Quantum Response at Finite Fields

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Abstract. We study the response of driven, time dependent, quantum system whose instantaneous Hamiltonian has gaps in its spectrum. We consider the question if the breakdown of quantum transport, and in particular, the breakdown of the Integer Quantum Hall conductance, is a phase transition. We prove that, under mild conditions, quantum response has an *isolated essential singularity* and a vanishing holomorphic piece at zero fields, and that there are no singularities at finite field. We also study the quantum response in Floquet states. We argue, and give numerical evidence, that Chern numbers for eigenvectors of finite dimensional Floquet operators are generically zero.

1. Introduction

One motivation for the present work is the question: Is the breakdown of the Integer quantum Hall effect a (quantum) phase transition? One can actually argue both ways. It is known [17, 16] that in linear response theory, the Hall conductance is related to a Chern number. Chern numbers, being integers, depend discontinuously, if at all, on parameters in the Hamiltonian. Discontinuity and more generally, loss of analyticity, of the response at zero temperature, is what we shall mean by a phase transition. Now, if the strength of the external electric field was just like any other parameters in the Hamiltonian, one would expect the breakdown to be a discontinuous at non-zero field. One can argue that the strength of the electric field is not like an ordinary parameter in the Hamiltonian. For, if it was, then the Hall conductance would remain precisely quantized also for small but finite values of the electric field. This would say that there are no corrections to adiabatic transport – not even exponentially small correction. Common wisdom is that while there are no power corrections to linear response there are exponentially small corrections for the integer quantum Hall effect [11, 14]. This argument, which singles out the electric field as a special parameter, leaves open the possibility that the breakdown is a phase transition of higher order at finite fields.

To address the breakdown as a quantum phase transition, a handle on the Hall conductance at finite driving fields is needed. This goes beyond linear response and Kubo's formulation.

The notion of breakdown of the Hall effect we are using is not standard. Normally, the breakdown of the quantum Hall effect refers to the sharp increase in the *dissipative*

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(longitudinal) resistance [7, 4, 5, 10, 15] while we focus on the *non-dissipative* (Hall) conductance. For theories of the breakdown of the dissipative resistance see e.g [19] and references therein.

We study rather general, time dependent, Hamiltonians, which include models of the quantum Hall effect as a special case. The general structure is as follows: $H(\phi, k)$ is a self-adjoint quantum Hamiltonian that depend periodically on two real, dimensionless parameters, ϕ and k, with period 2π. $H(\phi, k)$ may be associated with a *finite multiparticle* system, where ϕ and k are true external parameters. Alternatively, $H(\phi, k)$ may be a Bloch type Hamiltonian describing *infinitely many non-interacting* electrons, where ϕ and k are the two Bloch momenta. In either case, we shall require that for fixed ϕ and k the Hamiltonian $H(\phi, k)$ has discrete spectrum with no eigenvalue crossing.

For the sake of simplicity we assume that the ϕ and k dependence is smooth and that $H(\phi, k)$ is a bounded operator such as a tight binding model and its multiparticle generalizations. But, much of what we do generalizes also to Schrödinger operators.

The time dependence comes from a time dependence of ϕ on a time scale τ. That is, $\phi\left(\frac{t}{\tau}\right)$. It is convenient to rescale time so that $t = s\tau$, with t the physical time. With this notation, the Schrödinger equation takes the form

$$\mathrm{i}\dot{\psi}_\tau(s) = \tau\, H(\phi(s), k)\, \psi_\tau(s)\,. \tag{1}$$

We shall be interested in two types of time dependence of $\phi(s)$. The first is where $\phi(s)$ is a monotonically non-decreasing function of s with $\phi(s) = 0$ in the past, $s < 0$, and $\phi(s) = 2\pi$, in the future, $s \geq 2\pi$. In this case, H is time dependent on a finite interval of (physical) time $[0, 2\pi\tau]$. This setting interpolates between the sudden limit where $\tau \to 0$ and the adiabatic limit where $\tau \to \infty$ [17, 16].

The second type of time dependence we shall consider is where $\phi(s) = s$ for all times. This corresponds to periodically driven quantum systems, which are intimately related to Floquet theory.

It turns out that in both cases the strength of the driving field $\mathcal{E}$ can be related to the time scale τ via

$$\mathcal{E} = \frac{1}{\tau}\,. \tag{2}$$

This identification is central to our considerations and turns out to have consequences for quantum transport. In particular, it follows from the particularly simple τ dependence of the Schrödinger equation, that $\mathcal{E}$ is, indeed, a distinguished parameter.

In the case that $\phi(s)$ is the flux carried by an Aharonov-Bohm flux tube threading the quantum system, $\mathcal{E}$ is a characteristic emf around the tube. When $\phi(s)$ is a component of a Bloch momentum, $\mathcal{E}$ is a characteristic acceleration, which is proportional to the external force. When $\phi(s)$ is the coordinate of an object in the system, such as an impurity, or the core of a Aharonov-Bohm flux tube, $\mathcal{E}$ is the characteristic velocity of dragging this object.

The role of k is to define the observable, $I = \frac{\partial H}{\partial k}$, whose response one is interested in. The physical meaning of I can be identified from the principle of virtual work. For example, when k is the flux carried by an Aharonov-Bohm flux-tube, I is the current around the tube; When k is a Bloch momentum, I is a velocity operators, and when k is the coordinate of a distinguished point such as an impurity, I is the force acting on it.

It turns out to be convenient to consider (the k-average of) the time integral of the response I. Namely,

$$Q(\tau;\psi) = \frac{\tau}{2\pi}\int_0^{2\pi} \mathrm{d}k \int_0^{2\pi} \mathrm{d}s \left\langle \psi_\tau \left| \frac{\partial H}{\partial k} \right| \psi_\tau \right\rangle , \tag{3}$$

where ψ_τ is a solution of the quantum evolution equation (1) with initial condition ψ.

For example, in the case that ϕ and k are the fluxes carried by two distinct Aharonov Bohm flux tubes, Q is a conductance relating the (k-averaged) charge transported around k-th flux as the other flux, ϕ, is increased by 2π. In the case that ϕ and k are two orthogonal Bloch momenta, Q is the Hall conductance [18], etc.

