An equivariant branching lemma for relative equilibria

Giuseppe Gaeta[0]

Departamento de Matemáticas
Universidad Carlos III de Madrid
28911 Leganés (Spain)
gaeta@math.uc3m.es

Summary. The "equivariant branching lemma" of Cicogna and Vanderbauwhede, extended by Golubitsky and Stewart to Hopf bifurcation, is set in group-orbit space; this permits to consider, and obtain results on, the branching of relative equilibria via bifurcation in equivariant dynamical systems.
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1. Introduction

The Equivariant Branching Lemma (EBL) [1, 2, 3] is a fundamental tool in equivariant bifurcation theory [4, 5, 6, 7]; this was first formulated independently by Cicogna [1, 2] and by Vanderbauwhede [3] for the case of bifurcation of stationary solutions, and then extended in a number of ways, in particular to include Hopf bifurcation [8] and problems whose symmetry corresponds to a nonlinear group action [9]; see also [10]. For a comprehensive discussion see [5].

The basic idea behind the EBL in its various versions is to use the symmetry properties of the system (or class of systems) under study to guarantee the dynamical invariance of a lower-dimensional space or manifold, and to apply on the restriction of the initial system to this invariant space standard (usually topological) tools to guarantee the existence of a bifurcating branch of solutions; thus any version of the EBL is strictly connected to a suitable version of the Reduction Lemma (RL) [5, 8].

In the present short note we want to remark that a simple use of a classical result by L. Michel [11] allows to obtain, under certain condition, an EBL for the bifurcation of branches of *relative equilibria* out of stationary solutions. The main result given here could also be obtained in a different, albeit related, way, i.e. combining the standard EBL (and the Golubitsky-Stewart results on bifurcation of solutions invariant under a maximal isotropy subgroup) with the possibility of projecting equivariant dynamics to the "orbit space" (this is defined below) recently emphasized by Chossat and collaborators [12, 13, 14, 15].

[0]Address after 1 Oct 99: Dipartimento di Fisica, Universitá di Roma, 00185 Roma (Italy); *gaeta@roma1.infn.it*

It is maybe appropriate to stress that the use of orbit space techniques for equivariant dynamics is becoming increasingly popular, following the realization that general equivariant dynamics (and not only gradient one) can be projected to the orbit space, despite the fact this is not a manifold[1]; we quote, beside [12, 13, 14, 15] mentioned above, also e.g. [23, 24] in the general case, and [20, 25, 26] in the Hamiltonian case.

Relative equilibria – i.e. group orbits which are invariant under the dynamics – received in recent years extended attention, also in the context of bifurcation theory [12, 13, 27], leading to a good understanding of them; I do not claim the result given here is more powerful than those which are present in the literature (see e.g. [5, 14, 27, 28, 29, 30, 31]), but its derivation – and the check that a given system satisfies the assumptions required for its validity – is so simple that it can well be useful in practice, and I think it can be of interest to report on it. Moreover, it is different in spirit from the standard approach, and thus it could be of help when the latter encounters some difficulties.

2. Michel theorem

In his work [11], physically motivated by the theory of fundamental interactions [32, 33], Michel proved a remarkable general theorem, valid for the action of compact groups on finite dimensional manifolds. His investigation was naturally set in the variational frame, but he actually proved results on general equivariant vector fields, and I would like to quote his results in this language in view of the application to relative equilibria.

We denote by G a *compact* Lie group acting on the *finite dimensional* manifold M[2]; the quotient M/G is also called the orbit space (in the physical literature dealing with gauge theories – where G and M are considerably more complicate than in our present setting – this is also called the configuration space) and denoted by Ω (see e.g. [5, 7, 14, 23] for its relevance in nonlinear dynamics and equivariant bifurcation theory). In general, Ω is not a manifold.

In more abstract terms, Ω is the set of classes of equivalence on M under the group action. The orbit $G(p)$ of a point in M will also be denoted as $\omega(p)$ or ω_p.

