Dynamical Localization for the Random Dimer Schrödinger Operator

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Abstract

We study the one-dimensional random dimer model, with Hamiltonian $H_\omega = \Delta + V_\omega$, where for all $x \in \mathbb{Z}, V_\omega(2x) = V_\omega(2x+1)$ and where the $V_\omega(2x)$ are i.i.d. Bernoulli random variables taking the values $\pm V$, $V > 0$. We show that, for all values of V and with probability one in ω, the spectrum of H is pure point. If $V \leq 1$ and $V \neq 1/\sqrt{2}$, the Lyapounov exponent vanishes only at the two critical energies given by $E = \pm V$. For the particular value $V = 1/\sqrt{2}$, respectively $V = \sqrt{2}$, we show the existence of additional critical energies at $E = \pm 3/\sqrt{2}$, resp. $E = 0$. On any compact interval I not containing the critical energies, the eigenfunctions are then shown to be semi-uniformly exponentially localized, and this implies dynamical localization: for all $q > 0$ and for all $\psi \in \ell^2(\mathbb{Z})$ with sufficiently rapid decrease:

$$\sup_t r^{(q)}_{\psi,I}(t) \equiv \sup_t \langle P_I(H_\omega)\psi_t,\ |X|^q P_I(H_\omega)\psi_t \rangle \ < \infty.$$

Here $\psi_t = e^{-iH_\omega t}\psi$, and $P_I(H_\omega)$ is the spectral projector of H_ω onto the interval I. In particular if $V > 1$ and $V \neq \sqrt{2}$, these results hold on the entire spectrum (so that one can take $I = \sigma(H_\omega)$).

1 Introduction

We study a one-dimensional discrete Schrödinger operator, known as the random dimer model, introduced in [8]. More precisely, the family of Hamiltonians H_ω ($\omega \in \Omega = \{0,1\}^{\mathbb{Z}}$) that we consider is defined as follows. For $u \in \ell^2(\mathbb{Z})$,

$$(H_\omega u)(x) = u(x-1) + u(x+1) + V_\omega(x)u(x), \; x \in \mathbb{Z}, \tag{1.1}$$

where $V_\omega(2x+1) = V_\omega(2x)$, and the $(V_\omega(2x))_{x\in\mathbb{Z}}$ are independent and identically distributed random variables, with $\mathbb{P}(V_\omega(0) = -V) = p$, $0 < p < 1$ and $V > 0$. Note that the on-site potential takes only two values and takes the same value on pairs of sites, whence the name of the model which has attracted considerable attention in the physics literature since it seems to display an interesting localization-delocalization phenomenon [8] [9] [19] that we now briefly explain.

When $V \leq 1$, it is easy to see that, due to a resonance phenomenon, there is perfect transmission at two critical energies $E_c = \pm V$. In other words, at these energies, the model has a delocalized eigenstate [10]. It is then argued in [8] that, when considering the model constrained to a box of size N, the inverse localization length (Lyapounov exponent) of the eigenfunctions behaves as $\gamma(E) \sim |E - E_c|^2$ (a result confirmed by a perturbative calculation in [2] [10]), such that roughly $\sqrt{N}$ of the N eigenfunctions have a localization length of the order of the size of the box. Using these observations on the eigenfunctions, the authors of [8] argue that $\langle \psi_t, X^2 \psi_t \rangle$ behaves like $t^{3/2}$ when ψ_0 is a state initially localized at the origin, a result they confirm with numerical computations. In other words, according to those results, the random dimer model is a simple model in which a diverging localization length at isolated energies in the band could lead to superdiffusive behaviour.

This conclusion has been be contested on several grounds. It is argued in [14] that the behaviour in $t^{3/2}$ is only a transient effect, that would disappear if one explored $\langle \psi_t, X^2 \psi_t \rangle$ numerically over much longer times than was done in [8]. Their objections are essentially based on the way the $N \to \infty$ and $t \to \infty$ limits are taken in [8], and on the observation that the fraction of delocalized states over localized states behaves as $1/\sqrt{N}$, so that the role of the delocalized states may vanish in the infinite lattice model. This latter argument is already proposed in [16], in the context of other, similar models.

