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Eigenvalues as Dynamical Variables [1]

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Abstract

We review some work done with C. Rovelli on the use of the eigenvalues of the Dirac operator on a curved spacetime as dynamical variables, the main motivation coming from their invariance under the action of diffeomorphisms. The eigenvalues constitute an infinite set of "observables" for general relativity and can be taken as variables for an invariant description of the gravitational field dynamics.

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1 Introduction

A (generalized) Dirac operator D is the main actor in Alain Connes program of noncommutative geometry [9]. This operator codes the full information about spacetime geometry in a way usable for describing the dynamics of the latter. Not only the geometry is reconstructed from the (normed) algebra generated by D and the smooth functions on spacetime, but the Einstein-Hilbert action of the Standard Model coupled to gravity is approximated by the trace of a simple function of D [11, 8]. One should stress that the model obtained is both classical and Euclidean. But there is a new emphasis and a new conceptual interpretation of particle physics. The latter is used to unravel the fine geometric structure of spacetime pointing to a noncommutative structure at short distance scales and to an intrinsic coupling between gravity and other fundamental interactions. Recently [12] there has been a step in the direction of quantum field theories and it has been suggested that the spacetime itself and its geometrical structure should be regarded as a concept which is derived from properties of quantum field theory.

The previous attitude also suggests the possibility of taking the eigenvalues λ_n of D as "dynamical variables" for general relativity. They form an infinite family of diffeomorphism[2] invariant quantities and are therefore, truly *observables* for general relativity. It is a central point of the latter theory that fundamental physics is invariant under diffeomorphisms: there is no fixed non-dynamical structure with respect to which location or motion could be defined. Consequently, a fully diffeomorphism invariant description of the geometry has long been sought [6] and would be extremely useful also for quantum gravity [18]. Although this noncommutative approach has limitations, notably its 'Euclidean' character, it definitely opens new paths in the study of the dynamics of spacetime.

As a first step for the use of these ideas in classical and/or quantum theories, an expression for the Poisson brackets of the Dirac eigenvalues has been derived [21, 14]. Surprisingly, the brackets can be expressed in terms of the energy-momentum tensors of the Dirac eigenspinors. These tensors form the Jacobian matrix of the change of coordinates between metric and eigenvalues. The brackets are quadratic with a kernel given by the propagator of the linearized Einstein equations. The energy-momentum tensors of the Dirac eigenspinors provide the key tool for analyzing the representation of spacetime geometry in terms of Dirac eigenvalues.

In [21] we also study the Chamseddine-Connes spectral action. As given in [11, 8] it is rather unrealistic as a pure gravity action, because of a huge cosmological term implying that geometries for which the action approximates the Einstein-Hilbert action are *not* solutions of the theory. We introduce a minor modification which eliminates the cosmological term. The equations of motion, derived directly from the (modified) spectral action, are solved if the energy momenta of the high mass eigenspinors scale linearly with the mass. This scaling requirement approximates the vacuum Einstein equations. These results suggest that the Chamseddine-Connes gravitational theory can be viewed as a manageable theory possibly with powerful applications to classical and quantum gravity.

[2] In fact, the eigenvalues of the Dirac operator are invariant only under diffeomorphisms which preserve the spin structure[7].

2 Noncommutative Geometry and Gravity

We refer to [20, 23, 27] for friendly introductions to noncommutative geometry. In Connes' program [9], noncommutative C^*-algebras are the dual arena for noncommutative topology. We remind that a C^*-algebra $\mathcal{A}$ is an algebra over the complex numbers $\mathbb{C}$, which is complete with respect to a norm $||\cdot|| : \mathcal{A} \to \mathbb{C}$. Furthermore, there is an involution $^* : \mathcal{A} \to \mathcal{A}$ and these two structures are related by suitable compatibility conditions. The (commutative) Gel'fand-Naimark theorem provides a geometric interpretation for commutative C^*-algebras and concludes that there is a complete equivalence between the category of (locally) compact Hausdorff spaces and the dual category of commutative C^*-algebras (not necessarily with a unit). Any commutative C^*-algebra is realized as the C^*-algebra of complex valued continuous functions over a (locally) compact Hausdorff space, endowed with the *sup* norm. And the points of the space are seen as the maximal ideals (or equivalently, the irreducible representations or the pure states) of the algebra. A noncommutative C^*-algebra will now be thought of as an algebra of operator valued, continuous functions on some 'virtual noncommutative space'. The attention will be switched from spaces, which in general do not even exist 'concretely', to algebras of functions. This fact allows one to treat on the same footing 'continuum' and discrete spaces. It also permits one to address problems associated with spaces of orbits or spaces of foliations or even fractal sets for which the usual notion of space is inadequate.

