Mixed percolation as a bridge between site and bond percolation

Lincoln Chayes* Roberto H. Schonmann†

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Abstract

By using mixed percolation as a bridge between site and bond percolation, we derive a new inequality between the critical points of these processes, which is optimal in a certain sense, and we also extend a result on the crossover exponent of bond diluted Potts models to site diluted Potts models. Some new results about the critical line in mixed percolation are also proved.

1 Introduction

Bond and site percolation are similar processes in various respects, and it is easy to overlook their differences and miss the fact that their comparison may sometimes be interesting and

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challenging. The issues addressed in this paper include a non-trivial relation between the critical points of these two processes, and a non-trivial extension to the site percolation setting of a result on diluted Potts models that was only available in the bond percolation setting. For these purposes, we study mixed site-bond percolation, which can be seen as a smooth bridge between site and bond percolation. As a by-product of our approach, we also derive some new results on the critical line in mixed percolation.

1.1 Comparison between critical points. We will consider site or bond percolation on an infinite, locally finite, connected graph $G = (V, E)$. Here V is the set of vertices (sites) and E is the set of edges (bonds). The degree of a vertex is the number of edges incident to it. If the degree of each site is bounded above by a common finite constant the graph G is said to be of bounded degree. Write $Aut(G)$ for the group of graph automorphisms of G, i.e., the class of one-to-one transformations of V onto itself that preserve the graph structure. A graph $G = (V, E)$ is called transitive if for any $x, y \in V$ there exists a graph automorphism of G which maps x to y. In other words G is transitive if $Aut(G)$ acts transitively on V, i.e., it produces a single orbit. Intuitively, a graph is transitive if all its vertices play exactly the same role. A graph $G = (V, E)$ is called quasi-transitive if $Aut(G)$ acting on V produces a finite number of orbits. Intuitively, a graph is quasi-transitive if it has a finite number of types of vertices and vertices of the same type play exactly the same role.

In the site percolation model each site is kept independently of the others with probability s, while the other sites are removed along with all the bonds incident to them. We denote by $\pi^{\mathrm{S}}(s)$ the probability that the remaining graph contains some infinite component (i.e., the probability that percolation occurs). In the bond percolation model each bond is kept independently of the others with probability b, while the other bonds are removed. We denote by $\pi^{\mathrm{B}}(b)$ the probability that the remaining graph contains some infinite component (i.e., the probability that percolation occurs). The critical points are defined by:

$$\begin{aligned} s_c &= \inf\{s \in [0,1] : \pi^{\mathrm{S}}(s) > 0\} = \inf\{s \in [0,1] : \pi^{\mathrm{S}}(s) = 1\}, \\ b_c &= \inf\{b \in [0,1] : \pi^{\mathrm{B}}(b) > 0\} = \inf\{b \in [0,1] : \pi^{\mathrm{B}}(b) = 1\}, \end{aligned}$$

where the second equality in each definition is a consequence of Kolmogorov's 0-1 law.

The inequality

$$b_c \leq s_c, \tag{1}$$

which saturates for trees, is due to [14]. In particular, if $s_c < 1$, then $b_c < 1$. In contrast, there are locally finite graphs (of unbounded degree) for which $b_c < 1$, while $s_c = 1$. For this see [12]. For graphs of bounded degree the following inequality, from [11], shows that this cannot happen:

$$s_c \leq 1 - (1 - b_c)^{D-1}, \tag{2}$$

where D is the maximal degree of the graph. (Relations which are weaker than (2) but sufficient to show that for bounded degree graphs $b_c < 1$ implies $s_c < 1$ can be obtained from [17] and [18] (Remark 6.2); see also [12].)

In [11] the authors implicitly asked if (2) is optimal, or not. The theorem below shows that this is not the case when b_c is close to 1, and provides an inequality which is optimal "up to the value of a constant prefactor". Recall that $\lfloor x \rfloor$ denotes the integer part of x.

Theorem 1.1 *For each integer $D \geq 3$ the following holds.*

a) For any infinite connected graph of maximal degree D,

$$s_c \leq 1 - \frac{1}{2^{2\lfloor D/2 \rfloor + 1}} (1 - b_c)^{\lfloor D/2 \rfloor}.$$

b) There is an infinite connected quasi-transitive graph of maximal degree D for which

$$s_c \geq 1 - 2^{\lfloor D/2 \rfloor + 1} (1 - b_c)^{\lfloor D/2 \rfloor}.$$

Note that an infinite graph of maximal degree at most 2 has $b_c = s_c = 1$, so that the assumption that $D \geq 3$ is harmless. The optimal aspect of Theorem 1.1 is the identification of the exponent $\lfloor D/2 \rfloor$, which is the same in parts (a) and (b). The value of the constant which multiplies $(1 - b_c)^{\lfloor D/2 \rfloor}$ in parts (a) and (b) can be slightly improved, using our techniques with somewhat more work. But since these techniques do not seem to be sufficient to identify the optimal value of this constant, we satisfied ourselves with the statements above.

Question 1. How would Theorem 1.1 be modified if we restricted ourselves to the class of transitive graphs of degree D?

