Non-Monotoneous Random Schrödinger Operators: The Anderson Model

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Abstract

We prove exponential localization at all energies for one-dimensional continuous Anderson-type models with single site potentials of changing sign. A periodic background potential is allowed. The main problem arises from non-monotonicity, i.e. the operator does not depend monotoneously in form sense on the random parameters. We show that the method of "two-parameter spectral averaging", recently devised by Buschmann and Stolz to prove localization for Poisson and random displacement models, can be modified to work for the type of Anderson model considered here.

1 Introduction

We study a one-dimensional continuous Anderson type model with periodic background: Let $f \neq 0$ be real-valued, locally square-integrable and supported in $[0,1]$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and q_n $(n \in \mathbb{Z})$ real independent and identically distributed random variables on Ω with $\mathbb{E}(q_n^2) < \infty$, where $\mathbb{E}$ denotes expectation with respect to $\mathbb{P}$.

Let V_{per} be real, locally square-integrable, and 1-periodic on $\mathbb{R}$, i.e. $V(x+1) = V(x)$ for all $x \in \mathbb{R}$.

Under these assumptions (actually for much more general f, see e.g. [5]) the Schrödinger operator

$$H_\omega = -\frac{d^2}{dx^2} + V_{per}(x) + \sum_{n\in\mathbb{Z}} q_n(\omega) f(x-n) \tag{1.1}$$

in $L^2(\mathbb{R})$ is almost surely essentially self-adjoint on $C_0^\infty(\mathbb{R})$.

It also follows from the general theory of ergodic operators that there are sets Σ_{ac}, Σ_{sc} and Σ_{pp} such that almost surely

$$\sigma_{ac}(H_\omega) = \Sigma_{ac}, \quad \sigma_{sc}(H_\omega) = \Sigma_{sc}, \quad \sigma_{pp}(H_\omega) = \Sigma_{pp},$$

where σ_{ac}, σ_{sc} and σ_{pp} denote the absolutely continuous, singular continuous and point spectrum, respectively. For this and general background on random operators including extensive lists of references see [2] and [10].

Our main result is

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Theorem 1 *Let f and q_n be given as above and assume that the distribution of the q_n has a non-trivial absolutely continuous component. Then $\Sigma_{ac} = \Sigma_{sc} = \emptyset$, thus H_ω almost surely has pure point spectrum. The eigenfunctions of H_ω decay exponentially at the rate of the Lyapunov exponent.*

H_ω is non-deterministic in the sense of Kotani's theory [7, 8, 5] since f is compactly supported. Thus it follows that the Lyapunov exponent is positive and that $\Sigma_{ac} = \emptyset$. We only need to prove that $\Sigma_{sc} = \emptyset$. Exponential decay of the eigenfunctions then also follows from general results [9].

Theorem 1 is known for the case of a continuous single site potential f of *fixed sign*, where it has been proven by Kotani and Simon [9]. Our main point here is that localization at all energies for one-dimensional Anderson models is a very universal phenomenon which holds without having to assume a definite sign or regularity for f. We are aware of only two works where localization for Anderson models without sign restriction on the single site potential is studied, both showing localization near spectral boundaries in multi dimensional models under some additional assumptions: In [6], Klopp uses a weak version of Lifshitz tails for the density of states near spectral boundaries, which is not easily established for f with changing sign. Wang's work [13] on Landau hamiltonians holds for large magnetic fields.

If f has fixed sign, then the operator H_ω from (1.1) depends monotoneously (in form sense) on the coupling constants $q_n(\omega)$. This is used in most of the proofs of existing results on localization, in particular it enters the proofs of tools like Wegner estimates and spectral averaging. The non-monotonicity arising from sign changes in f requires new ideas. In [1] the method of *two-parameter spectral averaging* was developed in order to prove localization for one-dimensional Poisson and displacement models, which also lack monotonicity properties.

A somewhat different variant of two-parameter spectral averaging also works for the Anderson-type model (1.1) with indefinite f, and our goal here is to use this method to prove Theorem 1. The method consists in fixing all but two (rather than the traditional all but one) of the random parameters q_n in H_ω and then to average the spectral measure of H_ω with respect to Lebesgue measure in the remaining two parameters, say q_0 and q_1. More precisely, the averaging is done locally with respect to the energy: For every energy $E_0 \in \mathbb{R}$ one may average over a suitable parameter range for q_0 and q_1 (which in particular has non-zero probability), such that the resulting averaged measure is absolutely continuous in a neighborhood of E_0. This is the content of Theorem 2, proven in Section 2.