A metric structure is constructed out of a *real spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ [3]. Now $\mathcal{A}$ is a noncommutative *-algebra (indeed, in general not necessarily a C^*-algebra); $\mathcal{H}$ is a Hilbert space on which $\mathcal{A}$ is realized as an algebra of bounded operators; and D is a self-adjoint unbounded operator on $\mathcal{H}$ with suitable additional properties and which contains all (relevant) 'geometric' information. With any closed n-dimensional Riemannian spin manifold M there is associated a canonical spectral triple. The algebra is $\mathcal{A} = C^\infty(M)$, the algebra of complex valued smooth functions on M. The Hilbert space is $\mathcal{H} = L^2(M, S)$, the Hilbert space of square integrable sections of the irreducible spinor bundle over M, its rank being $2^{[n/2]}$ [4]. The scalar product in $L^2(M, S)$ is the usual one of the measure $d\mu(g)$ associated with the metric g,

$$(\psi, \phi) = \int d\mu(g) \overline{\psi(x)} \phi(x), \tag{1}$$

with bar indicating complex conjugation and scalar product in the spinor space being the natural one in $\mathbb{C}^{2^{[n/2]}}$. Finally, D is the Dirac operator associated with the Levi-Civita connection $\omega = dx^\mu \omega_\mu$ of the metric g. If $(e_a, a = 1, \ldots, n)$ is an orthonormal basis of vector fields which is related to the natural basis $(\partial_\mu, \mu = 1, \ldots, n)$ via the n-beins, with components e^μ_a, the components $\{g^{\mu\nu}\}$ and $\{\eta^{ab}\}$ of the curved and the flat metrics respectively, are related by [5]

$$g^{\mu\nu} = e^\mu_a e^\nu_b \eta^{ab} \ , \quad \eta_{ab} = e^\mu_a e^\nu_b g_{\mu\nu} \ . \tag{2}$$

[3] In fact, when constructing gauge theories one needs a 'quintuple' $(\mathcal{A}, \mathcal{H}, D, \Gamma, J)$, with Γ a grading operator on $\mathcal{H}$ and J a antilinear isometry on $\mathcal{H}$ [10, 11]. We shall not dwell upon these in this paper.

[4] The symbol $[k]$ indicates the integer part in k.

[5] Curved indices $\{\mu\}$ and flat ones $\{a\}$ run from 1 to n and as usual we sum over repeated indices. Curved indices are lowered and raised by the curved metric g, while flat indices are lowered and raised by the flat metric η.

The coefficients $(\omega_{\mu a}{}^{b})$ of the Levi-Civita (metric and torsion-free) connection of the metric g, defined by $\nabla_\mu e_a = \omega_{\mu a}{}^{b} e_b$, are the solutions of the equations

$$\partial_\mu e^a_\nu - \partial_\nu e^a_\mu - \omega_{\mu b}{}^{a} e^b_\nu + \omega_{\nu b}{}^{a} e^b_\mu = 0 \ . \tag{3}$$

Also, let $C(M)$ the the Clifford bundle over M whose fiber at $x \in M$ is the complexified Clifford algebra $Cliff_{\mathbb{C}}(T^*_x M)$ and $\Gamma(M, C(M))$ be the module of corresponding sections. We have an algebra morphism into bounded operators $\mathcal{B}(\mathcal{H})$ on $\mathcal{H}$,

$$\gamma : \Gamma(M, C(M)) \to \mathcal{B}(\mathcal{H}) \ , \tag{4}$$

defined by

$$\gamma(dx^\mu) =: \gamma^\mu(x) = \gamma^a e^\mu_a \ , \quad \mu = 1, \dots, n \ , \tag{5}$$

and extended as an algebra map and by requiring $\mathcal{A}$-linearity. The curved and flat gamma matrices $\{\gamma^\mu(x)\}$ and $\{\gamma^a\}$, which we take to be Hermitian, obey the relations

$$\begin{aligned} &\gamma^\mu(x)\gamma^\nu(x) + \gamma^\nu(x)\gamma^\mu(x) = -2g(dx^\mu, dx^\nu) = -2g^{\mu\nu} \ , \quad \mu, \nu = 1, \dots, n \ ; \\ &\gamma^a\gamma^b + \gamma^b\gamma^a = -2\eta^{ab} \ , \quad a, b = 1, \dots, n \ . \end{aligned} \tag{6}$$

The lift ∇^S of the Levi-Civita connection to the bundle of spinors is then

$$\nabla^S_\mu = \partial_\mu + \omega^S_\mu = \partial_\mu + \frac{1}{4}\omega_{\mu ab}\gamma^a\gamma^b \ , \tag{7}$$

while the Dirac operator, defined by

$$D = \gamma \circ \nabla^S \ , \tag{8}$$

can be written locally as

$$D = \gamma(dx^\mu)\nabla^S_\mu = \gamma^\mu(x)(\partial_\mu + \omega^S_\mu) = \gamma^a e^\mu_a(\partial_\mu + \frac{1}{4}\omega_{\mu ab}\gamma^a\gamma^b) \ . \tag{9}$$