Without settling the question of the $t^{3/2}$ behaviour, we provide in this letter some rigorous results on the random dimer model that should help to clarify the situation. First, one expects that in the infinite model, whatever the value of V, the Hamiltonian has pure point spectrum with exponentially localized eigenfunctions. Second, when $V > 1$, the $t^{3/2}$-behaviour should be completely suppressed in the sense that $\sup_t \langle \psi_t, X^2 \psi_t \rangle < \infty$, a property we refer to as "dynamical localization". This is is indeed proven in Theorem 2.3 ($V \neq \sqrt{2}$).

It is furthermore agreed on by all authors that, in the case $V < 1$, the superdiffusive behaviour – if any – can only come from contributions of the eigenstates close to the critical energies. We give a precise content to this statement and a proof of it in Theorem 2.2.

To obtain these results, we proceed as follows. We first show that for all energies E away from the critical energies, the corresponding eigenfunctions are semi-uniformly exponentially localized (this notion is introduced in [6]), *i.e.*:

$$|\psi_E(x)| \leq C_\varepsilon \exp |x_E|^\varepsilon \exp -\gamma_E |x - x_E|,$$

with $\varepsilon > 0$, where x_E is a point where ψ_E reaches its maximum and γ_E is the (strictly positive) Lyapounov exponent. This, together with the results of [12] implies in turn dynamical localization. This result has been announced in [13].

We insist once again that our results do not imply the absence of the superdiffusive behaviour observed by [8] when the disorder is low ($V < 1$): we actually feel this model should indeed display such behaviour, but to prove it requires lower bounds on the eigenfunctions close to the critical energies, rather than the above upper bounds. It would be interesting, since it would provide a random model with pure point spectrum in which $\langle \psi_t, X^2 \psi_t \rangle$ has a non-trivial lower bound at all times t.

We also exhibit the existence of new critical energies (in the sense that the Lyapunov exponent vanishes) for the special values $V = 1/\sqrt{2}$ and $V = \sqrt{2}$. This is the content of Theorem 2.4. To our opinion, the nature of these energies is different from the one of $E = \pm V$, and should not lead to a delocalization phenomenon, but we did not prove this (see section 3 for more details).

2 Theorems and Localization

We first rewrite the eigenvalue equation $H_\omega u = E\, u$ as follows:

$$\begin{pmatrix} u(x+1) \\ u(x) \end{pmatrix} = S^E_{V_\omega(x)} \begin{pmatrix} u(x) \\ u(x-1) \end{pmatrix}, \text{ where } S^E_v = \begin{pmatrix} E - v & -1 \\ 1 & 0 \end{pmatrix},$$

is the usual one-step transfer matrix. In the present case the structure of the potential leads us to consider the two-step random transfer matrices

$T_v^E = (S_v^E)^2$, *i.e.*:

$$T_v^E = \begin{pmatrix} (E-v)^2 - 1 & -(E-v) \\ (E-v) & -1 \end{pmatrix}.$$

Definition 2.1. *We'll say that H_ω, as in (1.1), is dynamically localized on a spectral interval I, iff with probability one, for all $q > 0$ and for all exponentially decaying initial state $\psi \in \ell^2(\mathbb{Z})$:*

$$\sup_t r_{\psi,I}^{(q)}(t) \equiv \sup_t \langle P_I(H_\omega)\psi_t,\ |X|^q P_I(H_\omega)\psi_t\rangle\ < \infty.$$

Here $\psi_t = e^{-iH_\omega t}\psi$, and $P_I(H_\omega)$ is the spectral projector of H_ω onto the interval I.

Our results are the following:

Theorem 2.2. *Let $(H_\omega)_{\omega\in\Omega}$ be as in (1.1) and $V \in]0,1]\backslash\{1/\sqrt{2}\}$. Then, with probability 1 in ω the Lyapounov exponent*

$$\gamma(E) = \lim_{x\to\infty} \frac{1}{|x|} \ln \left\| T^E_{V_\omega(x)} T^E_{V_\omega(x-1)} \cdots T^E_{V_\omega(1)} \right\|$$

exists, is independent of ω, and :
(i) $\gamma(E = \pm V) = 0$ and $\gamma(E \neq \pm V) > 0$;
(ii) H_ω has pure point spectrum;
(iii) Let $\varepsilon > 0$ and let I be a compact energy interval $I \subset \sigma(H_\omega) = [-V-2, V+2]$ with $\pm V \notin I$. Then, for all $0 < \gamma < \gamma(I) \equiv \inf\{\gamma(E), E \in I\}$ there exists a constant $C(\omega, \varepsilon, \gamma)$ and, for each eigenfunction $\varphi_{n,\omega}$ with energy $E_{n,\omega} \in I$, a "center" $x_{n,\omega} \in \mathbb{Z}$, such that

$$\forall x \in \mathbb{Z},\ \ |\varphi_{n,\omega}(x)| \leq C(\omega,\varepsilon,\gamma) e^{|x_{n,\omega}|^\varepsilon} e^{-\gamma|x - x_{n,\omega}|}; \tag{2.2}$$

