Semiclassical Eigenvalue Asymptotics for a Schrödinger Operator with a Degenerate Potential

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Abstract

We give the asymptotic behaviour, when h tends to zero, of the number of eigenvalues less then a fixed energy λ, of the Schrödinger operator: $\widehat{H}_h = -h^2\Delta + f(x)g(y),\ (x,y) \in \mathbf{R}^d = \mathbf{R}^n \times \mathbf{R}^m$. f and g are continuous and positive functions tending to infinity at infinity, $g(y)$ is homogeneous and $f(x) > 0$.

1 Introduction

We are interested in the spectrum of the Schrödinger operator $\widehat{H}_h = -h^2\Delta + V$ on $L^2(\mathbf{R}^d)$ for a nonnegative, real and continuous potential V on $\mathbf{R}^d$. $h \in]0,1]$ is a small parameter. It is well known, (see [Re-Si], that $\widehat{H}_h$ is essentially selfadjoint on $L^2(\mathbf{R}^d)$ and has discret spectrum when $V(X) \mapsto +\infty$ as $|X| \mapsto \infty$. Then for any fixed $\lambda \in \mathbf{R}$,

$$N(\lambda; \widehat{H}_h) = rk(E_{\widehat{H}_h}(]-\infty, \lambda[) \tag{1.1}$$

is finite, (for un operator P, $rk(P)$ denotes its rank, and when P is selfadjoint, $E_P(I)$ denotes its spectral projection on the borel subset I of $\mathbf{R}$).

The asymptotic behaviour of $N(\lambda; \widehat{H}_h)$, (when $h \mapsto 0$), is

$$N(\lambda; \widehat{H}_h) \sim \omega_d h^{-d} \int_{\mathbf{R}^d} [\lambda - V(X)]_+^{d/2} dX, \quad (\omega_d = (2\pi)^{-d} \int_{\{X \in \mathbf{R}^d;\ |X|<1\}} dX), \tag{1.2}$$

see [Re-Si] for the particular case $\lambda - V(X) \in L^{d/2}(\mathbf{R}^d)$, [He-Ro-2] for a proof with a remainder estimate, in the C^∞ case, and the book [Ivr] for the history of this asymptotic formula, (see also [Khu] for up to date C^∞ semiclassical results).

In this paper, we are interested in the asymptotic behaviour of $N(\lambda; \widehat{H}_h)$ when the potential $V(X)$ is degenerated, (does not tend to infinity with $|X|$), but $\widehat{H}_h$ is still with compact resolvent. For such a potential, there are some results on the asymptotic behaviour of $N(\lambda; \widehat{H}_1)$ when λ tends to infinity, see [Rob], [Sim], [Sol] and [Mo-No]. Our model is closer to the one of [Rob], it is a model of the ones studied in [He-Sj] of a potential V such that $\{X; V(X) = 0\}$ is a submanifold.

Here we consider the special case, (for $X = (x,y) \in \mathbf{R}^n \times \mathbf{R}^m = \mathbf{R}^d$),

$$V(X) = f(x)g(y), \quad f \in C(\mathbf{R}^n; \mathbf{R}_+^\star) \ \text{ and } g \in C(\mathbf{R}^m; \mathbf{R}_+). \tag{1.3}$$

with g homogeneous

$$\exists\ a > 0 \ \ s.t. \ \ g(\mu y) = \mu^a g(y),\ \forall\ (\mu, y) \in \mathbf{R}_+^\star \times \mathbf{R}^m, \ \text{ and } g(y) > 0 \text{ if } y \neq 0. \tag{1.4}$$

If we consider the Schrödinger operator $-\Delta + g(y) = D_y^2 + g(y)$ on $L^2(\mathbf{R}^m)$, then its spectrum is discret and positive:

$$\sigma(D_y^2 + g(y)) = \{\mu_j;\ j \in \mathbf{N}\},\quad 0 < \mu_j \leq \mu_{j+1}\ \forall j \in \mathbf{N}\ \text{ and} \tag{1.5}$$
$$rk(E_{\{D_y^2+g(y)\}}(]-\infty,\lambda[) = \sharp\{j;\ \mu_j < \lambda\} < \infty\ \forall \lambda \in \mathbf{R}.$$