For each point $p \in M$ we can consider the set of elements in G which leave p unchanged,

$$G_p \;:=\; \{g \in G \mid gp = p\} \;; \tag{1}$$

this is called the isotropy subgroup of p. We introduce another equivalence relation in M as follows: we consider points with isotropy subgroups conjugated in G as having the same *isotropy type*, and denote the set of points having the

[1] This is possible because the orbit space is, at least under the present hypotheses (compact Lie groups acting on finite dimensional manifolds) a *stratified manifold* [16, 17], and singular strata are invariant under any invariant dynamics, precisely thanks to Michel theorem. Taking advantage of this fact – also in the context of relative equilibria – requires also a use of *slice theory* [18, 19, 20], as was already clearly stated in [21, 22].

[2] These assumptions on G and M could be somewhat relaxed, as we discuss later on, but this simplest setting will make the basic argument clearer

same isotropy type as a *stratum* in M:

$$\sigma(p) \ := \ \{q \in M \mid \exists g \in G \ : \ G_q = g G_p g^{-1} \} \ . \tag{2}$$

Under our present hypotheses on G and M, strata are submanifolds in M.

Notice that $q = gp$ implies $G_q = g G_p g^{-1}$, so that necessarily

$$\omega(p) \ \subseteq \sigma(p) \tag{3}$$

for any point $p \in M$; this means also that we can define strata in Ω, i.e. that th eequivalence relation in M used in (2) can be projected to Ω; the stratum of an orbit $\omega \in \Omega$ will be denoted as $\Sigma(\omega)$.

When the group action in M is regular (this is automatically true if G is compact), the topology induced by the group action is compatible with the topology on M, and thus the topology in M induces a natural topology in Ω. In particular, it is then possible to define properly a distance $d(\omega_1, \omega_2)$ between group orbits (e.g. as the minimal distance in M between points $p \in \omega_1$ and $q \in \omega_2$).

We say that an orbit ω is *isolated in its stratum* if there is an $\varepsilon > 0$ such that $d(\omega, \mu) > \varepsilon$ for all the $\mu \neq \omega$ in $\Sigma(\omega) \subset \Omega$.

We refer to [33, 34, 35, 36, 37] for further detail on this geometric construction and its applications, in particular to Physics.

3. Relative equilibria

Let us now consider an equivariant vector field X on M; that is, a field which for all $Y \in \mathcal{G}$ (the Lie algebra of G) satisfies $[X, Y] = 0$.

We will, for ease of language, think of M as embedded in R^N, and use R^N coordinates $\{x_1, ..., x_N\}$ in M; thus X can be written in these coordinates as corresponding to the Dynamical System

$$\dot{x} \ = \ f(x) \tag{4}$$

where $f : M \to \mathrm{T}M$. The equivariance of f means that for any $x \in M$ and all $g \in G$,

$$f(gx) \ = \ \varphi_g[(D_g f)(x)] \ . \tag{5}$$

Here $\varphi : \mathrm{T}M \to \mathrm{T}M$ is the isomorphism induced by g on tangent spaces, i.e. φ_g maps $\mathrm{T}_x M$ into $\mathrm{T}_{gx} M$.

It was proved by Michel [11] that

$$f(x) \ \in \ \mathrm{T}_x \sigma(x) \ ; \tag{6}$$

this also means that, as it was already proved by Bierstone and Schwarz [34, 38, 39] the vector field $X : M \to \mathrm{T}M$ can be projected to a vector field $X_\Omega : \Omega \to \mathrm{T}\Omega$ defined on the orbit space (recall each stratum is a smooth manifold). The relevance of this fact for nonlinear dynamics has been recently emphasized by Chossat, Koenig and Lauterbach [13, 14, 23], see also the forthcoming book [15].

It should be noticed that (6) also means that the closure of a stratum is a globally invariant set under any equivariant dynamics, and we can consider the restriction of (4) to it. This is a general "reduction lemma" in the spirit of Golubitsky and Stewart [5, 8]. Now, from (6) it also follows that if an orbit ω is isolated in its stratum, then for all $x \in \omega$

$$f(x) \ \in \ \mathrm{T}_x \omega \ . \tag{7}$$

As noticed by Michel, in the case of invariant variational problem and under the conditions discussed above one can conclude that (Michel's theorem): *for ω isolated in its stratum, all the $x \in \omega$ are critical for* any *invariant potential* [11, 33]; indeed, the gradient is necessarily orthogonal to the group orbit, and thus in this case it must vanish.