Moreover if ψ decays exponentially with mass $\theta > 0$ and if $q > 0$, there exists a constant $C_{\psi,\omega}(I)$ so that :

$$\sup_t r_{\psi,I}^{(q)}(t)\ \leq\ C_{\psi,\omega}(I)\quad \mathbb{P}\, a.s. \tag{2.3}$$

In particular, H_ω is dynamically localized on I.

Remark: A careful analysis of Lemma 3.5 and 3.6 of [12] shows that our estimate fails (*i.e.* $C_{\psi,\omega}(I)$ grows to infinity) if the distance between I and the energies $\pm V$ decreases ($\gamma \to 0$): this is of course as it should be if one believes that the observed $t^{3/2}$ does indeed occur.

These results are completed by the two following theorems:

Theorem 2.3. *Let $(H_\omega)_{\omega\in\Omega}$ be as in (1.1), $V > 1$ and $V \neq \sqrt{2}$. Then, for almost all ω, $\gamma(E)$ exists and $\gamma(E) > 0$ for all E, the spectrum is pure point and (iii) of Theorem 2.2 holds with $I = \sigma(H_\omega)$.*

Theorem 2.4. *Let $(H_\omega)_{\omega\in\Omega}$ be as in (1.1) and $V = \sqrt{2}/2$ (respectively $V = \sqrt{2}$). Then the same conclusions as in Theorem 2.2 (resp. Theorem 2.3) hold except at the energies $E_c = \pm 3/\sqrt{2}$ (resp. $E_c = 0$). In particular (iii) hold for intervals I such that $\pm V, \pm 3/\sqrt{2} \notin I$ (resp. $0 \notin I$). In addition $E_c = \pm 3\sqrt{2}/2$ (resp. $E_c = 0$) is a critical energy in the sense that $\gamma(E_c) = 0$.*

We shall prove Theorems 2.2 and 2.3 simultaneously in this section, and then, in section 3, we prove Theorem 2.4 which deals with the critical couples ($V = 1/\sqrt{2}$, $E_c = \pm 3/\sqrt{2}$) and ($V = \sqrt{2}$, $E_c = 0$).

Proof of Theorems 2.2 and 2.3: That (2.2) implies (2.3) is not too hard to see, and is at any rate shown in [12], section 2 (see also [11]). To prove (2.2), it will be sufficient to show strict positivity of the Lyapunov exponent. Using Theorem 4.1 of [3] with the transfer matrix T_v^E, this will indeed imply the Wegner estimate, which is the ingredient needed to make the multiscale analysis function (see the appendix of [7], or [4] [18]). As a result, one can apply the proof of Theorem 3.1 in [12], or equivalently arguments developed in [11], to conclude.

We therefore turn to the proof of (i). We first recall it is well known [1] [5] that thanks to the Furstenberg and Kesten Theorem the Lyapunov exponent γ is well defined on a set Ω_0 of full measure, and is independent of $\omega \in \Omega_0$.

Consider first the energy $E = V$. The two possible transfer matrices are

$$T_{-V}^V = \begin{pmatrix} 4V^2 - 1 & -2V \\ 2V & -1 \end{pmatrix} \text{ and } T_V^V = -Id.$$

For $\omega \in \Omega_0$ and $x \in \mathbb{N}$, let $n_x = \sharp\{y \in \mathbb{Z}, 1 \leq y \leq x, V_\omega(2y) = -V\}$ (this is the number of times $-V$ is obtained after x trials). Using the following three simple facts:

- $\mathbb{P}$ a.s. $\dfrac{n_x}{x} \to p$;
- $\lim\limits_{x\to+\infty} \left\|(T_{-V}^V)^x\right\|^{1/x} = \rho(T_{-V}^V)$, where $\rho(T_{-V}^V)$ denotes the spectral radius of T_{-V}^V;
- $\rho(T_{-V}^V) = 1$, if $V \in]0, 1]$, and $\rho(T_{-V}^V) > 1$ if $V > 1$;

one immediately obtains that $\gamma(E=V)=0$ if $V\in]0,1]$ and $\gamma(E=V)>0$ if $V>1$. One proceeds similary for the energy $E=-V$.