($D_y = -i(\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_m})$ and $\sigma(P)$ denotes the spectrum of the operator P.)
From the homogeneity of g we get that, for any fixed $x \in \mathbf{R}^n$,

$$\int_{\mathbf{R}^m} [h^2|D_y u(y)|^2 + f(x)g(y)|u(y)|^2]dy \geq h^{2a/(2+a)} f^{2/(2+a)}(x)\mu_0 \int_{\mathbf{R}^m} |u(y)|^2 dy,\ \ \forall u \in C_0^\infty(\mathbf{R}^m),$$

so $\widehat{H}_h \geq h^{2a/(2+a)} f^{2/(2+a)}(x)\mu_0$, and then

$$\widehat{H}_h \geq \epsilon f(x)g(y) + (1-\epsilon)h^{2a/(2+a)} f^{2/(2+a)}(x)\mu_0,\ \forall\ \epsilon\ \in [0,1]. \tag{1.6}$$

This shows that $\widehat{H}_h$ has compact resolvent if

$$f(x)\ \mapsto\ +\infty\ \text{ as }\ |x|\ \mapsto\ +\infty. \tag{1.7}$$

If f is also homogenuous of degree $b > 0$, then $N(\lambda; \widehat{H}_h) = N(h^{-2(a+b)/(2+a+b)}\lambda; \widehat{H}_1)$ and its asymptotic estimate is given in [Sol]. Here, in our case f is strictly positive, more over we assume some uniformly locally regularity on f:

$$\exists\ b,\ c\ > 0\ \ s.t.\ \ c^{-1} \leq f(x)\ \text{ and } |f(x) - f(x')| \leq cf(x)|x - x'|^b \tag{1.8}$$

$\forall\ x,\ x'\ \in \mathbf{R}^m,\ |x - x'| \leq 1.$

Theorem 1.1 *Under the assumptions (1.3), (1.4), (1.7) and (1.8),there exists* $\tau_1,\ \tau_2\ \in]0,1[$ *such that, for any given* $\lambda > 0$*, one can find* $h_0\ \in]0,1[,\ C_1,\ C_2\ > 0$ *to get*

$$(1 - h^{\tau_1}C_1)n_{h,f}(\lambda - h^{\tau_2}C_2) \leq N(\lambda; \widehat{H}_h) \leq (1 + h^{\tau_1}C_1)n_{h,f}(\lambda + h^{\tau_2}C_2),\ \forall\ h\ \in]0,h_0]. \tag{1.9}$$

if

$$n_{h,f}(\lambda) = h^{-n}\omega_n \int_{\mathbf{R}^n} \sum_{j\in\mathbf{N}} [\lambda - h^{\frac{2a}{2+a}} f^{\frac{2}{2+a}}(x)\mu_j]_+^{n/2} dx \tag{1.10}$$

with $\omega_n = (2\pi)^{-n} \int_{\{x\in\mathbf{R}^n;\ |x|<1\}} dx.$

Corollary 1.2 *Under the assumptions of Theorem (1.1), and the one below*

$$\exists\ C_3 > 0\ \ s.t.\ \int_{\{x;\ f(x)<2\mu\}} f^{-m/a}(x)dx \leq C_3 \int_{\{x;\ f(x)<\mu\}} f^{-m/a}(x)dx,\ \ \forall\ \mu > 1, \tag{1.11}$$

then one can take $C_2 = 0$ *in (1.9):*

$$(1 - h^{\tau_1}C_1)n_{h,f}(\lambda) \leq N(\lambda; \widehat{H}_h) \leq (1 - h^{\tau_1}C_1)n_{h,f}(\lambda),\ \forall\ h\ \in]0,h_0]. \tag{1.12}$$

Moreover, if $f^{-m/a}\ \in\ L^1(\mathbf{R}^m)$*, (or equivalently* $[\lambda - V(X)]_+^{d/2} \in L^1(\mathbf{R}^d)$*), and if* $g\ \in\ C^1(\mathbf{R}^m \setminus \{0\}$*, then*

$$N(\lambda; \widehat{H}_h) \sim h^{-d}\omega_d \int_{\mathbf{R}^d} [\lambda - V(X)]_+^{d/2} dX,\ \ \text{as}\ \ h\ \mapsto\ 0. \tag{1.13}$$

Similar formulas, as (1.10), where a trace of some operator is related to the asymptotic of the number of eigenvalues are current for operators on $L^2(\mathbf{R}^d)$ with "principal symbol" which can degenerate on some non bounded submanifold of $T^\star(\mathbf{R}^d)$, see for example [Sol] and [Rob]. The well-known case is the Schrödinger with a magnetic field $(D_X - A(X))^2$ which degenerates on $\{(X, \Xi) \in T^\star(\mathbf{R}^d);\ \Xi = A(X)\}$. Its asymptotic formula of the number of eigenvalues was established by Y. Colin de Verdière [Col]. Similar formulas as (1.12) and (1.13) are still valid for the asymptotic of $N(\lambda; \widehat{H}_h)$ when $\lambda \mapsto \infty$ for the case $n = m = 1$ established in [Rob].