For this canonical triple Connes' construction gives back the usual differential calculus on M together with a metric structure. First of all, exterior forms on M are represented as bounded operators on $L^2(M, S)$. Elements of $C^\infty(M)$ act as multiplicative operators on $\mathcal{H}$ and for any function f it makes sense to consider the commutator $[D, f] = \gamma^\mu \partial_\mu f$, which results into a multiplicative and a fortiori bounded operator, and which realizes the exterior derivative df. From this Connes proceeds to obtain forms of higher degree. In this algebraic framework, the usual geodesic distance between any two points p and q of M is expressed as

$$d(p, q) = \sup_{f \in \mathcal{A}} \{ |f(p) - f(q)| \ : \ ||[D, f]|| \leq 1 \} \ , \tag{10}$$

where the norm $||[D, f]||$ is the operator norm. The formula (10) does not make use of curves on the manifold M. As it stands, for a general triple, it will provide a distance on the state space of the C^*-algebra $\bar{\mathcal{A}}$, the norm closure of the algebra $\mathcal{A}$, once any point $p \in M$ is thought of as a state on the algebra of functions and one writes $p(f)$ for $f(p)$

(remember that a point is the same as a representation of the algebra of functions). In a sense, formula (10) identifies the infinitesimal unit of length as the *bare* Dirac propagator,

$$ds = D^{-1} \ , \tag{11}$$

the ambiguity coming from possible zero modes being inconsequential (one can always add a mass term) [6].

What is more, the Einstein-Hilbert action of general relativity is obtained as the *noncommutative integral* (also known as the Wodzicki residue) of the infinitesimal unit of 'area' $ds^{n-2} = D^{2-n}$ [11, 19],

$$\begin{aligned} Res_W(D^{2-n}) &=: \frac{1}{n(2\pi)^n} \int_{S^*M} tr(\sigma_{-n}(x,\xi)) dx d\xi \\ &= c_n \int_M R dx \ , \\ c_n &= \frac{(2-n)}{12} \frac{2^{[n/2]-n/2}}{(2\pi)^{n/2}} \Gamma(\frac{n}{2}+1)^{-1} \ . \end{aligned} \tag{12}$$

Here,

$$\sigma_{-n}(x,\xi) = \text{part of order } -n \text{ of the total symbol of } D^{2-n} \ , \tag{13}$$

R is the scalar curvature of the metric of M and tr is a normalized Clifford trace. This result follows from the realization that $Res_W(D^{2-n})$ is (proportional) to the integral of the second coefficient of the heat kernel expansion of D^2. Furthermore, the result does not depend upon extra contributions coming from couplings to gauge potentials like $U(1)$ which are present, for instance, in a spin^c structure.

It may be worth noticing that the dimension n itself can be extracted from the operator D as well, the Weyl formula giving $\lambda_k(|D|) \sim k^{1/n}$ for large values of the index k.

3 From the Metric to the Eigenvalues

The idea that the phase space of a physical theory should be identified with the space of solutions of the equations of motion (modulo gauge transformations) can be traced back to Lagrange and has been given a new emphasis in more recent work [22]. In the case of general relativity, gauge transformations are diffeomorphisms of the space(-time) which are connected to the identity. Thus, the phase space Γ of general relativity is the space of the metric fields that solve Einstein equations, modulo diffeomorphisms (Ricci flat geometries). Corresponding observables are functions on Γ [17].

The Dirac operator allows one to define an infinite family of observables. The operator D is a self-adjoint operator on $\mathcal{H}$ admitting a complete set of real eigenvalues λ_n and "eigenspinors" ψ_n. The manifold M being compact, the spectrum is discrete

$$D\psi_n = \lambda_n \ \psi_n \ , \tag{14}$$

[6] In fact, formula (11) shows all its classical character since quantum effects will necessarily dress the bare propagator. That the dressed propagator will produce quantum effects on the geometry is a challenging and fascinating suggestion [12].

The eigenvalues are labeled so that $\lambda_n \leq \lambda_{n+1}$, with repeated multiplicity. Here n is integer (positive and negative) and we choose λ_0 to be the positive eigenvalue closest to zero. As already mentioned, for simplicity we assume that there are no zero modes. The eigenvalues have dimension of an inverse length.

We shall denote the space of smooth metric fields as $\mathcal{M}$ and the space of the orbits of the gauge group in $\mathcal{M}$ as $\mathcal{G}$ (geometries). To stress the dependence upon a metric g of the Dirac operator and of its eigenvalues, we shall also write $D[g]$ and $\lambda_n[g]$. The latter then, define a discrete family of real-valued functions on $\mathcal{M}$, $\lambda_n : g \longmapsto \lambda_n[g]$. Equivalently, we have a function λ from $\mathcal{M}$ into the space of infinite sequences R^∞

$$\lambda : \mathcal{M} \longrightarrow R^\infty \ , \quad g \longmapsto \{\lambda_n[g]\} \ , \tag{15}$$

the image $\lambda(\mathcal{M})$ of $\mathcal{M}$ under this map being contained in the cone $\lambda_n \leq \lambda_{n+1}$ of R^∞. As we shall also see explicitly later on, the functions λ_n are invariant under diffeomorphisms (in fact, the invariance is only under diffeomorphisms which preserve the spin structure; however, only large diffeomorphisms can change the spin structure). Therefore they are well defined functions on $\mathcal{G}$. In particular, they are well defined on the phase space Γ. Thus, they are *observables* of general relativity.