After spectral averaging is established, the proof of $\Sigma_{sc} = \emptyset$ follows quite easily by using known methods, in particular Kotani theory and Gilbert and Pearson's subordinacy theory (which was already used in [1] as a substitute for results on polynomial boundedness of generalized eigenfunctions). Theorem 2 leads to the fact that a neighborhood of E_0 contains no singular continuous spectrum with positive probability. Since E_0 is arbitrary and spectral types are non-random this implies $\Sigma_{sc} = \emptyset$, see Section 3.

2 Spectral Averaging

The general results in this section naturally hold under L^1_{loc} assumptions on the potentials (L^2_{loc} is used in Section 3 only to have a convenient reference for Kotani theory). For background on Sturm-Liouville respectively Weyl-Titchmarsh theory see e.g. [3] or [14], for form methods see [11].

Let $V \in L^1_{loc}(\mathbb{R})$ be real-valued and such that the differential expression $-d^2/dx^2 + V$ is limit point at $+\infty$ and $-\infty$. For example, this holds if $\int_{-N}^{N} |V(x)|\, dx = O(N^3)$ as $N \to \infty$. Let $W_1, W_2 \in L^1_{loc}(\mathbb{R})$ be real-valued, compactly supported, $W_1 \neq 0$, $W_2 \neq 0$ (in L^1-sense) and such that $\sup(\operatorname{supp} W_1) \leq \inf(\operatorname{supp} W_2)$.

Under these assumptions, the maximal operator $H_{\lambda,\mu}$ defined by $-\frac{d^2}{dx^2} + V + \lambda W_1 + \mu W_2$ is selfadjoint in $L^2(\mathbb{R})$ for arbitrary $\lambda, \mu \in \mathbb{R}$. By $\rho_{\lambda,\mu}$ we will denote the Weyl-Titchmarsh spectral measure for $H_{\lambda,\mu}$, i.e. the trace of the 2×2-matrix-valued spectral measure associated with $H_{\lambda,\mu}$ in Weyl-Titchmarsh theory, e.g. [3]. The absolutely continuous, singular continuous and pure point spectrum of $H_{\lambda,\mu}$ are the topological supports of the corresponding parts of the Lebesgue decomposition of $\rho_{\lambda,\mu}$.

Theorem 2 *Let $E_0 \in \mathbb{R}$ be fixed. Then there exists a discrete set $M \subset \mathbb{R}$ such that for every $\lambda_0 \in \mathbb{R} \setminus M$ there is a discrete set $N(\lambda_0) \subset \mathbb{R}$ with the following property:*

If $\mu_0 \in \mathbb{R} \setminus N(\lambda_0)$ then there are $\varepsilon > 0$ and $\delta > 0$ such that the Borel measure μ defined by

$$\mu(B) = \int_{\lambda_0-\delta}^{\lambda_0+\delta} \int_{\mu_0-\delta}^{\mu_0+\delta} \rho_{\lambda,\mu}(B)\, d\mu\, d\lambda \tag{2.1}$$

is absolutely continuous on $(E_0 - \varepsilon, E_0 + \varepsilon)$.

The proof of Theorem 2 will be prepared by several lemmas. Our main tool will be Prüfer variables. If $u = u_a(\cdot, \theta, E, Q)$ denotes the unique solution of the initial value problem $-u'' + Qu = Eu$, $u(a) = \sin\theta$, $u'(a) = \cos\theta$, then the Prüfer amplitude and phase are

$$r_a(\cdot, \theta, E, Q) = (u^2 + u'^2)^{1/2}, \tag{2.2}$$

and

$$\phi_a(\cdot, \theta, E, Q) = \arctan \frac{u}{u'} \quad (\text{resp. } \operatorname{arccot} \frac{u'}{u}). \tag{2.3}$$

The mod π ambiguity of the phase is removed by setting $\phi_a(a, \theta, E, Q) = \theta$ and requiring that $\phi_a(\cdot, \theta, E, Q)$ is continuous.

For the first lemma, let $b > a$, $q \in L^1(a, b)$ and $w \in L^1(a, b)$ real-valued, $\theta \in [0, \pi)$ and $E \in \mathbb{R}$ fixed. We will temporarily use $\phi_a(x, \lambda)$ for $\phi_a(x, \theta, E, q + \lambda w)$, similarly for r_a and u_a.