Colin de Verdière work [Col] shows that the famous method of Courant and Hilbert, the so called min-max method (see chapter XIII.15 of [Re-Si]), is still adequate for Schrödinger operator with degenerated symbol, (see also [Tru] for more recent developments), so we will perform it to get Theorem (1.1) and its Corollary.

Let us remark the well-known asymptotic formula, (see [Roz] or [Re-Si]),

$$N(\mu;\ D_y^2 + g(y)) \sim \omega_m \mu^{m(2+a)/(2a)} \int_{\mathbf{R}^m} [1 - g(y)]_+^{m/2} dy \ \text{ as } \ \mu \ \mapsto \ +\infty. \tag{1.14}$$

From (1.10) and (1.14) we get $\forall\ \lambda > 0,\ \epsilon\ \in]0, \lambda/2[,\ \exists\ C_\epsilon > 1\ s.t.$

$$h^{-d} C_\epsilon^{-1} \int_{\{x;\ f(x) < h^{-a}(\mu_0^{-1}(\lambda-\epsilon))^{(2+a)/2}\}} f^{-m/a}(x) dx \leq n_{h,f}(\lambda) \leq \tag{1.15}$$

$$h^{-d} C_\epsilon \int_{\{x;\ f(x) < h^{-a}(\mu_0^{-1}(\lambda+\epsilon))^{(2+a)/2}\}} f^{-m/a}(x) dx.$$

From this estimate we get that, if there exists $k > 0$ and $C_4 > 0$ s.t. $C_4^{-1}|x|^k \leq f(x) \leq C_4|x|^k$, if $|x| > 1$, then

$$N(\lambda; \widehat{H}_h) \approx h^{-n-ma/k} \ \text{ if } \ k < a, \quad N(\lambda; \widehat{H}_h) \approx h^{-d} log(h^{-1}) \ \text{ if } \ k = a \tag{1.16}$$

and

$$N(\lambda; \widehat{H}_h) \approx h^{-d} \ \text{ if } \ k > a. \tag{1.17}$$

2 Proof of the Theorem

If $\Omega \subset \mathbf{R}^n$ is open and bounded, we will denote by $-\Delta_{\Omega,D}$ and $-\Delta_{\Omega,N}$ the Dirichlet and Newman self adjoint operator on $L^2(\Omega)$ (associated to the Laplace operator $-\Delta = D_x^2$).

We will denote Q the unit cube of $\mathbf{R}^n$, $Q = (] - \frac{1}{2}, \frac{1}{2}[)^n$, and $Q_r(\alpha) = \alpha + rQ$ the cube with center α and side $r > 0$.

Lemma 2.1 *There exists a constant $C_n > 0$ such that, $\forall\ (\mu, \epsilon)\ \in\ \mathbf{R}_+ \times]0, \frac{1}{2}[$,*

$$(1-\epsilon)^{n/2}[\mu - \frac{C_n}{\epsilon}]_+^{n/2} \omega_n \leq N(\mu;\ -\Delta_{Q,D}) \leq N(\mu;\ -\Delta_{Q,N}) \leq (1+\epsilon)^{n/2}(\mu + \frac{C_n}{\epsilon})^{n/2} \omega_n, \tag{2.1}$$

The estimate comes easily from the well known one, $\forall\ \mu > 1$,

$$\mu^{n/2} \omega_n (1 - C_{n,1} \mu^{-1/2}) \leq N(\mu;\ -\Delta_{Q,D}) \leq N(\mu;\ -\Delta_{Q,N}) \leq \mu^{n/2} \omega_n (1 + C_{n,1} \mu^{-1/2}); \tag{2.2}$$

($C_{n,1}$ is a constant). For the proof of (2.2), see for example chapter XIII.15 of [Re-Si].

An elementary proof of (2.1) is given in [Roz], let us sketch it.