1.2 Crossover exponent for diluted Potts models. We suppose that the reader is familiar with the basic notions and terminology on the statistical mechanics of lattice systems. For an introduction to this subject, consult, e.g., [7]. The q-Potts model on a locally finite (not necessarily infinite or connected) graph $G = (V, E)$ is defined by associating to each site $x \in V$ a random variable (spin) $\sigma(x)$ which takes values $1, 2, ..., q$. To each configuration $\sigma \in \{1, 2, ..., q\}^V$ an energy is formally associated by the expression

$$H_J(\sigma) = -J \sum_{\{x,y\} \in E} \delta_{\sigma(x),\sigma(y)},$$

where $J \geq 0$ is a parameter and $\delta_{a,b} = 1$ if $a = b$, $\delta_{a,b} = 0$ if $a \neq b$. There is a critical value J_c (possibly 0 or ∞) such that if $J < J_c$ the model has a unique Gibbs distribution, but if $J > J_c$ the model has more than one Gibbs distribution. (If G is not connected, a Gibbs distribution on it is obtained by associating an independent Gibbs distribution to each connected component, so that the value of J_c is then the infimum of those of its connected components.) For an introduction to the Potts model and the associated random-cluster models, see, e.g., [8]. It is of relevance to recall that the case $q = 2$ corresponds to the Ising model.

Given an infinite, locally finite, connected graph $G = (V, E)$ one can consider the bond-diluted and the site-diluted q-Potts models on this graph. In each case one studies the q-Potts model on the random graph resulting from performing bond or site percolation on G. (Note that this resulting graph is in general not connected, but as observed in the last paragraph, this is not an important issue.) The study of statistical mechanics models on diluted graphs is a chapter in the study of systems in random environment, a very central area of research in physics and in mathematical physics. For some of the background on the subject, consult [1], [13] and references therein.

One can show, using Kolmogorov's 0-1 law and Theorem 7.33 of [7], or Theorem 3.1 of [4], that for each graph the critical value $J_c^B(b)$ (*resp.* $J_c^S(s)$) for the q-Potts model on the corresponding bond-diluted (*resp.* site-diluted) graph is well defined and not random.

In [1] the following was proved (see displays (9) and (10) in that paper).

Theorem 1.2 *For each infinite, locally finite, connected graph G which has $0 < b_c < 1$, and each q, there exist constants, $0 < A_1^B \leq A_2^B < \infty$ such that for the bond-diluted q-Potts model on G,*

$$A_1^B (b - b_c) \leq \exp(-J_c^B(b)) \leq A_2^B (b - b_c),$$

for $b > b_c$. Given $\delta > 0$, one can take $A_1^B = (1 - \delta)/(qb_c)$ and $A_2^B = 1/b_c$, provided that we restrict ourselves to values of b close enough to b_c.

In particular the crossover exponent Φ^B, defined by the relation

$$\exp(-J_c^B(b)) \sim (b-b_c)^{\Phi^B} \quad \text{in the logarithmic sense,} \quad \text{as} \quad b \searrow b_c,$$

takes the value $\Phi^B = 1$.

The methods in [1] are based on the relation between the Potts model and the random-cluster model (a certain dependent bond percolation model) and on comparisons between the random-cluster model and independent bond percolation. The method does not extend to site diluted Potts models, because the random-cluster model is not immediately comparable to independent percolation. Using the techniques from the current paper, we can fill this gap and prove the following.

Theorem 1.3 *For each infinite connected graph of bounded degree G which has $b_c < 1$, and each q, there exist constants, $0 < A_1^S \leq A_2^S < \infty$, which depend only on q and D, such that for the site-diluted q-Potts models on G,*

$$A_1^S(s-s_c) \leq \exp(-J_c^S(s)) \leq A_2^S(s-s_c),$$

for $s > s_c$. Given $\delta > 0$, one can take $A_1^S = (1-\delta)/(qs_c)$ and $A_2^S = (1+\delta)\, 2^D/(s_c(1-s_c)^{D-2})$, where D is the maximal degree of G, provided that we restrict ourselves to values of s close enough to s_c.

In particular the crossover exponent Φ^S, defined by the relation

$$\exp(-J_c^S(s)) \sim (s-s_c)^{\Phi^S} \quad \text{in the logarithmic sense,} \quad \text{as} \quad s \searrow s_c,$$

takes the value $\Phi^S = 1$.

As far as we can tell, the comparisons between independent bond and site percolation available in the literature (see [17], [18] (Remark 6.2) and [11]), when combined with the arguments from [1], would only allow one to prove a weaker result, which would only imply that if Φ^S exists then it is positive and finite. Results of this type are contained in [5] and [6].

Regarding the motivation behind the study of systems in random environment, it seems natural to consider site-diluted Potts models as more interesting than bond-diluted ones, since site-dilution corresponds to the presence of impurities in a crystal. We stress also that while

Theorem 1.3 is being stated in the context of fairly arbitrary graphs, as far as we know, it is a new result even in the special case of cubic lattices.

