C^*-Multipliers, crossed product algebras, and canonical commutation relations

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Abstract

The notion of a multiplier of a group X is generalized to that of a C^*-multiplier by allowing it to have values in an arbitrary C^*-algebra $\mathcal{A}$. On the other hand, the construction of the crossed product algebra $\mathcal{A}\times_\tau X$ is generalized by replacing the action τ of X in $\mathcal{A}$ by a projective action of X as linear transformations of the space of continuous functions with compact support in X and with values in $\mathcal{A}$. The generalizations are done in such a way that a one-to-one correspondence exists between C^*-multipliers and projective actions. The results are applicable in mathematical physics. Quantum spacetime is discussed as an example.

1 Introduction

Starting point for this work is the observation that the construction of the crossed product of a C^*-algebra with a group is very similar to the construction of twisted group algebras and of the C^*-algebra of canonical commutation relations (CCR). The convolution of the group algebra $\mathcal{L}_1(X)$ of a locally compact group X can be deformed in two ways. In the first case the integrable functions of X are allowed to have values in a C^*-algebra $\mathcal{A}$, and the deformation involves a representation τ of X as homomorphisms of $\mathcal{A}$. This is the basis for the definition of the crossed product algebra $\mathcal{A}\times_\tau X$. In the other case a multiplier $\xi : X\times X\rightarrow \mathbf{C}$ is used, which leads to the notion of a twisted group algebra. By a slight generalization both constructs can be unified.

The state of the art for the topic of this paper is found in [1]. A short history of the crossed product algebra can be found in the introduction of [2]. Its physical importance was brought out in [3]. The twisted group algebra was studied in [4]. A special case of twisted group algebra, of relevance in mathematical physics, is the algebra of CCR, introduced in [5], and, independently, in [6]. An introductory treatment is found in [7].

The generalization studied in the present paper unifies both ways of deforming the convolution product. Seen from the point of view of the crossed product algebra the action σ of X in $\mathcal{A}$ needs not to be a representation but can be deformed. Seen from the point of view of the twisted group algebra the complex valued functions are replaced by functions with values in $\mathcal{A}$. The obvious way to proceed with the generalization of the twisted group algebra is to generalize the notion of multiplier into that of a map ξ with values in the group of unitary elements of (the multiplier algebra of) $\mathcal{A}$. This path is followed here.

An important generalization, orthogonal to the present one, is obtained by replacing the group X by a groupoid. See [8], or [9], Ch. II, Sec. 1. The combination of both generalizations is not considered here.

The structure of the paper is as follows. In the next section the main results are stated. Then theorem 1 is proved. In section 4 the crossed product algebra is constructed. Theorem 2 is proved in section 5. The final section discusses an application in quantum mechanics.

2 Main results

Recall that a multiplier ξ of a group X is a map $\xi : X \times X \to \mathbf{C}_1 \equiv \{\alpha \in \mathbf{C} : |\alpha| = 1\}$ satisfying $\xi(x, e) = \xi(e, y) = 1$ for all $x \in X$, and satisfying the *cocycle property*

$$\xi(x, y)\xi(xy, z) = \xi(x, yz)\xi(y, z), \qquad x, y, z \in X \tag{1}$$

(e is the unit element of X). The notion of multiplier can be generalized as follows.

Definition 1 *A C^*-multiplier of a group X acting in a C^*-algebra $\mathcal{A}$ is a map ξ of $X \times X$ into the unitary elements of the multiplier algebra $M(\mathcal{A})$ of $\mathcal{A}$ satisfying the following axioms.*

(M1) $\xi(x, e) = \xi(e, y) = 1$ for all $x, y \in X$.

(M2) There exists a map σ of X into the automorphisms of $\mathcal{A}$ such that σ_e is the identity transformation and one has

$$\sigma_x \xi(y, z) = \xi(x, y)\xi(xy, z)\xi(x, yz)^*, \qquad x, y, z \in X \tag{2}$$

(M3) σ satisfies

$$\sigma_x \sigma_y a = \xi(x, y)(\sigma_{xy} a)\xi(x, y)^*, \qquad x, y \in X, a \in \mathcal{A} \tag{3}$$

(M4) The map $y \in X \to a\xi(x, y)$ is continuous for all $x \in X$ and $a \in \mathcal{A}$. The maps $x \in X \to a\xi(x, x^{-1})$ and and $x \in X \to \sigma_x a$ are continuous for all $a \in \mathcal{A}$.

ξ is also called an $\mathcal{A}$-multiplier.

If $\mathcal{A} = \mathbf{C}$ then σ is trivial and (2) reduces to (1). If the *-algebra generated by the range of ξ is dense in $M(\mathcal{A})$ then σ is uniquely determined by (2). In that case (M3) follows from (M2). In general, σ is not a group representation, but is twisted by means of the C^*-multiplier, according to (3). For convenience, the map σ is called a *twisted representation* associated with ξ.

Notation 1 *Let $\mathcal{C}_c(X)$ denote the space of complex continuous functions with compact support in X. Similarly, let $\mathcal{C}_c(X, \mathcal{A})$ denote the space of continuous functions with compact support in X and values in the C^*-algebra $\mathcal{A}$.*

Given a C^*-multiplier ξ and an associated twisted representation σ, one can define a map τ of X into the linear transformations of $\mathcal{C}_c(X, \mathcal{A})$ by

$$\tau_x f(y) = (\sigma_x f(x^{-1}y))\xi(x, x^{-1}y), \qquad x, y \in X, f \in \mathcal{C}_c(X, \mathcal{A}) \tag{4}$$

It is straightforward to verify that τ satisfies the following axioms.

(A1) τ_e is the identity transformation of $\mathcal{C}_c(X, \mathcal{A})$.

(A2) For all $g, h \in \mathcal{C}_c(X, \mathcal{A})$ and $y, z \in X$ holds

$$\tau_y(g(z)\tau_z h) = (\tau_y g)(yz)\tau_{yz} h \tag{5}$$

(A3) Given $f \in \mathcal{C}_c(X, \mathcal{A})$, the function g defined by $g(x) = (\tau_x f)(e)^*$ belongs to $\mathcal{C}_c(X, \mathcal{A})$ and satisfies $(\tau_x g)(y) = (\tau_y f)(x)^*$ for all $x, y \in X$.

(A4) For all $x, y \in X$ and $f \in \mathcal{C}_c(X, \mathcal{A})$ is $||(\tau_x f)(y)|| = ||f(x^{-1}y)||$.

For convenience, any map τ satisfying the above conditions is called a *projective action* of X in $\mathcal{C}_c(X, \mathcal{A})$.

The following characterization of multipliers is proved.

Theorem 1 *Let X be a locally compact Hausdorff group and $\mathcal{A}$ a C^*-algebra. There is a one-to-one correspondence between pairs (ξ, σ), consisting of an $\mathcal{A}$-multiplier ξ of X and an associated twisted representation σ, and projective actions τ of X in $\mathcal{C}_c(X, \mathcal{A})$.*

Recall that any group action τ of X as automorphisms of $\mathcal{A}$ can be used to construct the crossed product algebra $\mathcal{A} \times_\tau X$. It is shown here that this construction can be generalized to the projective actions associated with C^*-multipliers. In fact, the conditions (A1) to (A4) are precisely what is needed for this construction.

