Quasi-periodic Motions of a Rigid Body II
Implications for the Original System

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Abstract

This is a sequel to [Hanßmann;97]. The original system, while being an ε-perturbation of the Euler top, is ε^2-close to its normal form approximation. The normal form automatically 'removes the degeneracy' of the superintegrable Euler top and KAM-theory allows to conclude that a large part of the phase space is filled by Cantor families of invariant 3-tori. The way these 3-tori are distributed in phase space is determined by persisting invariant 2-tori, serving as 'landmarks' in the same way as the equilibria did for the one-degree-of-freedom systems treated in [Hanßmann;97]. The rigid body motion along such 2-tori closely follows the rotational-precessional motion of the Euler top.

1. The normal form

Let us describe the averaged flow on a fixed energy shell. Indeed, the energy shells $\{\, \bar{H}(|\mu|, \xi, \Im, \rho, \mu_1, \mu_2, \mu_3) = h \,\}$ fill up the phase space in a trivial way. Recall from [Hanßmann;97] that the co-ordinate functions $|\mu|, \xi, \Im, \rho, \mu_1, \mu_2, \mu_3$ on the phase space $T^*SO(3)$ are subject to the relation $|\mu|^2 = \mu_1^2 + \mu_2^2 + \mu_3^2$ and that the normal form $\bar{H}$ is given by

$$\begin{aligned} \bar{H}(|\mu|, \xi, \Im, \rho, \mu_1, \mu_2, \mu_3) \;\; &= \;\; \frac{|\mu|^2}{2I_1} \; - \; \frac{I_3 - I_1}{I_1 I_3} \frac{\Im^2}{2} \; + \; \frac{a+b+c}{2} (I_1 + (I_3 - I_1) \frac{\Im^2}{|\mu|^2}) \\ &+ \; s_3 \frac{\Im}{|\mu|} (\alpha \frac{\mu_1}{|\mu|} + \beta \frac{\mu_2}{|\mu|} + \gamma \frac{\mu_3}{|\mu|}) \; - \; \frac{I_3 - I_1}{2} \frac{3\Im^2 - |\mu|^2}{|\mu|^2} (a \frac{\mu_1^2}{|\mu|^2} + b \frac{\mu_2^2}{|\mu|^2} + c \frac{\mu_3^2}{|\mu|^2}) \qquad . \end{aligned}$$

To fix thoughts we concentrate on the case $(\alpha, \beta, \gamma) \approx (a, b, c)$ where the constant and the linear part of the perturbing force field are of comparable size. The corresponding h-level of the set $\Sigma \subseteq \mathbb{R}^3$ of critical values of the energy-momentum mapping

$$\mathcal{EM} := (\bar{H}, |\mu|, \Im) : \quad T^*SO(3)^{\backslash SO(3)} \longrightarrow \mathbb{R}^3 \qquad .$$

is depicted in *Figure 1.*

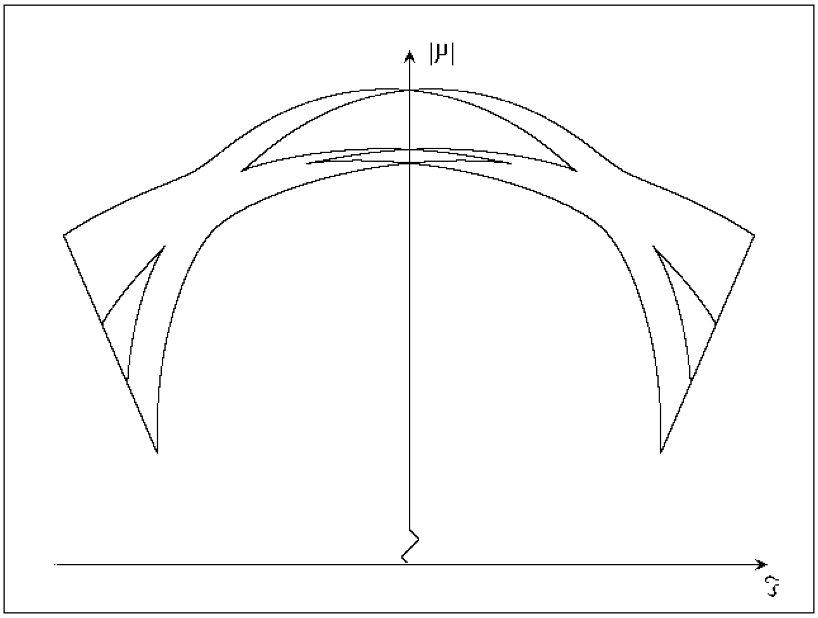

Figure 1: The energy level of the set Σ of critical values in the 'middle' case $(\alpha, \beta, \gamma) \approx (a, b, c)$.

We start our description of the flow on the energy shell with one of the three elliptic periodic orbits which occur at $\Im = |\mu|$. As predicted by Lyapunov's theorem on small oscillations, there are two families of invariant 2-tori originating from each periodic orbit. These two families correspond to the two pairs of (imaginary) Floquet exponents of the periodic orbit. When a 2-torus approaches the periodic orbit, its frequencies converge to the frequency of the periodic orbit and the imaginary part of the corresponding Floquet exponent, respectively. Both families consist of normally elliptic invariant 2-tori. One of the families is parametrised by Σ_+ (or Σ_- , depending on the periodic orbit we start with), confer Theorem 4.1 of [Hanßmann;97]. This family of invariant 2-tori corresponds to the larger imaginary eigenvalue, so we get a "fast" motion. The figure axis ζ precesses around the angular momentum μ_0 , which itself remains fixed. These "fast" invariant 2-tori correspond to our surviving Eulerian tori.

The other family of invariant 2-tori is parametrised by Σ_r and corresponds to the smaller imaginary eigenvalue. Here only the length $\|\mu_0\|$ of the angular momentum remains fixed. The direction of μ slowly varies along the trajectory with energy h of the reduced motion on $S^2_{|\mu_0|} \times \{\Im\}$. This means that μ circles around the direction μ_0 of the periodic orbit. During this motion the figure axis ζ points parallel to μ ; following [Benettin,Fassò,Guzzo;97] we speak of a *gyroscopic rotation*. Let us remain on the (three-dimensional!) invariant submanifold $\{\Im = |\mu|\}$ of the energy shell. We view this $SO(3)$ as an S^1-bundle over S^2 , whence the flow on the fibres is periodic. The flow on the base space is equivalent to the one depicted in *Figure 3* of [Hanßmann;97]. In particular, the "slow" period of the gyroscopic rotations diverges to infinity near the separating (un)stable manifold of the hypoelliptic periodic orbit.