As $N(\mu;\ -\Delta_{Q,D}) = \sharp I_D(\mu)$ and $N(\mu;\ -\Delta_{Q,N}) = \sharp I_N(\mu)$ with $I_D(\mu) = \{\alpha \in (\mathbf{N}^\star)^n,\ \pi^2\alpha^2 < \mu\}$ and $I_N(\mu) = \{\alpha \in \mathbf{N}^n,\ \pi^2\alpha^2 < \mu\}$, then $N(\mu;\ -\Delta_{Q,D}) = \int_{\Omega_D(\mu)} dx$ and $N(\mu;\ -\Delta_{Q,N}) = \int_{\Omega_N(\mu)} dx$ if $\Omega_D(\mu) = \cup_{\alpha\in I_D(\mu)}\overline{Q}_1(\alpha - (\frac{1}{2},\dots,\frac{1}{2}))$ and $\Omega_N(\mu) = \cup_{\alpha\in I_N(\mu)}\overline{Q}_1(\alpha + (\frac{1}{2},\dots,\frac{1}{2}))$.
But $\{x \in (\mathbf{R}^+)^n;\ |x| < [\frac{\sqrt{\mu}}{\pi} - \sqrt{n}]_+\} \subset \Omega_D(\mu)$, (if this last one is not empty), and $\Omega_N(\mu) \subset \{x \in (\mathbf{R}^+)^n;\ |x| < \frac{\sqrt{\mu}}{\pi} + \sqrt{n}\}$.

Then using that $(1-\epsilon)[\mu - C_n/\epsilon]_+ \leq (\sqrt{\mu} - \pi\sqrt{n})^2$ and $(\sqrt{\mu}+\pi\sqrt{n})^2 \leq (1+\epsilon)(\mu + C_n/\epsilon)$ if $C_n \geq 2n\pi^2$, (and for any $\epsilon \in]0,1[$), we get (2.1) from the last inclusions •

From now on, C will denote any positive constant independent of h, (depending only on $n,\ d,\ a$ in (1.4), $b,\ c$ of (1.8) and λ).

Let now $r \in]0,1]$ be fixed. As in chapter VIII.15 of [Re-Si], we use the min-max principle to get

$$\sum_{\gamma\in\mathbf{Z}^n} N(\lambda; -h^2\Delta_{Q_r(r\gamma),D} + h^2D_y^2 + f(x)g(y)) \leq N(\lambda; \widehat{H}_h) \leq \tag{2.3}$$

$$\sum_{\gamma\in\mathbf{Z}^n} N(\lambda; -h^2\Delta_{Q_r(r\gamma),N} + h^2D_y^2 + f(x)g(y)).$$

Using once more the min-max principle and (1.4), (1.5) and (2.3), then we get

$$\sum_{\gamma\in\mathbf{Z}^n}\sum_{j\in\mathbf{N}} N(\lambda - (h^a f(x^+(\gamma)))^{2/(2+a)}\mu_j; -h^2\Delta_{Q_r(r\gamma),D}) \leq N(\lambda; \widehat{H}_h) \leq \tag{2.4}$$

$$\sum_{\gamma\in\mathbf{Z}^n}\sum_{j\in\mathbf{N}} N(\lambda - (h^a f(x^-(\gamma)))^{2/(2+a)}\mu_j; -h^2\Delta_{Q_r(r\gamma),N});$$

if $x^\pm(\gamma) \in \overline{Q}_r(r\gamma)$ and $f(x^+(\gamma)) = Max\{f(x);\ x \in Q_r(r\gamma)\}$, $f(x^-(\gamma)) = min\{f(x);\ x \in Q_r(r\gamma)\}$.

The properties of the Laplace operator and (2.4) entail

$$\sum_{\gamma\in\mathbf{Z}^n}\sum_{j\in\mathbf{N}} N(h^{-2}r^2[\lambda - (h^a f(x^+(\gamma)))^{2/(2+a)}\mu_j]; -\Delta_{Q,D}) \leq N(\lambda; \widehat{H}_h) \leq \tag{2.5}$$

$$\sum_{\gamma\in\mathbf{Z}^n}\sum_{j\in\mathbf{N}} N(h^{-2}r^2[\lambda - (h^a f(x^-(\gamma)))^{2/(2+a)}\mu_j]; -\Delta_{Q,N}).$$