Recall that a representation π of $\mathcal{A}$ in a Hilbert space $\mathcal{H}$ is X-covariant if there exists a unitary representation $x \to U(x)$ of X such that $\pi(\sigma_x a) = U(x)\pi(a)U(x)^*$ holds for all $x \in X$ and $a \in \mathcal{A}$. The crossed product algebra has the basic property that there is a relation between covariant *-representations of $\mathcal{A}$ and arbitrary *-representations of $\mathcal{A} \times_\tau X$. This property is preserved in the following form.

Theorem 2 *Let X be a locally compact Hausdorff group. Let $\mathcal{A}$ be a C^*-algebra. Let ξ be an $\mathcal{A}$-multiplier of X, σ a twisted representation of X associated with ξ, and let τ be the projective action determined by ξ and σ via (4). $\mathcal{A}$ can be identified with a sub-C^*-algebra of the multiplier algebra $M(\mathcal{A} \times_\tau X)$. There exists a map W of X into the unitary elements of $M(\mathcal{A} \times_\tau X)$ satisfying*

$$\tau_x f = W(x) f, \qquad x \in X, f \in \mathcal{A} \times_\tau X \tag{6}$$

and

$$\sigma_x a = W(x) a W(x)^*, \qquad x \in X, a \in \mathcal{A} \tag{7}$$

and

$$W(x)W(y) = \xi(x,y)W(xy), \qquad x,y \in X \tag{8}$$

The $W(x)$ generalize the Weyl operators, see e.g. [7] and the discussion below.

Assume now that a *-representation π of $\mathcal{A} \times_\tau X$ in a Hilbert space $\mathcal{H}$ with cyclic vector Ω is given. Then, by the previous theorem, there exists a map $x \to U(x)$ of X into the unitary operators of $\mathcal{H}$ such that

$$\pi(\tau_x f) = U(x)\pi(f), \qquad x \in X, f \in \mathcal{A} \times_\tau X \tag{9}$$

and

$$\pi(\sigma_x a) = U(x)\pi(a)U(x)^*, \qquad x \in X, a \in \mathcal{A} \tag{10}$$

and

$$U(x)U(y) = \pi(\xi(x,y))U(xy), \qquad x,y \in X \tag{11}$$

Indeed, it suffices to take $U(x) = \pi(W(x))$ for all $x \in X$. This shows that any *-representation of $\mathcal{A} \times_\tau X$ with cyclic vector defines a (generalized) covariant representation of $\mathcal{A}$. The obvious way to apply the previous theorem in quantum mechanics is for the construction of covariant representations. It can also be used to construct representations of the CCR, as is shown below.

Recall that a symplectic space H, s is a real vector space H equipped with a symplectic form s, i.e., a real bilinear antisymmetric form which is non-degenerate: $s(x,y) = 0$ for all $y \in H$ implies $x = 0$. A quantization of H is a map $x \in H \to W(x)$ of H into a C^*-algebra satisfying the following CCR.

$$W(x)W(y) = e^{is(x,y)}W(x+y), \qquad x,y \in H \tag{12}$$

The unique C^*-algebra generated by the unitary elements $W(x)$, $x \in X$, is called the C^*-algebra of CCR. The vector space H can be considered as an abelian group for the addition. It is locally compact for the discrete topology. A multiplier ξ of H with values in $\mathbf{C}$ is defined by

$$\xi(x,y) = e^{is(x,y)} \tag{13}$$

The associated twisted representation σ is trivial. Hence (8) is a generalization of the standard CCR.

A simple but nontrivial example of this quantization is the quantum space-time of [10], [11]. This example is worked out in the last section of the present paper.

3 Proof of Theorem 1

First assume that ξ and σ are given, and let τ be defined by (4). The continuity (M4) is needed to show that τ_x maps $\mathcal{C}_c(X, \mathcal{A})$ into itself. The properties (A1) to (A4) can be verified by straightforward calculation. Note that, in order to prove that the function g of (A3) is continuous one needs again (M4).

The remainder of this section deals with the proof of theorem 1 in the other direction. It is assumed in subsequent subsections that a map τ of X into the linear transformations of $\mathcal{C}_c(X,\mathcal{A})$, satisfying the axioms (A1) to (A4), is given. The study of its properties will lead to the proof that τ is a projective action associated with a C^*-multiplier ξ.

3.1 Projective actions

Notation 2 *Given $\lambda \in \mathcal{C}_c(X)$ and $a \in \mathcal{A}$, let $W^a(\lambda)$ denote the function given by*

$$W^a(\lambda)(x) = \lambda(x)a \tag{14}$$

Clearly, $W^a(\lambda)$ belongs to $\mathcal{C}_c(X,\mathcal{A})$.

Proposition 1 *For each $x \in X$ and $f \in \mathcal{C}_c(X,\mathcal{A})$ there exists $g \in \mathcal{C}_c(X,\mathcal{A})$ such that $f = \tau_x g$.*

Proof
Take $z = y^{-1}$ in (5). One obtains

$$\tau_x(h(x^{-1})\tau_{x^{-1}}f) = (\tau_x h)(e)f \tag{15}$$

Fix $a \in \mathcal{A}$ and let us construct a function h for which $(\tau_x h)(e) = a$ holds. Let $\lambda \in \mathcal{C}_c(X)$ be such that $\lambda(x) = 1$. Let

$$h(y) = \tau_y W^a(\lambda)(e)^*, \qquad y \in X \tag{16}$$

Then by axiom (A3) one obtains

$$\tau_x h(e) = W^a(\lambda)(x)^* = \overline{\lambda(x)}a = a \tag{17}$$

This shows that for each $a \in \mathcal{A}$ there exists a function $g \in \mathcal{C}_c(X,\mathcal{A})$ such that $af = \tau_x g$. Now let $(u_\alpha)_\alpha$ be an approximate unit of $\mathcal{A}$ and let g_α be such that $u_\alpha f = \tau_x g_\alpha$. From the following estimate, obtained using (A4),

$$\begin{aligned} ||g_\alpha(y) - g_\beta(y)|| &= ||\tau_x g_\alpha(xy) - \tau_x g_\beta(xy)|| \\ &= ||(u_\alpha - u_\beta)f(xy)|| \end{aligned} \tag{18}$$

follows that the elements $(g_\alpha(y))_\alpha$ converge to some function g. The support $S(g)$ of g equals $x^{-1}S(f)$. The function g is continuous as is obvious from

$$\begin{aligned} ||g(y) - g(z)|| &= \lim_\alpha ||g_\alpha(y) - g_\alpha(z)|| \\ &= \lim_\alpha ||\tau_x g_\alpha(xy) - \tau_x g_\alpha(xz)|| \\ &= \lim_\alpha ||u_\alpha(f(xy) - f(xz))|| \\ &= ||f(xy) - f(xz)|| \end{aligned} \tag{19}$$

Hence g belongs to $\mathcal{C}_c(X,\mathcal{A})$. Finally, from

$$\lim_\alpha ||\tau_x g_\alpha(y) - \tau_x g(y)|| = \lim_\alpha ||g_\alpha(x^{-1}y) - g(x^{-1}y)|| = 0 \tag{20}$$

follows that $f = \tau_x g$. This ends the proof.