Our main concern is the analytic properties of $Q(\tau;\psi)$ as a function of τ. Alternatively, the analytic properties of $\tilde{Q}(\mathcal{E};\psi) = Q(\tau;\psi)$ as function of the driving field $\mathcal{E}$. The analytic properties depend on the choice of the initial state ψ. We consider two natural choices that correspond to the two types of time dependence we consider. In the case that $\dot{H}$ is supported on the finite interval of time, $[0, 2\pi\tau]$, we take the initial state to be an eigenvector of the Hamiltonian at $t = 0$. We shall denote this choice of initial condition by ψ^{H}. We shall refer to the corresponding $\tilde{Q}(\mathcal{E};\psi^{\mathrm{H}})$ as *non-adiabatic* transport. In the case that H varies periodically with time we shall take the initial state to be an eigenstate of the associated Floquet evolution operator, F. We shall denote this initial state as ψ^{F} and refer to the corresponding $\tilde{Q}(\mathcal{E};\psi^{\mathrm{F}})$ as the transport in Floquet states.

In the case of non-adiabatic transport we show that $\tilde{Q}(\mathcal{E};\psi^{\mathrm{H}})$ has an isolated essential singularity at zero field, and the holomorphic part vanishes identically. There are no other singularities in the complex plane and in particular, the function vanishes at infinity. It follows from this that the breakdown of quantum transport is not associated with a phase transition at finite fields.

The transport in Floquet states has a different character. We extend results of Ferrari [8] that $\tilde{Q}(\mathcal{E};\psi^{\mathrm{F}})$ is quantized for all (real) value of $\mathcal{E}$ and is related to a Chern number. This seems to suggest that the breakdown of the Hall conductance in this case is a phase transition at finite fields. Unfortunately, this is not the case. We argue that generically, the Chern numbers for Floquet operators vanish, and in particular, vanish for all $\mathcal{E} \neq 0$. We explain why non-zero Chern numbers tend to be unstable and describe some numerical evidence that supports this.

2. Analyticity of non-Adiabatic Transport

Because of the very simple way in which τ enters into the evolution equation (1), there is a lot one can say about the analyticity of Q in the adiabaticity parameter τ, with only mild assumptions about the Hamiltonian. Since the k variables play no role in this part of the analysis, let us simplify the notation and consistently drop k.

Let $U_\tau(s)$ be the quantum evolution operator determined by:

$$\mathrm{i}\dot{U}_\tau(s,k) = \tau\, H(\phi(s),k)U_\tau(s,k)\ , \quad U_\tau(0,k) = 1\ . \tag{4}$$

Two simple facts about the evolution are:

Proposition 1 *If $H(s)$ is bounded and self-adjoint, then $U_\tau(s)$ is an entire function of τ and for real values of τ one has $U^\dagger_\tau(s) = U_{-\tau}(s)$.*

Proof. The solution to the initial value problem (4) can be written as Dyson series

$$\begin{aligned} &U_\tau(s) = \sum_{n=0}^{\infty} (-\mathrm{i}\tau)^n U_n(s)\ , \\ &U_0(s) = 1 \quad U_n(s) = \int_0^s H(s')U_{n-1}(s')\,\mathrm{d}s'\ , \end{aligned} \tag{5}$$

with

$$\|U_n(s)\| \le \frac{s^n}{n!}h^n, \quad h = \max_{s'\in[0,s]} \|H(s')\|\ . \tag{6}$$

This can be seen by induction:

$$\begin{aligned} \|U_n(s)\| &\le h\left\|\int_0^s U_{n-1}(s')\,\mathrm{d}s'\right\| \le h\int_0^s \|U_{n-1}(s')\|\,\mathrm{d}s' \\ &\le h\int_0^s \frac{(s')^{n-1}}{(n-1)!}h^{n-1}\,\mathrm{d}s' = \frac{s^n}{n!}h^n\ . \end{aligned} \tag{7}$$

The analyticity of U_τ follows from the fact that the series (5) converges absolutely for all τ

$$\|U_\tau(s,k)\| \le \sum_{n=0}^{\infty} |\tau|^n\, \|U_n(s,k)\| \le \sum_{n=0}^{\infty} \frac{|\tau s|^n}{n!}h(k)^n = \mathrm{e}^{|\tau s|\, h(k)}\ . \tag{8}$$

The second fact is immediate. ■

It follows that:

Corollary 2 *Suppose that I is a τ independent bounded operator, and ψ a τ independent state then:*

$$I(\tau,s) = \langle\psi\,|\,U_{-\tau}(s)\,I\,U_\tau(s)\,|\,\psi\rangle \tag{9}$$

is an entire function of τ.

In non-adiabatic transport we take the initial state to be an eigenstate of the instantaneous Hamiltonian at $t = 0$. Since τ multiplies H, the instantaneous eigenvector, ψ^{H}, is τ independent. This is why the field strength is not like any other parameter in the Hamiltonian, and the ultimate reason why the breakdown of adiabatic transport is not a quantum phase transition.

As a consequence of the τ independence of ψ we have:

Theorem 3 *Let $H(\phi, k)$ be bounded, continuously differentiable and periodic in $k \in \mathbb{R}$ and $\phi \in \mathbb{R}$, self-adjoint, and without eigenvalue crossing. Let ψ^{H} be an instantaneous eigenvectors of $H(0, k)$, then $\tilde{Q}(\mathcal{E}; \psi^{\mathrm{H}})$ is either identically zero, or is an analytic function of $\mathcal{E}$ with one isolated essential singularity at the origin, $\mathcal{E} = 0$, which vanishes at infinity, and takes an integral value as $\mathcal{E} \to 0$ along the real axis whose Laurent expansion is*

$$\tilde{Q}(\mathcal{E}; \psi^H) = \sum_{n=1}^{\infty} \frac{a_n}{\mathcal{E}^n} \,. \tag{10}$$