1.3 Mixed percolation. In the mixed percolation model each bond is kept independently of anything else with probability b and each site is kept independently of anything else with probability s; the other bonds and sites are removed along with all the bonds incident to removed sites. We denote by $\pi^{\mathrm{BS}}(b,s)$ the probability that the remaining graph contains some infinite component (i.e., the probability that percolation occurs). It is natural to define, for $0 \le b \le 1$,

$$s_c(b) \;=\; \sup\{s \in [0,1] : \pi^{\mathrm{BS}}(b,s) = 0\}.$$

Theorem 1.4 *For any infinite connected graph of bounded degree which has $b_c < 1$, the function $s_c(\cdot)$ has the following properties:*

a) There exist $0 < C_1 \le C_2 < \infty$, which depend only on the maximal degree of the graph, such that for any $b', b'' \in [b_c, 1]$,

$$C_1|b' - b''| \le |s_c(b') - s_c(b'')| \le C_2|b' - b''|.$$

b) $s_c(\cdot)$ is a Lipschitz continuous function on $[0,1]$.

c) $s_c(b) < 1$ iff $b \in (b_c, 1]$.

d) If $b_c \le b' < b'' \le 1$, then $s_c(b'') < s_c(b')$, i.e., $s_c(\cdot)$ is strictly decreasing on $[b_c, 1]$.

It is also natural to define

$$b_c(s) = \inf\{b \in [0,1] : \pi^{\mathrm{BS}}(b,s) > 0\} = \inf\{b \in [0,1] : \pi^{\mathrm{BS}}(b,s) = 1\}.$$

Theorem 1.4 can be rephrased in terms of the function $b_c(\cdot)$. Note also that from this theorem we obtain that the function $s_c(\cdot)$ restricted to the domain $[b_c, 1]$ is the inverse of the function $b_c(\cdot)$ restricted to the domain $[s_c, 1]$. Moreover the boundary between the percolative and the non-percolative regions in the plane (b,s) is precisely the curve

$$\{(b, s_c(b)) : b_c \le b \le 1\} = \{(b_c(s), s) : s_c \le s \le 1\}.$$

An important piece of information contained in Theorem 1.4 refers to the edges of this curve. It implies that for $b_c \leq b \leq 1$,

$$C_1(b - b_c) \leq 1 - s_c(b) \leq C_2(b - b_c) \tag{3}$$

and

$$C_1(1 - b) \leq s_c(b) - s_c \leq C_2(1 - b). \tag{4}$$

In particular the critical exponents defined by

$$1 - s_c(b) \sim (b_c - b)^{\Phi'} \quad \text{in the logarithmic sense,} \quad \text{as} \quad b \searrow b_c$$

and

$$s_c(b) - s_c \sim (1 - b)^{\Phi''} \quad \text{in the logarithmic sense,} \quad \text{as} \quad b \nearrow 1$$

take the value $\Phi' = \Phi'' = 1$. The relations in (4) are the ones that will allow us to prove Theorem 1.3.

Question 2. How smooth is the function $s_c(\cdot)$, with domain restricted to $[b_c, 1]$? (Presumably it is analytic.) Is it concave?

In Section 2 we will derive a differential inequality which will be used in Section 3 to derive estimates on the function $s_c(\cdot)$, and so provide the proof of Theorem 1.4. In Section 4, these estimates on $s_c(\cdot)$ will be used to prove part (a) of Theorem 1.1; also the example which proves part (b) of this theorem will be presented there. Finally in Section 5 the proof of Theorem 1.3, based on (4) will be presented.

2 Differential inequality

In this section we consider mixed percolation on an infinite connected graph G of bounded degree. A chain is a finite sequence of distinct sites, $x_0, ..., x_n$, such that for $i = 0, ..., n-1$, x_i and x_{i+1} are joined by an edge. These edges are called the edges of the chain. The length of such a chain is n, and its end-points are x_0 and x_n. The distance between two sites is the minimal length of the chains which have these sites as end-points. An arbitrary site of the graph is chosen as its root, and denoted by r. The ball of radius n centered at the root, denoted $V(n)$, is the set of sites within distance n of the root. We also set $\tilde{V}(n) = V(n) \backslash \{r\}$.

Clusters are the connected components of the graph obtained from G after sites and bonds have been deleted in the percolation process. We denote by $\theta^{\mathrm{BS}}(b,s)$ the probability that for mixed percolation with parameters (b,s) on G, the root belongs to an infinite cluster. It is well known and easy to see that $\pi^{\mathrm{BS}}(b,s) = 1$ iff $\theta^{\mathrm{BS}}(b,s) > 0$, and that in particular, this last statement is independent of the choice of the root of G.

As usual, bonds and sites which are kept in the percolation process will be said to be occupied and those which are removed will be said to be vacant.

A chain is said to be internally occupied if all its sites, except possibly for its end-points, and all its edges are occupied. Two sites are said to be almost connected if there is an internally occupied chain which has these sites as end-points. We denote by A_n the event that the root is almost connected to a site which is at distance $n+1$ from the root. Set $\tilde{\theta}_n^{\mathrm{BS}}(b,s) = P(A_n)$. Denote, as usual, by $\theta_n^{\mathrm{BS}}(b,s)$ the probability that A_n happens and also the root is occupied. Then it is clear that

$$\lim_{n\to\infty} s\,\tilde{\theta}_n^{\mathrm{BS}}(b,s) = \lim_{n\to\infty} \theta_n^{\mathrm{BS}}(b,s) = \theta^{\mathrm{BS}}(b,s). \tag{5}$$

Proposition 2.1 *For any infinite connected graph of degree bounded above by $D < \infty$, the following differential inequality holds*

$$\frac{\partial \tilde{\theta}_n^{BS}(b,s)}{\partial s} \le \tilde{C}(b,s)\,\frac{\partial \tilde{\theta}_n^{BS}(b,s)}{\partial b}, \tag{6}$$

where $\tilde{C}(b,s)$ can be chosen as $2^{\lfloor D/2\rfloor}/(s\,(1-b)^{\lfloor D/2\rfloor-1})$ or as $2^D/(s\,(1-s)^{D-2})$.