$\mathbf{\Omega}$

Corrolary 1 *If for a given $x \in X$ and $a, b \in \mathcal{A}$ one has $a\tau_x f(e) = b\tau_x f(e)$ for all $f \in \mathcal{C}_c(X,\mathcal{A})$ then $a = b$ follows.*

Proof
As a consequence of the previous proposition one may assume that $x = e$. Now take $f = W^c(\lambda)$ with $c \in \mathcal{A}$ and $\lambda \in \mathcal{C}_c(X)$. There follows $\lambda(e)ac = \lambda(e)bc$. Since c and λ are abitrary there follows $a = b$.

$\mathbf{\Omega}$

3.2 Existence of the C^*-multiplier

The properties of τ proven in the previous subsection can be used to establish the existence of a C^*-multiplier ξ. The proof that ξ satisifes all axioms defining a C^*-multiplier is completed later on.

Proposition 2 *There exists a map $\xi : X \times X \to M(\mathcal{A})$ such that for all $f \in \mathcal{C}_c(X,\mathcal{A})$ holds*

$$\tau_y\tau_z f = \xi(y,z)\tau_{yz} f, \qquad y, z \in X \tag{21}$$

Proof
For a fixed z one can select a $\lambda \in \mathcal{C}_c(X)$ for which $\lambda(z) \neq 0$. Let $(u_\alpha)_\alpha$ be an approximate unit of $\mathcal{A}$, and define $\xi(y,z)$ by

$$\xi(y,z)a = \lambda(z)^{-1} \lim_\alpha (\tau_y W^{u_\alpha}(\lambda))(yz)a, \qquad a \in \mathcal{A} \tag{22}$$

That the limit converges follows from the following estimates. Let $h \in \mathcal{C}_c(X,\mathcal{A})$ be such that $\tau_{yz}h(e) = a$ (see the proof of proposition 1). Then, using (A2) and (A4), one obtains

$$\begin{aligned}
||\tau_y\left(W^{u_\alpha}(\lambda) - W^{u_\beta}(\lambda)\right)(yz)a|| &= ||\tau_y\left(W^{u_\alpha}(\lambda) - W^{u_\beta}(\lambda)\right)(yz)\tau_{yz}h(e)|| \\
&= ||\tau_y\left(\left(W^{u_\alpha}(\lambda) - W^{u_\beta}(\lambda)\right)(z)\tau_z h\right)(e)|| \\
&= ||\left(W^{u_\alpha}(\lambda) - W^{u_\beta}(\lambda)\right)(z)\tau_z h(y^{-1})|| \\
&= |\lambda(z)|\,||(u_\alpha - u_\beta)\tau_z h(y^{-1})||
\end{aligned} \tag{23}$$

Hence $(\tau_y W^{u_\alpha}(\lambda)(yz)a)_\alpha$ forms a Cauchy net converging to $\lambda(z)\xi(y,z)$.
It is straightforward to prove that $\xi(y,z)$ belongs to $M(\mathcal{A})$.
From (A2) one obtains for arbitrary $f \in \mathcal{C}_c(X,\mathcal{A})$

$$\begin{aligned}
(\tau_y W^{u_\alpha}(\lambda))(yz)\tau_{yz} f &= \tau_y(W^{u_\alpha}(\lambda)(z)\tau_z f) \\
&= \tau_y(\lambda(z)u_\alpha\tau_z f)
\end{aligned}$$

$$= \lambda(z)\tau_y(u_\alpha \tau_z f) \tag{24}$$

From (A4) follows that $\tau_y(u_\alpha\tau_z f)(x)$ converges to $(\tau_y\tau_z f)(x)$. Hence (24) implies (21).
Note that $\xi(y,z)$ does not depend on the choice of λ. Indeed, if $\lambda(z)\neq 0$ and $\mu(z)\neq 0$ then

$$\left[\lambda(z)^{-1}(\tau_y W^{u_\alpha}(\lambda))(yz) - \mu(z)^{-1}(\tau_y W^{u_\alpha}(\mu))(yz)\right]\tau_{yz} f = 0 \tag{25}$$

for all f, because of (24). By corrolary 1 this implies

$$\lambda(z)^{-1}(\tau_y W^{u_\alpha}(\lambda))(yz) - \mu(z)^{-1}(\tau_y W^{u_\alpha}(\mu))(yz) = 0 \tag{26}$$

This shows that both candidates for the r.h.s. of (22) coincide, i.e. the definition of ξ does not depend on the choice of λ.

$\boldsymbol{\Omega}$

Proposition 3 *The map ξ of the previous proposition satisfies*

$$\xi(y,z)^*\xi(y,z) = 1, \qquad y,z\in X \tag{27}$$

Proof
From (A4) and (21) follows

$$\begin{aligned} ||\xi(y,z)\tau_{yz} f(x)|| &= ||\tau_y\tau_z f(x)|| \\ &= ||f(z^{-1}y^{-1}x)|| \\ &= ||\tau_{yz} f(x)|| \end{aligned} \tag{28}$$

Take $a\in\mathcal{A}$ and $\lambda\in\mathcal{C}_c(X)$ arbitrarily, and let f be defined by $f(x) = \tau_x W^{a^*}(\lambda)(e)^*$. Then (A3) implies that

$$\tau_{yz} f(e) = W^{a^*}(\lambda)(yz)^* = \overline{\lambda(yz)}a \tag{29}$$

Equation (28) becomes

$$|\lambda(yz)|\,||\xi(y,z)a|| = |\lambda(yz)|\,||a|| \tag{30}$$

Since λ is arbitrary one concludes that $\xi(y,z)$ is an isometry of $\mathcal{A}$.

$\boldsymbol{\Omega}$

Note that the map ξ satisfies

$$\xi(e,y) = \xi(x,e) = \mathbf{1}, \qquad x,y\in X \tag{31}$$

This is an immediate consequence of (21) and corrolary 1.

3.3 Existence of the associated twisted representation

One can identify $\mathcal{A}$ with a class of linear transformations of $\mathcal{C}_c(X,\mathcal{A})$ in the following way.

Notation 3 *For each $a \in \mathcal{A}$ introduce a linear transformation ζ^a of $\mathcal{C}_c(X, \mathcal{A})$ by*

$$\zeta^a f(x) = af(x), \qquad x \in X, f \in \mathcal{C}_c(X, \mathcal{A}) \tag{32}$$

These linear transformations will later on become elements of the multiplier algebra of the crossed product algebra $\mathcal{A} \times_\tau X$.

Let us now show that there exists a deformed representation σ of X in $\mathcal{A}$.

Proposition 4 *There exists a map σ of X into the *-homomorphisms of $\mathcal{A}$ such that $\sigma_e = 1$, and*

$$\sigma_x f(x^{-1}y) = \tau_x f(y)\xi(x, x^{-1}y), \qquad x, y \in X, f \in \mathcal{C}_c(X, \mathcal{A}) \tag{33}$$

and

$$\tau_x \circ \zeta^a = \zeta^{\sigma_x a} \circ \tau_x, \qquad x \in X \tag{34}$$

For each $a \in \mathcal{A}$ the map $x \in X \to \sigma_x a$ is continuous.