Proof. From proposition 1, $Q(\tau, \psi^{\mathrm{H}})$ is an entire function which vanishes at $\tau = 0$ (since $\partial H/\partial k$ is bounded). This gives the Laurent expansion equation (10). The $\tau \to \infty$ limiting behavior, (along the real axis), follows from standard results of adiabatic transport [17, 16]. Since the limit along the real axis is finite, it follows that $\tilde{Q}(\mathcal{E}, \psi^{\mathrm{H}})$ can not have a pole of finite order at $\mathcal{E} = 0$. ■

Remarks:

(i) The essential parts of the theorem extend to a large class of Schrödinger operators for which H is unbounded. ψ_τ is an entire function of τ provided

$$\|H^n \psi\| \le C^n \, \|\psi\| \ , \quad \forall n \,. \tag{11}$$

(ii) The vanishing of $Q(\tau, \psi^{\mathrm{H}})$ at $\tau = 0$, however, does not hold for Schrödinger operators, in general. For the Landau Hamiltonian $Q(\tau; \psi) = 1$ for each full Landau level.

(iii) For the Harper model $Q(\tau, \psi) = O(\tau^2)$ as $\tau \to 0$. This follows from a special property of Harper, namely that $\partial_k H(\phi, k) = \partial_k H(0, k)$. In this case the Laurent expansion in equation (10) starts with $n = 2$.

(iv) The vanishing of $Q(\tau; \psi)$ for $\tau \to 0$ is intuitively clear. At the same time the vanishing of $\tilde{Q}(\mathcal{E}; \psi)$ for strong driving where $\mathcal{E} \to \infty$ is counterintuitive. This may cast some doubt on the identification of $\mathcal{E}$ with $1/\tau$. However, the study of models, e.g. the Harper model of the next section, support this identification.

This theorem has an interesting implication to the question what are the power law corrections to linear response. It follows from the absence of a holomorphic part at zero field, that if there is an expansion in powers of $\mathcal{E} \in \mathbb{R}$ at $\mathcal{E} = 0$ then the expansion must vanish identically. Moreover, since a prototype function that satisfies the conditions on $\tilde{Q}$ is $\tilde{Q}(\mathcal{E}, \psi^{\mathrm{H}}) = \exp(-1/\mathcal{E}^2) - 1$, one may wonder if the absence of power law corrections to linear response is a valid conclusion of equation (10).

Absence of power corrections to linear response has been proven rigorously for the quantum Hall effect by Klein and Seiler [11]. Their proof uses an adiabatic theorem to all orders, and a clever trick that shows that power corrections are associated with Chern numbers. The form of the Laurent expansion (10), which does not have positive powers of $\mathcal{E}$, is consistent with this fact, but uses only the adiabatic theorem to lowest order.

The Laurent expansion indeed implies that if a power law expansion exists, then all positive powers vanish, however such an expansion may not exist. An example is $\tilde{Q} = \mathcal{E}\sin(\mathcal{E}^{-2})$ which satisfies all the conditions on $\tilde{Q}$ but nevertheless is such that $\mathcal{E}\sin(\mathcal{E}^{-2}) = O(\mathcal{E})$ for small real $\mathcal{E}$. In other words, one needs the adiabatic theorem to all orders (which requires smoothness of $H(\phi, k)$ and smoothness of $\phi(s)$), to prove that an expansion exists. This is the hard part. The easy part is the vanishing result.

It follows that there is no phase transition in $\mathcal{E}$ for any $\mathcal{E} \neq 0$ and the breakdown of the quantum transport is smooth.

3. Example: Hall Conductance in the Harper Model

This section describes the Hall conductance of the Harper model. The purpose of this is two fold. First, we want to display a model which satisfies all the features of the general framework postulated in the previous sections. In particular, in this model, which is driven by an external, time independent, electric field $\mathcal{E}$, *not necessarily small*, the identification of $\mathcal{E}$ with $1/\tau$ holds. The second purpose is to display numerical results about the Hall conductance of the Harper model for all field strengths.

The Harper model is associated with a square lattice, $\mathbb{Z}^2$; an external *homogeneous* magnetic field B, and *homogeneous* electric field $\mathcal{E}$ pointing in the x direction. We choose a gauge so that the electric field is described by a time dependent vector potential. After separation of variable, the model is described a Hamiltonian on $\mathbb{Z}$ parameterized by one Bloch momentum, k. The Hamiltonian action on the vector $\Psi \in \ell^2(\mathbb{Z})$ is:

$$e^{i\mathcal{E}t}\Psi_{x+1} + e^{-i\mathcal{E}t}\Psi_{x-1} + 2\cos(Bx + k)\Psi_x\,, \quad k \in [-\pi, \pi]\,, \quad x \in \mathbb{Z}\,. \tag{12}$$

For rational magnetic field $B = 2\pi p/q$ with $p, q \in \mathbb{Z}$ the Hamiltonian is periodic in x, with period q. One then classifies the solutions by a second Bloch momentum, $\ell \in [-\pi, \pi]$ so that $\Psi_{x+q} = \exp(-i\ell)\Psi_x$. Fixing periodic boundary conditions is achieved by the unitary transformation $\Psi_x \to e^{i\frac{\ell}{q}x}\Psi_x$. So, finally, the requisite form of the Harper Hamiltonian we shall study is:

$$\begin{aligned}
\Big(H(\phi, k)\Psi\Big)_x &= e^{i\phi}\Psi_{x+1} + e^{-i\phi}\Psi_{x-1} + 2\cos(2\pi\frac{p}{q}x + k)\Psi_x\,, \\
\phi(t) &= \mathcal{E}t + \frac{\ell}{q}, \quad \Psi_{x+q} = \Psi_x\,.
\end{aligned} \tag{13}$$

This corresponds to a $q\times q$ hermitian matrix, periodic in ϕ and k and with the structure described in the previous section. In particular, we see that the driving electric field, $\mathcal{E}$, is related to the adiabaticity parameter τ, by (2).

We have studied the Hall conductance of this model numerically. For $p = 1,\ q = 3$, the Harper Hamiltonian is a 3×3 matrix. Its Chern numbers are $\{3, 3, -6\}$. Figure 1 shows the Hall conductance as function of the applied electric field. There is no sharp breaking of the Hall effect as the graph is smooth. Substantial deviation from integral quantization occur near $\tau = 5$.