Under the same assumptions one can also prove a differential inequality of the form

$$\frac{\partial \theta_n^{\mathrm{BS}}(b,s)}{\partial s} \le C(b,s)\,\frac{\partial \theta_n^{\mathrm{BS}}(b,s)}{\partial b}. \tag{7}$$

But the corresponding estimate on $C(b,s)$ turns out not to be good enough to prove Theorem 1.1(a) (it would be good enough, though, to prove Theorems 1.4 and 1.3).

Note that two alternative expressions for $\tilde{C}(b,s)$ appear in Proposition 2.1. The first one is important in the proof of Theorem 1.1(a) (the power of $1-b$ which appears in this expression is of major relevance then). The second one is important in the proof of Theorem 1.3 (because it does not blow up as b approaches 1). Both expressions are needed to prove Theorem 1.4 (because they allow one to bound $\tilde{C}(b,s)$ above, uniformly on a domain which excludes a neighborhood of the point $(b,s)=(1,1)$ and a neighborhood of the line $s=0$).

Under the same assumptions of Proposition 2.1, one can also prove the following complementary differential inequalities:

$$\frac{\partial \tilde{\theta}_n^{\rm BS}(b,s)}{\partial b} \leq \frac{D}{b}\,\frac{\partial \tilde{\theta}_n^{\rm BS}(b,s)}{\partial s}, \tag{8}$$

and

$$\frac{\partial \theta_n^{\rm BS}(b,s)}{\partial b} \leq \frac{D}{b}\,\frac{\partial \theta_n^{\rm BS}(b,s)}{\partial s}. \tag{9}$$

All the applications that we would have for them are nevertheless better accomplished by using (1).

Differential inequalities in the spirit of (6), (7), (8) and (9) have appeared in [19], [2], [3], and [10]. They have been used in order to prove strict inequalities between critical points, equalities between critical exponents obtained by approaching a critical point from different directions, and Lipschitz continuity of critical lines. The main applications in our paper are nevertheless of a different nature. In particular, as far as we know, in previous uses of such inequalities estimates on the values of the constants which played the role of our $\tilde{C}(b,s)$ were not a crucial issue.

Proof of Proposition 2.1: We will use the notation E_x to denote the set of edges incident to the vertex x, and V_x to denote the set of vertices which share an edge with x. Let also $E(n)$ be the set of edges which have at least one end-point in the ball $V(n)$. In order to prove the proposition, we only have to consider mixed percolation on the finite graph $G_n=(V(n),E(n))$. A realization of this process is a function $\omega: V(n)\times E(n)\to\{0,1\}$, where 0's are associated to vacant bonds and sites and 1's are associated to occupied bonds and sites.

Given a site $x\in\tilde{V}(n)$, let $\delta_x A_n$ be the event that x is pivotal for A_n, i.e., the event that if x is occupied then A_n happens, but if x is vacant then A_n does not happen. Given a bond $e\in E(n)$, the event $\delta_e A_n$, that e is pivotal for A_n, is defined in an analogous way.

Russo's formula (see, e.g., [9]), states that

$$\frac{\partial \tilde{\theta}_n^{\rm BS}(b,s)}{\partial s} = \sum_{x\in\tilde{V}(n)} P(\delta_x A_n) \qquad \frac{\partial \tilde{\theta}_n^{\rm BS}(b,s)}{\partial b} = \sum_{e\in E(n)} P(\delta_e A_n).$$

The proof will therefore be concluded once we argue that for each $x \in \tilde{V}(n)$

$$s\,P(\delta_x A_n) \le \left(\frac{2}{1-b}\right)^{\lfloor D/2\rfloor -1} \sum_{e \in E_x} P(\delta_e A_n), \tag{10}$$

$$s\,P(\delta_x A_n) \le \frac{2^{D-1}}{(1-s)^{D-2}} \sum_{e \in E_x} P(\delta_e A_n), \tag{11}$$

and use the fact that each bond in $E(n)$ obviously contains at most 2 end-points in $\tilde{V}(n)$.

In order to prove (10), first note that $s\,P(\delta_x A_n) = P(A_{x,n})$, where $A_{x,n}$ is the event that the site x is occupied and that $\delta_x A_n$ holds. We will introduce a transformation $F_x^B : A_{x,n} \to \cup_{e\in E_x} \delta_e A_n$, with the following properties:

i) $P(F_x^B(\omega)) \ge (1-b)^{\lfloor D/2\rfloor -1}\, P(\omega)$.

ii) Each element of $\cup_{e\in E_x} \delta_e A_n$ is the image of at most $2^{\lfloor D/2\rfloor -1}$ elements of $A_{x,n}$.