On account of the pair of imaginary Floquet exponents, there is also one family of invariant 2-tori originating from the hypoelliptic periodic orbit. These motions again correspond to that of an Eulerian torus. A motion homoclinic to the invariant 2-torus combines such a precession with a slow motion of μ . For both positive and negative infinite time μ approaches the fixed direction of a normally hyperbolic invariant 2-torus.

As $\Im \to |\mu|$ the invariant 3-tori in $T^*SO(3)$ shrink in the same way down to the gyroscopic rotations as the surviving Eulerian 2-tori shrink down to the periodic orbits. Also the (un)stable manifold of the hyperbolic 2-tori shrinks to the homoclinic orbits of the hypoelliptic periodic orbits.

For $\frac{\Im}{|\mu|}$ near $\frac{1}{\sqrt{3}}$ the flow on the base space is equivalent to the one shown in *Figure 2* of [Hanßmann;97], *i.e.* in each $\frac{\Im}{|\mu|}$-level of the energy shell a family of 3-tori shrinks down to two invariant 2-tori. The normally hyperbolic 2-torus has met one of the normally elliptic 2-tori in a quasi-periodic centre-saddle bifurcation. At the same time also one of the arcs of its (un)stable manifold shrunk to the invariant 2-torus, the other arc converged to the (un)stable manifold of the normally parabolic 2-torus and then turned into an invariant 3-torus.

We further lower $\frac{\Im}{|\mu|}$ and pass two other quasi-periodic centre-saddle bifurcations. An $\frac{\Im}{|\mu|}$-level set now consists of five families of 3-tori, the singular fibres of this ramified torus bundle are two normally hyperbolic invariant 2-tori (together with their (un)stable manifolds) and four normally elliptic invariant 2-tori. The 3-torus family that extends between the (un)stable manifolds of the two normally hyperbolic 2-tori vanishes at $\Im = 0$. Here the two normally hyperbolic 2-tori become connected by heteroclinic orbits, confer *Figure 4* of [Hanßmann;97]. For negative $\Im$ the situation resembles that of positve $\Im$. However, the (un)stable manifolds now separate different families of 3-tori. In particular, a normally hyperbolic invariant 2-torus does not meet the "same" normally elliptic 2-torus when vanishing in the quasi-periodic centre-saddle bifurcation.

2. Persistence of invariant 2-tori

What are the features of the flow of $X_{\bar{H}}$ that survive the perturbation from $\bar{H}$ to H ? Invariant tori that foliate into periodic orbits or into lower dimensional tori generically break up. On the other hand, diophantine 3-tori survive the perturbation by the KAM-theorem. The distribution of the surviving Cantor families is determined by the surviving invariant lower dimensional tori, to which this section is devoted. For elliptic and hypoelliptic periodic orbits the answer is straightforward : they persist by means of the implicit mapping theorem.

In the present context the maximal tori are 3-dimensional, while the sole lower dimensional tori are invariant 2-tori. This allows to treat persistence of normally elliptic and normally hyperbolic 2-tori at the same time. Indeed, there is no "lack-of-parameter problem" for the normally elliptic invariant tori as there would be in higher co-dimensions. The persistence problem for lower dimensional tori has been addressed in [Mel'nikov;65,68], [Moser;67], [Lieberman;72], [Graff;74,83], [Zehnder;75,76], [Eliasson;88], [Pöschel;89,96], [Broer,Huitema,Takens;90], [Kuksin;93], [Chierchia,Falcolini;96] and [Broer,Huitema,Sevryuk;95,96]. We need the following formulation of their results.

Theorem 2.1

Let T *be an* n*-torus,* Y *a domain in* $\mathbb{R}^n$ *and* $\mathsf{S} \subseteq \mathbb{R}^2$ *an open neighbourhood of the origin. Supply* $\mathsf{T} \times \mathsf{Y} \times \mathsf{S}$ *with the symplectic structure* $\sum_{i=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}y_i + \mathrm{d}q \wedge \mathrm{d}p$ *, where* $x \in \mathsf{T}$ *,* $y \in \mathsf{Y}$ *and* $(q,p) \in \mathsf{S}$ *. Consider an analytic Hamilton function* K *on* $\mathsf{T} \times \mathsf{Y} \times \mathsf{S}$ *that does not depend on* x *. As a Taylor series in* q *and* p *it starts*

$$(\omega(y) \mid y) \; + \; \Omega(y) \, \frac{p^2 \pm q^2}{2}$$

with $\Omega(y) \neq 0 \; \forall_{y \in \mathsf{Y}}$ *. Furthermore let the mapping*

$$\begin{array}{rccc} \nu : & \mathsf{Y} & \longrightarrow & \mathbb{RP}^n \\ & y & \mapsto & [\omega_1(y) : \ldots : \omega_n(y) : \Omega(y)] \end{array}$$

be a submersion. Let H *be a small analytic perturbation of* K *. Then the flow defined by* X_H *has an* n*-dimensional Cantor family of invariant* n*-tori.*

Proof :

Writing $Y(y,q,p,\lambda) := \lambda \cdot X_K(y,q,p)$ and $Z(x,y,q,p,\lambda) := \lambda \cdot X_H(x,y,q,p)$ we introduce an additional parameter λ . This corresponds to a reparametrisation of time.

On the Cantor family of invariant n-tori $\mathsf{T} \times \{y_0\} \times \{(0,0)\} \times \{\lambda_0\}$ satisfying the diophantine condition

$$\bigwedge_{k \in \mathbb{Z}^n \setminus \{0\}} \quad \bigwedge_{l \in \{0, \pm 1, \pm 2\}} \quad |\, (\lambda_0\, \omega(y_0) \,|\, k) + \lambda_0\, \Omega(y_0)\, l \,| \;\geq\; \frac{\gamma}{|k|^\tau} \tag{1}$$

(where $\gamma > 0$ and $\tau > n - 1$) Corollary 6.2 of [Broer,Huitema,Takens;90] yields a C^∞-near-identity conjugacy Φ from (the restriction of) the flow defined by Y into the flow of Z , preserving the normal linear behaviour $\lambda_0 \begin{pmatrix} 0 & \Omega(y_0) \\ \mp\Omega(y_0) & 0 \end{pmatrix}$. At $\lambda = 1$ we obtain the desired invariant Cantor family ; **q.e.d.**

Note that Φ does not leave the λ-level sets invariant, so we do not get a conjugacy but only an equivalence between the diophantine n-tori of X_K and X_H . This equivalence is (Whitney-)smooth in y and analytic in x . Preservation of the symplectic normal linear behaviour means that normally elliptic n-tori of X_K give rise to normally elliptic n-tori of X_H and normally hyperbolic n-tori get perturbed to normally hyperbolic n-tori. The rescaling of time is needed to get rid of one of the $n + 1$ "invariants" $\omega_1, \ldots, \omega_n, \Omega$. In [Broer,Huitema,Takens;90] the result of Theorem 2.1 is called weak quasi-periodic stability of X_K on the diophantine n-tori.