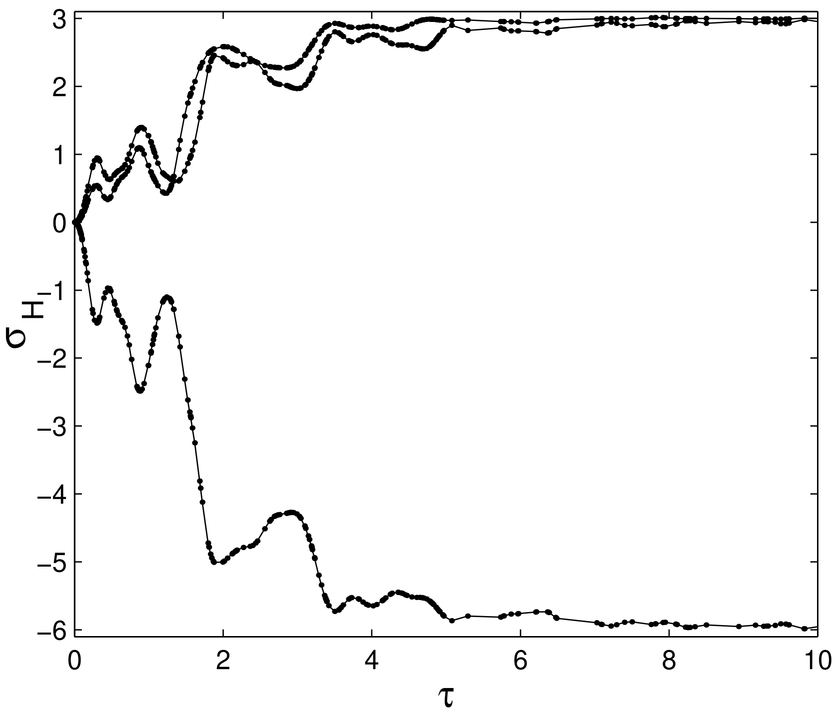

Figure 1. The Hall conductance of the 3×3 Harper Hamiltonian as function of the adiabaticity parameter τ

4. Transport in Floquet States

In this section we study transport for eigenstates of the Floquet operator. Choosing the initial state to be eigenstates of the Floquet operator leads to different qualitative behavior of quantum transport because Floquet eigenstates are τ dependent.

Hall conductance of Floquet eigenstates has been studied by Ferrari [8]. He showed that, provided the eigenvalues of the Floquet operator do not cross, the corresponding transport coefficient is a Chern number, for all τ. We remove this condition and rederive Ferrari results without it.

We start with a short review of Floquet theory [6] and then prove several results about the transport in these states. Consider a periodic, self-adjoint, finite dimensional matrix Hamiltonian $H(s + 2\pi, k) = H(s, k)$, which depends analytically on s and k. Floquet operator is the unitary evolution over one cycle

$$F_\tau(s, k) = U_\tau(s + 2\pi, k)U_{-\tau}(s, k)\ . \tag{14}$$

Floquet theorem can be expressed as

$$F_\tau(s + 2\pi, k) = F_\tau(s, k)\ . \tag{15}$$

We assume that F_τ is a finite dimensional matrix. It therefore has discrete spectrum and its spectral representation is

$$F_\tau(s,k) = \sum_n e^{\mathrm{i}E_n(s,k;\tau)}\mathcal{P}_n(s,k;\tau)\,. \tag{16}$$

E_n are the *quasienergies*, and $\mathcal{P}$ are the eigenprojections. We shall denote by $|\psi^{\mathrm{F}}_\tau\rangle$ a unit eigenvector of F_τ.

Lemma 4 *For a Floquet operator generated by a bounded and analytic $H(s,k)$,*

(1) The quasienergies are independent of s, $E_n(s,k;\tau) = E_n(k;\tau)$.

(2) The eigenfunctions and eigenprojections obey the Schrödinger equations

$$\mathrm{i}\partial_s|\psi^{\mathrm{F}}_\tau(s,k)\rangle = \tau H(s,k)|\psi^{\mathrm{F}}_\tau(s,k)\rangle\,, \tag{17}$$

$$\mathrm{i}\partial_s\mathcal{P}(s,k;\tau) = \tau[H(s,k),\mathcal{P}(s,k;\tau)]\,. \tag{18}$$

(3) $F_\tau(s,k)$ is an entire function of τ and analytic function of s and k.

Proof. The Floquet operator satisfies

$$F_\tau(s,k) = U_\tau(s,k)F_\tau(0,k)U_{-\tau}(s,k)\,. \tag{19}$$

so its eigenvalues are independent of s. The eigenfunctions and eigenprojections transform in the same way

$$|\psi^{\mathrm{F}}_\tau(s,k)\rangle = U_\tau(s,k)|\psi^{\mathrm{F}}_\tau(0,k)\rangle\,, \tag{20}$$

$$\mathcal{P}(s,k;\tau) = U_\tau(s,k)\mathcal{P}(0,k;\tau)U_{-\tau}(s,k)\,, \tag{21}$$

from which the second assertion follows. The third item follows from proposition 1. ∎

F depends on s,τ and k. Because of equations (20) and (21) the s variable is uninteresting, and the relevant variables are τ and k. It is a basic fact of perturbation theory [9] that if a (normal) operator depends analytically on two variables (or more), one can choose, in general, at most, one variable so that the eigenvectors and eigenvalues are real analytic. We choose real analyticity in the initial data, that is, in the k variable. The price one has to pay for this is two fold: First, the τ dependence will, in general, be discontinuous at crossings‡ and the second is that the 2π periodicity in k may be lost. (One can not assume that $\mathcal{P}$ is both real analytic and 2π periodic in k.) But, it is $2\pi N$ periodic for some finite N which is determined by the dimension of F (if there are no crossing we can choose $N=1$).