The existence of such a transformation clearly implies (10). To produce F_x^B, we suppose the bonds in E_x to be ordered in some arbitrary fashion, and we proceed as follows.

Given $\omega \in A_{x,n}$, let $E_{x,\omega}^{\mathrm{in}}$ be the set of bonds in E_x which in the configuration ω have the following property. Their end-point distinct from x is either the root, or else it is a site which is occupied and connected to the root by an internally occupied chain which does not pass through x. Let $E_{x,\omega}^{\mathrm{out}}$ be the set of bonds in E_x, which in the configuration ω have the following property. Their end-point distinct from x is either a site at distance $n+1$ from the root, or else it is a site which is occupied and connected to a site at distance $n+1$ from the root by an internally occupied chain which does not pass through x. Since $\omega \in \delta_x A_n$, we have $E_{x,\omega}^{\mathrm{in}} \ne \emptyset$, $E_{x,\omega}^{\mathrm{out}} \ne \emptyset$ and $E_{x,\omega}^{\mathrm{in}} \cap E_{x,\omega}^{\mathrm{out}} = \emptyset$. If $|E_{x,\omega}^{\mathrm{in}}| \le |E_{x,\omega}^{\mathrm{out}}|$, set $E_{x,\omega} = E_{x,\omega}^{\mathrm{in}}$; otherwise set $E_{x,\omega} = E_{x,\omega}^{\mathrm{out}}$. Since $|E_x| \le D$, obviously $|E_{x,\omega}| \le \lfloor D/2\rfloor$. Define $F_x^B(\omega)$ as the configuration obtained from ω by making vacant the first $|E_{x,\omega}|-1$ bonds of $E_{x,\omega}$. In the configuration $F_x^B(\omega)$ the last bond in $E_{x,\omega}$ is pivotal for A_n, so that F_x^B indeed maps $A_{x,n}$ to $\cup_{e\in E_x} \delta_e A_n$. It is clear that also (i) holds. To see that (ii) holds, we can argue as follows. The transformation F_x^B can only modify the configuration at bonds incident to x. Therefore, by looking at $F_x^B(\omega)$ restricted to the other bonds and sites in $E(n)$ and $\tilde{V}(n)$ one can find what the set $E_{x,\omega}$ is. If $F_x^B(\omega') = F_x^B(\omega'')$, ω' and ω'' can only differ at the bonds in $E_{x,\omega'} = E_{x,\omega''}$, with the exception of the last such bond. Since the cardinality of this set where they can differ is at most $\lfloor D/2\rfloor - 1$, this proves (ii).

In order to prove (11), we proceed in a similar fashion. We will introduce a transformation $F_x^S : A_{x,n} \to \cup_{e\in E_x} \delta_e A_n$, with the following properties:

i) $P(F_x^S(\omega)) \geq (1-s)^{D-2}\, P(\omega)$.

ii) Each element of $\cup_{e\in E_x} \delta_e A_n$ is the image of at most 2^{D-1} elements of $A_{x,n}$.

The existence of such a transformation clearly implies (11). To produce F_x^S, we suppose the sites in V_x to be ordered in some arbitrary fashion, except for the fact that if the root is contained in V_x, it will be the last site in the order. Given $\omega \in A_{x,n}$, let $V_{x,\omega}^{\mathrm{in}}$ be the set of sites in V_x which in the configuration ω have the following property. The site is either the root, or else it is occupied, the bond that it shares with x is occupied and it is connected to the root by an internally occupied chain which does not pass through x. Since $\omega \in \delta_x A_n$, we have $V_{x,\omega}^{\mathrm{in}} \neq \emptyset$. Let $\bar{V}_{x,\omega}^{\mathrm{out}}$ be the set of sites in V_x which in the configuration ω are connected to a site at distance $n+1$ from the root by an internally occupied chain which does not pass through x. Set $\bar{V}_{x,\omega}^{\mathrm{in}} = V_x \backslash \bar{V}_{x,\omega}^{\mathrm{out}}$. Note that since $\omega \in A_{x,n}$ we have $\bar{V}_{x,\omega}^{\mathrm{out}} \neq \emptyset$, $V_{x,\omega}^{\mathrm{in}} \subset \bar{V}_{x,\omega}^{\mathrm{in}}$ and hence $|V_{x,\omega}^{\mathrm{in}}| \leq |\bar{V}_{x,\omega}^{\mathrm{in}}| \leq D-1$. Define $F_x^S(\omega)$ as the configuration obtained from ω, by making vacant the first $|V_{x,\omega}^{\mathrm{in}}|-1$ sites of $V_{x,\omega}^{\mathrm{in}}$. Note that in the configuration $F_x^S(\omega)$ the bond in E_x which has as end-points x and the last site in $V_{x,\omega}^{\mathrm{in}}$ is pivotal for A_n, so that F_x^S indeed maps $A_{x,n}$ to $\cup_{e\in E_x} \delta_e A_n$. It is also again clear that (i) holds. To see that (ii) holds, we can argue as follows. The transformation F_x^S can only modify the configuration at sites in V_x. Therefore, by looking at $F_x^S(\omega)$ restricted to the other bonds and sites in $E(n)$ and $\tilde{V}(n)$ one can identify the set $\bar{V}_{x,\omega}^{\mathrm{in}}$. If $F_x^S(\omega') = F_x^S(\omega'')$, ω' and ω'' can only differ at the sites in $\bar{V}_{x,\omega'}^{\mathrm{in}} = \bar{V}_{x,\omega''}^{\mathrm{in}}$. Since the cardinality of this set where they can differ is at most $D-1$, this proves (ii). $\square$

3 Proof of Theorem 1.4

Let

$$\mathcal{P}_G = \{(b,s) \in [0,1]^2 : \pi^{\mathrm{BS}}(b,s) = 1\} = \{(b,s) \in [0,1]^2 : \theta^{\mathrm{BS}}(b,s) > 0\},$$

be the percolative phase of the mixed percolation model on a graph G.