Unfortunately, life is not easy: *we cannot (completely) hear the shape of a drum*, even if it is spinorial, namely the eigenvalues λ_n's need not be a set of coordinates for $\mathcal{G}$ and/or the phase space Γ. Two metric fields with the same collection of eigenvalues $\{\lambda_n\}$ are called *isospectral*. Isometric g fields are isospectral, but the converse needs not be true. There exists Dirac isospectral deformations: continuous 1-parameter family of mutually non-isometric metrics with the same Dirac spectrum [1]. They are of the form $M_s = G/F_s$, $s \in \mathbb{C}$, with G a nilpotent group (e.g. the Heisenberg group) and F_s a nilpotent subgroup. Also, there exist known examples of Laplace-isospectral 4-dimensional flat tori [13] which are also Dirac-isospectral, at least for the trivial spin structure [5]. Not even the topology is determined [4] [7]. Let us recall that a spherical space form is a manifold of the form S^n/F where S^n is the n-dimensional sphere and F is a finite fixed point free subgroup of $SO(n+1)$ (the group of orientation preserving isometries of S^n). Then, it has been proven in [4] that there exists two non-isometric spherical space form of dimension $4d-1$ with d an odd integer greater that 5, having the same Dirac spectrum and the same fundamental group. The smallest example would be in dimension 19! However, from what we understand, all the (counter)-examples constructed so far are very particular and by no means generic. The question of whether in the generic situation, the spectrum of the Dirac operator characterizes the metric is still open.

Before we proceed, let us mention another problem, namely the possibility of spectral flows [3, 5]: the map λ in (15) is only defined up to index shift: there may exist non-contractible loops in Γ such that by following the eigenvalues along the loop they come back with index shifted by some number. A possible way out could be to substitute the target space R^∞ by $R^\infty/(\text{index shift})$; however, the map λ would not be globally (continuous) defined. Locally, in a neighborhood of some geometry, things are fine.

Let us then proceed locally by working out the Jacobian of the transformation from metric to eigenvalues. The variation of λ_n for a variation of g can be computed using stan-

[7]Indeed, it is the interplay between the Dirac operator D and the algebra $\mathcal{A}$ that determines topological/geometric properties.

dard time independent quantum mechanics perturbation theory. For a self-adjoint operator $D(v)$ depending on a parameter v and whose eigenvalues $\lambda_n(v)$ are non-degenerate, we have

$$\frac{d\lambda_n(v)}{dv} = (\psi_n(v)| \left(\frac{d}{dv}D(v)\right) |\psi_n(v)). \tag{16}$$

This equation is well known for its application in elementary quantum mechanics. It can be obtained by varying v in the eigenvalue equation for $D(v)$, taking the scalar product with one of the eigenvectors, and noticing that the terms with the variation of the eigenvectors cancel. We now apply this equation to our situation, assuming generic metrics with non-degenerate eigenvalues (we refer to [7] for the general situation). We wish to compute the variation of $\lambda_n[g]$ for a small variation of the metric field g. Let $k(x) = (k_{\mu\nu}(x))$ be an arbitrarily chosen metric field and v a real parameter, and consider a 1-parameter family of metric fields g_v

$$g_v = g + vk. \tag{17}$$

Then, the variation $\delta\lambda_n[g]/\delta g_{\mu\nu}(x)$ of the eigenvalues under a variation of the metric, is the distribution defined by

$$\int d\mu(g)\ \frac{\delta\lambda_n[g]}{\delta g_{\mu\nu}(x)}\ k_{\mu\nu}(x) = \left.\frac{d\lambda_n[g_v]}{dv}\right|_{v=0} \tag{18}$$

Using (16), we have

$$\frac{d\lambda_n[g_v]}{dv} = (\psi_n[g_v]|\frac{dD[g_v]}{dv}|\psi_n[g_v]). \tag{19}$$

Explicitly

$$\frac{d\lambda_n[g_v]}{dv} = \int d\mu(g_v)\ \bar{\psi}_n[g_v]\ \frac{dD[g_v]}{dv}\ \psi_n[g_v]. \tag{20}$$

In $v = 0$ we have

$$\left.\frac{d\lambda_n[g_v]}{dv}\right|_{v=0} = \int d\mu(g)\ \bar{\psi}_n[g]\ \left.\frac{dD[g_v]}{dv}\right|_{v=0}\ \psi_n[g]. \tag{21}$$