Now we state the main result of this section:

Proposition 5 *Let $F_\tau(0,k)$ be a finite dimensional Floquet operator, which is unitary for real values of τ and k, and analytic in both. Fix τ and label the spectrum so that $\mathcal{P}_\tau(0,k)$ is real analytic in k with period $2\pi N$. Then*

(1) The quasienergies $E(k;\tau)$ and the associated eigenfunctions $|\psi^{\mathrm{F}}_\tau(s,k)\rangle$ are simultaneously real analytic in k and s.

‡ This can be seen already for 2×2 matrices

(2) The (k-average) charge transport is a Chern number multiple of $1/N$*:*

$$Q(\tau;\psi_\tau^{\mathrm{F}}) = \frac{1}{2\pi N}\ \mathrm{Im}\int_0^{2\pi} \mathrm{d}s \int_0^{2\pi N} \mathrm{d}k\, \langle \frac{\partial \psi_\tau^{\mathrm{F}}}{\partial s} | \frac{\partial \psi_\tau^{\mathrm{F}}}{\partial k} \rangle \in \frac{\mathbb{Z}}{N}\,. \tag{22}$$

(3) $Q(\tau;\psi_\tau)$ *is the winding number of the corresponding eigenvalue. Equivalently,*

$$Q(\tau;\psi_\tau) = -\frac{E(2\pi N;\tau) - E(0,\tau)}{2\pi N}\,. \tag{23}$$

(4) $\tilde{Q}(\mathcal{E},\psi_\tau^{\mathrm{F}}) = 0$ *for all* $\mathcal{E} > 2\max_{s,k\in[0,2\pi]} \|H(s,k)\|$.

Proof. The first assertion follows from the real analyticity of the projection and (20). The second assertion follows from:

$$\tau \left\langle \psi_\tau \left| \frac{\partial H}{\partial k} \right| \psi_\tau \right\rangle = \tau\, \frac{\partial}{\partial k}\, \langle \psi_\tau \,|\, H \,|\, \psi_\tau \rangle + \mathrm{Im}\langle \frac{\partial \psi_\tau}{\partial k} | \frac{\partial \psi_\tau}{\partial s} \rangle\,. \tag{24}$$

This is a consequence of integration by parts, the equation of motion, and the periodicity in k: Integrating over the period in k and using standard facts about Chern numbers gives the result. Using the identity

$$\left(F_\tau^\dagger \frac{\partial F_\tau}{\partial k} \right)(0,k) = -i\tau \int_0^{2\pi} U_\tau^\dagger(s,k) \left(\frac{\partial H}{\partial k} \right) U_\tau(s,k)\mathrm{d}s \tag{25}$$

and equation (16)

$$\left\langle \psi_\tau^{\mathrm{F}}(0,k) \,\middle|\, F_\tau^\dagger \partial_k F_\tau(0,k) \,\middle|\, \psi_\tau^{\mathrm{F}}(0,k) \right\rangle = \mathrm{i}\partial_k E(k;\tau) \tag{26}$$

gives assertion 3. Item 4 follows from the observation that the winding vanishes if $\max|E(k;\tau)| < \pi$. The maximum is achieved when all the U_n defined in (5) have the same eigenstate for their maximal eigenvalue and all the eigenvalues are summed up with the right signs

$$|E| \le \left| \arg\left(\sum_{n=0}^{\infty} (\mathrm{i}\tau)^n \frac{(2\pi h)^n}{n!} \right) \right| = \arg\left(\mathrm{e}^{2\pi\mathrm{i}\tau h} \right) = 2\pi\tau \max\|H\|\,. \tag{27}$$

Therefore we have vanishing winding for

$$\mathcal{E} > 2 \max_{s,k\in[0,2\pi]} \|H(s,k)\|\,. \tag{28}$$

■

A consequence of the relation between Chern numbers and the winding numbers is:

Theorem 6 *Let* $F_\tau(0,k)$ *be a* finite dimensional *Floquet operator, which is unitary for* τ, $k \in \mathbb{R}$*, and analytic in both. Label the spectrum of* F_τ *so that* $\mathcal{P}_\tau(0,k)$ *is real analytic in* k *with period* $2\pi N$*. If any Chern numbers is non-zero then* F_τ *has eigenvalue crossing for some value of* k*.*

Proof. By general principles, the sum of all the Chern numbers for any finite dimensional matrix, such as F_τ, must vanish. Hence, if there is a positive Chern number for one of the states of F_τ, there must also be a negative Chern number for some other state. The quasienergy with a positive winding must then cross the one with negative winding. ■

Proposition 5 shows that $\tilde{Q}(\mathcal{E}, \psi^{\mathrm{F}}_{\tau})$ is related to a Chern number for *all* values of τ. As such, it must change discontinuously, if at all. It is easy to construct models for which $\tilde{Q}(0, \psi^{\mathrm{F}}_{\infty})$ is a non-zero integer, and the Harper model of the previous section is such an example. This is because at $\mathcal{E} = 0$ eigenvector the Hamiltonian are also eigenvectors of the Floquet operator. On the other hand $\tilde{Q}(\mathcal{E}, \psi^{\mathrm{F}}_{\tau}) = 0$ for large enough $\mathcal{E}$. This may appear to be a proof that transport for Floquet states undergoes phase transition at finite values of $\mathcal{E}$.

There is a caveat in the above argument. Namely, the transition may occur at $\mathcal{E} = 0_+$. At first, it appears that one can argue that the transition must occur for some finite value of $\mathcal{E}$ because Chern numbers do not change under small deformations of the bundle of eigenstates, and so if the Chern number is non-zero at zero field, should it not be non-zero also for small fields?

To resolve this issue, we first consider a puzzle that seem to follow from theorem 6. The puzzle is a conflict that has Chern on one side, and Wigner von-Neumann (WVN) on the other: Wigner and von-Neumann [13, 12] say that eigenvalue crossing tend to be unstable, unless the dimension of parameter space is three or more§. The relevant parameter space for the Floquet operators is the τ-k space which has dimension two. This implies that non-zero Chern numbers are unstable. On the other hand Chern numbers give a topological characterization of the bundle of eigenstates and are stable under continuous deformations of the bundle. Who wins?