As mentioned above Michel was mainly interested, due to physical motivation [32], in the analysis of the variational case, and limited his attention to this (and the relevant physical consequences); here we are instead focusing on the general equivariant dynamics case. Essentially, this adds the possibility of motion along the group orbits [40], obviously impossible in the gradient case.

We recall that a group orbit $\gamma = G(x) \subseteq M$ is a *relative equilibrium* for the flow (4) if it is a dynamically invariant submanifold, i.e. if $f : \gamma \to \mathrm{T}\gamma$. Notice that in this case the dynamical orbit $\Phi(x)$ of the point x is contained in $G(x)$, but neither this nor its closure do necessarily coincide with the whole group orbit; the rational torus flow provides a simple example of this situation.

In the case of general equivariant vector fields, (7) can be interpreted as follows (as obvious from our discussion, this is just an easy corollary of Michel's theorem):

Lemma 1. *If a G-orbit ω on the G-manifold M is isolated in its stratum, then it is a relative equilibrium for any equivariant vector field X defined on M.*

4. Bifurcations

Let us now consider the case where the equivariant vector field X depends on a control parameter $\lambda \in \Lambda$ (Λ an interval on the real line), and $X^{(\lambda)}$ is G-equivariant uniformly in λ; in other words, (4) is now replaced by

$$\dot{x} \ = \ f(x; \lambda) \tag{4'}$$

and (5) holds for all $\lambda \in \Lambda$.

We assume moreover that `(a)` $x \equiv x_0$ (call ω_0 the corresponding G-orbit) is a solution for all $\lambda \in \Lambda$, and that it is stable for $\lambda < \lambda_0$, and unstable for $\lambda > \lambda_0$ (where λ_0 is assumed to be in the interior of Λ). To avoid explosive behaviour, we assume also that `(b)` there is an open set $\mathcal{B}$, containing x_0 in its interior and of positive radius (in the metric defined on M), such that $X^{(\lambda)}$ points inward on $\partial\mathcal{B}$, for all values of $\lambda \in \Lambda$. As standard in bifurcation theory, we also assume that `(c)` the real part $\sigma_i^r(\lambda)$ of the critical eigenvalues of $L(\lambda) := (Df)(x_0; \lambda)$ satisfies

$$\left(\frac{\mathrm{d}\sigma_i^r}{\mathrm{d}\lambda} \right)_{\lambda_0} \ > \ 0 \ . \tag{8}$$

These assumptions guarantee in a standard way [41, 42] that x_0 bifurcates at $\lambda = \lambda_0$.

Assume now that there is a stratum $\Sigma_0 \subset \Omega$ which is a one-dimensional manifold: as $X_\Omega^{(\lambda)}$ is tangent to strata, it is directed along Σ_0 at all $\omega \in \Sigma_0$; moreover the above assumptions guarantee that for $\lambda > \lambda_0$ there is a branch of zeroes of $X_\Omega^{(\lambda)}$ lying precisely in the intersection of Σ_0 with $\mathcal{B}$ and distinct from ω_0. A zero of $X_\Omega^{(\lambda)}$ corresponds, as already remarked, to a relative equilibrium for the dynamics on M, described by $X^{(\lambda)}$; as there is a branch of equilibria for $X_\Omega^{(\lambda)}$ bifurcating off ω_0, we can conclude there is a branch of relative equilibria for $X^{(\lambda)}$ bifurcating off x_0.

We can express the result of our discussion as follows; here we assume that x_0 is invariant under the full G:

Lemma 2. *Let us consider a one-parameter family of vector fields $X^{(\lambda)}$ on M, and let $x_0 \in M$ with $G_{x_0} = G$ be an isolated zero of $X^{(\lambda)}$ for all $\lambda \in \Lambda$; let assumptions (a),(b),(c) above be satisfied. Then, there is a local branch of relative equilibria bifurcating from x_0 at $\lambda = \lambda_0$.*

Notice that if $G_{x_0} \neq G$, then ω_0 does not reduce to x_0 alone, and x_0 is a degenerate zero of X; however, the discussion in terms of X_Ω does still hold, and we can again deduce the bifurcation of a branch of relative equilibria. Also, even if x_0 is not a zero of X, but ω_0 is a zero of X_Ω and we focus on the dynamics transversal to $\omega_0 \subset M$, we can again conduct the same discussion in terms of X_Ω, and deduce the existence of a branch of relative equilibria bifurcating off relative equilibria (notice relative equilibria can correspond to different subgroups of G, so that we can have a symmetry breaking leading from a type of relative equilibrium to another).