We can rewrite this equation as

$$\begin{aligned}
\left.\frac{d\lambda_n[g_v]}{dv}\right|_{v=0} &= \left.\frac{d}{dv}\right|_{v=0} \int d\mu(g_v)\ \bar{\psi}_n[g]\ D[g_v]\ \psi_n[g] \\
&\qquad - \int \left.\frac{d}{dv}(d\mu(g_v))\right|_{v=0}\ \bar{\psi}_n[g]\ D[g]\ \psi_n[g] \\
&= \left.\frac{d}{dv}\right|_{v=0} \int d\mu(g_v)\ \bar{\psi}_n[g]\ D[g_v]\ \psi_n[g] \\
&\qquad - \int \left.\frac{d}{dv}(d\mu(g_v))\right|_{v=0}\ \bar{\psi}_n[g]\ \lambda_n[g]\psi_n[g] \\
&= \left.\frac{d}{dv}\right|_{v=0} \int d\mu(g_v)\ (\bar{\psi}_n D[g_v]\psi_n - \lambda_n\bar{\psi}_n\psi_n).
\end{aligned} \tag{22}$$

The last formula gives the variation of the action of a spinor field with 'mass' λ_n under a variation of the metric, (computed for the n-th eigenspinor of the operator $D[g]$). But the variation of the action under a variation of the metric is a well known quantity: it provides the general definition of the energy momentum tensor $T^{\mu\nu}(x)$. Indeed, the Dirac energy-momentum tensor is defined in general by

$$T^{\mu\nu}(x) =: \frac{\delta}{\delta g_{\mu\nu}(x)} S_{\text{Dirac}}, \tag{23}$$

where $S_{\text{Dirac}} = \int d\mu(g)\ (\bar{\psi}D\psi - \lambda\bar{\psi}\psi)$ is the Dirac action of a spinor with "mass" λ. (Since there is no Planck constant in the Dirac action, λ has dimensions of an inverse length, rather than of a mass.) See for instance [15], where the explicit form of this tensor is also given. By denoting the energy momentum tensor of the eigenspinor ψ_n as $T_n{}^{\mu\nu}(x)$, we obtain, from (18), (22) and (23), that

$$\frac{\delta\lambda_n[g]}{\delta g_{\mu\nu}(x)} = T_n{}^{\mu\nu}(x). \tag{24}$$

This equation gives the variation of the eigenvalues λ_n under a variation of the metric $g_{\mu\nu}(x)$, namely the Jacobian matrix of the map λ in (15). The matrix elements of this Jacobian are given by the energy momentum tensor of the Dirac eigenspinors. This fact suggests that we can study the map λ locally in the space of the metrics, by studying the space of the eigenspinor's energy-momenta. As far as we know, little is known on the topology of the space of solutions of Euclidean Einstein's equations on a compact manifold. A local analysis on Γ would of course miss information on disconnected components of Γ.

It is now easy to prove that the eigenvalues λ_n are invariant under the action of diffeomorphisms in the connected component of the identity in the sense that their variation vanishes when we vary the metric g by the action of any such a diffeomorphism. If ξ is a vector field on M, the variation of the metric under the action of the infinitesimal diffeomorphism generated by ξ is given by

$$(\delta_\xi g)_{\mu\nu} = (\mathcal{L}_\xi g)_{\mu\nu} = 2\xi_{(\mu;\nu)}\ . \tag{25}$$

Here $\mathcal{L}$ denotes Lie derivative, the semicolon denotes covariant derivative with respect to the Levi-Civita connection and the round brackets denote symmetrization. Then, by using (24) and integrating by parts, we get

$$\delta_\xi\lambda_n = \int d\mu(g)\ \frac{\delta\lambda_n[g]}{\delta g_{\mu\nu}}\ (\delta_\xi g)_{\mu\nu} = 2\int d\mu(g)\ T_n{}^{\mu\nu}\xi_{(\mu;\nu)} = -2\int d\mu(g)\ T_n{}^{\mu\nu}{}_{;\nu}\xi_\mu \tag{26}$$

and this expression vanishes by the very 'equation of motion' for the spinor field ψ_n, $(D\psi_n - \lambda_n\psi_n) = 0$, which just state that ψ_n is an eigenspinor with eigenvalue λ_n.

It is worth stressing that the quantities λ_n are not invariant under *arbitrary* changes of the metric fields, i.e. the left hand side of (24) does not vanish in general.

Finally, we mention that the above derivations would go through for several other operators, beside the Dirac operator. In [24] a formula similar to (24) has been derived for any second order elliptic self-adjoint operator.

4 Action and Field Equations

We now turn to the gravitational sector of the spectral action introduced in [11, 8]. This action contains a cutoff parameter l_0 with units of a length, which determines the scale at which the defined gravitational theory departs from general relativity. We may assume that l_0 is the Planck length $l_0 \sim 10^{-33} cm$ (although we make no reference to quantum phenomena in the present context). We use also $m_0 = 1/l_0$, which has the same dimension as D and the eigenvalues λ_n. The action depends also on a dimensionless cutoff function $\chi(u)$, which vanishes for large u. The spectral action is then defined as

$$S_G[D] = \kappa \, Tr \left[\chi(l_0^2 \, D^2) \right] . \tag{27}$$

Here κ is a multiplicative constant to be chosen to recover the right dimensions of the action and the multiplicative overall factor.