Proof
Let $\lambda \in \mathcal{C}_c(X)$ be such that $\lambda(e) \neq 0$. Define σ_x by

$$\sigma_x a = \frac{1}{\lambda(e)} \tau_x W^a(\lambda)(x), \qquad x \in X, a \in \mathcal{A} \tag{35}$$

Note that the definition of σ_x does not depend on the choice of λ. Indeed, using (A2), one obtains for any $f \in \mathcal{C}_c(X, \mathcal{A})$

$$\begin{aligned}\frac{1}{\lambda(e)} \tau_x W^a(\lambda)(x) \tau_x f &= \frac{1}{\lambda(e)} \tau_x (W^a(\lambda)(e) f) \\ &= \tau_x(af)\end{aligned} \tag{36}$$

By corrolary 1 this implies that the definition of $\sigma_x a$ does not depend on the choice of λ. Equation (36) shows at once that (34) holds.
Obviously, one has

$$\sigma_e a = \frac{1}{\lambda(e)} \tau_e W^a(\lambda)(e) = a \tag{37}$$

This proves that σ_e is the identity transformation of $\mathcal{A}$.
Now take any $f \in \mathcal{C}_c(X, \mathcal{A})$ and calculate

$$\begin{aligned}\sigma_x f(x^{-1}y) &= \frac{1}{\lambda(e)} \tau_x W^{f(x^{-1}y)}(\lambda)(x) \\ &= \frac{1}{\lambda(e)} \tau_x (f(x^{-1}y) \tau_{x^{-1}y} h)(x)\end{aligned} \tag{38}$$

with h such that $W(\lambda) = \tau_{x^{-1}y} h$ (this exists because of proposition 1). Using (A2) there follows

$$\sigma_x f(x^{-1}y) = \frac{1}{\lambda(e)} \tau_x f(y) \tau_y h(x)$$

$$
\begin{aligned}
&= \frac{1}{\lambda(e)}\tau_x f(y)\xi(x,x^{-1}y)\tau_x W(\lambda)(x) \\
&= \tau_x f(y)\xi(x,x^{-1}y) \qquad (39)
\end{aligned}
$$

This is (33).
Because of (A3) the map $x \to \tau_x f(y)$ is continuous for all $y \in X$ and $f \in \mathcal{C}_c(X,\mathcal{A})$. Hence also the map

$$
x \to \sigma_x a = \frac{1}{\lambda(e)}\tau_x W^a(\lambda)(x) \qquad (40)
$$

is continuous.
Finally, let us show that σ_x is a *-homomorphism of $\mathcal{A}$. Clearly, σ_x is a linear transformation.
Given $a,b \in \mathcal{A}$, using (A2), one finds

$$
\begin{aligned}
\sigma_x(ab) &= \frac{1}{\lambda(e)}\tau_x W^{ab}(\lambda)(x) \\
&= \frac{1}{\lambda(e)^2}\tau_x(W^a(\lambda)(e)W^b(\lambda)(x) \\
&= \frac{1}{\lambda(e)^2}\tau_x W^a(\lambda)(x)\tau_x W^b(\lambda)(x) \\
&= (\sigma_x a)(\sigma_x b) \qquad (41)
\end{aligned}
$$

This shows that σ_x is a homomorphism.
Let $g(x) = \tau_x W^{a^*}(\lambda)(e)^*$. Then by (A3) one has

$$
\sigma_x a^* = \frac{1}{\lambda(e)}\tau_x W^{a^*}(\lambda)(x) = \frac{1}{\lambda(e)}\tau_x g(x)^* \qquad (42)
$$

For any $h \in \mathcal{C}_c(X,\mathcal{A})$ holds, using (A2) twice,

$$
\begin{aligned}
\tau_x g(x)\tau_x h &= \tau_x(g(e)\tau_x h) \\
&= \tau_x(W^a(\overline{\lambda})(e)\tau_x h) \\
&= \tau_x W^a(\overline{\lambda})(x)\tau_x h \qquad (43)
\end{aligned}
$$

By corrolary 1 there follows that $\tau_x g(x) = \tau_x W^a(\overline{\lambda})(x)$. Hence (42) becomes

$$
\begin{aligned}
(\sigma_x a^*)^* &= \frac{1}{\overline{\lambda(e)}}\tau_x W^a(\overline{\lambda})(x) \\
&= \sigma_x a \qquad (44)
\end{aligned}
$$

This ends the proof of the proposition.

Ω

Corrolary 2 *For each $a \in \mathcal{A}$ and $x \in X$ the map $y \to a\xi(x,y)$ is continuous.*

Proof

Let f be such that $\tau_x f = W^a(\lambda)$ (see proposition 1). Then (33) implies

$$\sigma_x f(x^{-1}y) = W^a(\lambda)(y)\xi(x, x^{-1}y), \qquad y \in X \tag{45}$$

Let $z = x^{-1}y$. Then, for all z for which $\lambda(xz) \neq 0$, one has

$$a\xi(x, z) = \frac{1}{\lambda(xz)} \sigma_x f(z) \tag{46}$$

The latter expression is clearly continuous in z.

Ω

Corrolary 3 *The map $x \to a\xi(x, x^{-1})$ is continuous for all $a \in \mathcal{A}$.*

Proof
Let $\lambda \in \mathcal{C}_c(X)$ satisfy $\lambda(e) = 1$ and let $f = W^{a^*}(\lambda)$. One obtains, with $g(x) = \tau_x f(e)^*$, using (33),

$$\begin{aligned} a\xi(x, x^{-1}) &= \frac{1}{\lambda(x)} f(x)^* \xi(x, x^{-1}) \\ &= \frac{1}{\lambda(x)} \tau_x g(e) \xi(x, x^{-1}) \\ &= \frac{1}{\lambda(x)} \sigma_x g(x^{-1}) \end{aligned} \tag{47}$$

The latter is continuous for all x for which $\lambda(x) \neq 0$. Since λ is arbitrary, except for the condition $\lambda(e) = 1$, there follows that $x \to a\xi(x, x^{-1})$ is continuous everywhere.

Ω

Proposition 5 *σ satisfies (2).*

Proof
Let $a \in \mathcal{A}$. From the definition (35) follows

$$\sigma_x(\xi(y, z)a) = \frac{1}{\lambda(e)} \tau_x W^{\xi(y,z)a}(\lambda)(x) \tag{48}$$

for any λ for which $\lambda(e) \neq 0$. We may assume as well that $\lambda(z) \neq 0$. Then, by (22),

$$W^{\xi(y,z)a}(\lambda) = \frac{1}{\lambda(z)} \lim_\alpha \tau_y W^{u_\alpha}(\lambda)(yz) W^a(\lambda) \tag{49}$$

Hence (48) becomes

$$\sigma_x(\xi(y, z)a) = \frac{1}{\lambda(e)} \frac{1}{\lambda(z)} \lim_\alpha \tau_x(\tau_y W^{u_\alpha}(\lambda)(yz) W^a(\lambda))(x) \tag{50}$$

By proposition 1 there exists a function h in $\mathcal{C}_c(X, \mathcal{A})$ such that $W^a(\lambda) = \tau_{yz} h$. Using this together with (A2) gives

$$\sigma_x(\xi(y, z)a) \quad = \quad \frac{1}{\lambda(e)} \frac{1}{\lambda(z)} \lim_\alpha \tau_x \left(\tau_y W^{u_\alpha}(\lambda)(yz) \tau_{yz} h\right)(x)$$

$$\begin{aligned}
&= \frac{1}{\lambda(e)}\frac{1}{\lambda(z)}\lim_\alpha \tau_x\tau_y W^{u_\alpha}(\lambda)(xyz)\tau_{xyz}h(x) \\
&= \frac{1}{\lambda(e)}\frac{1}{\lambda(z)}\lim_\alpha \xi(x,y)\tau_{xy}W^{u_\alpha}(\lambda)(xyz)\xi(x,yz)^*\tau_x W^a(\lambda)(x) \\
&= \xi(x,y)\xi(xy,z)\xi(x,yz)^*\sigma_x a \qquad (51)
\end{aligned}$$

Because a is arbitrary, (2) follows.