In the diophantine condition (1) we assume $\tau > n - 1$ to be fixed. The "diophantine parameter" $\gamma > 0$, on the other hand, limits the size of the perturbation. Writing $\beta := \inf_{y \in \mathsf{Y}} |\Omega(y)| > 0$ there is a constant $\delta > 0$, independent of β and γ , such that the smallness condition on the perturbation H of K reads

$$\|H - K\|_{\mathsf{T} \times \mathsf{Y} \times \mathsf{S}} \;<\; \beta\, \gamma^2\, \delta \quad .$$

In other words, the closer H is to K , the smaller we may choose γ and the more n-tori satisfy (1) and survive the perturbation. On a "small" complex neighbourhood $\mathcal{D}_\epsilon := U_\kappa(\mathsf{T}) \times U_\epsilon(y_0) \times U_\epsilon(0,0)$ of a diophantine n-torus the constant $\delta > 0$ in the smallness condition $\|H - K\|_{\mathcal{D}_\epsilon} < \beta \gamma^2 \delta$ is not independent of this neighbourhood. The relation between δ and ϵ is $\epsilon = \delta^{\mathsf{q}}$, with $\mathsf{q} \in \,]\frac{1}{3}, \frac{2}{3}[$, confer [Graff;74,83] and [Broer,Huitema,Takens;90].

It is instructive to analyse the geometry of the Cantor set defined by the diophantine condition (1). If (ω, Ω) satisfies this condition then this also holds true for all $(s\omega, s\Omega)$, $s \geq 1$. For $l = 0$ we have the usual Cantor set of lines in ω-space, hence a Cantor set of planes in (ω, Ω)-space. These planes are transversal to the resonance hyperplanes $(k, l)^\perp$ with $k \neq 0$ and $l \in \{\pm 1, \pm 2\}$. Thus (1) defines a Cantor set of lines in (ω, Ω)-space. As a result the normally elliptic n-tori are parametrised by Cantor dust.

For normally hyperbolic n-tori we may restrict ourselves to $l = 0$ in the diophantine condition (1), *i.e.* only the diophantine condition

$$\bigwedge_{k \in \mathbb{Z}^n \setminus \{0\}} \quad |\, (\omega(y_0) \,|\, k) \,| \;\geq\; \frac{1}{\lambda_0} \frac{\gamma}{|k|^\tau}$$

on the internal frequencies is needed. Recall from the above proof that λ varies near 1 , *e.g.* $\lambda \in [\frac{1}{2}, 2]$. It is not surprising that the normal "frequency" Ω does not enter the small denominators as one may as well first restrict to a centre manifold. We can replace the frequency ratio $[\omega_1 : \ldots : \omega_n : \Omega] \in \mathbb{RP}^n$ by the frequency vector $\omega \in \mathbb{R}^n$ and work in Theorem 2.1 with a local diffeomorphism $y \mapsto \omega(y)$. In particular, the parametrising Cantor set of the normally hyperbolic 2-tori is locally the product of Cantor dust with one-dimensional lines. Furthermore, the flow of X_K on the diophantine normally hyperbolic n-tori is conjugate to the flow of X_H . In [Broer,Huitema,Takens;90] this is called strong quasi-periodic stability.

The persisting normally hyperbolic n-tori have stable and unstable manifolds. As shown in [Graff;74] and [Zehnder;76], the stable (unstable) manifold of a persistent torus is a continuation of the stable (unstable) manifold in the unperturbed system.

Let us show by a rough calculation that the way we obtained the normal form $\bar{H}$ is not sufficient for our present purposes. Indeed, Theorem 2.1 of [Hanßmann;97] yields a co-ordinate transformation ψ such that $\|H \circ \psi - \bar{H}\|_{D_\varepsilon} \leq const. \cdot \varepsilon^2$ on a neighbourhood $D_\varepsilon = \mathsf{T} \times U_\varepsilon(|\mu|_0, \Im_0) \times \mathsf{S}$ of a diophantine Eulerian torus $\mathsf{T} \times \{(|\mu|_0, \Im_0)\} \times \{(q_0, p_0)\}$. Here the constant ε measures the size of the perturbation, and since the higher order terms $H \circ \psi - \bar{H}$ also involve $|\mu| - |\mu|_0$ and $\Im - \Im_0$ we have to restrict to an ε-small neighbourhood of $(|\mu|_0, \Im_0)$. In the smallness condition $\|H \circ \psi - \bar{H}\|_{D_\varepsilon} < \beta\gamma^2\delta$ we met above the size of this neighbourhood is related to the required estimate by $\varepsilon = \delta^{\mathsf{q}}$, with $\mathsf{q} \in]\frac{1}{3}, \frac{2}{3}[$. As we shall see later, the lower bound β of the normal frequency is of order ε , too. Hence, we cannot obtain this smallness condition on the perturbation $H \circ \psi$ of $\bar{H}$ from the inequality $|H \circ \psi - \bar{H}| \leq const. \cdot \varepsilon^2$ on D_ε .

In the present special situation of a dynamically symmetric rigid body in an affine force field we can improve Theorem 2.1 of [Hanßmann;97] because the Fourier series of H in the angles $x \in \mathsf{T}$ turns out to be a trigonometric polynomial of degree 4 . This implies that only resonances up to order four are leading to "small denominators" in the Hamilton function K that defines the co-ordinate transformation ψ . Recall from Subsection 2 of [Hanßmann;97] that the domain of the Andoyer variables on $T^*SO(3)$ can be embedded in $\mathsf{T} \times \hat{\mathsf{Y}} \times \mathsf{P}$, where $\mathsf{T} = \{ (\xi, \rho) \mid \xi \in S^1, \rho \in S^1 \}$ is a 2-torus, $\hat{\mathsf{Y}} = \{ (|\mu|, \Im) \mid \Im \in]-|\mu|, |\mu|[\}$ is a subset of $\mathbb{R}^2$ and $\mathsf{P} = \{ (\mu_1, \mu_2, \mu_3) \in \mathbb{R}^3 \mid (\mu_1, \mu_2) \neq (0,0) \}$ is the complement of the third axis in $\mathbb{R}^3$; this embedding is given by the relation $\mu_1^2 + \mu_2^2 + \mu_3^2 = |\mu|^2$ between the co-ordinates.