5. Relations with the Golubitsky–Stewart theory

We can actually refine (6) – and the above remark – a little further: following Golubitsky and Stewart[5, 8], let us define for each $H \subseteq G$ the submanifold

$$M^{(H)} \ := \ \{x \in M \mid H \subseteq G_x\} \ . \tag{9}$$

Now, if in (5) we consider $g \in G_x$, we have immediately that

$$f(x) \ = \ (D_g f)(x) \quad \forall g \in G_x \tag{10}$$

and thus that

$$f(x) \ \in \ \mathrm{T}_x M^{(G_x)} \ . \tag{11}$$

Combining (7) and (11) we conclude immediately that for an orbit ω isolated in its stratum we have, for all $x \in \omega$,

$$f(x) \ \in \ \mathrm{T}_x \omega \cap \mathrm{T}_x M^{(G_x)} \tag{12}$$

(notice in this case we could as well consider the manifold of points whose isotropy subgroup is exactly equal to G_x, as on ω we cannot have a G_y which strictly contains G_x).

However, $T_x\omega$ is by definition equivalent to the linear space spanned by the Lie algebra $\mathcal{G}$ of G acting on x, i.e. is isomorphic to $\mathcal{G}/\mathcal{G}_x$ (where $\mathcal{G}_x$ is the Lie algebra of G_x): this is the subspace of $\mathcal{G}$ spanned by the orthogonal complement (under the natural scalar product in $\mathcal{G}$) of $\mathcal{G}_x$.

Thus the flow $\Phi_f(x;t)$ under an equivariant vector field will take x into $y \in \omega(x)$ such that $G_y = G_x$; such y can be written, as we are on $\omega(x)$, as $y = hx$ for some $h \in G$, and we conclude that necessarily $h \in N_G[G_x]$, the normalizer of G_x in G. Notice that, since G_x acts as the identity, we can actually consider $h \in D_x^0$, where $D \subseteq G$ is defined as

$$D_x \ := \ N_G[G_x]/G_x \tag{13}$$

and D^0 is the connected component of the identity in D. Thus, necessarily

$$f(x) \ \in \ \mathrm{T}_x[D^0(x)] \ . \tag{14}$$

The above construction and discussion are due to Golubitsky and Stewart; when G_x is a maximal isotropy subgroup of G, they obtained – as well known – powerful general results for linear actions: based on a homological classification of possible D^0, they obtained a complete classification of elementary bifurcations [5, 8]. From the physical point of view, this is a complete classification of elementary spontaneous symmetry breaking via second order phase transitions; for the specific features of the Hamiltonian case, see [43, 5].

In the present context, the above discussion shows that in order to investigate the possible relative equilibria arising by the bifurcation mechanism we are considering, we have to focus attention on the subgroups D^0; as shown by Golubitsky and Stewart, the only three possibilities in elementary bifurcations are $D^{(0)} = \{e\}, SO(2), SU(2)$[3] [5, 6, 7, 8].

In particular, if $D^0 = \{e\}$, then the relative equilibria whose existence is granted from Lemma 2 are actually equilibria *tout court*. Similarly, if $D^0 = SO(2)$ we are guaranteed that these relative equilibria are actually periodic orbits (possibly degenerate, i.e. stationary); for $D^0 = SU(2)$ we have "quaternionic" branches of bifurcating periodic (again possibly degenerate, i.e. stationary) solutions [5, 7, 44].

It should also be stressed that all the relative equilibria lying on the same G-orbit must be isomorphic: this poses some apriori restrictions on the possible kind of relative equilibria depending on the structure of $\omega \subset M$. E.g., we can have periodic orbits only if ω admits a fibration in circles S^1, quaternionic branches only if it admits fibration in spheres S^3, and n-periodic orbits only if ω admits a fibration in tori $\mathbf{T}^n$.