Let $\epsilon \in]0,1/2[$ be given and let us use (2.1) of Lemma (2.1) in (2.5), then we get

$$\omega_n(1-\epsilon)^{n/2}h^{-n}\sum_{\gamma\in\mathbf{Z}^n}\sum_{j\in\mathbf{N}} r^n[\lambda - h^2r^{-2}\frac{C_n}{\epsilon} - (h^a f(x^+(\gamma)))^{2/(2+a)}\mu_j]_+^{n/2} \leq N(\lambda; \widehat{H}_h) \tag{2.6}$$

$$\leq \omega_n(1+\epsilon)^{n/2}h^{-n}\sum_{\gamma\in\mathbf{Z}^n}\sum_{j\in\mathbf{N}} r^n[\lambda + h^2r^{-2}\frac{C_n}{\epsilon} - (h^a f(x^-(\gamma)))^{2/(2+a)}\mu_j]_+^{n/2}.$$

We will choose r and ϵ such that

$$h^2r^{-2}\frac{C_n}{\epsilon} \leq 1. \tag{2.7}$$

Then, from the hypothesis (1.8) on $f(x)$, from our choice of $r \in]0,1]$ and from the fact that (2.7) entails $(h^a f(x^{\pm}(\gamma)))^{2/(2+a)}\mu_j \leq C$ if $0 \leq \lambda \pm h^2 r^{-2}\frac{C_n}{\epsilon} - (h^a f(x^{\mp}(\gamma)))^{2/(2+a)}\mu_j$, we get

$$\int_{Q_r(r\gamma)} [\lambda - Cr^b - h^2 r^{-2}\frac{C_n}{\epsilon} - (h^a f(x))^{2/(2+a)}\mu_j]_+^{n/2} dx \leq \tag{2.8}$$

$$r^n[\lambda - h^2 r^{-2}\frac{C_n}{\epsilon} - (h^a f(x^+(\gamma)))^{2/(2+a)}\mu_j]_+^{n/2}$$

and

$$r^n[\lambda + h^2 r^{-2}\frac{C_n}{\epsilon} - (h^a f(x^-(\gamma)))^{2/(2+a)}\mu_j]_+^{n/2} \leq \tag{2.9}$$

$$\int_{Q_r(r\gamma)} [\lambda + r^b C + h^2 r^{-2}\frac{C_n}{\epsilon} - (h^a f(x))^{2/(2+a)}\mu_j]_+^{n/2} dx.$$

Then we choose

$$\epsilon = h^{\tau_1}, \quad r = h^{\tau_3}, \quad \tau_1,\ \tau_3\ > 0 \quad s.t. \quad \tau_2 := \min\{2 - \tau_1 - 2\tau_3,\ b\tau_3\} > 0, \tag{2.10}$$

so (2.7) is satisfied if h is small enough; then (1.9) comes easily from (2.6), (2.8), (2.9) and (2.10) •

3 Proof of the Corollary

As in the proof of Theorem (1.1), C will denote any constant independent of h.

<u>Proof of</u> (1.12)

By the definitions (1.5) and (1.10), we get

$$|n_{h,f}(\lambda \pm h^{\tau_2}C_2) - n_{h,f}(\lambda)|$$

$$\leq h^{-n}C\int_{\mathbf{R}^n}\sum_j |[\lambda \pm h^{\tau_2}C_2 - \mu_j(h^a f(x))^{2/(2+a)}]_+^{n/2} - [\lambda - \mu_j(h^a f(x))^{2/(2+a)}]_+^{n/2}|dx$$

$$\leq h^{-n+\tau_{n,2}}C\int_{\{x\in\mathbf{R}^n;\ f(x)<h^{-a}\mu_0^{-1}(\lambda+h^{\tau_2}C_2)^{(2+a)/2}\}} N((\lambda+h^{\tau_2}C_2)(h^a f(x))^{-2/(2+a)}\mu_0^{-1}; D_y^2+g(y))dx,$$

with $\tau_{n,2} = \tau_2$ if $n > 1$ and $\tau_{1,2} = \tau_2/2$. Then using (1.14), we have the estimate

$$|n_{h,f}(\lambda \pm h^{\tau_2}C_2) - n_{h,f}(\lambda)| \leq \tag{3.1}$$

$$h^{-d+\tau_{n,2}}C\int_{\{x\in\mathbf{R}^n;\ f(x)<h^{-a}(\mu_0^{-1}(\lambda+h^{\tau_2}C_2))^{(2+a)/2}\}} f^{-m/a}(x)dx.$$

The hypothesis (1.11) and the estimates (1.9), (1.15) and (3.1) entail (1.12), (with a new τ_1, the minimum of the old one and $\tau_{n,2}$ •

<u>Proof of the second part of the Corollary</u> (1.13)

We will need the asymptotic formula (1.14) with a remainder estimate.