We now turn to others energies $E\neq\pm V$, and prove that $\gamma(E\neq\pm V)>0$ for all E belonging to the spectrum of H_ω. Let G be the smallest closed subgroup of $\mathrm{SL}(2,\mathbb{R})$ generated by the matrices T_V^E and T_{-V}^E. Recall that there is a natural action of $\mathrm{SL}(2,\mathbb{R})$ on $P(\mathbb{R}^2)$, the set of all the directions of $\mathbb{R}^2$. A matrix $T\in G$ is then seen as an homography acting on $P(\mathbb{R}^2)$. According to the Furstenberg Theorem (see Theorem I.4.4 of [1]), the conclusion will follow if G is not compact and if either there is no probability measure on $P(\mathbb{R}^2)$ that is invariant under the action of G, or equivalently if the orbit $G\cdot\tilde{x}\equiv\{T\cdot\tilde{x}, T\in G\}$ of each direction $\tilde{x}\in P(\mathbb{R}^2)$ contains at least three elements (Proposition I.4.3 in [1]).

In order to alleviate the notations, let's define $\alpha=E-V$ and $\beta=E+V$. Note that in the present case $\alpha\neq 0$ and $\beta\neq 0$. We will also rename $T_V^E=T_\alpha$ and $T_{-V}^E=T_\beta$, *i.e.*

$$T_X=\begin{pmatrix} X^2-1 & -X \\ X & -1\end{pmatrix} \text{ with } X=\alpha,\,\beta.$$

We recall that a matrix T is said to be elliptic if $|\operatorname{tr}T|<2$, parabolic if $|\operatorname{tr}T|=2$ and hyperbolic if $|\operatorname{tr}T|>2$. The proof is reduced to the study of three cases: a) both the matrices T_α and T_β are elliptic; b) T_α is parabolic; c) T_α is hyperbolic. These clearly cover all the possible cases since the problem is symmetric in α and β. Note that in cases b) and c) the group G is clearly not compact.

Case a). Suppose T_α and T_β are both elliptic, *i.e.* $|\alpha|,|\beta|\in]0,2[$. In that case they do not commute, since $E\neq V$. Since the commutator $T=T_\alpha T_\beta\left(T_\alpha\right)^{-1}\left(T_\beta\right)^{-1}$ of two non-commuting elliptic elements is known to be hyperbolic ($|\operatorname{tr}T|>2$) - see the proof of Proposition 2.8 in [15] - it follows that G is not compact. We will show $G\cdot\tilde{x}$ contains at least three points provided $\alpha^2\neq 2$ or $\beta^2\neq 2$.

To that end, note first that $\mathrm{tr}T_X^2=X^4-4X^2+2$, so that if $X^2\in]0,4[$ and $X^2\neq 2$, then T_X^2 is elliptic. Hence, if $\alpha^2\neq 2$ or $\beta^2\neq 2$, then T_α and T_α^2 or T_β and T_β^2 are elliptic. Since elliptic elements have no fixed points in $\mathbb{P}(\mathbb{R}^2)$, it follows easily that for any $\tilde{x}\in\mathbb{P}(\mathbb{R}^2)$, $G\cdot\tilde{x}$ contains at least the three points $\tilde{x}$, $T_X\cdot\tilde{x}$, $T_X^2\cdot\tilde{x}$, with $X=\alpha$ or β.

If, on the other hand, $\alpha^2=2$ and $\beta^2=2$, then $E=0$ and $V=\sqrt{2}$, which is one of the two critical couple described in Theorem 2.4, and to be dealt with in section 3.