Corollary 2.2

Let $H = T + V_\varepsilon$ be the Hamilton function of a rigid body, fixed at one point, with principal moments of inertia $I_1 = I_2 \neq I_3$ and centre of mass $s_1\vec{e}_1 + s_3\vec{e}_3$, and moving in a conservative affine force field. Denote by Y *a set of $(|\mu|, \Im) \in \mathbb{R}^2$ with $\Im \in [-|\mu|, |\mu|]$*

for which the frequency vector

$$\omega(|\mu|, \Im) = \begin{pmatrix} \frac{1}{I_1} \cdot |\mu| \\ \\ -\frac{I_3 - I_1}{I_1 I_3} \cdot \Im \end{pmatrix}$$

is bounded away from resonances up to order four and let the 2-torus T *and the Poisson space* P *be as given above. Then there is a Poisson co-ordinate transformation* ψ *on* $\mathsf{T} \times \mathsf{Y} \times \mathsf{P}$ *such that*

$$\|H \circ \psi - \bar{H}\|_{\mathsf{T}\times\mathsf{Y}\times\mathsf{P}} \quad \leq \quad c_{\mathsf{Y}}\, \varepsilon^2$$

for some constant c_{Y} *depending on the distance to the above resonances.*

Proof :

We simplify and unify the notation by setting $y_1 := |\mu|$, $y_2 := \Im$, $x_1 := \xi$, $x_2 := \rho$ and $z := (\mu_1, \mu_2, \mu_3)$. The coefficients of the affine force field can be written as $(\alpha, \beta, \gamma, a, b, c) = \varepsilon \cdot (\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{a}, \tilde{b}, \tilde{c})$ with $\tilde{\alpha}^2 + \ldots + \tilde{c}^2 = 1$. In the expansion

$$V_\varepsilon(x, y, z) \quad = \quad \sum_{h \in \mathbb{Z}^2} \varepsilon \cdot u_h(y, z)\, \mathrm{e}^{\mathrm{i}(h \,|\, x)}$$

the coefficient functions u_h with $|h| \geq 5$ vanish identically — this follows immediately from the form of V_ε . The function

$$K(x, y, z) \quad := \quad \sum_{1 \leq |h| \leq 4} \frac{u_h(y, z)}{\mathrm{i}(h \,|\, \omega(y))}\, \mathrm{e}^{\mathrm{i}(h \,|\, x)} \tag{2}$$

generates a Hamiltonian vector field X_K with flow ψ_t . For sufficiently small ε , depending on the bound $\inf\limits_{\substack{1 \leq |h| \leq 4 \\ y \in \mathsf{Y}}} |\, (h \,|\, \omega(y))\, |$, this flow is defined for all $(x, y, z, t) \in \mathsf{T} \times \mathsf{Y} \times \mathsf{P} \times [0, \varepsilon]$. The convex hull $\mathcal{D}$ of the union $\bigcup\limits_{t \in [0,\varepsilon]} \psi_t(\mathsf{T} \times \mathsf{Y} \times \mathsf{P})$ is still bounded away from the "low order" resonances. We use the time ε flow ψ_ε as our Poisson co-ordinate transformation. Writing $\varphi_t := \psi_t - (\mathrm{id} + tX_K)$ we obtain

$$H \circ \psi_\varepsilon - \bar{H} \quad = \quad V_\varepsilon \circ \psi_\varepsilon - \bar{V}_\varepsilon \; - \; (\omega \,|\, \varepsilon \frac{\partial K}{\partial x}) \; + \; (\omega \,|\, \pi_{\mathsf{Y}} \circ \varphi_\varepsilon) \; + \; \tau \circ (\pi_{\mathsf{Y}} \circ \psi_\varepsilon - \pi_{\mathsf{Y}})$$

with $\tau(\eta) = \frac{\eta_1^2}{2I_1} - \frac{I_3 - I_1}{I_1 I_3} \cdot \frac{\eta_2^2}{2}$ and $\pi_{\mathsf{Y}}(x, y, z) = y$. Here ω is a function of y (and not a constant as in the proof of Theorem 2.1 in [Hanßmann;97]). Next to $|\tau(\eta)| \leq \frac{I_1 + I_3}{2I_1 I_3} \|\eta\|^2$ we use the inequalities

$$\begin{aligned} \|\psi_t - \mathrm{id}\|_{\mathsf{T}\times\mathsf{Y}\times\mathsf{P}} \quad &\leq \quad t\, \|X_K \circ \psi_t\|_{\mathsf{T}\times\mathsf{Y}\times\mathsf{P}} \quad \leq \quad \varepsilon\, \|X_K\|_{\mathcal{D}} \\ \|\varphi_t\|_{\mathsf{T}\times\mathsf{Y}\times\mathsf{P}} \quad &\leq \quad \int_0^t \|X_K \circ \psi_s - X_K\|_{\mathsf{T}\times\mathsf{Y}\times\mathsf{P}}\, \mathrm{d}s \quad \leq \quad \varepsilon^2\, \|DX_K\|_{\mathcal{D}}\, \|X_K\|_{\mathcal{D}} \end{aligned}$$

and

$$\|V_\varepsilon \circ \psi_\varepsilon - V_\varepsilon\|_{\mathsf{T}\times\mathsf{Y}\times\mathsf{P}} \;\leq\; \varepsilon\, \|DV_\varepsilon\|_{\mathcal{D}}\, \|X_K\|_{\mathcal{D}}$$

to conclude that all remaining terms contain the factor ε^2 ; **q.e.d.**

A closer look at the potential energy V_ε shows that also some of the coefficient functions u_h with $|h| \leq 4$ vanish. As a result the sum (2) only involves indices $h \in \{0, \pm 1, \pm 2\}^2$. It is therefore sufficient to bound the frequency vector $\omega(|\mu|, \Im)$ away from these "low order resonances".

We now can show that the main features of this averaged flow survive the perturbation from $\bar{H}$ to H .