Proposition 3.1 *Suppose that G is an infinite connected graph of degree bounded above by $D < \infty$. Suppose also that for some constants $C > 0$, $0 \le b_1 < b_2 \le 1$, $0 \le s_1 < s_2 \le 1$, the inequality*

$$\frac{\partial \tilde{\theta}_n^{BS}(b,s)}{\partial s} \le C \frac{\partial \tilde{\theta}_n^{BS}(b,s)}{\partial b} \tag{12}$$

holds for arbitrary n and arbitrary $(b,s) \in [b_1,b_2] \times [s_1,s_2]$.

i) If $(b,s) \in ([b_1,b_2] \times [s_1,s_2]) \cap \mathcal{P}_G$, then for all $x \ge 0$ such that $(b+x, s-x/C) \in [b_1,b_2] \times [s_1,s_2]$ we have $(b+x, s-x/C) \in \mathcal{P}_G$.

ii) If $(b,s) \in ([b_1,b_2] \times [s_1,s_2]) \cap (\mathcal{P}_G)^c$, then for all $x \ge 0$ such that $(b-x, s+x/C) \in [b_1,b_2] \times [s_1,s_2]$ we have $(b-x, s+x/C) \in (\mathcal{P}_G)^c$.

Proof: From (12), we have under the conditions in (i):

$$\tilde{\theta}_n^{\mathrm{BS}}(b,s) \le \tilde{\theta}_n^{\mathrm{BS}}(b+x, s-x/C).$$

Multiplying by s, letting $n \to \infty$ and using (5), gives $\theta^{\mathrm{BS}}(b,s) \le \theta^{\mathrm{BS}}(b+x, s-x/C)$, which proves the claim in (i).

The proof of (ii) is analogous. □

In combination with Proposition 2.1, Proposition 3.1 provides estimates on the behavior of the function $s_c(\cdot)$. To obtain estimates in the opposite direction, one can use the complementary inequalities (8) or (9). For our purposes one can do better, though, using (1). This is done in next proposition, in which we are recalling results and arguments from [15].

Proposition 3.2 *Suppose that G is an infinite, locally finite, connected graph.*

i) If $(b,s) \in \mathcal{P}_G$, then for all $0 < \alpha \le 1$ such that $(b\alpha, s/\alpha) \in [0,1]^2$ we have $(b\alpha, s/\alpha) \in \mathcal{P}_G$.

ii) If $(b,s) \in (\mathcal{P}_G)^c$, then for all $0 < \alpha \le 1$ such that $(b/\alpha, s\alpha) \in [0,1]^2$ we have $(b/\alpha, s\alpha) \in (\mathcal{P}_G)^c$.

Proof: To prove (i), consider the random graph G' obtaining by performing mixed percolation on G with parameters $(b, s/\alpha)$. By performing site percolation on G' with parameter α (independently of the process that randomly originated G'), we obtain a random graph G'' which has the law of a random graph obtained from performing mixed percolation with parameters (b, s) on G. Since we are supposing that $(b, s) \in \mathcal{P}_G$ we must have that G'' contains a.s. some infinite component. This means that G' supports site percolation at parameter α. From (1) it follows that G' also supports bond percolation at parameter α. In other words, the random graph G''' obtained by performing bond percolation on G' with parameter α (independently of the process that randomly originated G') contains a.s. some infinite component. But G''' has the law of a random graph obtained from performing mixed percolation with parameters $(b\alpha, s/\alpha)$ on G. This proves (i).

The claim in (ii) follows from that in (i). $\square$

Proof of Theorem 1.4: This theorem is immediate from Propositions 2.1, 3.1, and 3.2. $\square$

4 Proof of Theorem 1.1

Proof of part (a) of Theorem 1.1: We will suppose, with no loss, that $b_c < 1$. We will apply Proposition 3.1 with $b_1 = b_c$, $b_2 = (b_c + 1)/2$, $s_1 = 1/2$, $s_2 = 1$, and

$$C = \frac{2^{\lfloor D/2 \rfloor}}{s_1(1 - b_2)^{\lfloor D/2 \rfloor - 1}} = \frac{2^{2\lfloor D/2 \rfloor}}{(1 - b_c)^{\lfloor D/2 \rfloor - 1}}.$$

Note that this choice of C is possible thanks to Proposition 2.1.