Lemma 3.1 *As* $\mu \mapsto +\infty$,

$$N(\mu; D_y^2 + g(y)) = \mu^{m(2+a)/(2a)}\theta(g)(1 + \mathcal{O}(\mu^{-1/4})). \tag{3.2}$$

with $\theta(g) = \omega_m \int_{\mathbf{R}^m} [1 - g(y)]_+^{m/2} dy.$

A sharp remainder estimate (with $\mu^{-(2+a)/(2a)}$ if $m > 1$ and $\mu^{-(2+a)/(2a)} log(\mu)$ if $m = 1$, in place of $\mu^{-1/4}$) has been established in [He-Ro-1], (see also in [Moh]), but with $g \in C^\infty(\mathbf{R}^m \setminus \{0\})$, and the proof is not elementary. For our purpose we need only a remainder estimate $\mathcal{O}(\mu^{-\delta})$, with $\delta > 0$, but the best one we can easily get is $\delta = 1/4$.

We give a simple proof of (3.2) using min-max principle.

We use the same strategy as in the proof of Theorem (1.1).

The homogeneity of g and its regularity, $g \in C^1(\mathbf{R}^m \setminus \{0\})$, entails that

$$\mid g(y) - g(z) \mid \leq \mid y - z \mid g(y)C, \ \forall\ y,\ z\ \in\ \mathbf{R}^m,\ \mid y - z \mid \leq 1 \ \text{ and } \ \mid y \mid > 1,\ \mid z \mid > 1. \tag{3.3}$$

For any $\mu > 1$, we take a partition of $\mathbf{R}^m$ into cubes of side $r = \mu^{-1/4}$:

$$\mathbf{R}^m = \cup_{\alpha \in \mathbf{Z}^m} \overline{Q}_r(r\alpha).$$

Let $y^\pm(\alpha) \in \overline{Q}_r(r\alpha)$ such that

$$g(y^+(\alpha)) = Max\{g(y);\ y \in Q_r(r\alpha)\}, \quad g(y^-(\alpha)) = min\{g(y);\ y \in Q_r(r\alpha)\};$$

then

$$\sum_{\alpha \in \mathbf{Z}^m} N(\mu;\ -\Delta_{Q_r(r\alpha),D} + g(y^+(\alpha))) \leq N(\mu;\ D_y^2 + g(y)) \tag{3.4}$$

$$\leq \sum_{\alpha \in \mathbf{Z}^m} N(\mu;\ -\Delta_{Q_r(r\alpha),N} + g(y^-(\alpha))).$$

It comes from (2.1) of Lemma (2.1), with $\epsilon = \mu^{-1/4}$, and from (3.4) that

$$(1 - \mu^{-1/4})^{m/2} r^m \omega_m \sum_{\alpha \in \mathbf{Z}^m} [\mu - \mu^{3/4} C_m - g(y^+(\alpha))]_+^{m/2} \leq N(\mu;\ D_y^2 + g(y)) \tag{3.5}$$

$$\leq (1 + \mu^{-1/4})^{m/2} r^m \omega_m \sum_{\alpha \in \mathbf{Z}^m} [\mu + \mu^{3/4} C_m - g(y^-(\alpha))]_+^{m/2}.$$

Then we use (3.3), for the $Q_r(r\alpha) \subset \mathbf{R}^m \setminus 2Q$, (let us remind that $r = \mu^{-1/4}$ so on such $Q_r(r\alpha)$, $\mid g(y) - g(y^\pm(\alpha)) \mid \leq \mu^{-1/4} C g(y^\pm(\alpha))$), and if $Q_r(r\alpha) \cap (2Q) \neq \emptyset$ we use $0 \leq g \leq C$ on $2Q$ to get from (3.5) that

$$(1 - \mu^{-1/4})^{m/2} \omega_m \int_{\mathbf{R}^m} [\mu - \mu^{3/4} C - g(y)]_+^{m/2} dy \ \leq N(\mu;\ D_y^2 + g(y)) \tag{3.6}$$

$$\leq (1 + \mu^{-1/4})^{m/2} \omega_m \int_{\mathbf{R}^m} [\mu + \mu^{3/4} C - g(y)]_+^{m/2} dy.$$