The winners are Wigner and von-Neumann. The reason that Chern loses is the following. It is, of course, true that continuous deformation of the bundle keep the Chern number fixed. But, there is no reason why a continuous deformations of the Hamiltonian or a variation in τ, should lead to continuous deformations of the bundle of eigenstates. By standard perturbation theory a small deformation in the Hamiltonian can result in a discontinuous change of the bundle at points of crossing. Since non-zero Chern number for Floquet states always come with eigenvalue crossing, such bundles always lie at the boundary of the region of stability. Unfortunately, non-zero Chern numbers, are not stable under small variations in the Hamiltonian when $\mathcal{E} \neq 0$. As a consequence, generically, at least, Chern numbers for Floquet operators‖ are zero for all finite values of the field and the transition is indeed at $\mathcal{E} = 0_+$. In the next section we give supporting numerical evidence for this.

5. Numerical studies

The question whether finite dimensional Floquet operators do or do not have stable non-zero Chern numbers, is equivalent to the question whether they do or do not have stable eigenvalue crossing. To investigate this question we studied the Floquet operator

§ In the complex case, which is the case relevant here.

‖ At least, those of the class we study here, that depend on a single variable k

associated with the following toy model Hamiltonian represented by a 2×2 matrix:

$$H_2(s,k,\varepsilon) = \frac{1}{\sqrt{a^2+(b+\varepsilon)^2+c^2}} \begin{pmatrix} a & b-\mathrm{i}c+\varepsilon \\ b+\mathrm{i}c+\varepsilon & -a \end{pmatrix} \tag{29}$$

$$a = \cos(k)\ , \quad b = \cos(s)\ , \quad c = \cos(s+k)\ .$$

This Hamiltonian is inspired to some extent by the Harper model on a triangular lattice [1]. We have added an extra parameter, ε so that the associated Floquet operator, $F_\tau(0,k;\varepsilon)$, depends on three relevant variables, $\{\tau,k,\varepsilon\}$.

What should one expect for the Floquet operator based on WVN genericity argument? Since (the unitary) $F_\tau(0,k;\varepsilon)$ depends on three real variables, one expects isolated points in $\{\tau,k,\varepsilon\}$ space where there is crossing. In other words, for some special values of ε one expects to find crossing, and some non-zero Chern numbers, while for most values of ε, there should be no crossing and all and all Chern numbers should vanish. This turns out to be the case.

The model Hamiltonian has the following features:

(i) For all s,k,ε the spectrum is constant $Spec(H_2(s,k,\varepsilon)) = \{-1,1\}$.

(ii) The Chern numbers associated with the eigenvectors of $H_2(s,k,\varepsilon)$ are ±2 for $-1<\varepsilon<1$ and zero otherwise¶.

Even for this toy model the analysis of the Floquet spectrum can be done only numerically. This is because the Floquet operator is a solution of a first order differential equation involving H_2 which can only be integrated numerically.

A useful general fact about the quasienergies of Floquet operators with determinant one that are 2×2 matrices is that $E_1 = -E_2$. Hence, gap closure occur only at $E=0$ and $E=\pm\pi$. A stable non-zero Chern number requires a stable crossing at $E=\pm\pi$. This feature simplifies the numerical analysis considerably.

For $\varepsilon=0$ the crossing at $E=0$ is stable for all τ. The gap at $E=\pm\pi$ is plotted in figure 2. As can be seen, for most values of τ the gap is wider then the numerical tolerance. This gap is usually very small. It is easy to mistake these small gaps for crossing. This may have happened in Ferrari's analysis.

Near integers values of τ the gap width decreases rapidly. At these points the gaps seem to close, up to our numerical accuracy, and the winding numbers change from zero to ±2.

The model with $\varepsilon=0$ is in conflict with both the stability of Chern numbers, and the WVN instability of crossing. It is in conflict with Chern, as non-zero Chern numbers appear to occur at isolated points on the τ axis. It is also in conflict with WVN, for, by the no-crossing rule one expects no crossing in the $\tau-k$ plane, and the model has crossing points. The model would be consistent with WVN if the point $\varepsilon=0$ turns out to be a special point. Some indication for this can be seen from the fact that $\int H_2(s,\pi/2,0)\mathrm{d}s = 0$. Although we do not know why this vanishing condition leads to

¶ This can be seen by examining sections of the bundle.

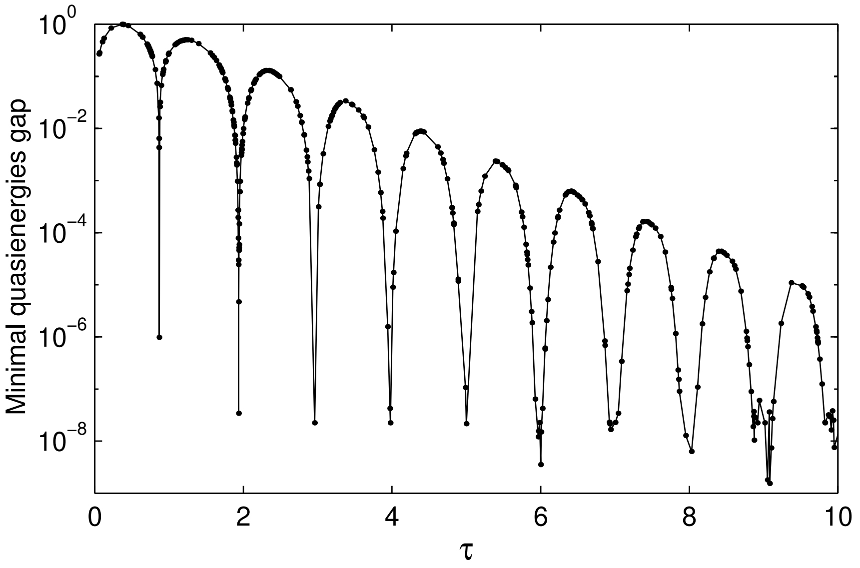

Figure 2. Plot of the minimal quasienergy gap vs. τ for $\varepsilon = 0$. Notice the logarithmic y–scale. The calculation accuracy was of order 10^{-8}. The circles represents the calculation points which are spaced non-uniformly.

the failure of the WVN rule, one expects WVN to fail for *some* ε, and a test would be to see if the crossing disappear when we wiggle away from $\varepsilon = 0$.