Given $\epsilon > 0$, set $b = b_c + \epsilon$, and $s = 1$. If ϵ is small enough, then $(b, s) \in ([b_1, b_2] \times [s_1, s_2]) \cap \mathcal{P}_G$ and $x = (1 - b_c)/2 - \epsilon > 0$. Since $x \leq 1$ and $C \geq 2$, we also have $(b + x, s - x/C) \in [b_1, b_2] \times [s_1, s_2]$. Proposition 3.1(i) gives now $(b + x, s - x/C) \in \mathcal{P}_G$. This can be rewritten as

$$\left(\frac{1 + b_c}{2}, 1 - \frac{(1 - b_c)^{\lfloor D/2 \rfloor}}{2^{2\lfloor D/2 \rfloor + 1}} + \frac{\epsilon}{C} \right) \in \mathcal{P}_G.$$

Since $\epsilon > 0$ can be taken arbitrarily small, we obtain

$$s_c \left(\frac{1 + b_c}{2} \right) \leq 1 - \frac{(1 - b_c)^{\lfloor D/2 \rfloor}}{2^{2\lfloor D/2 \rfloor + 1}}.$$

The proof is therefore complete, since $s_c = s_c(1) \leq s_c((1 + b_c)/2)$. $\square$

Proof of part (b) of Theorem 1.1: In order to construct our example, we need to first introduce various types of finite graphs, to be used as our building blocks. Each one of these finite graphs will have exactly 2 vertices which will be called its exterior vertices, the other vertices being called interior vertices.

A K-bridge is a finite graph with $K+2$ vertices, 2 of them being called exterior vertices and the others being called interior vertices. The set of edges is defined by declaring that each one of the 2 exterior vertices is connected by edges to each one of the interior vertices. Note that the graph has $2K$ edges, that the two exterior vertices play the same role of each other and have degree K, and that the K interior vertices play the same role of each other and have degree 2.

Suppose that $K_1 \leq K_2$. A (K_1, K_2)-double-bridge is a graph obtained by identifying one of the exterior vertices of a K_1-bridge with one of the exterior vertices of a K_2-bridge. The interior vertices of the two original graphs, as well as the vertex resulting from the identification just described are considered to be interior vertices of the new graph, while the other two vertices are considered to be exterior vertices of the new graph. Note that the vertex resulting from the identification has degree $K_1 + K_2$, the other interior vertices have degree 2, one of the exterior vertices has degree K_1 (it will be called the first exterior vertex of the graph) and the other one has degree K_2 (it will be called the second exterior vertex of the graph). (If $K_1 = K_2$, then the two exterior vertices play the same role of each other and the choices of the first one and the second one are arbitrary.)

An l-chain of (K_1, K_2)-double-bridges is a graph obtained as follows. Start with a sequence of l disjoint (K_1, K_2)-double-bridges. Now, for $i = 1, ..., l-1$ identify the second exterior vertex of the i-th graph in the sequence with the first exterior vertex of the $(i+1)$-th graph in the sequence. The interior vertices of the l original graphs, as well as the vertices resulting from the identifications just described are considered to be interior vertices of the new graph, while the other two vertices (the first exterior vertex of the first graph and the second exterior vertex of the last graph) are considered to be exterior vertices of the new graph. The first exterior vertex of the first graph is called the first exterior vertex of the new graph, and the second exterior vertex of the last graph is called the second exterior vertex of the new graph. Note that the degrees of the exterior vertices are K_1 and K_2, while the degrees of the interior vertices are 2 and $K_1 + K_2$.

A decorated l-chain of (K_1, K_2)-double-bridges is a graph obtained from an l-chain of (K_1, K_2)-double-bridges by adding two new vertices: v_- and v_+, the first of them is joined by an edge to the first exterior vertex of the l-chain of (K_1, K_2)-double-bridges, and the second one is joined by an edge to the second exterior vertex of the l-chain of (K_1, K_2)-double-bridges. The vertices v_- and v_+ are the only exterior vertices of the new graph, and are called respectively its first and second exterior vertex. Note that the degree of the exterior vertices is 1, while the degrees of the interior vertices are K_1+1, K_2+1, 2 and K_1+K_2.

We are ready now to introduce an infinite graph which will serve as our example. Its maximal degree will be D and for it to satisfy the desired relation between s_c and b_c, a certain parameter l will have to be taken large enough. For the moment, recall that $\lceil x \rceil$ is the smallest integer which is not smaller than x. Note that $\lfloor D/2 \rfloor + \lceil D/2 \rceil = D$. Let T_2 denote the binary tree, i.e., the tree in which each vertex has degree 3. Our graph will be denoted by $T_{2,D,l}$, and is obtained by replacing each edge of T_2 with a decorated l-chain of $(\lfloor D/2 \rfloor, \lceil D/2 \rceil)$-double-bridges, in the sense that the exterior vertices of the decorated l-chains play the role of the vertices of T_2. In order to assure that the graph $T_{2,D,l}$ is quasi-transitive, the orientation of the decorated l-chains of $(\lfloor D/2 \rfloor, \lceil D/2 \rceil)$-double-bridges is relevant, and a proper choice corresponds to identifying the second exterior vertex of each such decorated l-chain with the first exterior vertices of 2 other such decorated l-chains. Note that indeed the maximal degree of $T_{2,D,l}$ is D.