Ω

Corrolary 4 *For all $x, y \in X$ is*

$$\xi(x,y)\xi(x,y)^* = 1 \qquad (52)$$

Proof
Take $z = e$ in (2). Then (52) follows.

Ω

Proposition 6 *σ satisfies (3).*

Proof
One calculates

$$\begin{aligned}
\sigma_x\sigma_y a &= \frac{1}{\lambda(e)}\tau_x g^{\sigma_y a}(x) \\
&= \frac{1}{\lambda(e)}\tau_x(W(\lambda)\sigma_y a)(x) \\
&= \frac{1}{\lambda(e)}\tau_x\left(W(\lambda)\frac{1}{\lambda(e)}\tau_y W^a(\lambda)(y)\right)(x) \qquad (53)
\end{aligned}$$

From proposition 1 follows that $W(\lambda) = \tau_y h$ for some $h \in \mathcal{C}_c(X, \mathcal{A})$. Hence the previous expression becomes

$$\sigma_x\sigma_y a = \frac{1}{\lambda(e)}\tau_x\left(\frac{1}{\lambda(e)}\tau_y W^a(\lambda)(y)\tau_y h\right)(x) \qquad (54)$$

Now apply (A2) to obtain

$$\begin{aligned}
\sigma_x\sigma_y a &= \frac{1}{\lambda(e)^2}\tau_x\tau_y W^a(\lambda)(xy)\tau_{xy}h(x) \\
&= \xi(x,y)(\sigma_{xy}a)\xi(x,y)^* \qquad (55)
\end{aligned}$$

To obtain the latter, use is made of

$$\begin{aligned}
\frac{1}{\lambda(e)}\tau_{xy}h(x) &= \frac{1}{\lambda(e)}\xi(x,y)^*\tau_x\tau_y h(x) \\
&= \frac{1}{\lambda(e)}\xi(x,y)^*\tau_x W(\lambda)(x) \\
&= \xi(x,y)^* \qquad (56)
\end{aligned}$$

Ω

Corrolary 5 *σ_x is an automorphism of $\mathcal{A}$ for any $x \in X$.*

Proof
Proposition 4 shows that σ_x is a *-homomorphism. Invertibility follows from the previous proposition by taking $x = y^{-1}$. One obtains

$$(\sigma_y)^{-1} a = \xi(y^{-1}, y)^* \sigma_{y^{-1}} a \xi(y^{-1}, y), \qquad y \in X, a \in \mathcal{A} \tag{57}$$

Ω

4 Construction

Let be given a map τ of X into the linear transformations of $\mathcal{C}_c(X, \mathcal{A})$. Assume τ satisfies the axioms (A1) to (A4).

4.1 Product and involution

Consider $\mathcal{C}_c(X, \mathcal{A})$ as a linear space. In what follows a product law and an involution are defined which make $\mathcal{C}_c(X, \mathcal{A})$ into a *-algebra. In the next subsection this algebra is completed in norm. In the final subsections an approximate unit and the enveloping C^*-algebra are constructed.

Define a product on $\mathcal{C}_c(X, \mathcal{A})$ by

$$(fg)(x) = \int_X \mathrm{d}y \; f(y)(\tau_y g)(x) \tag{58}$$

Lemma 1 *If f and g belong to $\mathcal{C}_c(X, \mathcal{A})$ then also fg belongs to $\mathcal{C}_c(X, \mathcal{A})$.*

Proof
From proposition (A4) follows that the support of $\tau_y g$ satisfies $S(\tau_y g) = yS(g)$. Hence

$$S(fg) \subset \cup_{y \in S(f)} S(\tau_y g) = \cup_{y \in S(f)} \, yS(g) \tag{59}$$

From the continuity of multiplication and inverse and the fact that $S(f)$ and $S(g)$ are compact follows that also $\cup_{y \in S(f)} S(g)$ is compact. Hence fg has a compact support. Continuity of fg is obvious.

Ω

Proposition 7

$$\tau_x(fg) = (\tau_x f)g, \qquad f, g \in \mathcal{C}_c(X, \mathcal{A}), x \in X \tag{60}$$

Proof
Observe that, using (A2),

$$\begin{aligned}(\tau_x(fg))(y) &= \tau_x \left(\int_X \mathrm{d}z \; f(z)\tau_z g \right)(y) \\ &= \int_X \mathrm{d}z \; \tau_x \left(f(z)\tau_z g \right)(y)\end{aligned}$$

$$
\begin{aligned}
&= \int_X \mathrm{d}z\ (\tau_x f)(xz)(\tau_{xz} g)(y) \\
&= \int_X \mathrm{d}z\ (\tau_x f)(z)(\tau_z g)(y) \\
&= ((\tau_x f)g)(y)
\end{aligned}
\tag{61}
$$

Hence one concludes (60).

Ω

It is now easy to show that the product is associative. Indeed, one has

$$
\begin{aligned}
((fg)h)(x) &= \int_X \mathrm{d}y\ (fg)(y)(\tau_y h)(x) \\
&= \int_X \mathrm{d}y \int_X \mathrm{d}z\ f(z)(\tau_z g)(y)(\tau_y h)(x) \\
&= \int_X \mathrm{d}z\ f(z)((\tau_z g)h)(x) \\
&= \int_X \mathrm{d}z\ f(z)(\tau_z(gh))(x) \\
&= (f(gh))(x)
\end{aligned}
\tag{62}
$$

Next define an involution of $\mathcal{C}_c(X, \mathcal{A})$. The adjoint f^* of a function $f \in \mathcal{C}_c(X, \mathcal{A})$ is defined by

$$
f^*(x) = \Delta(x)^{-1} \tau_x f(e)^*, \qquad x \in X \tag{63}
$$

(the modular function of X is denoted Δ). f^* belongs again to $\mathcal{C}_c(X, \mathcal{A})$ by assumption (A3). One has

$$
f^{**}(x) = \Delta(x)^{-1} \tau_x \Delta^{-1} g(e)^* \tag{64}
$$

with $g(x) = \tau_x f(e)^*$. Using (A2) one proves that

$$
(\tau_x \Delta^{-1} g)(y) = \Delta(x) \tau_x g(y) \tag{65}
$$

Hence $f^{**}(x) = \tau_x g(e)^*$ follows. Using (A3) this implies $f^{**}(x) = f(x)$.