Lemma 3 *Let $w \neq 0$. Then the set*

$$\{\lambda \in \mathbb{R} :\ \partial_\lambda \phi_a(b, \lambda) = 0\}$$

is discrete.

Proof: For fixed x the solution $u_a(x, \lambda)$ and its derivative depend analytically on λ, e.g. [14, Ch.2], thus the same holds for ϕ_a. If the Lemma were not true, then $\partial_\lambda \phi_a(b, \lambda) = 0$ for all λ and therefore

$$\phi_a(b, \lambda) = \eta$$

constantly in λ. This implies that the boundary value problem

$$-u'' + (q + \lambda w)u = Eu, \quad u(a)\cos\theta - u'(a)\sin\theta = 0, \quad u(b)\cos\eta - u'(b)\sin\eta = 0 \tag{2.4}$$

has a non-trivial solution for all $\lambda \in \mathbb{R}$. Therefore E is an eigenvalue of the selfadjoint Sturm-Liouville operator T_λ corresponding to (2.4) for every λ. The operators T_λ are bounded from below, have simple discrete spectrum, and all eigenvalues depend analytically on λ. Since no eigenvalue crossings are allowed in dimension one, there is an integer N such that T_λ has exactly N eigenvalues below E for all λ.

We have $w \neq 0$. Thus there exist $N+1$ disjoint intervals $I_1, \dots, I_{N+1}$ in (a,b) such that (w.l.o.g.) $\int_{I_n} w < 0$. There also exist functions $f_1, \dots, f_{N+1} \in C_0^\infty(a,b)$ with f_i supported in I_i and $\int w|f_i|^2 < 0$.

The f_i belong to the form domain of T_λ and (in form sense) $\langle f_i, T_\lambda f_i\rangle = C_i + \lambda \int w|f_i|^2$. So there is $\Lambda \in \mathbb{R}$ sufficiently large such that $\langle f_i, T_\Lambda f_i\rangle < E\|f_i\|^2$ for all i, and thus $\langle v, Tv\rangle < E\|v\|^2$ for all $v \in \operatorname{lin}\{f_1, \dots, f_{N+1}\} \setminus \{0\}$. By the variational principle T_Λ has at least $N+1$ eigenvalues below E, a contradiction. $\square$

We note the formula (a proof is included in Section 4)

$$\partial_\lambda \phi_a(b,\lambda) = -\frac{1}{r_a(b,\lambda)^2} \int_a^b w(x) u_a(x,\lambda)^2 \, dx. \tag{2.5}$$

Thus, Lemma 3 says that if $\int_0^1 w(x) u_a(x,\lambda)^2\, dx = 0$ for all λ, then $w = 0$.

For the next lemma, let V, W_1 and W_2 be given as in Theorem 2. For simplicity of notation we will assume in the following that $\operatorname{supp} W_1 \subset [0,1]$ and $\operatorname{supp} W_2 \subset [1,2]$. We will look at the one-parameter family of potentials $V_\lambda := V + \lambda W_1$, and the two-parameter family $V_{\lambda,\mu} := V + \lambda W_1 + \mu W_2$.

Lemma 4 *Let $E_0 \in \mathbb{R}$ be fixed. Then there exists a discrete set $M \subset \mathbb{R}$ such that for every $\lambda_0 \in \mathbb{R} \setminus M$ there is a discrete set $N(\lambda_0) \subset \mathbb{R}$ with the following property:*

If $\mu_0 \in \mathbb{R} \setminus N(\lambda_0)$, then there exist closed sets $T_1 \subset \mathbb{R}$ and $T_2 \subset \mathbb{R}$ such that

(i) $T_1 \cup T_2 = \mathbb{R}$,

(ii) $\partial_\lambda \phi_0(1,\theta,E_0,V_\lambda) \neq 0$ *at* $\lambda = \lambda_0$ *for all* $\theta \in T_1$,

(iii) $\partial_\mu \phi_0(2,\theta,E_0,V_{\lambda_0,\mu}) \neq 0$ *at* $\mu = \mu_0$ *for all* $\theta \in T_2$.