Before we proceed, recall that from branching theory, one knows that the critical value for bond percolation on T_2 is 1/2. This allows us to write for $T_{2,D,l}$

$$b_c^2\left(1-(1-b_c^2)^{\lfloor D/2\rfloor}\right)^l\left(1-(1-b_c^2)^{\lceil D/2\rceil}\right)^l = \frac{1}{2} = s_c^{2+2l}\left(1-(1-s_c)^{\lfloor D/2\rfloor}\right)^l\left(1-(1-s_c)^{\lceil D/2\rceil}\right)^l. \tag{13}$$

Note, in particular, that

$$\lim_{l\to\infty} b_c = 1, \tag{14}$$

since otherwise the l.h.s. of (13) would become smaller than 1/2 for some large values of l.

Since $\lfloor D/2 \rfloor \le \lceil D/2 \rceil$ and $0 < s_c \le 1$, we can extract the following two inequalities from (13).

$$b_c^2\left(1-(1-b_c^2)^{\lfloor D/2\rfloor}\right)^{2l} \le \frac{1}{2} \le s_c^{2l}. \tag{15}$$

From the first inequality in (15) and from (14), we can conclude that for large l,

$$1-(1-b_c^2)^{\lfloor D/2\rfloor} \le \left(\frac{1}{\sqrt{2}b_c}\right)^{1/l} \le (b_c)^{1/l}. \tag{16}$$

From (15) and (16), we obtain now

$$s_c \;\ge\; (b_c)^{1/l}\left(1-(1-b_c^2)^{\lfloor D/2\rfloor}\right) \ge \left(1-(1-b_c^2)^{\lfloor D/2\rfloor}\right)^2$$

$$\geq \; 1-2(1-b_c^2)^{\lfloor D/2 \rfloor} = 1-2(1+b_c)^{\lfloor D/2 \rfloor}(1-b_c)^{\lfloor D/2 \rfloor} \geq 1-2^{\lfloor D/2 \rfloor+1}(1-b_c)^{\lfloor D/2 \rfloor}.$$

$\square$

Remark: We chose the binary tree in the construction above for convenience. This provided us with the particularly simple expression (13), on which the rest of the derivation was built. But there is great flexibility in the choice of the underlying graph. For instance, if one uses the Euclidean square lattice Z^2 instead, one produces a graph which is not only quasi-transitive, but is also a periodic graph in the sense of [16]. The first equality in (13) is then still true. It is also not hard to see that while the second equality in (13) is no longer true, it can, nevertheless, be replaced with the inequality

$$s_c^{2l}\left(1-(1-s_c)^{\lfloor D/2 \rfloor}\right)^l\left(1-(1-s_c)^{\lceil D/2 \rceil}\right)^l \geq \frac{1}{6}.$$

This is sufficient to follow the steps in the proof above, with minor modifications, and derive also in this case the inequality between b_c and s_c claimed in Theorem 1.1(b), when l is large.

5 Proof of Theorem 1.3

In this section, the notation will include an extra subscript, indicating the graph to which it refers.

In [1] the following result was proved via relations between Potts and random-cluster models and comparisons of those with independent bond percolation models. For an arbitrary infinite, locally finite graph H,

$$b_{c,H} \;\leq\; 1-\exp(-J_{c,H}) \;\leq\; \frac{q\, b_{c,H}}{1+(q-1)\, b_{c,H}}. \tag{17}$$

Given an infinite connected graph G of bounded degree, let G_s be the random graph obtained from G by performing site percolation with parameter s on it. Then, a.s.,

$$J_{c,G}^S(s) = J_{c,G_s}^S \qquad\qquad b_{c,G}(s) = b_{c,G_s}. \tag{18}$$

From (17), with $H = G_s$, and (18) we obtain

$$b_{c,G}(s) \;\leq\; 1-\exp(-J_{c,G}^S(s)) \;\leq\; \frac{q\, b_{c,G}(s)}{1+(q-1)\, b_{c,G}(s)}. \tag{19}$$

For the purpose of comparing this last display with (4), we rewrite (4) in terms of $b_{c,G}(s)$ (recall that $b_{c,G}(s)$ and and $s_{c,G}(b)$ are inverse functions for $b_c \le b \le 1$, $s_c \le s \le 1$):

$$1 - \frac{1}{C_1}(s - s_{c,G}) \;\le\; b_{c,G}(s) \;\le\; 1 - \frac{1}{C_2}(s - s_{c,G}). \tag{20}$$

From (19) and (20), we obtain the desired inequalities

$$\frac{1}{q\,C_2}(s - s_{c,G}) \;\le\; \exp(-J^S_{c,G}(s)) \;\le\; \frac{1}{C_1}(s - s_{c,G}).$$

The values of the constants $A_1^S = 1/(qC_2)$ and $A_2^S = 1/C_1$ which appear in the statement of the theorem are obtained from bounds on C_1 and C_2 in (4) which derive from Propositions 2.1, 3.1(ii) and 3.2(i). □

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Lincoln Chayes
Department of Mathematics, University of California, Los Angeles.
`lchayes@math.ucla.edu`

Roberto H. Schonmann
Department of Mathematics, University of California, Los Angeles.
`rhs@math.ucla.edu`
`http://www.math.ucla.edu/~rhs/`