\psi.$$

By Theorem 1.5, $\int \|w(t)\|^2\mathrm{d}t < \infty$. Moreover, since $[H,\mathbf{1}_0] = [H_0,\mathbf{1}_0]$, $\|w'(t)\| \leq 2\|H_0\|$. This yields that $\lim_{|t|\to\infty} w(t) = 0$ (see Exercise 62 in [RS]). Since $\mathcal{D}$ is dense in $\mathrm{Ran}\mathbf{1}_{(-c_d,c_d)}(H)$ the statement follows. □

We will also make use of the following elementary result (for the proof see [JL1])

Lemma 4.3 *For all* $R \geq 0$ *the projection* $\mathbf{1}_R$ *is* H_0*-smooth. In particular, there are constants* C_R *such that for all* $\psi \in \mathcal{H}$,

$$\int_{\mathbf{R}} \|\mathbf{1}_R\mathrm{e}^{-\mathrm{i}tH_0}\psi\|^2\mathrm{d}t \leq C_R\|\psi\|^2. \tag{4.33}$$

In the sequel we use the shorthand $\mathbf{1}_{\overline{0}} := \mathbf{1} - \mathbf{1}_0$. Let T be a linear operator defined by

$$T\delta_{(n,x)} = \begin{cases} -\delta_{(n,1)} & \text{if } x = 0 \\ \delta_{(n,0)} & \text{if } x = 1 \\ 0 & \text{if } x > 1. \end{cases}$$

The next result we need is

Lemma 4.4 $H_0 \mathbf{1}_{\bar{0}} - \mathbf{1}_{\bar{0}} H = T$.

Proof. Since $\mathbf{1}_{\bar{0}} V = 0$, we have to show that $[H_0, \mathbf{1}_{\bar{0}}] = T$. This relation follows by direct computation. □

Proof of Theorem 1.3 It follows from Lemmas 4.1 and 4.2 that to prove the statement it suffices to show that for a set of vectors ψ dense in $\mathrm{Ran}\mathbf{1}_{(-c_d,c_d)}(H)$, the limits

$$\lim_{t\to\mp\infty} \mathrm{e}^{\mathrm{i}tH_0} \mathbf{1}_{\bar{0}} \mathrm{e}^{-\mathrm{i}tH} \psi \tag{4.34}$$

exist. The proof of this fact follows closely Theorem XIII.24 in [RS].

Let $\mathcal{D}$ be as in Theorem 1.5 and $\psi \in \mathcal{D}$. Let

$$w(t) = \mathrm{e}^{\mathrm{i}tH_0} \mathbf{1}_{\bar{0}} \mathrm{e}^{-\mathrm{i}tH} \psi,$$

and let $\phi \in \mathcal{H}$ be arbitrary. Then, the function $t \mapsto (\phi|w(t)\psi)$ is differentiable and

$$\frac{\mathrm{d}}{\mathrm{d}t}(\phi|w(t)) = \mathrm{i}(\mathrm{e}^{-\mathrm{i}tH_0}\phi|T\mathrm{e}^{-\mathrm{i}tH}\psi),$$

where we used Lemma 4.4. Therefore, if $t > s$,

$$\begin{aligned} |(\phi|w(t) - w(s))| &\leq \textstyle\int_s^t \left|(\mathbf{1}_1 \mathrm{e}^{-\mathrm{i}\tau H_0}\phi|T\mathbf{1}_1\mathrm{e}^{-\mathrm{i}\tau H}\psi)\right| \mathrm{d}\tau \\ &\leq \left(\textstyle\int_{\mathbf{R}} \|\mathbf{1}_1 \mathrm{e}^{-\mathrm{i}\tau H_0}\phi\|^2 \mathrm{d}\tau\right)^{\frac{1}{2}} \left(\textstyle\int_s^t \|\mathbf{1}_1\mathrm{e}^{-\mathrm{i}\tau H}\psi\|^2\mathrm{d}\tau\right)^{\frac{1}{2}}, \end{aligned}$$

where we used that $T = \mathbf{1}_1 T \mathbf{1}_1$ and $\|T\| = 1$. It follows from Lemma 4.3 that for some constant C,

$$\|w(t) - w(s)\| \leq C\left(\int_s^t \|\mathbf{1}_1 \mathrm{e}^{-\mathrm{i}\tau H}\psi\|^2\mathrm{d}\tau\right)^{\frac{1}{2}}.$$

Since $\psi \in \mathcal{D}$, by Theorem 1.5 the integrand on the right-hand side of the last equation is in $L^1(\mathbf{R})$. Therefore the sequence $w(t)$ is Cauchy as $t \to \infty$ or $t \to -\infty$. Since $\mathcal{D}$ is dense in $\mathrm{Ran}\mathbf{1}_{(-c_d,c_d)}(H)$, this yields the statement. □

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