Proof: Let $E_0 \in \mathbb{R}$ be fixed, pick an arbitrary $\theta_0 \in [0,\pi)$ and choose

$$M := \{\lambda \in \mathbb{R} : \partial_\lambda \phi_0(1,\theta_0,E_0,V_\lambda) = 0\}.$$

M is discrete by Lemma 3. Let $\lambda_0 \in \mathbb{R} \setminus M$ and define

$$T(\lambda_0) := \{\theta \in \mathbb{R} : \partial_\lambda \phi_0(1,\theta,E_0,V_\lambda) = 0 \text{ at } \lambda = \lambda_0\}.$$

$\partial_\lambda \phi_0(1,\theta,E_0,V_\lambda)$ is analytic and π-periodic in θ (this can be seen from (2.5), noting that $u_a = u_a^N \sin\theta + u_a^D \cos\theta$, where $u_a^{N/D}$ are solutions with Neumann respectively Dirichlet boundary conditions at a). Since $\theta_0 \notin T(\lambda_0)$, this implies that $T(\lambda_0)$ is discrete and π-periodic.

For arbitrary θ, the set

$$M_\theta := \{\mu \in \mathbb{R} : \partial_\mu \phi_0(2,\theta,E_0,V_{\lambda_0,\mu}) = 0\}$$

is discrete by Lemma 3. Thus the finite union

$$N(\lambda_0) := \bigcup_{\theta \in T(\lambda_0) \cap [0,\pi)} M_\theta$$

is discrete. If $\mu_0 \in \mathbb{R} \setminus N(\lambda_0)$ then

$$\partial_\mu \phi_0(2,\theta,E_0,V_{\lambda_0,\mu}) \neq 0 \quad \text{at } \mu = \mu_0 \tag{2.6}$$

for all $\theta \in T(\lambda_0)$. By continuity of $\partial_\mu \phi_0$ in θ there is an $\eta > 0$ such that (2.6) holds for all θ with $\mathrm{dist}(\theta, T(\lambda_0)) \leq \eta$. If $\mathrm{dist}(\theta, T(\lambda_0)) \geq \eta$, then $\partial_\lambda \phi_0(1, \theta, E_0, V_\lambda) \neq 0$ at $\lambda = \lambda_0$ by definition of $T(\lambda_0)$. Thus, with the choice

$$\begin{aligned} T_1 &:= \{\theta \in \mathbb{R} : \mathrm{dist}(\theta, T(\lambda_0)) \geq \eta\}, \\ T_2 &:= \{\theta \in \mathbb{R} : \mathrm{dist}(\theta, T(\lambda_0)) \leq \eta\} \end{aligned}$$

we have proven Lemma 4. □

Lemma 5 *Let E_0, λ_0 and μ_0 be given as in Lemma 4. Then there exist $\varepsilon > 0$, $\delta > 0$ and $C = C(E_0, \lambda_0, \mu_0, \varepsilon, \delta)$ such that*

$$\int_{\lambda_0-\delta}^{\lambda_0+\delta} \int_{\mu_0-\delta}^{\mu_0+\delta} r_0(N, \theta, E, V_{\lambda,\mu})^{-2} \, d\mu \, d\lambda \leq C \tag{2.7}$$

uniformly in $N > 0$, $\theta \in [0, \pi)$, and $E \in [E_0 - \varepsilon, E_0 + \varepsilon]$.

Proof: We will frequently use that $r_0(N, \theta, E, V_{\lambda,\mu})$ is jointly continuous in all its parameters $(N, \theta, E, \lambda, \mu)$, a consequence of standard facts from ODE theory. Thus we may assume $N > 2$.

With T_1 and T_2 chosen as in Lemma 4, assume $\theta \in T_1$ first. From the definition of Prüfer variables we get (omitting the parameter E)

$$\begin{aligned} r_0(N, \theta, V_{\lambda,\mu})^{-2} &= r_0(1, \theta, V_\lambda)^{-2} r_1(N, \phi_0(1, \theta, V_\lambda), V_{\lambda,\mu})^{-2} \\ &\leq C_1 r_1(N, \phi_0(1, \theta, V_\lambda), V + \mu W_2)^{-2} \end{aligned} \tag{2.8}$$

locally uniformly in (λ, θ, E) (by continuity; note also that $V_{\lambda,\mu} = V + \mu W_2$ in $[1, \infty)$).