Obviously, the involution is compatible with the linear structure of $\mathcal{C}_c(X, \mathcal{A})$. It is also compatible with the product. Indeed, using (A2) and (A3),

$$
\begin{aligned}
(fg)^*(x) &= \Delta(x)^{-1} \tau_x (fg)(e)^* \\
&= \Delta(x)^{-1} \left(\tau_x \int_X \mathrm{d}y\ f(y) \tau_y g \right)(e)^* \\
&= \Delta(x)^{-1} \int_X \mathrm{d}y\ \big(\tau_x \big(f(y) \tau_y g\big)\big)(e)^* \\
&= \Delta(x)^{-1} \int_X \mathrm{d}y\ \big((\tau_x f)(xy)(\tau_{xy} g)(e)\big)^* \\
&= \Delta(x)^{-1} \int_X \mathrm{d}y\ (\tau_{xy} g)(e)^* (\tau_x f)(xy)^* \\
&= \Delta(x)^{-1} \int_X \mathrm{d}y\ (\tau_y g)(e)^* (\tau_x f)(y)^*
\end{aligned}
$$

$$
\begin{aligned}
&= \Delta(x)^{-1} \int_X \mathrm{d}y\ \Delta(y) g^*(y) (\tau_x f)(y)^* \\
&= \int_X \mathrm{d}y\ g^*(y) \Delta(x^{-1}y) (\tau_y h)(x) \qquad (66)
\end{aligned}
$$

with $h(x) = (\tau_x f)(e)^* = \Delta(x) f^*(x)$. There follows

$$
\begin{aligned}
(fg)^*(x) &= \int_X \mathrm{d}y\ g^*(y) \Delta(x^{-1}y) (\tau_y \Delta f^*)(x) \\
&= (g^* f^*)(x) \qquad (67)
\end{aligned}
$$

This shows the compatibility of the involution and the multiplication.

4.2 Norm completion

A norm is defined on $\mathcal{C}_c(X, \mathcal{A})$ by

$$
||f||_1 = \int_X \mathrm{d}x\ ||f(x)|| \qquad (68)
$$

Let us show that this norm is compatible with the product and the involution of $\mathcal{C}_c(X, \mathcal{A})$.

One verifies, using (A4), that

$$
\begin{aligned}
||fg||_1 &= \int_X \mathrm{d}x\ ||(fg)(x)|| \\
&= \int_X \mathrm{d}x\ || \int_X \mathrm{d}y\ f(y)(\tau_y g)(x)|| \\
&\leq \int_X \mathrm{d}x \int_X \mathrm{d}y\ ||f(y)||\, ||(\tau_y g)(x)|| \\
&= \int_X \mathrm{d}x \int_X \mathrm{d}y\ ||f(y)||\, ||g(y^{-1}x)|| \\
&= \int_X \mathrm{d}x \int_X \mathrm{d}y\ ||f(y)||\, ||g(x)|| \\
&= ||f||_1\, ||g||_1 \qquad (69)
\end{aligned}
$$

Hence the product is continuous for this norm. Again using (A4), one calculates

$$
\begin{aligned}
||f^*||_1 &= \int_X \mathrm{d}x\ ||f^*(x)|| \\
&= \int_X \mathrm{d}x\ \Delta(x)^{-1}\, ||\tau_x f(e)|| \\
&= \int_X \mathrm{d}x\ \Delta(x)^{-1}\, ||f(x^{-1})|| \\
&= \int_X \mathrm{d}x\ ||f(x)|| \\
&= ||f||_1 \qquad (70)
\end{aligned}
$$

Note that $||\tau_x f||_1 = ||f||_1$, i.e. the projective action τ leaves the $||\cdot||_1$-norm invariant.

One concludes that the completion of $\mathcal{C}_c(X,\mathcal{A})$ in the $||\cdot||_1$-norm is an involutive Banach algebra. It is denoted $\mathcal{L}_1(X,\mathcal{A},\tau)$, the algebra of integrable functions of X with values in $\mathcal{A}$. In many places, the notation $\mathcal{L}_1(\mathcal{A},X)$ is used. It is however more natural to see the C^*-algebra $\mathcal{A}$ as a generalization of the complex numbers and to adapt a notation compatible with $\mathcal{L}_1(X,\mathbf{C})$.

4.3 Approximate unit

Let $(u_\alpha)_\alpha$ be an approximate unit of $\mathcal{A}$. Select for each neighborhood v of $e\in X$ a continuous non-negative function δ_v with support in v and normalized to one:

$$\int_X \mathrm{d}x\,\delta_v(x)=1 \tag{71}$$

$\left(W^{u_\alpha}(\delta_v)\right)_{\alpha,v}$ is an approximate unit of $\mathcal{C}_c(X,\mathcal{A})$. Indeed, one has

$$\begin{aligned}
||W^{u_\alpha}(\delta_v)g-g||_1 &= \int_X \mathrm{d}x\ ||(W^{u_\alpha}(\delta_v)g)(x)-g(x)||\\
&= \int_X \mathrm{d}x\ ||\int_X \mathrm{d}y\ \delta_v(y)u_\alpha(\tau_y g)(x)-g(x)||\\
&\le \int_X \mathrm{d}x\ \int_X \mathrm{d}y\ \delta_v(y)||u_\alpha(\tau_y g)(x)-g(x)||\\
&\le \int_X \mathrm{d}x\ \int_X \mathrm{d}y\ \delta_v(y)||u_\alpha[(\tau_y g)(x)-g(x)]||\\
&+ \int_X \mathrm{d}x\ \int_X \mathrm{d}y\ \delta_v(y)||(u_\alpha-1)g(x)||\\
&\le \int_X \mathrm{d}x\ \int_X \mathrm{d}y\ \delta_v(y)||(\tau_y g)(x)-g(x)||\\
&+ \int_X \mathrm{d}x\ ||(u_\alpha-1)g(x)||
\end{aligned} \tag{72}$$

The latter expression can be made arbitrary small (note that from (A3) follows that $y\rightarrow\tau_y g(x)$ is continuous; use further that $\tau_y g$ is continuous with compact support).

Using (4) one obtains

$$\begin{aligned}
&||gW^{u_\alpha}(\delta_v)-g||_1\\
=&\int_X \mathrm{d}x\ ||(gW^{u_\alpha}(\delta_v))(x)-g(x)||\\
=&\int_X \mathrm{d}x\ ||\int_X \mathrm{d}y\ g(y)\tau_y W^{u_\alpha}(\delta_v)(x)-g(x)||\\
=&\int_X \mathrm{d}x\ ||\int_X \mathrm{d}y\ g(y)\delta_v(y^{-1}x)\sigma_y(u_\alpha)\xi(y,y^{-1}x)-g(x)||\\
\le&\int_X \mathrm{d}x\ \int_X \mathrm{d}y\ \delta_v(y^{-1}x)||g(y)\sigma_y(u_\alpha)\xi(y,y^{-1}x)-\Delta(y^{-1}x)g(x)||\\
\le&\int_X \mathrm{d}x\ \int_X \mathrm{d}y\ \delta_v(y^{-1}x)||g(y)-g(x)||
\end{aligned}$$

$$
\begin{aligned}
&+ \int_X \mathrm{d}x \int_X \mathrm{d}y\ \delta_v(y^{-1}x)||g(x)[\sigma_y(u_\alpha) - 1]|| \\
&+ \int_X \mathrm{d}x \int_X \mathrm{d}y\ \delta_v(y^{-1}x)||g(x)[\xi(y, y^{-1}x) - \Delta(y^{-1}x)]||
\end{aligned}
\tag{73}
$$

Also this expression can be made arbitrary small. Note that convergence of $g(x)\xi(y, y^{-1}x)$ to $g(x)$ as $y^{-1}x \to e$ is used here (proposition 2).