Theorem 2.3

Let $H = T + V_\varepsilon$ be the Hamilton function of a dynamically symmetric rigid body, fixed at one point and moving in a generic small conservative affine force field. The centre of mass $\vec{s}$ has a nonzero component s_3 along the figure axis, and the principal moments of inertia $I_1 = I_2$ and I_3 satisfy $I_1 \neq I_3$. Then the flow on $T^*SO(3)$ has the following properties.

A measure-theoretically large part of the phase space is filled by 3-dimensional Cantor families of invariant 3-tori. In compact subsets of $T^*SO(3) \backslash SO(3)$ the measure of their complement goes to zero as $\varepsilon \to 0$. The motion on these 3-tori is quasi-periodic, with diophantine frequencies. The rigid body motion along the invariant 3-tori is a superposition of a "fast" rotational-precessional motion with a "slow" nutation.

Close to the separatrices of the integrable approximation $X_{\bar{H}}$ there are stable and unstable manifolds of Cantor families of normally hyperbolic invariant 2-tori. The normally hyperbolic 2-tori are Eulerian 2-tori, giving rise to a "fast" motion of the figure axis. The persisting normally elliptic Eulerian 2-tori form 2-dimensional Cantor families, their union having positive 4-dimensional measure in phase space.

Proof :

We write the coefficients of the affine force field as $(\alpha, \beta, \gamma, a, b, c) = \varepsilon \cdot (\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{a}, \tilde{b}, \tilde{c})$ with $\tilde{\alpha}^2 + \ldots + \tilde{c}^2 = 1$. From Corollary 2.2 we get the inequality

$$\|H \circ \psi - \bar{H}\|_{A(h)} \;\leq\; c(h)\, \varepsilon^2 \quad . \tag{3}$$

Here $A(h)$ denotes a subset of $T^*SO(3)$ on which the frequency vector $\omega(|\mu|, \Im)$ is bounded away from the "low order resonances". For sufficiently high values of the energy

$H = T + V_\varepsilon$ the kinetic part T can be bounded away from zero, thus avoiding the resonance $\frac{|\mu|}{I_1} = 0$. The other low order resonances $\frac{\Im}{|\mu|} = \frac{I_3}{I_3 - I_1} \frac{h_1}{h_2}$, $h_1 \in \{0, \pm 1, \pm 2\}$, $h_2 \in \{\pm 1, \pm 2\}$ lead to the exclusion of several strips from every energy shell.

The persistence of most invariant 3-tori follows from [Arnol'd;63], [Mazzocco;97]. The crucial point is the asymptotic behaviour of the frequency mapping as $\varepsilon \to 0$. The derivative of the frequency vector ω with respect to the actions has the structure

$$\begin{pmatrix} D^2 T & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix} + D^2 \bar{V}_\varepsilon \qquad .$$

First we have to check that the frequencies of the free motion have maximal rank as function of the corresponding actions. Indeed, $\det D^2 T(|\mu|, \Im) = -\frac{I_3 - I_1}{I_1^2 I_3} \neq 0$. The second "determinant" that should be nonzero concerns the remaining frequency ω_3 which is related to the "slow" motion. Being an analytic function, it suffices to show that ω_3 is not constant. Indeed, upon approaching a separatrix we have $\omega_3 \to 0$. The reduced system $X_{\bar{H}_{|\mu|, \Im}}$ on $S^2_{|\mu|} \times \{\Im\}$ depends on ε only through a global multiplication. Extracting some neighbourhood of the isolated zeros of the derivative of ω_3 with respect to the action P of the action-angle variables to $X_{\bar{H}_{|\mu|, \Im}}$, we get the necessary lower bound of order ε for the determinant of the derivative of the frequency vector. Hence, most invariant 3-tori survive the perturbation from $\bar{H}$ to $H \circ \psi$.

For the persistence of invariant 2-tori we use Theorem 2.1. The internal frequencies of the Eulerian tori in the averaged flow are $\omega_1 = \frac{\partial \bar{H}_{|\mu|, \Im}}{\partial |\mu|} = \frac{|\mu|}{I_1} + O(\varepsilon)$ and $\omega_2 = \frac{\partial \bar{H}_{|\mu|, \Im}}{\partial \Im} = -\frac{I_3 - I_1}{I_1 I_3} \Im + O(\varepsilon)$. The normal frequency is $\Omega = \frac{s_3}{\sqrt{2}} \frac{\sqrt{f_3(\lambda)}}{f_5(\lambda)} \frac{\Im}{|\mu|}$, where the relation between the parameter λ and $\frac{\Im}{|\mu|}$ is given by

$$\frac{\Im}{|\mu|} = \frac{\sqrt{f_3(\lambda)}}{6 \, f_5(\lambda)} \Big(\frac{-s_3}{I_3 - I_1} \pm \sqrt{\frac{s_3^2}{(I_3 - I_2)^2} + 12 \frac{(f_5(\lambda))^2}{f_2(\lambda)}} \; \Big)$$

and the f_i are as defined in Theorem 3.1 of [Hanßmann;97]. From the proof of Theorem 4.1 of [Hanßmann;97] it follows that Ω is neither zero nor infinity at the three roots of f_5 . Hence, the normal frequency vanishes exactly at the two roots λ_+ and λ_- of f_3 . Sufficiently far away from the corresponding quasi-periodic centre-saddle bifurcations we have a lower bound $|\Omega| \geq \kappa\varepsilon$ with $\kappa > 0$ independent of ε . As a result the smallness condition reads $\|H \circ \psi - \bar{H}\|_{A(h)} < \kappa \varepsilon \eta^2 \delta$ and is satisfied for $\varepsilon < \frac{\kappa \eta^2 \delta}{c(h)}$. Here $\eta > 0$ is the "diophantine parameter" appearing in the diophantine condition (in the present context the letter γ is already used to describe the affine force field), and $\delta > 0$ is independent of η, κ and ε . The derivative of (ω_1, ω_2) with respect to $|\mu|$ and $\Im$ is

$$\begin{pmatrix} \frac{1}{I_1} + O(\varepsilon) & O(\varepsilon) \\ O(\varepsilon) & -\frac{I_3 - I_1}{I_1 I_3} + O(\varepsilon) \end{pmatrix}$$

with determinant $-\frac{I_3-I_1}{I_1^2 I_3}+O(\varepsilon)$. Since this expression does not vanish (for ε sufficiently small) most normally hyperbolic 2-tori are diophantine and persist, together with their stable and unstable manifolds.