For non-zero values of ε both gaps closure at $E = 0, \pi$ indeed disappear, as is seen in figure 3, and Wigner and von-Neumann are vindicated. In conclusion, the numerical results for the toy model Hamiltonian are in agreement with WVN and with our argument that Chern numbers for Floquet operators are generically zero.

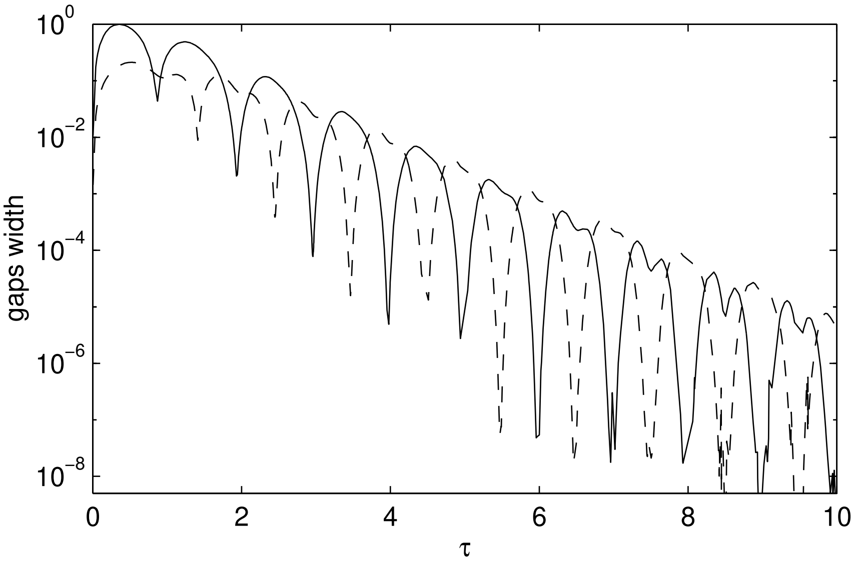

Figure 3. Gaps width of the Floquet operators from H_2 with $\varepsilon = 0.2$. The solid line is for the gap near $E = \pm\pi$, the dashed line is for the gap near $E = 0$

Similar results are obtained for the Hamiltonian

$$H_1(s,k) = \frac{1}{\sqrt{\sin(t)^2 + |a|^2}} \begin{pmatrix} \sin(t) & a \\ \bar{a} & -\sin(t) \end{pmatrix} \tag{30}$$

with

$$a = \cos(k + \frac{\pi}{4}) + \mathrm{i}\cos(k - \frac{\pi}{4}) + (1+\mathrm{i})\,[\cos(t) + 1]\ , \tag{31}$$

which has ± 1 Chern numbers. The gaps at $E = 0, \pi$ are plotted in figure 4.

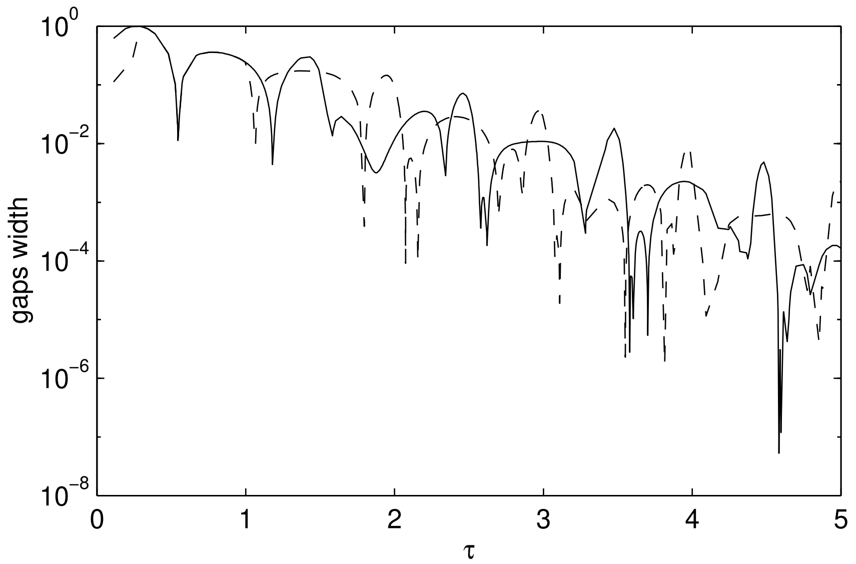

Figure 4. Gaps width of the Floquet operators from H_1. The solid line is for the gap near $E = \pm\pi$, the dashed line is for the gap near $E = 0$

We have also studied numerically the question of stability of Chern numbers for the Harper model. The canonical Harper model is free from extra parameters like ε in the toy model. It may be generic and it may fail to be generic. We have therefore resorted to a numerical study.

Telling a true crossing from an avoided crossing in a numerical study is often a challenge. Often one finds complicated pictures where it is difficult to tell with certainty which is really the case. This is illustrated by numerical results for the Harper model with $p = 1$ and $q = 3, 4$, see figures 5 and 6. Our numerical results are consistent with the assertion that the Harper model has some stable crossing. It is an open problem to prove this.

One reliable method to identify crossing of *self-adjoint* matrices, is to compute a Chern number (in the Hermitian case) or the Longuet-Higgins phase (in the symmetric case) [2, 3]. Unfortunately, this method does not work for Floquet operators, as we now proceed to explain.

Because of the s independence of the spectrum an isolated crossing would look like the line of constant k and τ as in figure 7. The associated Chern number involves the surface of integration, a 2-torus of (s,θ), as described in figure 7.