Case b). Suppose now that T_α is parabolic, *i.e.* $|\alpha| = 2$. We treat the case $\alpha = 2$ (the case $\alpha = -2$ is similar). The eigenvector of T_α is then given by $(1,1)$. Denoting by e_2 the orthogonal vector $(1,-1)$, the matrix T_α in the basis (e_1, e_2) can be written

$$\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}, \text{ and so } \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 4n \\ 0 & 1 \end{pmatrix}.$$

Taking a vector $x = x_1 e_1 + x_2 e_2$, and writing $\tilde{x}$ for its direction (*i.e.* its projection onto $P(\mathbb{R}^2)$), one concludes that $\lim_{n\to\infty} T_\alpha^n \cdot \tilde{x} = \tilde{e}_1$ (where T_α^n is seen here as a homography of $P^2(\mathbb{R})$). But now, if m is a probability measure that is invariant under the action of G, and if $f \in C_0^\infty(P(\mathbb{R}^2))$, using a Lebesgue dominated argument, one has

$$f(\tilde{e}_1) = \lim_{n\longrightarrow\infty} \int f(T_\alpha^n \cdot \tilde{x}) dm(\tilde{x}) = \langle m, f \rangle.$$

This means that $m = \delta_{\tilde{e}_1}$. But now one uses the second matrix T_β: it does not leave invariant the direction $\tilde{e}_1$ except for $\beta = 0$ or $\beta = 2 = \alpha$ (simple check), which is excluded since the first condition yields $E = -V$ and the second one $V = 0$. Thus we proved there is no invariant measure in case b).

Case c). Suppose now that T_α is hyperbolic ($|\alpha| > 2$). It is clearly sufficient to study the orbit of the eigendirections of T_α, namely $e_\varepsilon = (\alpha + \varepsilon\sqrt{\alpha^2 - 4}, 2)$, $\varepsilon = \pm 1$. Note that T_α and these T_β cannot have eigenvectors in common, since it is easy to show that it would imply $\alpha = \beta$ (and $V = 0$). Now, if T_β is hyperbolic then it is clear that the orbit of e_ε is infinite. If T_β is parabolic then we are again in case b). Finally, if T_β is elliptic then let's consider $\tilde{X} \equiv T_\beta \widetilde{e_\varepsilon}$. If $\tilde{X} \neq \widetilde{e}_{-\varepsilon}$ then $\tilde{X}$ cannot belong to the eigendirections of T_α and its orbit is then infinite.

Hence, the only case we still need to consider is the case where T_β is elliptic and exchanges these two directions (the orbit of these elements would then have cardinal 2). In that case $T_\beta e_\varepsilon$ and $e_{-\varepsilon}$, $\varepsilon = \pm 1$, have the same directions, and simple calculations lead to the two equations

$$(\beta^2 - 1)(\alpha + \varepsilon\sqrt{\alpha^2 - 4}) = 4\beta - (\alpha - \varepsilon\sqrt{\alpha^2 - 4}), \quad \varepsilon = \pm 1.$$

It trivially implies $\beta^2 = 2$ and $\alpha = 2\beta$, which means $V = \sqrt{2}/2$ and $E = -3\sqrt{2}/2$. The symmetric case where one assumes that T_β is hyperbolic leads naturally to $\alpha^2 = 2$ and $\beta = 2\alpha$, which means this time $V = \sqrt{2}/2$ and $E = 3\sqrt{2}/2$. Since, in Theorem 2.2 we have supposed $V \neq \sqrt{2}/2$, the proof is complete.

3 New critical cases

We now consider the two special cases which haven't been studied in the previous section and that are dealt with in Theorem 2.4, that is $(V = 1/\sqrt{2}, E_c = \pm 3/\sqrt{2})$ and $(V = \sqrt{2}, E_c = 0)$.

Proof of Theorem 2.4:

It clearly follows from the previous proof that the only thing that remains to be proven is that the Lyapunov exponent is zero at the critical energies E_c. Note first that in all cases E_c belongs to the spectrum of H_ω almost surely since $d(-3\sqrt{2}/2, -1/\sqrt{2}) = d(3\sqrt{2}/2, 1/\sqrt{2}) = d(0, \pm\sqrt{2}) = \sqrt{2} < 2$.

We first deal with the critical case $(V = 1/\sqrt{2}, E_c = \pm 3/\sqrt{2})$. The second one will then be easier to treat.

$\underline{(V = 1/\sqrt{2}, E_c = \pm 3/\sqrt{2})}$

Clearly it is enough to restrict ourselves to the case $E_c = -3/\sqrt{2}$. Using the notations and the results of the previous proof, we thus have, in the present case, $\beta^2 = 2$ and $\alpha = 2\beta$. The eigenvectors of T_α are then given by $(\beta + \varepsilon, 1)$, $\varepsilon = \pm 1$, and looking at the matrices in the basis of these two vectors we are reduced to considering products of matrices of the following two types:

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1-\beta \\ 1+\beta & 0 \end{pmatrix},$$

with $\lambda_1 \lambda_2 = 1$, $\lambda_1 > 1$, and $(1-\beta)(1+\beta) = -1$. With some abuse of notation, we will again denote those two matrices by T_α and T_β.