Let $\alpha := \phi_0(1, \theta, V_\lambda)$. Then $\partial_\lambda \alpha \neq 0$ at $\lambda = \lambda_0$, $E = E_0$ by Lemma 4. Continuity of $\partial_\lambda \alpha$ in (λ, θ, E) (use (2.5)) implies the existence of $\varepsilon > 0$ and $\delta > 0$ such that $|\partial_\lambda \alpha| \geq C_2 > 0$ uniformly in $\theta \in T_1$, $E \in [E_0 - \varepsilon, E_0 + \varepsilon]$ and $\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta]$. Thus we can estimate the λ-integral in (2.7) by using (2.8) and substituting α for λ to get

$$\int_{\lambda_0-\delta}^{\lambda_0+\delta} r_0(N, \theta, V_{\lambda,\mu})^{-2} \, d\lambda \leq \frac{C_1}{C_2} \int_{\alpha_1}^{\alpha_2} r_1(N, \alpha, V + \mu W_2)^{-2} \, d\alpha. \tag{2.9}$$

Here we have chosen an interval $[\alpha_1, \alpha_2]$ such that $\alpha([\lambda_0 - \delta, \lambda_0 + \delta]) \subset [\alpha_1, \alpha_2]$ uniformly in $\theta \in [0, \pi)$ and $E \in [E_0 - \varepsilon, E_0 + \varepsilon]$. Subdividing $[\alpha_1, \alpha_2]$ into intervals of length $\leq \pi$ and using the well known formula (e.g. Corollary 12 of [12])

$$\int_\theta^{\theta+\pi} r_0(x, \alpha, E, Q)^{-2} \, d\alpha = \pi \quad \text{for all } \theta, x, E, Q,$$

we get the estimate $(|\alpha_2 - \alpha_1| + \pi) C_1 / C_2$ for (2.9). Carrying out the now trivial μ-integral in (2.7) we get

$$\int_{\mu_0-\delta}^{\mu_0+\delta} \int_{\lambda_0-\delta}^{\lambda_0+\delta} r_0(N, \theta, V_{\lambda,\mu})^{-2} \, d\lambda \, d\mu \leq C < \infty \tag{2.10}$$

uniformly in $N > 2$, $\theta \in T_1$, and $E \in [E_0 - \varepsilon, E_0 + \varepsilon]$.

For $\theta \in T_2$ we will carry out the μ-integral first. Use

$$\begin{aligned} r_0(N, \theta, V_{\lambda,\mu})^{-2} &= r_0(2, \theta, V_{\lambda,\mu})^{-2} r_2(N, \phi_0(1, \theta, V_{\lambda,\mu}), V)^{-2} \\ &\leq C_3 r_2(N, \phi_0(2, \theta, V_{\lambda,\mu}), V)^{-2} \end{aligned}$$

locally uniformly in $(\lambda, \mu, \theta, E)$. With $\beta(\mu) := \phi_0(2, \theta, V_{\lambda,\mu})$ we have $\partial_\mu \beta \neq 0$ at (λ_0, μ_0, E_0) by Lemma 4. Continuity of $\partial_\mu \beta$ in $(\lambda, \mu, \theta, E)$ allows to argue as above and conclude the existence of $\varepsilon > 0$ and $\delta > 0$ such that

$$\int_{\lambda_0-\delta}^{\lambda_0+\delta} \int_{\mu_0-\delta}^{\mu_0+\delta} r_0(N, \theta, V_{\lambda,\mu})^{-2} \, d\mu \, d\lambda \leq C < \infty \tag{2.11}$$

uniformly in $N > 0$, $\theta \in T_2$ and $E \in [E_0 - \varepsilon, E_0 + \varepsilon]$. This implies (2.7) with the smaller of the two values found for ε and δ and the larger of the constants C in (2.10) and (2.11). □

To deduce absolute continuity of the averaged spectral measure (2.1) from (2.7) we will use "Carmona's formula" (e.g. [2, Thm.III.3.6]): If ρ is the Weyl-Titchmarsh spectral measure for the selfadjoint operator $-d^2/dx^2 + Q$ on $L^2(\mathbb{R})$ and $\rho_{-\infty}^{0,\theta}$ the Weyl-Titchmarsh spectral measure for the selfadjoint realization of $-d^2/dx^2 + Q$ on $L^2(-\infty, 0)$ with boundary condition $u(0)\cos\theta - u'(0)\sin\theta = 0$, then

$$\int f(E) \, d\rho(E) = \lim_{N\to\infty} \frac{1}{\pi} \int_0^\pi \int f(E) r_0(N, \theta, E, Q)^{-2} \, d\rho_{-\infty}^{0,\theta}(E) \, d\theta \tag{2.12}$$

for all continuous and compactly supported f.