4.4 The crossed product algebra

The crossed product of $\mathcal{A}$ with X is denoted

$$
\mathcal{A} \times_\tau X \tag{74}
$$

It is defined as the enveloping C^*-algebra of $\mathcal{L}_1(X, \mathcal{A}, \tau)$, see [12], Section 2.7. The C^*-norm is given by

$$
||f|| = \sup\{\omega(f^*f)^{1/2} :\ \omega \text{ positive normalized functional of } \mathcal{L}_1(X, \mathcal{A}, \tau)\} \tag{75}
$$

The proof that $||\cdot||$ is nondegenerate goes as follows. We first need

Proposition 8 *Fix $f \in \mathcal{C}_c(X, \mathcal{A})$. Any state ω_0 of $\mathcal{A}$ extends to a positive linear form ω_f of $\mathcal{L}_1(X, \mathcal{A}, \tau)$ by*

$$
\omega_f(g) = \omega_0((f^*gf)(e)) \tag{76}
$$

Proof
Linearity is obvious. Positivity follows from

$$
\begin{aligned}
\omega_f(g^*g) &= \omega_0((f^*g^*gf)(e)) \\
&= \int_X \mathrm{d}z\ \omega_0((gf)^*(z)\tau_z(gf)(e)) \\
&= \int_X \mathrm{d}z\ \Delta(z)^{-1}\omega_0(\tau_z(gf)(e)^*\tau_z(gf)(e)) \\
&\geq 0
\end{aligned}
\tag{77}
$$

It is straightforward to show that a constant $K(f)$ exists such that $|\omega_f(g)| \leq K(f)||g||_1$ holds for all $g \in \mathcal{C}_c(X, \mathcal{A})$. Hence, by continuity ω_f extends to a positive linear functional on $\mathcal{L}_1(X, \mathcal{A}, \tau)$.

$\mathbf{\Omega}$

Proposition 9 *The norm (75) is nondegenerate.*

Proof
Assume that $||g|| = 0$. Then $\omega_f(g^*g) = 0$ for all $f \in \mathcal{C}_c(X, \mathcal{A}$. By construction this implies $\omega_0((f^*g^*gf)(e)) = 0$. Now take ω_0 faithful. Then one concludes that $(gf)(e) = 0$ for all $f \in \mathcal{C}_c(X, \mathcal{A})$. This means

$$
\int_X \mathrm{d}y\ g(y)\tau_y f(e) = 0 \tag{78}
$$

$g = 0$ follows using the same argument as in corrolary 1. Hence $||\cdot||$ is nondegenerate.

$\mathbf{\Omega}$

Invariance of $||\cdot||$ under τ, i.e. $||\tau_x f|| = ||f||$ for all $x \in X$ and $f \in \mathcal{A} \times_\tau X$, follows from the following result.

Proposition 10

$$(\tau_x f)^* \tau_x g = f^* g, \qquad f, g \in \mathcal{C}_c(X, \mathcal{A}) \tag{79}$$

Proof
One calculates

$$\begin{aligned}
(\tau_x f)^* \tau_x g &= \int_X \mathrm{d}y \ (\tau_x f)^*(y) \tau_y \tau_x g \\
&= \int_X \mathrm{d}y \ \Delta(y)^{-1} \tau_y \tau_x f(e)^* \tau_y \tau_x g \\
&= \int_X \mathrm{d}y \ \Delta(y)^{-1} (\tau_{yx} f)(e)^* \xi(y,x)^* \xi(y,x) \tau_{yx} g \\
&= \int_X \mathrm{d}y \ \Delta(y)^{-1} \tau_{yx} f(e)^* \tau_{yx} g \\
&= \int_X \mathrm{d}y \ \Delta(x) f^*(yx) \tau_{yx} g \\
&= \int_X \mathrm{d}y \ f^*(y) \tau_y g \\
&= f^* g
\end{aligned} \tag{80}$$

This shows (79).

$\mathbf{\Omega}$

5 Proof of theorem 2

The following result shows that $\mathcal{A}$ can be identified with a sub-C^*-algebra of $M(\mathcal{A} \times_\tau X)$.

Proposition 11 *ζ extends to an injection of $\mathcal{A}$ into $M(\mathcal{A} \times_\tau X)$. One has $(\zeta^a)^* = \zeta^{a^*}$ for all $a \in \mathcal{A}$.*

Proof
By continuity ζ extends to a linear transformation of $\mathcal{A} \times_\tau X$. A straightforward calculation shows that

$$(\zeta^{a^*} f)^* g = f^* \zeta^a g, \qquad f, g \in \mathcal{C}_c(X, \mathcal{A}) \tag{81}$$

Hence ζ^a belongs to the multiplier algebra of $\mathcal{A} \times_\tau X$.

$\mathbf{\Omega}$

From now on ζ is omitted, i.e. $\mathcal{A}$ is considered to be a sub-C^*-algebra of $M(\mathcal{A} \times_\tau X)$.

Proposition 12 *The linear transformation $W(x)$ defined by*

$$W(x)f = \tau_x f, \qquad x \in X, f \in \mathcal{A} \times_\tau X \tag{82}$$

belongs to $M(\mathcal{A} \times_\tau X)$.

Proof
For all $f, g \in \mathcal{A} \times_\tau X$ is

$$\begin{aligned}
&\left(\xi(x^{-1},x)^* W(x^{-1}) f\right)^* g \\
=& \int_X \mathrm{d}z \ \left(\xi(x^{-1},x)^* W(x^{-1}) f\right)^* (z) \tau_z g \\
=& \int_X \mathrm{d}z \ \Delta(z)^{-1} \tau_z \left(\xi(x^{-1},x)^* W(x^{-1}) f\right)(e)^* \tau_z g \\
=& \int_X \mathrm{d}z \ \Delta(z)^{-1} \left(\sigma_z(\xi(x^{-1},x)^*) \tau_z (W(x^{-1})f)(e)\right)^* \tau_z g \\
=& \int_X \mathrm{d}z \ \Delta(z)^{-1} \tau_z (W(x^{-1})f)(e)^* \sigma_z(\xi(x^{-1},x)) \tau_z g \\
=& \int_X \mathrm{d}z \ \Delta(z)^{-1} \tau_z \tau_{x^{-1}} f(e)^* \xi(z,x^{-1}) \xi(zx^{-1},x) \tau_z g \\
=& \int_X \mathrm{d}z \ \Delta(z)^{-1} \tau_{zx^{-1}} f(e)^* \xi(zx^{-1},x) \tau_z g \\
=& \int_X \mathrm{d}z \ \Delta(z)^{-1} \tau_z f(e)^* \xi(z,x) \tau_{zx} g \\
=& \int_X \mathrm{d}z \ f^*(z) \tau_z \tau_x g \\
=& f^*(W(x)g)
\end{aligned} \tag{83}$$