To prove that $\gamma(V = \sqrt{2}/2, E = -3\sqrt{2}/2) = 0$, one has to analyse, roughly speaking, the behaviour of large products of matrices T_α and T_β. While the matrices T_α contribute to the growth of the norm of such a product, the T_β not only do not contribute (being a rotation) but in fact "destroy" this growth. Indeed one checks

$$\begin{aligned} T_\beta T_\alpha^n T_\beta &= -\begin{pmatrix} \lambda_2^n & 0 \\ 0 & \lambda_1^n \end{pmatrix} \\ \text{and} \quad T_\beta T_\alpha^{n_2} T_\beta T_\alpha^{n_1} &= -\begin{pmatrix} \lambda_1^{n_1-n_2} & 0 \\ 0 & \lambda_2^{n_1-n_2} \end{pmatrix}, \end{aligned} \tag{3.4}$$

since $\lambda_1\lambda_2 = 1$ and $(1-\beta)(1+\beta) = -1$. Noting that $T_\beta^2 = -Id$, a product of factors T_α and T_β is, up to a sign, a succession of $T_\alpha^{n_i}$ and T_β. One then easily understands, from (3.4), that the norm of a product $T_n \dots T_1$ can not grow fast enough to ensure the positivity of the Lyapunov exponent. This is exactly what we show below.

We will see a product $T_n \dots T_1$ as a sequence of $m(n)$ steps, where $m(n)$ is the number the matrices T_α contained in the chain $T_n \dots T_1$; in other words a "step" means that one matrix T_α has been met. So each step is a product of matrices of the form $T_\beta^{j_i} T_\alpha$. So $T_n \dots T_1$ will be written $\prod_{i=0}^{m(n)-1} \left(T_\beta^{j_i} T_\alpha \right)$. Looking at (3.7), it is clear that without loss of generality on can suppose $T_1 = T_\alpha$. Clearly, depending on the parity of j_i, the i^{th} step will contribute or not to the growth (in norm) of the total product $T_n \dots T_1$.

More precisely, in order to study the product of elements of the form $T_\beta^{j_i} T_\alpha$, we define two sequences u_k and V_k such that, after k steps,

$$\prod_{i=0}^{k-1} \left(T_\beta^{j_i} T_\alpha \right) = \pm T_\beta^{u_k} T_\alpha^{V_k}, \tag{3.5}$$

with $u_k \in \{0, 1\}$. This is clearly always possible using relations (3.4) and $T_\beta^2 = -Id$. Now it is easy to obtain recurrence relations for $\varepsilon(u_k) = (-1)^{u_k}$ and V_k:

$$\begin{aligned} T_\beta^{u_{k+1}} T_\alpha^{V_{k+1}} &= \pm \left(T_\beta^{j_k} T_\alpha \right) T_\beta^{u_k} T_\alpha^{V_k} \\ &= \pm \begin{cases} T_\beta^{j_k} T_\alpha^{V_k+1} \text{ if } \varepsilon(u_k) = 1 \\ T_\beta^{j_k+1} T_\alpha^{V_k-1} \text{ if } \varepsilon(u_k) = -1. \end{cases} \end{aligned}$$

And this leads to

$$\begin{cases} V_{k+1} = V_k + \varepsilon(u_k) \\ \varepsilon(u_{k+1}) = \varepsilon(j_k)\varepsilon(u_k). \end{cases} \tag{3.6}$$

Then define $\varepsilon_k = \varepsilon(j_{k-1})$ and $U_k = \varepsilon(u_k) = \varepsilon_k \cdots \varepsilon_1$, for $k \geq 1$. So ε_k is a sequence of independent and identically distributed random variables taking the two values ± 1, and such that $\varepsilon_k = +1$ if one meets an even(possibly zero) number of T_β between the $(k-1)^{\text{th}}$ and the k^{th} matrix T_α, and $\varepsilon_k = -1$ if not. Let's recall that $\mathbb{P}(T_\alpha) = 1 - p$ and $\mathbb{P}(T_\beta) = p$. So one has

$$P(\varepsilon_k = 1) = (1-p)(1 + p^2 + \dots) = \frac{1-p}{1-p^2} = \frac{1}{p+1}$$

and

$$P(\varepsilon_k = -1) = (1-p)(p + p^3 + \dots) = \frac{p(1-p)}{1-p^2} = \frac{p}{p+1}.$$