To complete the **proof of Theorem 2**, let E_0, λ_0, μ_0, ε and δ be chosen as in Lemma 4 and Lemma 5. Absolute continuity (in fact, Lipshitz continuity) of μ on $(E_0 - \varepsilon, E_0 + \varepsilon)$ will follow if we can show the existence of $C > 0$ such that

$$\int f(E) \, d\mu(E) \leq C \int f(E) \, dE \tag{2.13}$$

for all continuous non-negative f with support in $(E_0 - \varepsilon, E_0 + \varepsilon)$. Expressing μ by (2.1) and using (2.12) on each of the $\rho_{a,b}$ we get

$$\int f(E) \, d\mu(E) = \int_{\lambda_0-\delta}^{\lambda_0+\delta} \int_{\mu_0-\delta}^{\mu_0+\delta} \lim_{N\to\infty} \frac{1}{\pi} \int_0^\pi \int f(E) r_0(N, \theta, E, V_{\lambda,\mu})^{-2} \, d\rho_{-\infty}^{0,\theta}(E) \, d\theta \, d\mu \, d\lambda. \tag{2.14}$$

Uniformity in N, θ, E of the estimate (2.7) allows to exchange the (λ, μ)-integration and the N-limit in (2.14) by Fatou's lemma and to conclude

$$\int f(E) d\rho(E) \leq \frac{C}{\pi} \int_0^\pi \int f(E) d_{-\infty}^{0,\theta}(E) d\theta = C \int f(E) dE,$$

where finally spectral averaging over θ ("Kotani's trick") was used. □

3 Proof of Theorem 1

As mentioned in the introduction, $\Sigma_{ac} = \emptyset$ and exponential decay of eigenfunctions at the rate of the Lyapunov exponent are well known consequences of the fact that the family H_ω is ergodic and non-deterministic and thus has positive Lyapunov exponent [7, 8, 9, 5]. It remains to show that $\Sigma_{sc} = \emptyset$, which follows by arguments similar to those used in [1] for Poisson models and for displacement models:

Let ρ_ω be the Weyl-Titchmarsh spectral measure of H_ω and $\rho_\omega = \rho_\omega^{ac} + \rho_\omega^{sc} + \rho_\omega^{pp}$ be its Lebesgue decomposition. It follows from subordinacy theory ([4], see also [1, Sect.2]) that

$$\rho_\omega^{sc}(\mathbb{R} \setminus M_\omega^{sc}) = \emptyset, \tag{3.1}$$

where

$$M_\omega^{sc} = \{E \in \mathbb{R} : \exists \text{ solution } u_+ \text{ of } H_\omega u = Eu \text{ which is subordinate at } +\infty \text{ and } u_+ \notin L^2(0,\infty), \text{ or } \exists \text{ solution } u_- \text{ of } H_\omega u = Eu \text{ which is subordinate at } -\infty \text{ and } u_- \notin L^2(-\infty,0)\}\,.$$

Positivity of the Lyapunov exponent implies that [1, Theorem 3.1]

$$|M_\omega^{sc}| = 0 \qquad \text{almost surely,}$$

i.e. for almost every choice of the sequence of coupling constants $(q_n(\omega))_{n\in\mathbb{Z}}$. Let a sequence (q_n) with this property be fixed. Then the set M_ω^{sc} (depending only on asymptotic properties of solutions) does not depend on (q_0, q_1) (which effect the potential only locally). Thus there exists a set $M^{sc} = M^{sc}((q_n)_{n\in\mathbb{Z}\setminus\{0,1\}})$ such that $M^{sc} = M_\omega^{sc}$ for all (q_0, q_1). Below we will use the result from Section 2 to prove

Proposition 6 *For every $E_0 \in \mathbb{R}$ there exists an $\varepsilon > 0$ with the following property:*
If $(q_n)_{n\in\mathbb{Z}\setminus\{0,1\}}$ is given such that $|M_{sc}| = 0$, then

$$\mathbb{P}^{0,1}\{\sigma_{sc}(H^\omega) \cap (E_0-\varepsilon, E_0+\varepsilon) = \emptyset\} > 0, \tag{3.2}$$

where $\mathbb{P}^{0,1}$ denotes the joint distribution of (q_0, q_1).