6. Final comments

We would now like to briefly comment on what are the possible advantages of the equivariant branching lemma presented here (in the Ω setting) over the standard formulation (in the M setting).

[3] Their classification actually extends to the full D as well; in the case $D^{(0)} = \{e\}$, which corresponds to true equilibria rather than relative one, several possibiities exist, but for $D^{(0)} = SO(2)$ necessarily D is either $SO(2)$ or $0(2)$, and for $D^{(0)} = SU(2)$, this is also the full D.

It is quite clear from the discussion in previous section that in the cases covered by the Golubitsky-Stewart theory (and extensions thereof), our branching lemma is merely a reformulation in orbit space of well known results. Notice that although it is known that in some cases direct bifurcation to non-maximal isotropy subgroups is generically present [45] (a criterion for this can be set in terms of the relevant Weyl group), the standard Golubitsky-Stewart theory – in the M formulation – can cope perfectly with this case.

A first advantage of the orbit space formulation, admittedly depending on individual tastes, is the unification of several results in a geometrical way; this advantage is of aesthetical nature, unless such a geometrical formulation in orbit space could give something more than the known results. We would now mention some other advantages of the orbit space setting and discuss some points related to this approach.

1) Firstly, we would like to notice that in the same way as the "standard" equivariant branching lemma (EBL) can be extended from one-dimensional strata to higher dimensional cases by the use of topological arguments guaranteeing the existence of zeroes of a vector field on an invariant set Σ based on the topology of Σ [2, 46], the same kind of extensions can be obtained for our "relative equilibria EBL" (lemma 2): extensions of the standard EBL apply to bifurcations of stationary points for $X_\Omega^{(\lambda)}$, and these in turn correspond to relative equilibria of $X^{(\lambda)}$.

2) I would like to stress that although these extensions can only ensure the existence of relative equilibria, without providing a clear algorithm for their determination (as for "topological" extension of the EBL in standard space), nevertheless they can ensure the existence of a given number of relative equilibria with well determined *qualitative* property, in particular with a given symmetry.

3) Setting the dynamics in orbit space also allows to obtain with a reasonable effort information about solutions which would be much harder to obtain without exploiting the "splitting" [40] of the dynamics: let us consider for example the case in which we have an invariant subset $\Omega_0 \subset \Omega$ (whose dynamical invariance follows from the results discussed above) which is two-dimensional: applying Poincaré-Bendixson theorem to X_Ω we can then immediately guarantee the existence of a *relative periodic orbit*; Notice in the M formulation the corresponding invariant set will be of higher dimension and thus the Poincaré-Bendixson theorem cannot be applied.

Similar considerations result from applying standard topological reasoning to X_Ω in the presence of higher-dimensional invariant subspaces; e.g., if the invariant set Ω_0 is topologically a closed ball, then Brouwer's fixed point theorem guarantees the existence of a fixed point for X_Ω in Ω_0, and thus of a relative equilibrium in the corresponding subset of M.

4) In the case of systems corresponding to reducible representations of $D^{(0)}$, the informations provided directly by the standard theory are not so powerful as in the irreducible case (in the sense some extra work is needed to deal with this setting); the orbit space approach does not suffer from these limitations.

In some cases, one considers identical subsystems with identical symmetry related via some coupling which introduce an extra (discrete) symmetry relating the blocks corresponding to the subsystems. This setting would arise e.g. in the case of systems of coupled identical oscillators with a K-symmetric interaction, K a subgroup of the permutation group; a related symmetric variational problem with additional S_n symmetry has been considered using an orbit space approach in [47].

5) In the case of relative periodic orbits, notice that if there is a significantly different time scale for the motions along G-orbits and those in orbit space, the symmetry splitting would correspond to a dynamical splitting between fast and slow variables (moreover, if one adopts an adiabatic approximation, topological effects similar to Hannay angles could appear).