To be definite, we shall work in dimension 4, although much of what follows can be easily generalized. The action (27) approximates the Einstein-Hilbert action with a large cosmological term for "slowly varying" metrics with small curvature (with respect to the scale l_0). Indeed, the heat kernel expansion [8, 16], allows to write,

$$S_G(D) = (l_0)^{-4} f_0 \kappa \int_M \sqrt{g} \, dx \; + \; (l_0)^{-2} f_2 \kappa \int_M R \, \sqrt{g} \, dx \; + \; \dots \; . \tag{28}$$

The momenta f_0 and f_2 of the function χ are defined by

$$f_0 = \frac{1}{4\pi^2} \int_0^\infty \chi(u) u du \; , \qquad f_2 = \frac{1}{48\pi^2} \int_0^\infty \chi(u) du \; . \tag{29}$$

The other terms in the expansion (28) are of higher order in l_0.

The expansion (28) shows that the action (27) is dominated by the Einstein-Hilbert action with a Planck-scale cosmological term. The presence of this term is a problem for the physical interpretation of the theory because the solutions of the equations of motions would have Planck-scale Ricci scalar, and therefore they would *all* be out of the regime for which the approximation taken is valid! However, the cosmological term can be cancelled by replacing the function χ with $\widetilde{\chi}$ defined by,

$$\widetilde{\chi}(u) = \chi(u) - \epsilon^2 \chi(\epsilon u) \; , \tag{30}$$

with $\epsilon << 1$. Indeed, one finds for the new momenta $\widetilde{f}_0 = 0$, $\widetilde{f}_2 = (1-\epsilon) f_2$. The modified action becomes

$$\widetilde{S}_G(D) = \frac{\widetilde{f}_2 \kappa}{l_0^2} \int_M R \, \sqrt{g} \, dx \; + \; \dots \; . \tag{31}$$

We obtain the Einstein-Hilbert action in dimension four by fixing

$$\kappa = \frac{l_0^2}{16\pi G \widetilde{f}_2} \; . \tag{32}$$

If l_0 is the Planck length $\sqrt{\hbar G}$, then $\kappa = \frac{3}{2} h$, where h is the Planck constant, up to terms of order ϵ. Low curvature geometries, for which the expansion (28) holds *are now* solutions of the theory. Thus we obtain a theory that genuinely approximates pure general relativity at scales which are large compared to l_0.

Next, let us consider the equations of motion derived from the previous action when we regard the λ_n's as the gravitational variables. The action can easily be expressed in terms of these variables:

$$\widetilde{S}_G[\lambda] = \kappa \sum_n \; \widetilde{\chi}(l_0^2 \lambda_n^2). \tag{33}$$

However, we cannot obtain (approximate) Einstein equations by simply varying (33) with respect to the λ_n's. We must minimize (33) on the surface $\lambda(\mathcal{M})$, not on the entire R^∞. In other words, the λ_n's are not independent variables, there are relations among them and these relations among them code the complexity of general relativity. We can still obtain the equations of motion by varying $\widetilde{S}_G$ with respect to the metric field:

$$0 = \frac{\delta \widetilde{S}_G}{\delta g_{\mu\nu}} = \sum_n \frac{\partial \widetilde{S}_G}{\partial \lambda_n} \frac{\delta \lambda_n}{\delta g_{\mu\nu}} = \sum_n \frac{d\widetilde{\chi}(l_0^2 \lambda_n^2)}{d\lambda_n} \; T_{n\,I}^{\;\mu}. \tag{34}$$

By defining $f(u) =: \frac{d}{du}\widetilde{\chi}(u)$, equation (34) becomes

$$\sum_n f(l_0^2 \lambda_n^2) \; \lambda_n \; T_{n\,I}^{\;\mu} = 0. \tag{35}$$

These are the Einstein equations in the Dirac eigenvalues formalism.