For the normally elliptic Eulerian 2-tori the normal frequency also enters the small denominators. As formulated in Theorem 2.1 the frequency ratio $[\omega_1 : \omega_2 : \Omega]$ has to have maximal rank as function of the actions $|\mu|$ and $\Im$. Since Ω is nonzero we may as well work with

$$\frac{\omega_1}{\Omega} = \frac{\sqrt{2}}{s_3}\frac{f_5(\lambda)}{\sqrt{f_3(\lambda)}}\frac{|\mu|}{\Im}\Big(\frac{|\mu|}{I_1}+O(\varepsilon)\Big)$$

and

$$\frac{\omega_2}{\Omega} = \frac{\sqrt{2}}{s_3}\frac{f_5(\lambda)}{\sqrt{f_3(\lambda)}}\frac{|\mu|}{\Im}\Big(-\frac{I_3-I_1}{I_1 I_3}\Im+O(\varepsilon)\Big) \qquad .$$

The derivative of $(\frac{\omega_1}{\Omega},\frac{\omega_2}{\Omega})$ with respect to $|\mu|$ and $\Im$ has the determinant $-\frac{I_3-I_1}{I_1^2 I_3}(\Omega^{-1}+|\mu|\frac{\partial\Omega^{-1}}{\partial|\mu|}+\Im\frac{\partial\Omega^{-1}}{\partial\Im})$. This expression only vanishes on a set of codimension one, establishing the persistence of most $X_{\bar{H}}$-invariant Eulerian tori ; **q.e.d.**

The smaller we may choose the diophantine parameter η appearing in the diophantine conditions, the more Eulerian tori survive. Taking η as a function of the strength of the affine force field we ensure that in compact subsets of $T^*SO(3)\backslash SO(3)$ the (two-dimensional) measure of the complement of persisting 2-tori (within the 2-parameter family of $X_{\bar{H}}$-invariant Eulerian tori) goes to zero as $\varepsilon \to 0$. The persisting normally elliptic X_H-invariant Eulerian 2-tori lie close to the normally elliptic $X_{\bar{H}}$-invariant Eulerian 2-tori, in this sense we have a “global” persistence result.

In Theorem 2.3 we did not address the persistence of the periodic orbits and invariant 2-tori of $X_{\bar{H}}$ at $\Im = \pm|\mu|$. These regions of the phase space could even fall into two of the resonance strips we had to exclude, in which case the inequality (3) would not apply. Therefore, we impose the conditions $kI_1 \neq lI_3$ for $(k,l) \in \{1,2\}\times\{1,2,3\}$ on the principal moments of inertia. Then we can derive persistence of the elliptic and hypoelliptic periodic orbits inside the energy shells $\{H = h\}$, using the implicit mapping theorem. Next to the Floquet multipliers near $\mathrm{e}^{\pm 2\pi \mathrm{i}\frac{I_3}{I_3-I_1}}$ we have a pair $\mathrm{e}^{\pm\lambda\varepsilon}$, with λ bounded away from zero as $\varepsilon \to 0$. For a hypoelliptic periodic orbit λ is real, and for an elliptic periodic orbit λ lies on the imaginary axis. Hence, these periodic orbits persist if $|H\circ\psi - \bar{H}| < \Gamma\varepsilon$ on some neighbourhood, with Γ independent of ε . This holds true for $\varepsilon < \frac{\Gamma}{c(h)}$.

For the (normally elliptic) 2-tori near $\Im = \pm|\mu|$ the problem is more delicate. Recall from Section 4 in [Hanßmann;97] that these 2-tori are not Eulerian, in the schematic representation of *Figure 1* in [Hanßmann;97] they correspond to equilibria on the right hand sphere. For the Eulerian 2-tori, which come from equilibria on the left hand sphere in *Figure 1* of [Hanßmann;97], the internal frequencies ω_1 and ω_2 were of order 1 and the normal frequency was of order ε . We have seen in Theorem 2.3 how to control the asymptotic behaviour of the frequency ratio $[\omega_1 : \omega_2 : \Omega]$ and show persistence. For the normally elliptic 2-tori near $\Im = \pm|\mu|$ the normal frequency is of order 1 , but one of the internal frequencies is now of order ε .

Theorem 2.4

Let $H = T + V_\varepsilon$ be the Hamilton function of a dynamically symmetric rigid body, as in Theorem 2.3, but with I_1 and I_3 satisfying $kI_1 \neq lI_3$ for $(k,l) \in \{(2,l) \,|\, l \in \mathbb{N}\} \cup \{(3,l) \,|\, l = 2,\ldots,7\}$. Then the flow on $T^*SO(3)\backslash SO(3)$ has the following additional properties.

The $X_{\bar{H}}$-invariant families of invariant 2-tori at $\Im = \pm|\mu|$ give rise to a 2-dimensional Cantor family of X_H-invariant 2-tori. In compact subsets of $\{\,\Im = \pm|\mu| \neq 0\,\}$ the 4-dimensional measure of the complement of persisting 2-tori goes to zero as $\varepsilon \to 0$. The rigid body motion along these "non-Eulerian" invariant 2-tori consists of a "fast" rotation superposed by a "slow" movement of the figure axis.

Proof :

The $X_{\bar{H}}$-invariant 2-tori at $\Im = \pm|\mu|$ are normally elliptic. The normal frequency Ω is approximately $\mp\frac{I_3 - I_1}{I_1 I_3}|\mu|$ and does not vanish as $\varepsilon \to 0$. On the set $A(h)$ of (3) we can bound $|\Omega| \geq \kappa$ from below by a constant $\kappa > 0$ independent of ε . Next to the internal frequency $\omega_1 = \frac{|\mu|}{I_1} + O(\varepsilon)$ we have an internal frequency $\omega_2 = O(\varepsilon)$ related to the "slow" motion.

Our first step should be to show that the frequency ratio $[\omega_1 : \omega_2 : \Omega]$ has maximal rank as function of the two corresponding actions. But the derivative of $\frac{\omega_1}{\Omega} = \frac{\mp I_3}{I_3 - I_1} + O(\varepsilon)$ with respect to $|\mu|$ vanishes for $\varepsilon = 0$. Therefore we use a more direct approach.