So the homogeneity and (3.6) give

$$(1 - \mu^{-1/4})^{m/2} (1 - \mu^{-1/4} C)^{m(2+a)/(2a)} \mu^{m(2+a)/(2a)} \omega_m \int_{\mathbf{R}^m} [1 - g(y)]_+^{m/2} dy \tag{3.7}$$

$$\leq N(\mu;\ D_y^2+g(y)) \leq$$

$$(1+\mu^{-1/4})^{m/2}(1+\mu^{-1/4}C)^{m(2+a)/(2a)}\mu^{m(2+a)/(2a)}\omega_m\int_{\mathbf{R}^m}[1-g(y)]_+^{m/2}dy.$$

The Lemma (3.1) follows from (3.7) •

We will need a precise estimate of the semiclassical asymptotic of the moment of the eigenvalues of $\epsilon^{(2+a)/a}D_y^2+g(y)$, (see [Li-Th]),

$$\sum_j[\lambda-\epsilon\mu_j]_+^{n/2}\sim\epsilon^{-m(2+a)/(2a)}\lambda^{d/2+m/a}\Theta(g),\quad \text{as } \epsilon\ \mapsto\ 0; \tag{3.8}$$

if μ_j are the eigenvalues of $D_y^2+g(y)$ and if $\Theta(g)=\theta(g)\dfrac{m(2+a)}{2a}B(\dfrac{n}{2}+1,\dfrac{m}{2}+\dfrac{m}{a})$
so $\Theta(g)=\dfrac{m(2+a)}{2a^2}\omega_m B(\dfrac{m}{2}+1,\dfrac{m}{a})B(\dfrac{n}{2}+1,\dfrac{m}{2}+\dfrac{m}{a})\int_{S^{m-1}}g^{-m/a}(\omega)d\omega,$
with $B(u,v)=\int_0^1 s^{u-1}(1-s)^{v-1}ds=\Gamma(u)\Gamma(v)/\Gamma(u+v)$ is the Beta function.

Lemma 3.2 *There exist constants C and $\epsilon_0>0$ such that, for any $\epsilon\in]0,\epsilon_0]$,*

$$[\lambda-C\epsilon^{1/4}]_+^{d/2+m/a}\epsilon^{-m(2+a)/(2a)}\Theta(g)\leq\sum_{j\in\mathbf{N}}[\lambda-\epsilon\mu_j]_+^{n/2} \tag{3.9}$$

$$\leq[\lambda+C\epsilon^{1/4}]_+^{d/2+m/a}\epsilon^{-m(2+a)/(2a)}\Theta(g).$$

Proof of the Lemma From (3.2) of Lemma (3.1) with $\mu=\mu_j$, we get that the multiplicity m_j of μ_j is bounded by $C\mu_j^{m/2+m/a-1/4}$ and

$$|\mu_j-((\theta(g))^{-1}j)^{\frac{2a}{m(2+a)}}|\leq Cj^{\frac{3a}{2m(2+a)}}. \tag{3.10}$$

So if $0\leq\lambda-\epsilon\mu_j$, then

$$\lambda-C\epsilon^{1/4}-\epsilon((\theta(g))^{-1}j)^{\frac{2a}{m(2+a)}}\leq\lambda-\epsilon\mu_j\leq\lambda+C\epsilon^{1/4}-\epsilon((\theta(g))^{-1}j)^{\frac{2a}{m(2+a)}},$$

which proves

$$[\lambda-C\epsilon^{1/4}-\epsilon((\theta(g))^{-1}j)^{\frac{2a}{m(2+a)}}]_+^{n/2}\leq[\lambda-\epsilon\mu_j]_+^{n/2}\leq[\lambda+C\epsilon^{1/4}-\epsilon((\theta(g))^{-1}j)^{\frac{2a}{m(2+a)}}]_+^{n/2}. \tag{3.11}$$

It comes from (3.11)

$$\epsilon^{-m(2+a)/(2a)}\theta(g)\int_{\mathbf{R}}[\lambda-C\epsilon^{1/4}-C\epsilon-s^{\frac{2a}{m(2+a)}}]_+^{n/2}ds\leq\sum_{j\in\mathbf{N}}[\lambda-\epsilon\mu_j]_+^{n/2} \tag{3.12}$$

$$\leq\epsilon^{-m(2+a)/(2a)}\theta(g)\int_{\mathbf{R}}[\lambda+C\epsilon^{1/4}+C\epsilon-s^{\frac{2a}{m(2+a)}}]_+^{n/2}ds.$$

The Lemma (3.2) follows from (3.12) •

End of the proof of (1.13) Let $\tau\in]0,1[$ be fixed.