Up to now, the cutoff function $\chi(u)$ is arbitrary. The simplest choice is to take it to be smooth and monotonic on R^+ with

$$\chi(u) = \begin{cases} 1 & \text{if } u < 1 - \delta \\ 0 & \text{if } u > 1 + \delta \end{cases} \tag{36}$$

where $\delta << 1$. Namely $\chi(u)$ is the smoothed-out characteristic function of the interval $[0, 1]$. With this choice, the action (27) is essentially (κ times) the *number* of eigenvalues λ_n with absolute value smaller that m_0! (up to corrections of order δ). Then the function $f(u)$ vanishes everywhere except on two narrow peaks. A negative one (width 2δ and height $1/2\delta$) centered at one; and a positive one (width $2\delta/\epsilon$ and height $\epsilon^3/2\delta$) around the arbitrary large number $1/\epsilon =: s >> 1$. The first of these peaks gets contributions from λ_n's such that $\lambda_n \sim m_0$, namely from Planck scale eigenvalues. The second from ones such that $\lambda_n \sim sm_0$. Equations (35) are solved if the contributions of the two peaks cancel. This happens if below the Planck scale the energy momentum tensor scales as

$$\lambda_{n(m_0)}\rho(1) \; T_{n(m_0)}{}_I^{\mu}(x) = s^{-2}\lambda_{n(sm_0)}\rho(s) \; T_{n(sm_0)}{}_I^{\mu}(x), \tag{37}$$

Here $\rho(1)$ and $\rho(s)$ are the densities of eigenvalues of $l_0^2 D^2$ at the two peaks and the index $n(t)$ is defined by

$$l_0 \lambda^2_{n(t)} = t. \tag{38}$$

For large n the growth of the eigenvalues of the Dirac operator is given by the Weyl formula $\lambda_n \sim \sqrt{2\pi} V^{-1/4} n^{1/4}$, where V is the volume. Using this, one derives immediately the eigenvalue densities, and simple algebra yields

$$T_{n\,I}^{\;\mu}(x) = \lambda_n \; l_0 \; T_{0\,I}^{\;\mu}(x) \; . \tag{39}$$

for $n >> n(m_P)$, where $T_{0\,I}^{\;\mu}(x) = T_{n(m_0)}{}_I^{\mu}(x)$ is the energy momentum at the Planck scale. We have shown that *the dynamical equations for the geometry are solved if below the Planck length the energy-momentum of the eigenspinors scales as the eigenspinor's mass.* In other words, we have expressed the Einstein equations as a scaling requirement on the energy-momenta of the very-high-frequency Dirac eigenspinors. This scaling requirement yields vacuum Einstein equations at low energy scale [21].

5 Poisson Brackets for the Eigenvalues

A simplectic structure on the phase space Γ can be constructed in covariant form [2]. First of all, we recall that a vector field X on the space $\mathcal{S}$ of solutions of Einstein field equations can be written as a differential operator

$$X = \int d^4x \; X_{\mu\nu}(x)[g] \;\; \frac{\delta}{\delta g_{\mu\nu}(x)} \tag{40}$$

where $X_{\mu\nu}(x)[g]$ is any solution of the Einstein equations for the metric field, *linearized* over the background g. A vector field $[X]$ on Γ is given by an equivalence class of such vector fields X, modulo linearized gauge transformations of $X_{\mu\nu}(x)$. A linearized gauge transformation is given by

$$g \longmapsto g + \delta_\xi g = \mathcal{L}_\xi g, \tag{41}$$

where ξ is a vector field on the spacetime M (generating an infinitesimal diffeomorphism). Two linearized field X and Y (around the metric g) are gauge equivalent if

$$Y = X + \delta_\xi g, \tag{42}$$

for some vector field ξ.

The simplectic two-form Ω of general relativity is given by [2]

$$\Omega(X,Y) = \int_\Sigma d^3\sigma \; n_\rho \; (X_{\mu\alpha} \overset{\leftrightarrows}{\nabla}_\tau \; Y_{\nu\beta}) \; \epsilon^{\tau\alpha\beta}{}_{\upsilon} \, \epsilon^{\upsilon\rho\mu\nu} \tag{43}$$

where

$$(X_{\mu\alpha} \overset{\leftrightarrows}{\nabla}_\tau \; Y_{\nu\beta}) =: (X_{\mu\alpha} \; \nabla_\tau \; Y_{\nu\beta} - Y_{\mu\alpha} \; \nabla_\tau \; X_{\nu\beta}) \; . \tag{44}$$

Moreover, $\Sigma \ni \sigma \longmapsto x(\sigma) \in M$ is chosen to be a (compact non-contractible) three-dimensional surface, such that, topologically, $M = \Sigma \times S^1$ (so that it gives a non trivial 3-cycle of M), but otherwise arbitrary, and n_ρ its normal one-form.

Both sides of (43) are functions of the metric g, namely scalar functions on $\mathcal{S}$. The form Ω is degenerate precisely in the gauge directions, thus it defines a non-degenerate *simplectic* two form on the space of the orbits of the diffeomorphism group, namely on Γ. The coefficients of Ω form can be written as

$$\Omega^{\mu\nu;\alpha\beta}(x,y) = \int_\Sigma d^3\sigma \; n_\rho \; [\delta(x,x(\sigma)) \overset{\leftrightarrows}{\nabla}_\tau \delta(y,x(\sigma))] \; \epsilon^{\tau\alpha\beta}{}_{\upsilon} \, \epsilon^{\upsilon\rho\mu\nu}. \tag{45}$$