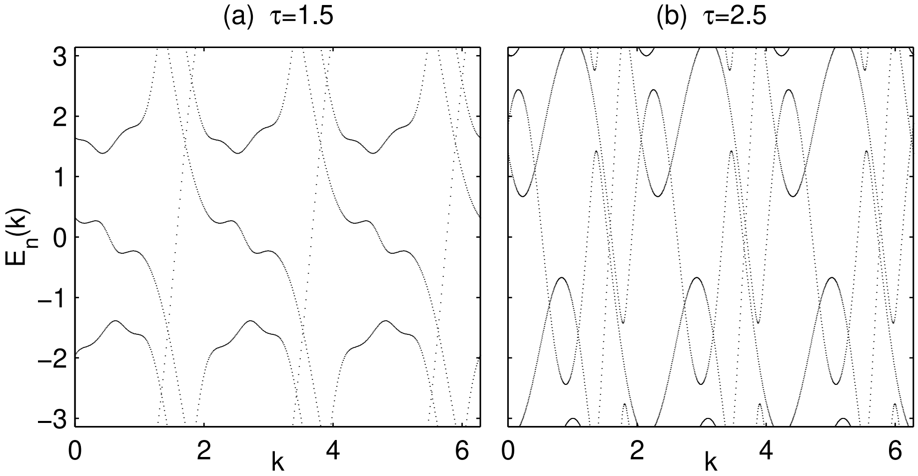

Figure 5. The quasienergies $E(k)$ of the Floquet operator associated with the 3×3 Harper Hamiltonian for $\tau = 1.5, 2.5$.

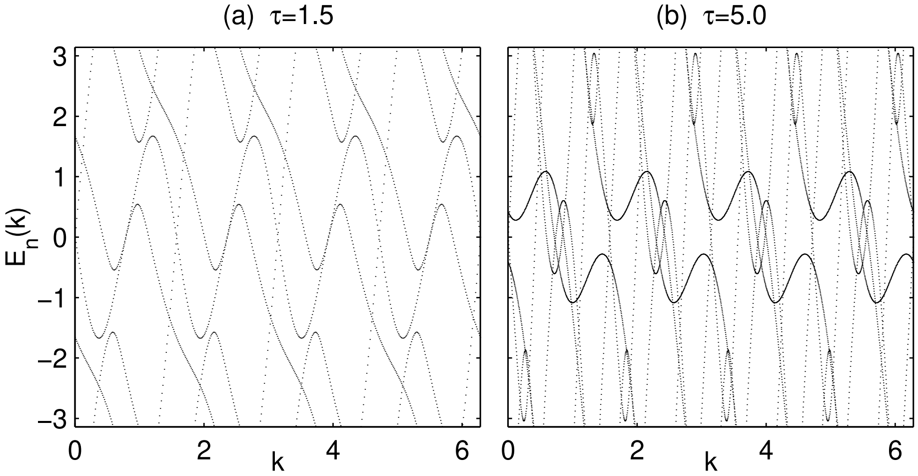

Figure 6. The quasienergies $E(k)$ of the Floquet operator associated with the 4×4 Harper Hamiltonian for $\tau = 1.5, 5.0$.

Using similar arguments as in equations (22) and (23) we find

$$\mathrm{ch}(\mathcal{P}) = -\frac{1}{2\pi}\int_0^{2\pi} \partial_\theta E(\theta)\,\mathrm{d}\theta = -\frac{E(2\pi)-E(0)}{2\pi}\,. \tag{32}$$

Since the surface lies away from any crossing, the winding number of all eigenvalues must be the same. Since the Chern numbers sum to zero, the winding must be zero too. Chern number are therefore not useful to identify the crossing of Floquet operators.

6. Conclusion

One of the amusing and surprising observation we have made is that one can learn much about adiabatic response, which is concerned with weak fields and the $\mathcal{E} \to 0$ limit, by expanding about ∞ fields. In particular, one learns form this expansion, that there are

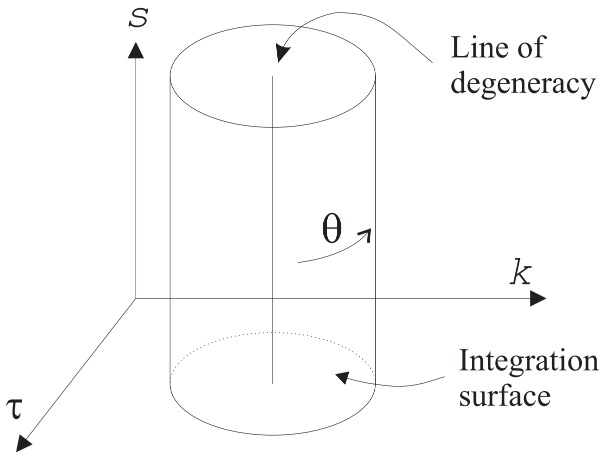

Figure 7. The 2–torus which is used to enclose the crossing line. Notice that s is a cyclic parameter, so the cylinder closes to a torus.

no power law corrections to linear response (for Hamiltonians with a gap condition) if such an expansion exists for $\mathcal{E} > 0$.

Transport coefficients *in linear response theory*, such as the Hall conductance, can, in general, undergo a phase transition as functions of *parameters* in the Hamiltonian. This is because, in the linear regime, the conductance can be expressed in terms of the spectral data of the Hamiltonian. Spectral data may, and in general will, lose smoothness near eigenvalue crossing and allow for the loss of analyticity.

The breakdown of the Hall effect in finite fields seen in figure 1 is not a phase transition and is not associated with any loss of analyticity. This is so because transport at finite fields is not a function of spectral data that depend on the field strength. The field strength does not affect the initial state at all, and affects the evolution analytically. As such there is no loss of analyticity as the field is increased.

For Hamiltonians that are periodic in time the study of transport reduces to a spectral problem for the Floquet operator. This holds for any value of the driving field, and not just in the linear response regime. Because of this, changing the field can, in principle, lead to a non-smooth behavior of the transport coefficients when eigenvalues of the Floquet operator cross.

We have extended results of Ferrari that relate transport coefficients to Chern numbers of Floquet operators. We argued that the Chern numbers of Floquet operators are generically zero, because non-zero Chern numbers necessarily come with eigenvalue crossing. Because of this, Chern numbers loose their stability to small variations in the Hamiltonian. It is open if some special model, such as Harper, have non-zero Chern numbers.

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