Hence W(x) belongs to the multiplier algebra of $\mathcal{A} \times_\tau X$ and one has

$$W(x)^* = \xi(x^{-1},x)^* W(x^{-1}), \qquad x \in X \tag{84}$$

Ω

Proposition 13

$$W(x)^* W(x) = W(x) W(x)^* = 1, \qquad x \in X \tag{85}$$

Proof
$W(x)^* W(x) = 1$ follows from proposition 79. In order to prove $W(x)W(x)^* = 1$ calculate, using (84),

$$\begin{aligned}
&(W(x)^* f)^* (W(x)^* g) \\
=& \int_X \mathrm{d}z \ (\xi(x^{-1},x)^* W(x^{-1}) f)^*(z) \tau_z (\xi(x^{-1},x)^* W(x^{-1}) g) \\
=& \int_X \mathrm{d}z \ \Delta(z)^{-1} \tau_z (\xi(x^{-1},x)^* \tau_{x^{-1}} f)(e)^* \sigma_z(\xi(x^{-1},x)^*) \tau_z \tau_{x^{-1}} g \\
=& \int_X \mathrm{d}z \ \Delta(z)^{-1} \tau_{zx^{-1}} f(e)^* \xi(z,x^{-1})^* \sigma_z(\xi(x^{-1},x))
\end{aligned}$$

$$
\begin{aligned}
& \times\sigma_z(\xi(x^{-1},x)^*)\xi(z,x^{-1})\tau_{zx^{-1}}g \\
= & \int_X \mathrm{d}z\ \Delta(z)^{-1}\tau_{zx^{-1}}f(e)^*\tau_{zx^{-1}}g \\
= & \int_X \mathrm{d}z\ \Delta(z)^{-1}\tau_z f(e)^*\tau_z g \\
= & \int_X \mathrm{d}z\ f^*(z)\tau_z g \\
= & f^*g
\end{aligned} \tag{86}
$$

Ω

The remainder of the proof of theorem 2 is straightforward. Equation (7) follows immediately from (34) and the unitarity of $W(x)$. Equation (8) follows from (21).

6 Application to Quantum Spacetime

Consider $X = \mathbf{R}^4, +$ as a locally compact group. Let Σ be the space of anti-symmetric 4-by-4 real matrices of the form

$$
\epsilon(e,m) = \begin{pmatrix} 0 & e_1 & e_2 & e_3 \\ -e_1 & 0 & m_3 & -m_2 \\ -e_2 & -m_3 & 0 & m_1 \\ -e_3 & m_2 & -m_1 & 0 \end{pmatrix} \tag{87}
$$

satisfying $|e|^2 = |m|^2$ and $e.m = \pm 1$. It is locally compact for the topology induced by the supremum norm of linear transformations of X. A motivation for this particular choice of Σ is given in [10], [11]. Let $\mathcal{C}_0(\Sigma, \mathbf{C})$ denote the C^*-algebra of complex continuous functions of Σ vanishing at infinity. A bicharacter $\xi:\ X \times X \to M(\mathcal{C}_0(\Sigma, \mathbf{C}))$ is defined by

$$
\xi(x,y)(\epsilon) = \exp\left(i\sum_{\mu,\nu=0}^{3} x_\mu \epsilon_{\mu,\nu} y_\nu\right), \qquad x,y \in X, \epsilon \in \Sigma \tag{88}
$$

It is easy to verify that ξ is a C^*-multiplier with values in $\mathcal{A} = \mathcal{C}_0(\Sigma, \mathbf{C})$, and with associated action σ which is trivial, i.e. $\sigma_x a = a$ for all $x \in X$ and $a \in \mathcal{A}$. The crossed product algebra $\mathcal{A} \times_\tau X$ coincides with the algebra constructed in [11].

Let ω be a state of $\mathcal{A} \times_\tau X$ for which the maps $\lambda \in \mathbf{R} \to \omega(g^*\tau_{\lambda x} f)$ are continuous for all $f, g \in \mathcal{A} \times_\tau X$ and $x \in X$ (this is a so-called *Weyl*-state). Let $(\pi, \mathcal{H}, \Omega)$ be the GNS-representation induced by ω. Then the map $\lambda \in \mathbf{R} \to \pi(W(\lambda x))$ is a strongly continuous one-parameter group of unitary operators. Hence, by Stone's theorem, there exists a self-adjoint operator $Q(x)$ which is the generator of this group. Let $e_\mu, \mu = 0,1,2,3$ be unit vectors of $\mathbf{R}^4$. Let $Q_\mu \equiv Q(e_\mu)$. On a dense domain one has $Q(x) = \sum_{\mu=0}^{3} x_\mu Q_\mu$. Now calculate

$$
\exp(ix_\mu Q_\mu)\exp(ix_\nu Q_\nu) \quad = \quad \pi(W(x_\mu e_\mu)W(x_\nu e_\nu))
$$

$$
\begin{aligned}
&= \pi(\xi(x_\mu e_\mu, x_\nu e_\nu) W(x_\mu e_\mu + x_\nu e_\nu)) \\
&= \pi(\xi(x_\mu e_\mu, x_\nu e_\nu)) \exp(iQ(x_\mu e_\mu + x_\nu e_\nu)) \\
&= \pi(\xi(x_\mu e_\mu, x_\nu e_\nu)) \exp(i(x_\mu Q_\mu + x_\nu Q_\nu)
\end{aligned} \tag{89}
$$

Using Stone's theorem one shows that there exist self-adjoint operators $\xi_{\mu,\nu}$ satisfying

$$
\pi(\xi(x_\mu e_\mu, x_\nu e_\nu)) = e^{ix_\mu x_\nu \xi_{\mu,\nu}} \qquad \text{and } \xi_{\mu,\nu} = -\xi_{\nu,\mu} \tag{90}
$$

They also commute with the operators Q_μ. Comparing (89) with

$$
e^{i(A+B)} = e^{iA} e^{iB} e^{\frac{1}{2}[A,B]}, \qquad [B,[A,B]] = 0 \tag{91}
$$

one concludes that on a dense domain one has

$$
i\big[Q_\mu, Q_\nu\big] = 2\xi_{\mu,\nu} \quad \text{ and } \big[Q_\mu.\xi_{\nu,\rho}\big] = 0 \tag{92}
$$

The Q_μ can be interpreted as the components of the position operators of a relativistic particle. They do not commute, but rather satisfy the commutation relations (92). The latter have been proposed in [10, 11] as a simplified model for quantum spacetime.

The example is rather simple in that both X and $\mathcal{A}$ are commutative. The present theory allows for non-commutative groups and algebras. Hence there are many possibilities for generalization. The example shows further that it is necessary to allow for C^*-algebras without unit. Indeed, the continuity of $y \to f\xi(x,y)$ is only true for functions f vanishing at infinity. Note that the Weyl-operators $W(x)$ have been introduced here taking into account the usual topology of $X = \mathbf{R}^4$. Common complaints against the algebra of the CCR are the necessity of considering the discrete topology on the underlying symplectic space, and the fact that relevant physical observables do not belong to it – see e.g. [1], IV.3.7 and the notes of IV.3.5. The conclusion reached in the present paper is that it is better to consider the crossed product algebra, as has been done in [11]. It takes the proper topology into account, and contains the Weyl-operators in its multiplier algebra.

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