Because of the degeneracy, Ω has no inverse on $\mathcal{S}$. However, let us fix a gauge (choose a representative field g for any four geometry, and, consequently, choose a field X in any equivalence class $[X]$). On the space of the gauge fixed fields, Ω is non degenerate and we can invert it. Let $P_{\mu\nu;\alpha\beta}(x,y)$ be the inverse of the simplectic form matrix on this subspace, namely

$$\int d^4y \int d^4z \; P_{\mu\nu;\alpha\beta}(x,y) \; \Omega^{\nu\rho;\beta\gamma}(y,z) \; F_{\rho\gamma}(z) = \int d^4z \; \delta(x,z) \; \delta^\rho_\mu \; \delta^\gamma_\alpha \; F_{\rho\gamma}(z) \tag{46}$$

for all solutions F of the linearized Einstein equations, satisfying the gauge condition chosen. Integrating over the delta functions, and using (45), we have

$$\int_\Sigma d^3\sigma \; n_\rho \; [P_{\mu\nu;\alpha\beta}(x,x(\sigma)) \overset{\leftrightarrows}{\nabla}_\rho F_{\tau\gamma}(x(\sigma))] \; \epsilon^{\rho\beta\gamma}{}_{\upsilon} \, \epsilon^{\upsilon\nu\tau\sigma} = F_{\mu\alpha}(x). \tag{47}$$

This equation, where F is any solution of the linearized equations, defines P, in the chosen gauge. Then, we can write the Poisson bracket between two functions f, g on $\mathcal{S}$ as

$$\{f,g\} = \int d^4x \int d^4y \;\; P_{\mu\nu;\sigma\tau}(x,y) \; \frac{\delta f}{\delta g_{\mu\sigma}(x)} \; \frac{\delta g}{\delta g_{\nu\tau}(y)}. \tag{48}$$

If the functions f and g are gauge invariant, i.e. are well defined on Γ, the r.h.s of (48) is independent of the gauge chosen. But equation (47) is precisely the definition of the propagator of the linearized Einstein equations over the background g, in the chosen gauge.

By combining (47,48) and (24) we obtain the Poisson brackets for any two eigenvalues of the Dirac operator as,

$$\{\lambda_n, \lambda_m\} = \int d^4x \int d^4y \; T_{[n}{}^{\mu\alpha}(x) \; P_{\mu\nu;\alpha\beta}(x,y) \; T_{m]}{}^{\nu\beta}(y) \; . \tag{49}$$

This equation gives the Poisson bracket of two eigenvalues in terms of the energy-momentum tensor of the two corresponding eigenspinors and of the propagator of the linearized Einstein equations. The right hand side does not depend on the gauge chosen.

6 Final Remarks

Recent work of Connes and Chamseddine on a spectral description of fundamental interactions and in particular of gravity, has suggested our attempt to describe gravity by means of the eigenvalues of the Dirac operator. This approach could open new paths in the exploration of the physics of spacetime and find applications in classical and quantum gravitation. The main obstacle for a full development of this approach is its natural euclidean character since, at the moment, there does not exist a satisfactory 'Lorentzian' version of Connes' program. Some interesting steps in the direction of a 'quantum spectral approach' have also been recently presented [12, 25].

We have analyzed some aspects of the dynamical structure of the theory in the λ_n variables by computing their Poisson algebra (49). In the way it is presented, the Poisson algebra is not in closed form, since the right hand side of equation (49) is not expressed in terms of the λ_n themselves, and it is unclear if this can be done in general. Still, representations of this algebra could give information on a diffeomorphism invariant quantum theory.

The central and important feature of the approach that we have presented is that the theory is formulated in terms of diffeomorphism invariant quantities. The λ_n's are a family of diffeomorphism invariant observables in euclidean general relativity, which is presumably complete or "almost complete" : it would fail to distinguish possible isospectral and not isometric geometries, although at the moment it is not clear what is the generic situation. Another remarkable aspect of the spectral approach is that there is a physical cutoff and an elementary physical length in the action that does not break diffeomorphism invariance. All high frequency modes are cuts off without introducing background structures, then in a diffeomorphic invariant manner. Since the number of the remaining modes is determined by the ratio of the spacetime volume to the Planck scale, one may expect that such a theory would have infrared divergences but not ultraviolet ones in the quantum regime.

The key open problem is, of course, a better (complete) understanding of the map λ given in (15) and its range. Namely a characterization of the constraints that a sequence of real numbers λ_n must satisfy, in order to represents the spectrum of the Dirac operator of some geometry. We have partially addressed this problem locally in the phase space of the theory by studying the *tangent* map to λ. This tangent map is given explicitly in terms of the eigenspinor's energy-momenta and of the propagator of the linearized Einstein equation. The constraints on the λ_n's are the core of the formulation of the gravitational theory that we have begun to explore here. They should be contained in Connes' axioms for D in its axiomatic definition of a spectral triple [9]. The equations in these axioms capture the notion of Riemannian manifold algebraically and they should code the constraints satisfied by the λ_n.

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