Moreover one checks $\mathbb{E}(\varepsilon_k) = (1-p)/(1+p) \in]0,1[$ since $p \in]0,1[$. Finally let us rewrite equations (3.6) as

$$U_k = \prod_{i=1}^{k} \varepsilon_i \quad \text{and} \quad V_m = \sum_{k=1}^{m} U_k.$$

To understand how these random sequences behave, note that $U_{k+1} = \varepsilon_{k+1} U_k \in \{-1,1\}$. So, if $\varepsilon_{k+1} = 1$ then $U_{k+1} + U_k = \pm 2$, but if $\varepsilon_{k+1} = -1$, then $U_{k+1} + U_k = 0$. As a result looking at the sum V_m, U_{k+1} destroys in the latter case the term before, and does not contribute to the growth of V_m.

Note that one can prove from (3.6) that

$$\mathbb{P}(U_k = 1) = \frac{1}{2}\left(1 + \left(\frac{1-p}{1+p}\right)^k\right) \quad \text{and} \quad \mathbb{P}(U_k = -1) = \frac{1}{2}\left(1 - \left(\frac{1-p}{1+p}\right)^k\right).$$

By construction V_m in turn is closely related to the exponential growth of the product $T_n \dots T_1$, as one can see from the following formula:

$$\begin{aligned} \ln \|T_n \dots T_1\| &= \ln \left\| \prod_{i=0}^{m(n)-1} T_\beta^{j_i} T_\alpha \right\| = \ln \left\| T_\beta^{u_{m(n)}} T_\alpha^{V_{m(n)}} \right\| \\ &\leq |V_{m(n)}| \ln \lambda_1 + \ln \|T_\beta\|. \end{aligned} \tag{3.7}$$

Since, by the Furstenberg and Kesten Theorem [1] [5], γ exists almost surely and is constant, and since $\frac{1}{n} \ln \|T_n \dots T_1\| \leq \max(\|T_\alpha\|, \|T_\beta\|)$, the Lebesgue dominated convergence Theorem gives

$$\gamma = \mathbb{E}\left(\lim_{n\to\infty} \ln \|T_n \dots T_1\|/n\right) = \ln \lambda_1 \lim_{n\to\infty} \mathbb{E}\left(|V_{m(n)}|/n\right).$$

It remains to evaluate the latter limit. Computing $V_{m(n)}^2$ one obtains that

$$\begin{aligned} V_{m(n)}^2 &= \sum_{k=1}^{m(n)} U_k^2 + 2 \sum_{1 \leq k < l \leq m(n)} U_k U_l \\ &= m(n) + 2 \sum_{1 \leq k < l \leq m(n)} \varepsilon_{k+1} \cdots \varepsilon_l, \end{aligned} \tag{3.8}$$

since $\varepsilon_k^2 = 1$. Moreover, using the independence of the ε_i, one has $\mathbb{E}(U_k U_l) = \mathbb{E}(\varepsilon_1)^{|k-l|}$; but $m(n)$ does also depend on ω (write $m(n,\omega)$). So one needs some control on how $m(n,\omega)$ depends on ω. This is provided by the following lemma, which just recalls well-known results about Bernoulli random variables (*e.g.* [17]).

Lemma 3.1. *Let $m(n,\omega)$ be the number of T_α contained in the product $T_n(\omega)\cdots T_1(\omega)$. One has*

$$\overline{m_n} \equiv \mathbb{E}(m(n,\omega)) = (1-p)n,$$

and

$$\mathrm{Var}(m(n,\omega)) = \mathbb{E}\left[(m(n,\omega)-\overline{m_n})^2\right] = p(1-p)n.$$