Since $|M^{sc}| = 0$ for almost every $(q_n)_{n\in\mathbb{Z}\setminus\{0,1\}}$, it follows from Proposition 6, independence of (q_0, q_1) from $(q_n)_{n\in\mathbb{Z}\setminus\{0,1\}}$ and Fubini that

$$\mathbb{P}\{\sigma_{sc}(H_\omega) \cap (E_0-\varepsilon, E_0+\varepsilon) = \emptyset\} > 0.$$

Since $\sigma_{sc}(H_\omega) = \Sigma_{sc}$ with probability one, this implies $\Sigma_{sc} \cap (E_0-\varepsilon, E_0+\varepsilon) = \emptyset$. E_0 is arbitrary. Thus $\Sigma_{sc} = \emptyset$.

It remains to **prove Proposition 6**: With $V = V_{per} + \Sigma_{n\in\mathbb{Z}\setminus\{0,1\}} q_n f(x-n)$, $W_1 = f$ and $W_2 = f(\cdot - 1)$ we are in the situation of Theorem 2. Let μ be the probability distribution of q_0 respectively q_1, and μ_{ac} its (non-vanishing) absolutely continuous component. Let M be the set provided by Theorem 2 and pick $\lambda_0 \in (\mathbb{R}\setminus M) \cap \operatorname{supp}\mu_{ac}$. Also, let $N(\lambda_0)$ be as given in Theorem 2 and $\mu_0 \in (\mathbb{R}\setminus N(\lambda_0)) \cap \operatorname{supp}\mu_{ac}$. Such λ_0, μ_0 exist by the discreteness of M and $N(\lambda_0)$. Finally, pick $\varepsilon > 0$ and $\delta > 0$ as in Theorem 2. Thus the measure $\mu = \int_{\lambda_0-\delta}^{\lambda_0+\delta}\int_{\mu_0-\delta}^{\mu_0+\delta} \rho_{\lambda,\mu}\, d\mu\, d\lambda$ is absolutely continuous in $(E_0-\varepsilon, E_0+\varepsilon)$.

In particular, $\mu(M_{sc} \cap (E_0-\varepsilon, E_0+\varepsilon)) = 0$ and therefore $\rho_{\lambda,\mu}(M_{sc} \cap (E_0-\varepsilon, E_0+\varepsilon)) = 0$ for Lebesgue a.e. $(\lambda,\mu) \in (\lambda_0-\delta, \lambda_0+\delta) \times (\mu_0-\delta, \mu_0+\delta)$. (3.2) now follows from λ_0, $\mu_0 \in \operatorname{supp}\mu_{ac}$.

We note that the choice of $\varepsilon > 0$ in Theorem 2 does not depend on $V|_{\mathbb{R}\setminus[0,2]}$ (see the proof of Lemma 5). Therefore (3.2) holds with the same interval $(E_0-\varepsilon, E_0+\varepsilon)$ for every sequence $(q_n)_{n\in\mathbb{Z}\setminus\{0,1\}}$ with $|M_{sc}| = 0$. □

4 Proof of equation (2.5)

For convenience, we include a proof of formula (2.5). The Prüfer variables satisfy the differential equations

$$\partial_x r = (1 + q + \lambda w - E) r \sin\phi\cos\phi$$

and

$$\partial_x \phi = \cos^2 \phi - (q + \lambda w - E) \sin^2 \phi,$$

i.e. also

$$\begin{aligned} \partial_x \partial_\lambda \phi &= -2(\partial_\lambda \phi)(1 + q + \lambda w - E) \sin \phi \cos \phi - w \sin^2 \phi \\ &= -2(\partial_\lambda \phi) \frac{\partial_x r}{r} - w \sin^2 \phi. \end{aligned} \tag{4.1}$$

This yields $\partial_x (r^2 \partial_\lambda \phi) = -wr^2 \sin \phi = -wu^2$ and therefore (2.5) since $(r^2 \partial_\lambda \phi)(a) = 0$.

We note that in (4.1) we have used that $\partial_x \partial_\lambda \phi = \partial_\lambda \partial_x \phi$. Under L^1_{loc}-assumptions on q and w this holds in distributional sense and thus for almost every (x, λ), which yields (2.5) for almost every λ. By continuity this extends to every λ.

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