In the proof of Theorem 2.3 we used $\det D(\frac{\omega_1}{\Omega}, \frac{\omega_2}{\Omega}) \neq 0$ to ensure that the diophantine condition (1) is fulfilled for most normally elliptic invariant tori. Now the resonances $(k \,|\, \omega) + l\,\Omega = 0$ between internal and normal frequencies only concern $l = 0, \pm1, \pm2$ (while k varies in the whole set $\mathbb{Z}^2\backslash\{0\}$). For $k_2 = 0$ we get the relations $k_1 \frac{|\mu|}{I_1} = \pm\frac{I_3 - I_1}{I_1 I_3}|\mu|$, $2k_1 \in \mathbb{Z}\backslash\{0\}$, and these are either explicitly excluded or contradict $I_1, I_3 > 0$. Hence, it is sufficient to check that the internal frequencies have maximal rank as function of the corresponding actions. The intersection of the five Cantor sets that the diophantine condition (1) yields for $l = 0$, $l = \pm1$ and $l = \pm2$ is the Cantor dust that parametrises the normally elliptic 2-tori.

Fixing the action $|\mu|$ we consider ω_2 as a function of the action P that parametrises the periodic orbits of the reduced system $X_{\bar{H}_{|\mu|,\pm|\mu|}}$. Let us show that this analytic function is not constant. Near the elliptic periodic orbits we have the two values $\omega_2 \approx \mp \frac{s_3}{\sqrt{2}} \frac{\sqrt{f_3(\lambda_1)}}{f_5(\lambda_1)}$ and $\omega_2 \approx \pm \frac{s_3}{\sqrt{2}} \frac{\sqrt{f_3(\lambda_2)}}{f_5(\lambda_2)}$ with $\lambda_1 < a$ and $\lambda_2 > c$, confer Theorem 3.1 of [Hanßmann;97]. If there are hypoelliptic periodic orbits on the $X_{\bar{H}}$-invariant manifold $\{\Im = \pm|\mu|\}$, the frequency ω_2 converges to zero as the invariant 2-tori approach a separatrix. In any case, ω_2 is not constant, so its critical points are isolated. Since furthermore $\frac{\partial \omega_1}{\partial |\mu|} = \frac{1}{I_1} + O(\varepsilon)$ and $\frac{\partial \omega_1}{\partial P} = O(\varepsilon)$ we conclude that for sufficiently small ε the determinant $\det D(\omega_1, \omega_2)$ only vanishes on a set of codimension one.

Letting $k_1 = 0 = l$ in the diophantine condition (1) we see that the diophantine parameter γ in (1) must be of the form $\varepsilon\eta$, with $\eta > 0$ independent of ε . As a result the smallness condition reads $|H \circ \psi - \bar{H}| < \kappa \varepsilon^2 \eta^2 \delta$ on some neighbourhood of the 2-tori in question. The estimate (3) is too weak to ensure this condition.

Therefore we now improve Corollary 2.2. We apply a second normalizing co-ordinate transformation φ to obtain

$$\| H \circ \psi \circ \varphi - \overline{H \circ \psi} \|_{B(h)} \quad \leq \quad \hat{c}(h)\, \varepsilon^3 \qquad .$$

The transformation ψ leads to additional resonances. As a result we have to avoid resonances $h_1 |\mu| = h_2 \frac{I_3 - I_1}{I_3} \Im$ with $(h_1, h_2) \in \{0, \pm 1, \ldots, \pm 4\}^2$. The conditions on I_1, I_3 ensure that there exists a neighbourhood $B(h)$ of $\{\Im = \pm|\mu|\} \cap \{H \geq h\}$ on which the frequency vector ω is bounded away from these resonances. The flow defined by the normal form of order two $\overline{H \circ \psi}$ has the same qualitative description as the flow of $X_{\bar{H}}$, due to the structural stability of the family $\bar{H}_{|\mu|, \Im}$. In particular, the normally elliptic manifold $\{\Im = \pm|\mu|\}$ (invariant under both $X_{\bar{H}}$ and $X_{\overline{H \circ \psi}}$) is also an $X_{\overline{H \circ \psi}}$-invariant ramified 2-torus bundle. For $\varepsilon < \frac{\kappa \eta^2 \delta}{\hat{c}(h)}$ the smallness condition $\| H \circ \psi \circ \varphi - \overline{H \circ \psi} \|_{B(h)} < \kappa \varepsilon^2 \eta^2 \delta$ is fulfilled and most of these invariant 2-tori survive the perturbation **;** **q.e.d.**

Theorem 4.1 of [Hanßmann;97] defines three cases, depending on whether the constant part of the force field is large as compared to the linear part, or of comparable size, or small with respect to the linear part. In all of these cases the normally elliptic Eulerian 2-tori shrink down to elliptic periodic orbits. The second family of normally elliptic invariant 2-tori originating from these elliptic periodic orbits are the 2-tori at $\Im = \pm|\mu|$. In the two cases where the linear part is sufficiently large there are also hypoelliptic periodic orbits, confer *Figures 8* and *9* of [Hanßmann;97]. From these originate normally hyperbolic Eulerian 2-tori.

In following the normally hyperbolic Eulerian 2-tori we are led to the quasi-periodic centre-saddle bifurcation these vanish in. Even in the case where the linear part of the affine force field is small with respect to the constant part there remain four (families of) quasi-periodic centre-saddle bifurcations in the flow of $X_{\bar{H}}$.

In [Hanßmann;98] a KAM-theorem on persistence of the quasi-periodic centre-saddle bifurcation is proven. However, to apply this theorem directly one would need that the lines where the quasi-periodic centre-saddle bifurcations take place lie transversal to the resonances of the two internal frequencies ω_1 and ω_2 , while in fact these lines are given by the same ratio $\frac{\Im}{|\mu|} = const.$ that also determines the frequency ratio $\frac{\omega_1}{\omega_2} = -\frac{I_3 - I_1}{I_3} \cdot \frac{\Im}{|\mu|}$. Thus, the whole bifurcation scenario may fall into a resonance hole. Therefore only a weaker result can be obtained, confer Theorem 3.3 of [Hanßmann;98], for *most* rigid bodies the occurring quasi-periodic centre-saddle bifurcations persist the perturbation from $X_{\bar{H}}$ to X_H .

There is a further organizing centre for the flow of $X_{\bar{H}}$. When $\Im = 0$ the two (families of) normally hyperbolic invariant 2-tori have the same energy and become connected by heteroclinic orbits. However, exactly at $\Im = 0$ the largest resonance hole opens up as $\frac{\omega_1}{\omega_2} = -\frac{I_3 - I_1}{I_3 \cdot |\mu|} \cdot \Im$. Thus, already the normally hyperbolic 2-tori involved are not likely to persist and this organizing centre falls into a resonance hole that cannot be removed.