Also, in the case of *approximate symmetries* one can adopt a perturbation approach based on a symmetric unperturbed system, which can be analyzed via orbit space; the non-symmetric perturbation will in general introduce a (weak) dependence of the dynamics in Ω on the flow along G-orbits.

6) It should be stressed that the problem in Ω can happen to have a variational structure even if the original problem in M lacked it; the presence of a variational structure does at once permit to use all the tools of the Calculus of Variations (and thus e.g. Morse theory, Lyusternik category, etc.) in the Ω problem.

In order to illustrate this, let us consider the following class of polynomial three dimensional systems, invariant under $G = SO(2) \times Z_2$, where $SO(2)$ acts as rotations in the (x, y) plane and Z_2 as reflections across the same plane:

$$\begin{cases} \dot{x} = \alpha(r^2, z^2)x - \beta(r^2, z^2)y \\ \dot{y} = \beta(r^2, z^2)x + \alpha(r^2, z^2)x \\ \dot{z} = \gamma(r^2, z^2)z \end{cases} \tag{15}$$

(where $r^2 = x^2 + y^2$); obviously a system of this form cannot be hamiltonian, and it can be (nonautonomous) hamiltonian in the subspace (x, y) only if $\alpha \equiv 0$; it is gradient (exact) only if $\beta \equiv 0$ and $(\partial\alpha/\partial z^2) = (\partial\gamma/\partial r^2)$.

In the orbit space Ω, in which we can use coordinates $\rho = r^2$ and $\zeta = z^2$, the system is written as

$$\begin{cases} \dot{\rho} \;=\; 2\,\rho\,\alpha(\rho,\zeta) \\ \dot{\zeta} \;=\; 2\,\zeta\,\gamma(\rho,\zeta) \end{cases} \tag{16}$$

which is gradient (exact) when $\rho(\partial\alpha/\partial\zeta) = \zeta(\partial\gamma/\partial\rho)$. The same problem is instead Hamiltonian when $\alpha + \rho\alpha_\rho = -\gamma - \zeta\gamma_\zeta$.

Thus (with simple algebra) the system in the orbit space is a gradient one when

$$\alpha(\rho,\zeta) \;=\; \zeta^2 \sum_{k=0}^{\infty} c_k\,(\zeta\rho)^k \quad , \quad \gamma(\rho,\zeta) \;=\; \rho^2 \sum_{k=0}^{\infty} c_k\,(\zeta\rho)^k \; ; \tag{17}$$

and Hamiltonian when

$$\alpha(\rho,\zeta) \;=\; \sum_{k=0}^{\infty} c_k\,(\zeta\rho)^k \;=\; -\gamma(\rho,\zeta) \; . \tag{18}$$

7) We would also like to recall that a large part of the "standard theory" refers to *compact* Lie groups with a *linear* action; in the case of compact groups acting on finite dimensional compact manifolds – or on manifolds which can be compactified, as e.g. finite dimensional linear spaces, which can be compactified via the one-point compactification (the choice of compactification raises some delicate points relative to the deformation of the linear action in the compactification process) – we know [48, 49] that any nonlinear action can[4] be equivariantly embedded into an orthogonal action in some $\mathcal{R}^k$ space, so that in principle (i.e. modulo the equivariant embedding), this covers the general case of nonlinear action as well. However, applications in general and Physics in particular abound with infinite dimensional spaces (whenever we are in the realm of Field Theory) and non-compact groups (just think of the Euclidean or Poincaré groups, or the group of scale transformations), so that it can be useful to have a tool which only uses the (local) structure of orbit space.

8) In this respect, it should be mentioned that Michel theory has been successfully applied to gauge theories (not necessarily of strict Yang-Mills type) [37], i.e. in a case where an infinite dimensional non-compact Lie group (locally modelled on a compact semisimple Lie group) acts on an infinite dimensional space of sections of a fiber bundle. Moreover, Michel theory can be set in terms of *equivariant retractions* and more generally of *slice theory* [18, 19] (see e.g. [34, 35]); this is a much more general setting, essentially needing only a G-compatible topology (slice theory can be formulated for topological groups, without requiring a differentiable structure), and thus it is possible it can be applied to much more general cases: e.g., we know that local slices exist in the space of Riemannian structures of a manifold [50].

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