For any $x\in\mathbf{R}^n$ such that $(h^af(x))^{2/(2+a)}\leq h^\tau$, let us use (3.9) with $\epsilon=(h^af(x))^{2/(2+a)}$, then we get

$$h^{-m}f^{-m/a}(x)\Theta(g)[\lambda-Ch^{\tau/4}]_+^{d/2+m/a}\leq\sum_{j\in\mathbf{N}}[\lambda-(h^af(x))^{2/(2+a)}\mu_j]_+^{n/2} \tag{3.13}$$

$$\leq h^{-m} f^{-m/a}(x)\Theta(g)[\lambda + Ch^{\tau/4}]_+^{d/2+m/a}.$$

Let $\Omega(h,\tau) = \{x \in \mathbf{R}^n;\ (h^a f(x))^{2/(2+a)} > h^\tau\}$. Then

$$\int_{\Omega(h,\tau)} \sum_{j\in\mathbf{N}} [\lambda - (h^a f(x))^{2/(2+a)}\mu_j]_+^{n/2} dx \leq C\int_{\Omega(h,\tau)} N(\lambda(h^a f(x))^{-2/(2+a)}; D_y^2 + g(y))dx,$$

so we can use (3.2) with $\mu = \lambda(h^a f(x))^{-2/(2+a)} > \lambda h^{-\tau}$ to get

$$\int_{\{x\in\mathbf{R}^n;\ f(x)>h^{-a+\tau(2+a)/2}\}} \sum_{j\in\mathbf{N}} [\lambda - (h^a f(x))^{2/(2+a)}\mu_j]_+^{n/2} dx \tag{3.14}$$

$$\leq h^{-m} C\int_{\{x\in\mathbf{R}^n;\ f(x)>h^{-a+\tau(2+a)/2}\}} f^{-m/a}(x)dx.$$

Then, from (1.12), (3.13) and (3.14), (with for exemple $\tau = a/(2+a)$), we get that

$$N(\lambda;\widehat{H}_h) \sim h^{-d}\lambda^{\frac{d}{2}+\frac{m}{a}}\Theta(f,g), \quad \text{when } h \mapsto 0, \tag{3.15}$$

if $f^{-m/a} \in L^1(\mathbf{R}^n)$, with $\Theta(f,g) = \Theta(g)\times\omega_n\times\int_{\mathbf{R}^n} f^{-m/a}(x)dx$, ($\Theta(g)$ is defined in (3.8)).

But $\omega_d\int_{\mathbf{R}^d}[\lambda - V(X)]_+^{d/2} = \lambda^{\frac{d}{2}+\frac{m}{a}}\omega_d\frac{1}{a}B(\frac{m}{a},\frac{d}{2}+1)(\int_{S^{m-1}} g^{-m/a}(\omega)d\omega)\times(\int_{\mathbf{R}^n} f^{-m/a}(x)dx)$. So (1.13) is exactly (3.15) if we check

$$\omega_n\omega_m(\frac{m}{2}+\frac{m}{a})B(\frac{n}{2}+1,\frac{m}{2}+\frac{m}{a})B(\frac{m}{a},\frac{m}{2}+1) = \omega_d B(\frac{m}{a},\frac{d}{2}+1) \tag{3.16}$$

or equivalently, (after writing the Beta functions into the Gamma ones and after simplifications),

$$|B_n|\times|B_m|\times\Gamma(\frac{n}{2}+1)\times\Gamma(\frac{m}{2}+1) = |B_{n+m}|\times\Gamma(\frac{n}{2}+\frac{m}{2}+1); \tag{3.17}$$

if B_k is the unit ball of $\mathbf{R}^k$ and $|B_k|$ its volume. The last equality (3.17) can be checked easily by writing that $|B_{n+m}| = |B_m|\times(\int_{\mathbf{R}^n}[1-|x|^2]_+^{m/2}dx)$ •

Remark 3.3 *If* $f^{-m/a}(x) \in L^1(\mathbf{R}^n)$ *and if there exist* C *and* $\rho > 0$ *such that*

$$\int_{\{x\in\mathbf{R}^n;\ f(x)>\mu\}} f^{-m/a}(x)dx \leq C\mu^{-\rho}, \quad \forall\ \mu > 1,$$

then, for some $\tau > 0$,

$$N(\lambda;\widehat{H}_h) = (1+\mathcal{O}(h^\tau))\omega_d\int_{\mathbf{R}^d}[\lambda - V(X)]_+^{d/2}dX, \quad \text{as } h \mapsto 0. \tag{3.18}$$

This comes easily from the proof of (1.13).

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