To prove persistence of invariant 2-tori we used explicitly that almost all terms in the Fourier series of the perturbation V_ε of T vanish. This is a very special situation which is not likely to hold for perturbations of superintegrable systems. But the proof of Theorem 2.4 also indicates how to proceed in the general case. The key idea is to combine Theorem 2.1 of [Hanßmann;97] and Theorem 2.1 in the following way. Truncating the Fourier series of the perturbation one can find a normalizing co-ordinate transformation on a "large domain", like in Corollary 2.2 . Instead of applying the result of Theorem 2.1 to this normal form, one only applies the first induction step of the KAM-procedure which lies at the basis of this theorem. Continuing this way, *i.e.* truncating the Fourier series of the (transformed) perturbation at some higher order, finding a second normalizing co-ordinate transformation, *etc.*, one should obtain a general persistence result for invariant tori of superintegrable systems. In fact, in [Arnol'd;63] the proof of the existence of maximal tori in perturbations of properly degenerate systems proceeds exactly along these lines.

The flow defined by an asymmetric free rigid body still makes the phase space $T^*SO(3)$ a ramified 2-torus bundle. The corresponding generalized action-angle variables involve elliptic functions, see [Sadov;70] or [Tantalo;93]. Therefore, writing the potential energy defined by an affine force field as a Fourier series in the two angles generating the Eulerian 2-tori now yields infinitely many terms. To compute the normal form in this case it is not necessary to express the Hamilton function in action-angle variables. Indeed, the central action is still one of the generators of the Eulerian 2-tori. Averaging the Hamilton function along the central action one can reduce to a two-degree-of-freedom problem on $S^2_{|\mu|} \times S^2_{|\mu|}$. The corresponding flow is governed by Euler's equations (the component of the vector field on the left hand sphere vanishes) and averaging along this periodic flow yields the normal form (of order one), confer [Chernous'ko;63]. In [Kozlov;74] it is shown that the kinetic energy remains a homogeneous function of degree two in the conjugate actions. In particular the frequency ratio remains fixed by the ratio of these actions. This means that the degeneracies we encountered in the case of dynamical symmetry $I_1 = I_2$ also appear when this "symmetry" is broken.

3. Persistence of maximal tori

In the previous section we have seen to what extent the flow of $X_{\bar{H}}$ persists when perturbed to the flow of X_H . While the persistent periodic orbits remain parametrised by "complete" lines, the diophantine 2- and 3-tori are parametrised by Cantor sets (of Hausdorff dimensions 2 and 3). It is well-known, see *e.g.* [Pöschel;82], that elliptic equilibria with non-vanishing second Birkhoff coefficient are density points of the Cantor sets that parametrise surviving maximal tori, and the same holds true for elliptic periodic orbits. This has been generalized in [Sevryuk;97] and [Jorba,Villanueva;97] to lower dimensional tori, allowing for the following result.

Theorem 3.1

Under the conditions of Theorem 2.3 the Cantor set parametrising the surviving Eulerian 2*-tori consists of density points of the Cantor set parametrising persistent maximal tori. Under the conditions of Theorem 2.4, the same holds true for the surviving* 2*-tori near* $\Im = \pm|\mu|$.

Proof :

We want to apply Theorem 4 of [Jorba,Villanueva;97]. The crucial point is again the asymptotic behaviour of the frequency mapping as $\varepsilon \to 0$. This has already been checked in the previous section, and no changes have to be made close to the invariant 2-tori. Note that the additional assumptions of Theorem 2.4 are (again) needed to prevent that $\{ \Im = \pm|\mu| \}$ falls into a resonance hole **;** **q.e.d.**

Other than addressing higher dimensional tori in the vicinity of lower dimensional ones, one obtains from [Jorba,Villanueva;97] also Nekhoroshev-like estimates for the (exponentially long) time the motion stays near persisting lower dimensional invariant tori. In [Benettin,Fassò,Guzzo;97] Nekhoroshev-like estimates are obtained for motions close to the gyroscopic rotations near $\Im = \pm|\mu|$ even if the invariant 2-torus in question is not persistent.

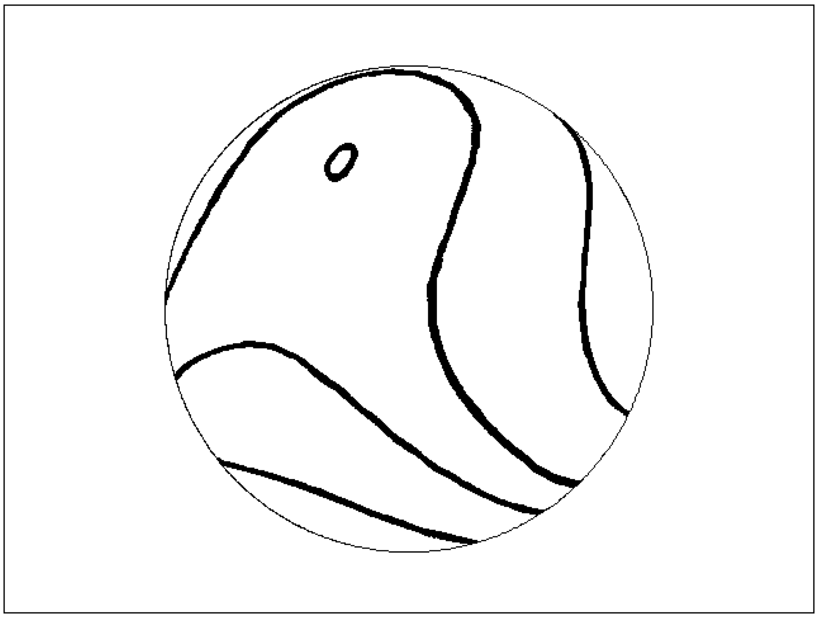

Figure 2: Motion of the angular momentum in space according to the flow of X_H for initial conditions on invariant tori.

We end with a figure illustrating our results. The motion of the angular momentum in space μ in *Figure 2* is superposed by a fast rotational-precessional motion of the rigid body, while the motion, if any, of the component $\Im$ of the angular momentum along the figure axis is too slow to be noticed during times used to produce the figure, confer [Benettin,Fassò;96] ; the same holding true for the total angular momentum $|\mu| = \sqrt{\mu_1^2 + \mu_2^2 + \mu_3^2}$. Note that in *Figure 2* no initial condition has been chosen close to the separatrices of *Figure 3* in [Hanßmann;97], a known generator of instablilities.

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