An immediate consequence of this lemma is that

$$\mathbb{E}\left[(m(n,\omega)-[\overline{m_n}])^2\right] \sim p(1-p)n \quad \text{as } n\to\infty, \tag{3.9}$$

where $[\overline{m_n}]$ denotes the integer part of $\overline{m_n}$. Then the result follows from

$$\begin{aligned}
&\mathbb{E}(|V_{m(n)}|/n)\\
&\le \frac{1}{n}\mathbb{E}\left(\left|V_{m(n,\omega)}-V_{[\overline{m_n}]}\right|\right)+\frac{1}{n}\mathbb{E}\left(\left|V_{[\overline{m_n}]}\right|\right)\\
&\le \frac{1}{n}\sqrt{\mathbb{E}\left((m(n,\omega)-[\overline{m_n}])^2\right)}+\frac{1}{n}\sqrt{[\overline{m_n}]+2\sum_{1\le k<l\le[\overline{m_n}]}\mathbb{E}(\varepsilon_1)^{|k-l|}}\\
&\le \frac{C}{\sqrt{n}},
\end{aligned}$$

for some constant $C>0$, where we used, successively, the Cauchy-Schwartz inequality, relations (3.8) and (3.9), and the facts that $\mathbb{E}(\varepsilon_1)<1$ and $[\overline{m}]\le n$. In conclusion it follows that $\mathbb{E}(\gamma(E=-3\sqrt{2}/2))=0$.

We now turn to the second special case.

$(V=\sqrt{2}, E_c=0)$

So $\alpha=-\beta=\pm\sqrt{2}$ and let us recall that

$$T_\alpha^2 = T_\beta^2 = -Id. \tag{3.10}$$

We shall follow the idea of the previous case, but the arguments are much simpler. Regrouping all the powers of T_α and T_β that appear in the product of the n first matrices $T_n\cdots T_1$ and taking (3.10) into account, the product $T_n\cdots T_1$ can be reduced (essentially) to some power V_n of the matrix $T_\alpha T_\beta$ which is hyperbolic. This would then lead to a strictly positive Lyapunov exponent (since the spectral radius of $T_\alpha T_\beta$ is strictly greater than 1) if V_n and n had the same order, which is however not the case.

Let us consider groups of two matrices in the product $T_n \cdots T_1$. Then it is easy to see that one can define a sequence V_k with $V_0 = 0$ and

$$T_{2k} \cdots T_1 = (T_\alpha T_\beta)^{V_k}.$$

Depending on the values of T_{2k+2} and T_{2k+1}, and noting that $(T_\alpha T_\beta)^{-1} = T_\beta T_\alpha$, one has

$$\begin{aligned} \mathbb{P}(V_{k+1} = V_k + 1) = p(1-p) = \mathbb{P}(V_{k+1} = V_k - 1), \\ \text{and} \quad \mathbb{P}(V_{k+1} = V_k) = p^2 + (1-p)^2. \end{aligned} \tag{3.11}$$

This situation is different from the previous one where the way the value of V_k changed (between the k^{th} and $(k+1)^{\text{th}}$ steps) was depending on what happened before. So let us define $U_k = V_{k+1} - V_k$. It is (unlike before) an i.i.d random sequence the law of which is given by (3.11). One easily computes $\mathbb{E}(U_k) = 0$ and $\mathbb{E}(U_k^2) = 2p(1-p)$. It is then immediate that

$$\begin{aligned} \mathbb{E}(V_n^2) &= \sum_{k=1}^{n} \mathbb{E}(U_k^2) + 2 \sum_{1 \le k < l \le n} \mathbb{E}(U_k U_l) \\ &= 2np(1-p), \end{aligned}$$

since $\mathbb{E}(U_k U_l) = \mathbb{E}(U_k)\mathbb{E}(U_l) = 0$ for $l \neq k$. The result then follows in the same way as previously. $\square$

Remark: the situation is, to our opinion, different from the one we met with the critical energies $E = \pm V$. It is worth to notice that if $E = \pm V$ then $\lim_{n\to+\infty} \frac{1}{n^\nu} \ln \|T_n \dots T_1\| = 0$ for all $\nu > 0$, since it is easy to see that in this case $\|T_n \dots T_1\|$ is bounded independently of n. We conjecture that this is not the case at the critical couples $(V = 1/\sqrt{2}, E_c = \pm 3/\sqrt{2})$ and $(V = \sqrt{2}, E_c = 0)$, where for $\nu < 1/2$ the limit is probably infinite (and zero for $\nu > 1/2$). If so it is reasonable to think that the eigenfunctions with energy E close to E_c should decay sub-exponentially (semi-uniformly) as $\exp -\gamma(E) n^\nu$ $(\nu < 1/2)$. This would still imply dynamical localization even on a spectral interval containing the critical energy E_c.

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